A DOZEN OF KNOT THEORETICAL PROBLEMS

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Abstract. This is a preliminary, partially commented list of problems, compiled from various papers (of myself and others). I concentrated on the ones which are easiest to formulate, and thus (expectedly) of greatest interest. I omit the bibliographic references for simplicity. Any comments are welcome!

Problem 1 *Does the Jones polynomial V admit only finitely many values of given span? What about the Q polynomial or the 2 variable polynomials (when fixing the span in both variables)?*

It is true for the skein polynomial when bounding the canonical genus (for which the Alexander degree of the skein polynomial is a lower bound by Morton), in particular it is true for the skein polynomial of homogeneous links. It is true for the Jones, Q and Kauffman F polynomial of alternating links (for F more generally for adequate links). One cannot bound the number of different links, at least for the skein and Jones polynomial, because of Kanenobu's series.

Problem 2 *Why are the unit norm complex numbers α for which the value $Q(\alpha)$ has maximal norm statistically concentrated around $e^{11\pi i/25}$?*

This was revealed by an experiment in an attempt to estimate the asymptotical growth of the coefficients of the Q polynomial. There seems no difference in the behaviour of alternating and non-alternating knots.

Problem 3 *Do positive links of given signature σ have bounded (below) maximal Euler characteristic χ?*

So far for general positive links only $\sigma > 0$ is known, and for positive knots $\sigma \geq 4$ if $2g = 1-\chi \geq 4$. For positive braid links the answer is positive, and also for special alternating links by Murasugi.

Problem 4 *(with Mark Kidwell) Let K be a non-trivial knot, let W_K be a Whitehead double of K, $P(l,m)$ the HOMFLY polynomial and m the Alexander variable of P, and $F(a,z)$ the Kauffman polynomial with Alexander variable z. Is then $\deg_m P(W_K) = 2\deg_z F(K)+2$?*

It is true for K up to 11 crossings. $\deg_m P(W_K)$ is independent on the twist of W_K if it is > 2 by a simple skein argument.

Problem 5 *(with E. Ferrand) Is for any alternating link L,*

$$\sigma(L) \geq \min\deg_l(P_L) \geq \min\deg_a(F_L)\,?$$

The second inequality is conjectured by Ferrand (see also comment on problem 12), and related to estimates of the Bennequin numbers of Legendrian knots. As for the first inequality, by Cromwell we have $\min\deg_l(P_L) \leq 1-\chi$ and classically $\sigma \leq 1-\chi$.

Problem 6 *(following N. Askitas) Can a knot of 4-genus g_s always be sliced (made into a slice knot) by g_s crossing switches?*

Clearly (at least) g_s crossing switches are needed, but sometimes more are needed to *unknot* the knot.

Problem 7 *If a prime knot K can be transformed into its mirror image by one crossing change, is K achiral or (algebraically?) slice?*

Smoothing out this crossing gives a link of zero signature(s) and zero Alexander polynomial. Many such links are slice, and then K would be slice also. But unlikely.

Problem 8 *Let n be an odd natural number, different from 1, 9, and 49, such that n is the sum of two squares. Is there a prime alternating achiral knot of determinant n?*

If there is an achiral knot of determinant n, then n is the odd sum of two squares. The converse is also true, and the achiral knot of determinant n can be chosen to be alternating *or* prime, but *not* always both. For $n = 1$, 9, and 49, there is no prime alternating achiral knot of determinant n. If there is another such n, then $n > 2000$ and n is not a square.

Problem 9 *(following Y. Shinohara) If $n = 4k+1$ with $k > 0$, is there a knot with determinant n and signature 4?*

The form $4k+1$ follows from Murasugi, and the condition $k \neq 0$ from a signature theorem for even unimodular quadratic forms over $\mathbb{Z}$. If a counterexample for $n > 1$ exists, then all prime divisors of n are of the form $24k+1$ and not smaller than 2857. If $\sigma_{4+8l,8l+5}$ is the elementary symmetric polynomial of degree $4+8l$ in $8l+5$ variables, then all values of $\sigma_{4+8l,8l+5}$ on *positive odd* arguments are no counterexamples, so the problem could "reduce" to showing that some of the $\sigma_{4+8l,8l+5}$ realizes almost all n on positive odd arguments. This appears number theoretically hard, however.

Problem 10 *(Boileau) Are there mutants of distinct unknotting numbers?*

There are mutants of distinct genera (Gabai) and slice genera (Livingston).

Problem 11 *If ∇_k is the coefficient of z^k in the Conway polynomial and $c(L)$ is the crossing number of a link L, is then*

$$\left|\nabla_k(L)\right| \leq \frac{c(L)^k}{2^k k!}\,?$$

The inequality is non-trivial only for L of $k+1, k-1, \dots$ components. It is also trivial for $k = 0$, easy for $k = 1$ (∇_1 is just the linking number of 2 component links) and proved by Polyak-Viro for *knots* and $k = 2$. There are constants C_k with

$$\left|\nabla_k(L)\right| \leq C_k\, c(L)^k\,,$$

following from the proof of the Lin-Wang conjecture for links, but determining C_k from the proof is difficult. Can the inequality be proved by Kontsevich-Drinfel'd, say at least for knots, using the description of the weight systems of ∇ of Bar-Natan and Garoufalidis? More specifically, one can ask whether the $(2,n)$-torus links (with parallel orientation) attain the maximal values of ∇_k. One can also ask about the shape of C_k for other families of Vassiliev invariants, like $V^{(k)}(1)$.

Problem 12 *Does for any link L hold* $\min\deg_a(F_L) \leq 1-\chi(L)$*? If* $u(K)$ *is the unknotting number of a knot K, does for any knot K hold* $\min\deg_a(F_K) \leq 2u(K)$*?*

For the common lower bound of $2u$ and $1-\chi$ for knots, $2g_s$, there is a 15 crossing knot with $2g_s < \min\deg_a(F)$. Morton conjectured long ago that $1-\chi(L) \geq \min\deg_l(P)$. There are recent counterexamples, but only of 19 to 21 crossings. Ferrand observed that very often $\min\deg_l(P) \geq \min\deg_a(F)$ (he conjectures it in particular always to hold for alternating K), so replacing '$\min\deg_a(F)$' for '$\min\deg_l(P)$' enhances the difficulty of Morton's problem (the counterexamples are no longer such).