ON DUAL TRIANGULATIONS OF SURFACES, LIE ALGEBRA INVARIANTS, AND ALTERNATING LINK VOLUMES

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Abstract. A famous theorem of Whitney asserts that spherical embeddings of any 2-connected 3-valent graph are interconvertible by flips. It is applied to characterize alternating and positive links with planar Seifert surfaces. Then we show that cellular embeddings of any 2-connected 3-valent planar graph G on any orientable compact surface $S \neq S^2$ are counterexamples to Whitney's theorem. Using this, we study special types of volume-maximizing sequences among links of given canonical Euler characteristic. We describe their maximal volume in terms of links associated to planar 3-connected 3-valent graphs. We investigate the relation between the volume of such links and the sl_N weight system of their graphs, coming from the theory of Vassiliev invariants.

Keywords: hyperbolic volume, triangulation, planar embedding, graph, link, Seifert surface, weight system

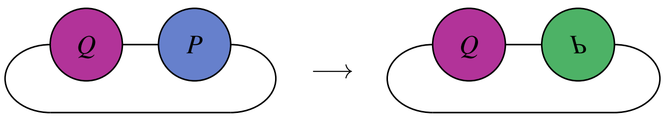
2020 AMS subject classification: 57K10, 05C10 (primary), 57Q15, 57M50, 17B10, 05-08, 57-04 (secondary)

Contents

1. Introduction and results

1.1. Graph embeddings

Whitney's theorem [W] gives a set of moves that interconvert any two planar embeddings of a graph G. We call a graph G *n-connected* if no deletion of at most $n-1$ edges disconnects G. (Thus G is connected iff it is 1-connected, and n'-connected implies n-connected for $n' > n$.) We will nearly exhaustively deal with 3-valent graphs G. For 2-connected 3-valent graphs G Whitney's moves reduce to a *flip*, which turns around one of the components of the complement of a 2-cut of G.



(1)

For a graph G, we say that $\chi(G)$ is the *Euler characteristic* of G. For a closed compact orientable surface S, we write $g(S)$ for the *genus* of S. We assume S' is such a surface which is not a sphere (so $g(S') > 0$).

We say that a trivalent graph G is *cellularly embedded* on S if G is the 1-skeleton of a cellular decomposition where each closed 2-cell is homeomorphic to a 2-disk, and *cellularly embeddable* if this type of embedding exists. The dual of such a cellular decomposition is a triangulation where each triangle is homeomorphic to a classical triangle (embedded on S). Note that the fewer 2-cells are added, the higher the genus of S. (Also, there is a parity condition on the number of such 2-cells.) We will be later in particular concerned with such embeddings where $S \setminus G$ is a single 2-cell (or the dual triangulation has one vertex; see [BV]) – and $g(S)$ is maximal. (Further terminology is clarified below.)

In this paper we show that most such graphs provide counterexamples to Whitney's theorem for embeddings on any surface of higher genus, even when restricted to a natural subclass of such embeddings, and for a potentially much larger set of moves.

Theorem 1.1 Fix $\chi < -1$. Consider a 3-valent 2-connected planar graph G with $\chi(G) = \chi$. Assume G is cellularly embeddable on a compact oriented surface S', that is, such that G is the 1-skeleton of a cellular decomposition, the dual of a triangulation T of S'. Then for any g with $1 \le g \le g(S')$, the graph G has a pair (p_1, p_2) of embeddings on a closed compact oriented surface S of genus $g = g(S)$ with the following properties:

1) $p_i(G)$ are cellular embeddings with n 2-cells (where $n + \chi = 2 - 2g$).

2) Let O_i be the vertex orientation induced from the embeddings p_i of G. Then the number of vertices v in G with $O_1(v) = O_2(v)$ is odd.

Note that since contraction and decontraction does not change essentially the planar embedding, trivalency is not a strong restriction, as far as embeddability is concerned. However, for higher valence graphs there is a problem to define vertex orientation. Similarly one can lift the 2-connectedness, but then such embeddings exist already on a sphere, and the statement is not interesting. Since for given S and general G there may be few (possibly no or a single) embedding(s), certainly some condition must be imposed on G, which in our case is planarity. On the opposite side, we must assume that an initial cellular embedding on S' exists, since even 3-connected planar 3-valent (cubic) graphs may not have one for $2g(S') \le 1 - \chi(G)$, which is an obvious homological restriction (see proposition 6.1). Most graphs G have, though, such an embedding. Its existence can be checked for each G, but a general explicit criterion to decide it is not easy to find.

On the opposite side, the theorem says that not only flips, but any set of orientation-preserving moves is insufficient to interconvert different embeddings, even among cellular ones. Without latter property, the theorem follows immediately from its (much easier to prove) special case for the torus.

A direct consequence of Whitney's theorem is that 3-connected planar graphs are uniquely planarly embeddable (as they don't admit flips). Negami [Ne] extended this result to the torus, and then in [Ne2] to other surfaces, by determining the minimal connectivity of graphs needed to ensure unique embeddability. Our result has, however, little to do with Negami's. It goes in the opposite direction and can at best be understood vaguely related to his construction of non-uniquely embeddable graphs of smaller connectivity. (In the special case of the torus this was done also in [Lv].) We also apply very different tools.

1.2. Hyperbolic volume

The reason why theorem 1.1 was of interest to us has in fact *a priori* nothing to do with graph embeddings. Theorem 1.1 is related to the enumeration [St] and maximal hyperbolic volume [Br] of alternating links of given number of components and genus.

We use our previous work [SV, St8] to study the maximal hyperbolic volume v_χ of alternating links of given Euler characteristic (or arbitrary links of given canonical Euler characteristic) χ. For knots, we related this to certain algebraic objects named Wicks forms [Wi, CE, Cu, BV] and the construction of such forms in [BV, V]. We will specify these in §3.3.

We give a characterization of v_χ (theorem 4.1) in terms of volumes of links L_G assigned to trivalent (or cubic) 3-connected planar graphs G, similar to links considered recently by v.d. Veen [vdV]. A detailed study of the complement decomposition of these links is used to obtain good upper and lower bounds on the maximal volume (corollaries 4.1 and 4.3), and facilitate its practical calculation (see the proof of proposition 4.3).

In particular we can determine the asymptotic growth of v_χ when $\chi \to -\infty$. (See §4.3 for the proof.)

Theorem 1.2 With the values (22) and (23), there exists the 'stable volume-χ ratio'

$$\delta = \lim_{\chi \to -\infty} \frac{v_\chi}{(-2-6\chi)V_8} = \sup_{\chi < 0} \frac{v_\chi}{(-2-6\chi)V_8}\,, \qquad (2)$$

and

$$\delta = \frac{2}{3} + \frac{5}{3}\frac{V_4}{V_8} \approx 1.12836\,. \tag{3}$$

An important issue that has to be dealt with thereby is: for which graph G do links of what number of components occur? This returns us to the embeddability problems in the main theorem. The property G to be dual to a 1-vertex-triangulation (or to admit a knot marking, as we will paraphrase it) can be decided by an exact recursive description from [BV]. Such a description then easily follows also for 2 vertices (see theorem 6.1). But this criterion is not helpful in practice. Thus we are led to consider an explicit property, cyclic 4-connectedness (definition 4.3), which was conjectured to ascertain maximal genus embeddability (or minimal markings). This conjecture was later proved in [St10] (see theorem 6.2 below). Its consequence is that the maximal alternating link volume $v_{n,\chi}$ for given Euler characteristic χ does not depend on the number n of link components.

Theorem 1.3 We have $v_{n,\chi} = v_\chi$, for any $1 \le n \le 2-\chi$ with $n+\chi$ even. I.e., the maximal volume does not depend on the number of components.

In table 2, we see that we can compute the maximal volume for, say, (alternating) knots up to genus 10 with a manageable overhead. From our graph theoretic setting we also obtain a statement concerning the pairs (n,χ) for which the maximal hyperbolic volume $v_\chi = v_{n,\chi}$ is attained by links of both crossing number parities (theorem 4.4).

Our approach here allows us to classify all pairs (n,χ) for which the number of non-split alternating links of n components and Euler characteristic χ of even and odd number of crossings are asymptotically equivalent (theorem 7.1). The main exception occurs if $n = 2-\chi$, for which an exact description of such links is possible (see remark 7.1 and corollary 7.2) and shows that they all have even crossing number.

1.3. The sl_N polynomial

The embeddability problems are also closely related to the sl_N weight system polynomial $W_N(G)$, and more precisely its calculation using thickened surfaces (reviewed in §3.2), which we reformulate in terms of marked graphs. The sl_N weight system occurs in the theory of Vassiliev invariants [BN] and assigns to a trivalent graph G a polynomial $W_N(G)$ in N. It was used in the noteworthy work of Bar-Natan on the Four color theorem [BN2], but seems otherwise little studied. Our main result (theorem 1.1) means that in the thickened surface calculation of $W_N(G)$, for planar G cancellations occur in all degrees in which terms occur, except the maximal one (considered by Bar-Natan in [BN2]). Bar-Natan's work provided some of the motivation for theorem 1.1.

Albeit we establish a natural link from our approach to both the weight system and hyperbolic volume, both theoretical and experimental evidence (discussed in detail in §5.2) mounts that a much more direct relation may exist. In a way, the hyperbolic volume of the 3-valent graph G (in a sense analogous to that in [vdV]; see §4.3) behaves like a certain "logarithm" of, and in particular is at least very closely determined by, G's weight system polynomial (question 5.1). Further work is outlined in §5.4.

Then, we include (§9) a short description of the compilation of maximal knot generators of genus up to 6.

A brief treatment of non-orientable surfaces (§10) from the point of view of our main result concludes the work.

2. General definitions and preliminaries

We begin with introducing many notations and recalling previous results that will be used throughout the paper. Most of these notations and results are well-known, but some are more specific, and build on our own previous work. They are given in the next section.

2.1. Polynomials

Let $[Y]_{t^a} = [Y]_a$ be the *coefficient* of t^a in a polynomial $Y \in \mathbb{Z}[t^{\pm 1}]$. Let for $Y \in \mathbb{Z}[t^{\pm 1}]$

$$\min\deg Y = \min\{\, a \in \mathbb{Z} : [Y]_a \neq 0 \,\}, \quad \max\deg Y = \max\{\, a \in \mathbb{Z} : [Y]_a \neq 0 \,\}, \quad \operatorname{span} Y = \max\deg Y - \min\deg Y$$

be the *minimal* and *maximal degree* and *span* (or breadth) of Y, respectively. Finally, define the *leading coefficient* of Y to be

$$\max\operatorname{cf}_x Y := [Y]_{x^{\max\deg Y}},$$

and similarly $\min\operatorname{cf}_x Y$.

2.2. Miscellanea

The expressions $\#S$ or $|S|$ denote the cardinality of a (finite) set S. (The former should not confused with the *binary* operation on graphs of definition 4.1 later.) For any $S \subset \mathbb{R}$, we denote by $\sup S$ the supremum of S, with the natural convention that $\sup \varnothing = -\infty$. Let

$$\chi\%2 = 2 \text{ for } \chi \text{ even and } \chi\%2 = 1 \text{ if } \chi \text{ is odd.}$$

We use '∂' to indicate boundary.

2.3. Properties of knot and link diagrams

Knot and link diagrams are generally assumed to be oriented. Orientation (in particular for link diagrams) is essential; only at very few places like table 4 (which addresses knots) has orientation been ignored. Further instances where orientation is not specified (and not needed) are the links L_G in (20) and L'_G in (39).

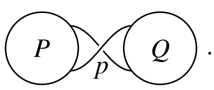
Crossings in diagrams can be positive, negative or smoothed out, as depicted in the following figure.

positive negative smoothed out (4)

Smoothing out is the replacement of a (positive or negative) crossing by a smoothed out crossing. A crossing change is the replacement of a positive crossing to a negative or vice versa. Smoothing out all crossings of D one obtained the *Seifert circles* of D. The *valency* of a Seifert circle s is the number of crossings attached to s. We call such crossings also *adjacent* to s.

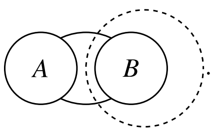
The obverse (mirror image) of a knot K is denoted by $!K$, the inverse(ly oriented knot) by $-K$. Let $\overline{T}_k$ for $k \in 2\mathbb{Z} \setminus \{0\}$ be the $(2,k)$-torus link with reverse orientation, with the mirroring convention that $\overline{T}_k$ is positive for $k \geq 2$. Thus $\overline{T}_2$ is the (positive) Hopf link. Knots and links of ≤ 10 crossings will be denoted, mirroring convention including, according to Rolfsen's tables [Ro, appendix], and knots of ≥ 11 crossings according to Hoste and Thistlethwaite [HT].

A crossing p in a link diagram D is called *reducible* (or *nugatory*) if D can be represented in the form



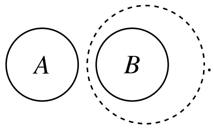
A diagram D is called reducible if it has a reducible crossing, otherwise it is called *reduced*.

A link diagram D is *composite*, if there is a closed curve γ intersecting (transversely) the curve of D in two points, such that both in- and exterior of γ contain crossings of D, that is, D has the form



Otherwise D is *prime*. A link is prime if in any composite diagram replacing one of A and B by a trivial (0-crossing) arc gives an unknot diagram.

The diagram is *split*, if there is a closed curve not intersecting it, but which contains parts of the diagram in both its in- and exterior:

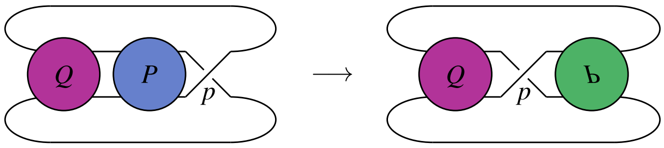


Otherwise D is *connected* or *non-split*. A link is split if it has a split diagram, and otherwise non-split.

A prime diagram is called *special* if no Seifert circle contains other Seifert circles in its interior and exterior (i.e., is *separating*).

A *region* of a link diagram is a connected component of the complement of the (plane curve of) the diagram. An *edge* or *segment* of D is the part of the plane curve of D between two crossings (clearly each edge bounds two regions). At each crossing p, exactly two of the four adjacent regions contain a part of the Seifert circles near p, as obtained by smoothing out (4). We call these the *Seifert circle regions* of p. The other two regions are called the *non-Seifert circle regions* of p. If the diagram is special, each Seifert circle coincides with (the boundary of) some region. We call the regions accordingly Seifert circle regions or non-Seifert circle regions (without regard to a particular crossing).

A *flype* is the move



(5)

The *canonical Euler characteristic* of a link diagram D is called the Euler characteristic of D's canonical Seifert surface, for which we have

$$\chi(D) = -c(D) + s(D),$$

where $c(D)$ is the number of crossings of D, and $s(D)$ the number of its Seifert circles. The *canonical genus* $g(D)$ is given by

$$g(D) = \frac{1}{2}\big(2 - n(D) - \chi(D)\big),$$

where $n(D)$ is the number of components of (the link represented by) D, and D is connected. The canonical Euler characteristic and canonical genus of a link L are the maximal canonical Euler characteristic resp. minimal canonical genus of any diagram D of L.

The (classical, or Seifert) *Euler characteristic* $\chi(L)$ resp. *genus* $g(L)$ of a link L is called the maximal Euler characteristic resp. minimal genus of all its (not necessarily canonical for some diagram) Seifert surfaces. From their definition, we have the inequalities $\chi(L) \ge \chi_c(L)$ and $g(L) \le g_c(L)$. By the classical Crowell-Murasugi theorems, when D is alternating, then $\chi(D) = \chi_c(L) = \chi(L)$.

In [St9] we called a diagram *k-almost positive*, if D has exactly k negative crossings. A link L is k-almost positive, if it has a k-almost positive diagram, but no l-almost positive one for any $l < k$. We call a diagram or link *positive*, if it is 0-almost positive (see [Cr, O, Yo, Zu]), and *almost positive* if it is 1-almost positive [St5]. Similarly one defines *k-almost negative*, and in particular *almost negative* and *negative* links and diagrams to be the mirror images of their k-almost positive (or almost positive or positive) counterparts.

It is well known that a special diagram is alternating if and only if it is positive or negative. These *special alternating* diagrams and their (homonymous) links have been studied [Mu, N].

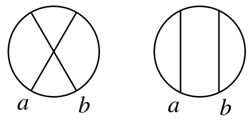
The *crossing number* $c(L)$ of a link L is the minimal number of the crossing number $c(D)$ of all diagrams D of L. Kauffman-Murasugi-Thistlethwaite's theorem asserts that $c(L) = c(D)$ when D is alternating and reduced.

By $n(D) = n(L)$ we denote the number of components of D or L.

Finally, we add some clarification regarding *Gauß code* of *knot* diagrams. When D is a knot diagram, one can record the passing of crossing points (two per each crossing) in the cyclic order of the parametrizing circle by the same (or mutually reverse) letter(s). The resulting (cyclic) word is called *Gauß code*. It will not be relevant here to distinguish the over- and underpass by using a letter and its inverse or the like. (For the Wicks forms, §3.3, the convention is to use a letter and its inverse once, without distinction which one where.)

There is a geometric version of Gauß codes, the *Gauß diagrams*, in which a baseline circle is depicted, and over- and underpass of the same crossing (or equal letters in the Gauß code) are connected by a chord. See, e.g., [STV, St2, St5]. Here again we leave chords undirected, since we do not (need to) distinguish over- and underpass.

In the terminology of [St2], a crossing pair a,b of D is *linked*, if its chords in the Gauß diagram intersect, i.e., the cyclic order of their letters in the Gauß code is $\cdots a^{\pm 1} \cdots b^{\pm 1} \cdots a^{\pm 1} \cdots b^{\pm 1}$. Otherwise a and b are *unlinked*; this corresponds to the cyclic order $\cdots a^{\pm 1} \cdots a^{\pm 1} \cdots b^{\pm 1} \cdots b^{\pm 1}$.



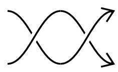
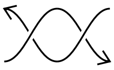
(6)

linked unlinked

Note that, equivalently, linked means that smoothing out a and b gives a knot diagram, while unlinked means that this smoothing out gives a 3-component link diagram. We will further take the freedom to talk about (un)linked chords in the Gauß diagram, in the sense that they (do not) intersect; thus 'intersecting' and 'linked' will mean the same in their context. It is a consequence of the Jordan curve theorem that

every crossing is linked with an even number of other crossings, (7)

a property called in [St2] *even valence*.

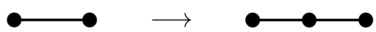
As an illustration, note that a crossing pair in a *parallel clasp*  is linked, while one in a *reverse clasp*  is unlinked. Furthermore, when a and b form a clasp (parallel or reverse), then a and b are linked with exactly the same set of crossings outside $\{a,b\}$. Thus we can talk of clasps being (mutually) linked or unlinked. This status is, in general, independent of whether the clasps are parallel (i.e., their crossings are internally linked) or reverse (i.e., internally linked).

2.4. Graphs

A graph G will have for us possibly multiple edges but usually no *loop edges* (edges connecting one and the same vertex). If G has no multiple edges, we call G *simple*. A multiple edge should best be understood as a set of simple edges connecting the same two crossings. These simple edges may be treated separately. When vertices v and w are connected by an edge e, we say that v and w are *adjacent*, and that e is *incident* to v (and w) *as well as* that v (and w) is incident to e.

$V(G)$ will be the set of vertices of G, and $E(G)$ the set of edges of G (each multiple edge regarded as a set of single edges). We write $v(G)$ and $e(G)$ for the number of vertices and edges of G (multiple edges counted by the number of simple ones), respectively. By $v_k(G)$ we denote the number of vertices of G of valence k. We call G to be $\geq k$*-valent* if $v_l(G) = 0$ when $l < k$. More particularly, G is k*-valent* if $v_l(G) = 0$ when $l \neq k$; sometimes 3-valent graphs are also called *cubic*.

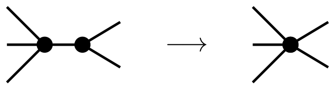
For a graph, let the operation



(adding a vertex of valence 2) be called *bisecting* and its inverse (removing such a vertex) *unbisecting* (of an edge). We call a graph G' a *bisection* of a graph G, if G' is obtained from G by a sequence of edge bisections. We call a bisection G' *reduced*, if it has no adjacent vertices of valence 2 (that is, each edge of G is bisected at most once if G is $\geq$ 3-valent). Contrarily, if G' is a graph, its *unbisected graph* G is the graph with no valence-2-vertices, of which G' is a bisection.

Unless otherwise noted, G will be a 3-connected 3-valent planar graph, and G' a reduced bisection of G, with some further properties that will be specified in each situation. (Usually G' will be the Seifert graph of some generator; see definition 2.1 below.) Note that these designations are *opposite* to [St8]. Here they are chosen so for technical reasons.

Similarly, a *contraction* is the operation

 ,

and a *decontraction* its inverse.

A *cut vertex* is a vertex which disconnects a graph, when removed together with all its incident edges.

A graph is *n-connected*, if at least n edges need to be removed from it to disconnect it. (Thus connected means 1-connected.) Such a collection of edges is called an *n-cut*. A graph thus has an n'-cut for some $n' \leq n$ if it is not $(n+1)$-connected.

For every n-cut of a planar graph we can draw a (closed) *cut curve* or *cut circle* γ in the plane, which intersects G only in interior points of the edges in the cut. This curve γ is determined up to isotopies of the plane which avoid intersection with vertices of G. We will often for convenience identify a cut with its cut curve.

A *(cyclic) orientation* O of a graph G can be described as a map

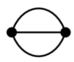
$$O : V(G) \to \bigcup_{n=0}^{e(G)} E(G)^n/\mathbb{Z}_n ,$$

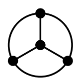
with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ acting by cyclic permutation on $E(G)^n$. If $O(v) = (e_1, \dots, e_n)$, then we demand that $n = \mathrm{val}_G(v)$ is the valence of v in G, and that e_i are the edges incident to v (with $e_i \neq e_j$ for $i \neq j$). The *opposite* orientation $-O$ is defined by $-O(v) = (e_n, \dots, e_1)$. Any embedding p of G on an oriented surface S (in particular, any planar embedding of G) defines a *canonical orientation* O_p of G (corresponding to this embedding), given by listing the edges incident to v in counterclockwise order.

An embedding p of G on a surface S is *cellular* if all components of $S \setminus p(G)$ are discs. If G is trivalent, the dual of $p(G)$ is then the 1-skeleton of a triangulation of S.

We will define a few more properties of graphs at an appropriate place below. For now let us finish related definitions by fixing symbols for two graphs which will continuously occur in the sequel.

Let

θ =  the theta-curve (8)

τ =  the tetrahedral graph (9)

The letters θ and τ will retain this meaning throughout the paper.

Finally, we mention the connection to link diagrams.

Definition 2.1 When D is a link diagram, the *Seifert graph* $\Gamma(D)$ has vertices being the Seifert circles of D and edges between two vertices connected by a crossing, which carry a $\pm$ marking according to the sign of the crossing in (4).

It is known that $\Gamma(D)$ is bipartite. It has no cut vertex if and only if D is special and prime. In that case, one can reconstruct D from the planar embedding of $\Gamma(D)$,

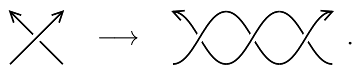
$$\textit{up to simultaneous reversal of orientation of all components of } D. \qquad (10)$$

This means that we will regard links up to isotopies that may interchange components, and either preserve or reverse all component orientations.

3. Generators, Markings, Wicks forms

3.1. Generators

Now, we consider the move of link diagrams we call a $\bar{t}'_2$ *twist* or $\bar{t}'_2$ *move*. Up to mirroring this is given by



(11)

We call diagrams that cannot be reduced by flypes (5) and inverses of the move (11) *generating* or $\bar{t}'_2$*-irreducible*. That is, a diagram is $\bar{t}'_2$-irreducible, if after flypes one cannot make it have the fragment on the right of (11).

We say that two crossings p,q in a link diagram are $\sim$*-equivalent* if they have the same pair of non-Seifert circle regions. Recall that in [St] we called two crossings $\sim$-equivalent, if after a sequence of flypes they can be made to form a reverse clasp (as shown below (7); of course, one also assumes a crossing $\sim$-equivalent to itself). It is an exercise to check that this is an equivalent definition.

Similarly two crossings p,q in a link diagram are $\overset{\sim}{*}$*-equivalent* if they have the same pair of Seifert circle regions. In particular, crossings of a reverse clasp are $\sim$-equivalent and of a parallel one are $\overset{\sim}{*}$-equivalent. A diagram is $\bar{t}'_2$-irreducible, as defined above, if and only if every $\sim$-equivalence class has not more than (i.e., 1 or) 2 crossings.

We have noticed that, for *knot* diagrams, there is a reformulation using Gauß codes: p,q are $\sim$-equivalent if they (equal or) are unlinked among each other and linked with the same set of crossings outside $\{p,q\}$. They are $\overset{\sim}{*}$-equivalent if they are linked among each other and linked with the same set of other crossings. In that form one can easily use even valence (7) to infer that these are equivalence relations.

It can be seen further that for a connected link diagram of more than 2 crossings, a non-trivial (i.e., more-than-one-element) $\sim$-equivalence class and $\overset{\sim}{*}$-equivalence class are disjoint. The property "$\sim$-equivalent or $\overset{\sim}{*}$-equivalent" is thus also an equivalence relation, and this is the *twist equivalence*, as used, e.g., in [La].

We know (in the case of knots from [St], and then for links from [St8]; see theorem 3.2 below) that reduced diagrams D of given $\chi(D)$ decompose into finitely many equivalence classes under flypes, $\bar{t}'_2$ twists and their inverses. We call these collections of diagrams *series*. We attempted their classification for knot diagrams. Since the number of series grows rapidly with the genus g (where $1-\chi=2g$; see table 4), this was practically possible so far only up to genus 4. In manageable form, the list of generators can be given for genus 1 and 2 [St, St4].

Theorem 3.1 ([St, St4]) A genus one knot diagram is (modulo crossing changes) a rational diagram $C(p,q)$ with $p,q>0$ even, or a pretzel diagram $P(p,q,r)$, with $p,q,r>0$ odd. That is, it can be obtained via $\bar{t}'_2$ moves and crossing changes from the alternating trefoil and figure eight knot diagram.

A prime genus two knot diagram can be obtained via $\bar{t}'_2$ moves and crossing changes from an alternating diagram of the following 24 knots: 5_1, 6_2, 6_3, 7_5, 7_6, 7_7, 8_{12}, 8_{14}, 8_{15}, 9_{23}, 9_{25}, 9_{38}, 9_{39}, 9_{41}, 10_{58}, 10_{97}, 10_{101}, 10_{120}, 11_{123}, 11_{148}, 11_{329}, 12_{1097}, 12_{1202}, and 13_{4233}.

Genus 3 was also discussed in [St4], and the compilation for genus 4 was explained in [St8]. For general genus, and also for links, we obtained in [St8] rather sharp estimates on the maximal number of crossings and $\sim$-equivalence classes of generators. This was a consequence of a detailed study of the special diagram algorithm of Hirasawa [Hr] and myself [St3, §7].

Theorem 3.2 ([St8]) In a connected link diagram D of canonical Euler characteristic $\chi(D)\le 0$ there are at most

$$\begin{cases} -3\chi(D) & \text{if } \chi(D)<0 \\ 1 & \text{if } \chi(D)=0 \end{cases}$$

$\sim$-equivalence classes of crossings. If D is $\bar{t}'_2$-irreducible and has $n(D)$ link components, then

$$c(D)\le\begin{cases} 4 & \text{if } \chi(D)=-1 \text{ and } n(D)=1\,, \\ 2 & \text{if } \chi(D)=0\,, \\ -6\chi(D) & \text{if } \chi(D)<0 \text{ and } n(D)=2-\chi(D)\,, \\ -5\chi(D)+n(D)-3 & \text{else}\,. \end{cases} \tag{12}$$

This, together with the examples in [SV], settles the problem to determine the maximal crossing number of a generator for knots.

Corollary 3.1 The maximal crossing number of a knot generator of genus $g \geq 2$ is $10g-7$.

The finiteness of generators, together with the Flyping theorem [MT], shows:

Theorem 3.3 (see [St]) Let $a_{c,g}$ be the number of prime alternating knots K of genus $g(K)=g$ and crossing number $c(K)=c$ (with orientation and mirroring ignored). Then for $g \geq 1$

$$\sum_c a_{c,g} x^c = \frac{R_g(x)}{(x^{p_g}-1)^{d_g}},$$

for some polynomial $R_g \in \mathbb{Q}[x]$, and $p_g, d_g \in \mathbb{N}$. Alternatively, this statement can be written also in the following form: there are numbers p_g (period), c_g (initial number of exceptions) and polynomials $P_{g,1}, \dots, P_{g,p_g} \in \mathbb{Q}[c]$ with $a_{c,g} = P_{g,c \bmod p_g}(c)$ for $c \geq c_g$.

This was explained roughly in [St], and then in more detail in [SV], where we made effort to characterize the leading coefficients of these polynomials $P_{g,i}$. Even if the polynomials vary with a very large period p_g, the leading coefficients depend only on the parity of c. Let these coefficients be $C_{g,e}$ and $C_{g,o}$, where g is a natural number and 'e/o' are formal symbols. (It has transpired that composite alternating knots do not affect $C_{g,*}$ so that, in that context, primality can be automatically assumed.)

The degrees of all $P_{g,i}$ are also the same, and equal to 1 less than the maximal number of $\sim$-equivalence classes of diagrams of canonical genus g. There is the exception $g=1$, where this degree is 1 or 2 depending on whether i is even or odd.

Let a genus g generator be *maximal* if it has the maximal number, $6g-3$, of $\sim$-equivalence classes.

Note: unlike what could be suggested by corollary 3.1, maximality of generators will always be understood with regard to number of $\sim$-equivalence classes, *not* crossings. (We know from the proofs in [St8] that, except for the figure-8-knot, crossing number maximality implies maximality, but the converse is far from true, as displayed in table 4.)

These generators were studied in detail in [SV]. For purposes of normalization, let then $C_{g,e}, C_{g,o}$ be defined as follows.

Let $a_{c,g}$ be the number of prime alternating knots K of genus $g(K)=g$ and crossing number $c(K)=c$ (see §3.1 for more details). Set

$$C_{g,e} = 2^{6g-4}(6g-4)! \lim_{c\to\infty} \frac{a_{2c,g}}{(2c)^{6g-4}}, \quad C_{g,o} = 2^{6g-4}(6g-4)! \lim_{c\to\infty} \frac{a_{2c+1,g}}{(2c+1)^{6g-4}}. \tag{13}$$

They are non-zero, except for $C_{1,e}$. (Table 4 gives these numbers for $g \leq 6$.) Then in [BV] it was established that $C_{g,*}$ are the numbers of maximal even/odd generators of genus g.

A rough explanation of the role of maximal generators in such enumeration (following details discussed in [SV]) is thus. A maximal generator does not admit a flype, so it has a unique alternating diagram D'. An alternating diagram D of c crossings in the series $\langle D' \rangle$ of D' is obtained by $1/2(c-c(D'))\ \bar{t}'_2$-twists on D'. The number of diagrams $D \in \langle D' \rangle$ (and, by [MT], their knots) is determined by the number of distributions of $1/2(c-c(D'))\ \bar{t}'_2$-twists among the $6g-3$ $\sim$-equivalence classes of D which, in the leading asymptotic term, is

$$\binom{(c-c(D'))/2+6g-4}{6g-4} \sim \frac{c^{6g-4}}{2^{6g-4}(6g-4)!}. \tag{14}$$

This explains the normalization factors in (13). Symmetries of D do not affect the asymptotics unless they pass on to symmetries of D', which can be understood as symmetries of a $\mathbb{Z}_2$-vertex marked graph (see §3.2).

The numbers $C_{g,*}$ will be studied in detail in §7.3. Let their sum

$$C_g = C_{g,o} + C_{g,e} \tag{15}$$

be the number of maximal generators of genus g. See table 4. It follows from (13) that

$$C_g = \lim_{c\to\infty} \frac{2^{6g-3}(6g-3)!}{c^{6g-3}} \sum_{c'=1}^{c} a_{c',g}\,. \tag{16}$$

Formulas (13) and (16) remain true if we replace counting alternating knots by positive ones, by [St11, corollary 6.5].

We note in passing that C_g also has a meaning as a different limit. Let $\tilde{a}_{b,g}$ be the number of alternating knots of genus g and *braid index* at most b. Then it is possible to prove

$$C_g = \lim_{b\to\infty} \frac{2^{6g-3}(6g-3)!\tilde{a}_{b,g}}{(2b)^{6g-3}} = \lim_{b\to\infty} \frac{(6g-3)!\tilde{a}_{b,g}}{b^{6g-3}}\,.$$

See [St7, corollary 6.2]; however, notice that the corollary needs this correction: replace $6g-4$ by $6g-3$ and allow braid index *at most b*. The stated form would still hold if the Graph Index conjecture (see [St8, Conjecture 2.9.5]) is true.

We remark that in this counting, mirror images and mutually reverse knots are identified. For the explanation of this convention, which generalizes to links, see §5.1.

3.2. Markings

The difference between even and odd crossing number generators with the maximal number of $\sim$-equivalence classes is also related to a quite different object, arising in the theory of Vassiliev invariants [BN]. We will now explain this relation. To express ourselves nicely, we need some terminology. Most of it was already introduced in [SV].

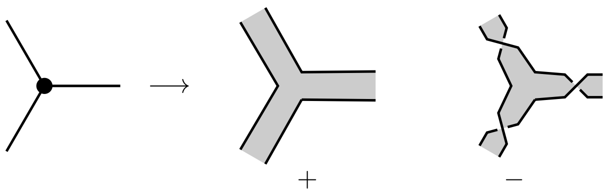
Definition 3.1 Take a 3-connected 3-valent planar graph G in a particular planar embedding p_0, which we keep in mind, but do not write. Let $D_{G,O}$ be the special alternating diagram corresponding to G with choice of vertex orientation O. The diagram $D_{G,O}$ can be defined by being special, having as Seifert graph a reduced bisection of G, and the orientation of the Seifert circles corresponding to the vertices of G being given by O. This specifies $D_{G,O}$ as an oriented link diagram up to reversal of orientation of all components of $D_{G,O}$. This ambiguity (which also affects table 4, for instance) will not cause any problem.

We denote the orientation $O(v)$ of $v \in V(G)$ by $+$ or $-$.

We call then

$$O : V(G) \to \{+,-\}$$

also a *marking* of G. We will often not distinguish between a marking O and its diagram $D_{G,O}$ to simplify our language. Let $L_{G,O}$ be the link represented by $D_{G,O}$. We call O an *n-component marking* (or *knot marking* for $n=1$), if $n(L_{G,O}) = n$. The marking O is said to be *even* or *odd* depending on the parity of $c(L_{G,O})$. Let $T_{G,O}$ be the *thickening* of (G,O), i.e. the canonical Seifert surface of $D_{G,O}$ with boundary $\partial T_{G,O} = L_{G,O}$.



(17)

Notice that the above picture specifies properly the orientation of $T_{G,O}$ near each vertex of G, but a pair of crossings has to be added for each edge between positive marked vertices to obtain $D_{G,O}$.

Whenever a marking O is given, it induces an *edge coloring* of G into *even* and *odd* edges, depending on whether the two vertices connected have the same or opposite marking.

Note that then one can alternatively (and equivalently) define a marking to be even or odd according to the parity of the number of its odd edges.

Definition 3.2 We say that O is *even or odd* depending on whether its odd edges are an even or odd number. Note that this is equivalent to whether O has an even or odd number of positive (or equivalently, negative) vertices.

Recall that for any Lie algebra with *ad*-invariant non-degenerate scalar product, one can associate a *weight system*, an integer-valued invariant of 3-valent graphs subject to certain local relations (see [BN]). The calculation of the weight system $W_N(G)$ of sl_N on a 3-valent graph G is described[1] in [BN, §6.3.6]. It uses a construction very reminiscent to the even-odd coloring of edges in G, and can in our language be written as follows:

$$W_N(G) = W_{N,+}(G) - W_{N,-}(G), \tag{18}$$

with

$$W_{N,+}(G) = \sum_{O \text{ even}} N^{n(L_{G,O})}, \quad \text{and} \quad W_{N,-}(G) = \sum_{O \text{ odd}} N^{n(L_{G,O})}.$$

Here the total number of summands of both sums is equal to the number $2^{-2\chi(G)}$ of choices of orientation O of the (Seifert circles of $D_{G,O}$ corresponding to the) $-2\chi(G)$ vertices of G. As in [BN], it is useful to regard herein N as a variable rather than as some given number, so the W_N become polynomials in N. We extend $W_{N,*}$ linearly on sums of graphs.

As far as the issue of cyclic vertex orientation (and the antisymmetry relation in [BN]) goes, working with planar graphs G, we can assume that the cyclic vertex ordering is given by a(ny) planar embedding. (As discussed in [BN2], Whitney's theorem implies that flips change the cyclic edge ordering at an even number of vertices.)

It was explained in [BN2] (for the sphere, but higher genus is completely analogous), that an n-component marking O of G gives rise to a cellular embedding p of G on an oriented surface S, s.t. $p(G)$ is the 1-skeleton of the dual (of a) triangulation of S with n vertices. To obtain p, glue (abstractly) disks into the boundary components of $T_{G,O}$. On the opposite side, given p, one can recover $T_{G,O}$ by recording the cyclic orientation of any $v \in V(G)$ induced by p. Given a planar embedding p_0 of G, define O by putting a $+$ or $-$ in $v \in V(G)$ depending on whether $p_0(v) = p(v)$ or $p_0(v) = -p(v)$. Then $T_{G,O}$ is homeomorphic to a neighborhood of $p(G)$ on S.

3.3. Wicks forms

As outlined, in continuing the setting of [SV], we will work with certain algebraic objects named (maximal) Wicks forms.

A *maximal Wicks form* w is a cyclic word in the free group over an alphabet with the following 3 conditions:

1) Each letter a appears exactly once in w, and so does its inverse a^{-1}.
2) w has no subwords of the form $a^{\pm 1}a^{\mp 1}$.
3) If $a^{\pm 1}b^{\pm 1}$ and $b^{\pm 1}c^{\pm 1}$ are subwords of w (for some independently to choose signs), then $c^{\pm 1}a^{\pm 1}$ is also a subword of w (for proper to be chosen signs).

Two forms are *equivalent* if a dihedral (i.e., cyclic and/or inversive) permutation and a permutation of the letters (and between letters and their inverses) transforms the one form into the other.

Such words were first considered in [Wi]. Later they were studied in several contexts, e.g. [CE, Cu], but most relevant here will be their description as duals of 1-vertex triangulations of oriented surfaces [BV].

The number of letters of a maximal Wicks form w is always $6g-3$ for some $g > 0$. Such a form w gives rise to a triangulation of an oriented surface S. First label the edges of a $6g-3$-gon X by the letters of w and reverse the orientation induced from the one of X on edges corresponding to inverses of letters. Then identify the edges labelled by each letter and its inverse according to their orientation. The surface S thus obtained from X is orientable and of genus g. We call g also the *genus* of the Wicks form. The boundary of X gives a certain 3-valent graph G embedded on S, which is the 1-skeleton of a 1-face cell complex (or dual of a 1-vertex triangulation). The edges of G correspond

[1] Bar-Natan remarks that the *ad*-invariant non-degenerate scalar product on sl_N is unique up to scalars, so that the construction below is valid for a proper choice of constants.

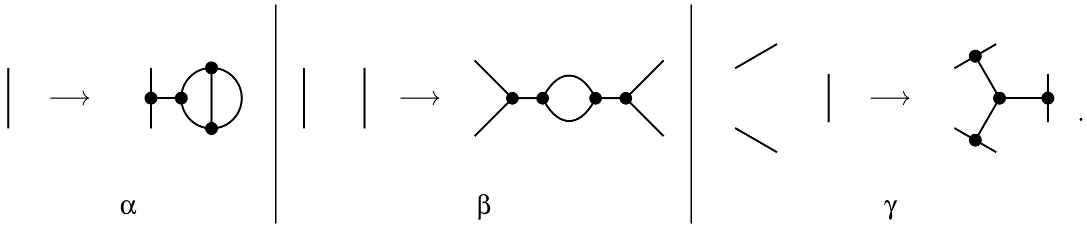


Figure 1: The three Vdovina constructions. (The segments on the left of the moves β and γ do not necessarily belong to different edges.)

to letters $\{a, a^{-1}\}$ of w, while the vertices to triples of such pairs occurring as in property 3) of the above description of Wicks forms. Thus G comes from a Wicks form if and only if it admits a knot marking.

In [V] three elementary operations to construct Wicks forms of genus $g+1$ out of Wicks forms of genus g were introduced. They were called α, β *and* γ *construction (or transformation)*. The effect of these operations on the graphs of the Wicks form are given in figure 1 (see also Figure 1 of [BV]). We will call these graph moves also graphic α, β and γ construction (or transformation).

In [SV] we defined *maximal planar Wicks forms* to be those, whose graph G is planar and 3-connected. We showed that maximal planar Wicks forms bijectively correspond to maximal knot generators.

We made clear that a maximal planar Wicks form bijectively corresponds to a knot marking of its 3-valent graph G. (The reverse map is easily described: take the Gauß code of the marking and remove all cyclic occurrences of $b^{\pm 1}$ in $\cdots a^{\pm 1} b^{\pm 1} \cdots b^{\pm 1} a^{\pm 1} \cdots$.) We also introduced, in [STV], the Gauß diagram ([St2, St5]) version of the form and its knot marking, which will be preferably used in the proofs.

Thus the enumeration of such forms is directly related to the coefficients of the polynomials $P_{g,i}$ of theorem 3.3. We obtained estimates for these coefficients (see §7.3).

In this paper *we will slightly deviate* from our previous definition of maximality of a Wicks form by not necessarily demanding G to be 3-connected. Whenever we wish it to be so, we will specify this. (Compare the remark below definition 8.1, or the convention at the beginning of section §8.1.)

It should be emphasized that maximality of a (planar) Wicks form just means that its graph is (planar and) 3-valent. Maximality of a generator means, though, that the graph of its Wicks form is planar 3-valent *and 3-connected*. It is that the notions of maximality were introduced for both objects independently before the connection was established.

As a small illustration, the following table breaks down the 927 maximal genus 3 Wicks forms, compiled by Vdovina, by their connectivity (maximal k for which their graph is k-connected) and planarity.

connectivity	1	2	≥ 3
planar	53	371	158
non-planar	3	113	229

(19)

4. Maximal hyperbolic volume of links for given Euler characteristic

4.1. Limit links

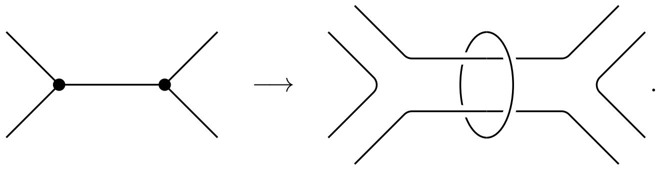
When considering the canonical Euler characteristic, it is worth making a few remarks on the hyperbolic volume. It was related to our context by a preprint of Brittenham [Br], in which he came up simultaneously and independently from us with the generator approach for the canonical genus. His motivation was to show that, in the case $n = 1$ of knots, the supremum

$$v_{n,\chi} := \sup \{ \operatorname{vol}(L) : n(L) = n, \chi_c(L) = \chi \}$$

is finite, that is, the canonical genus bounds the hyperbolic volume of the knot.

The value $v_{1,-1}$ was determined in [Br], and $v_{1,-3}$ in [St4]. The approach we take in this subsection and §4.3 (and some computer calculation) will yield many more values of $v_{n,\chi}$; see §5.3. Before computing such values, we will first clarify an important point, the independence of n, and then explain the relation between this calculation and the sl_N weight system.

Define for a 3-connected 3-valent planar graph G the (*unoriented*) link L_G by

 (20)

The transformation is understood so that it is applied on each edge of G, and the components drawn unclosed on the right globally close to give the boundary of a planarly embedded thickening of G. (See the right diagram in (39) for an example.)

To incorporate $\chi = -1$ and $G = \theta$, one can allow G to have multiple edges. We did not specify whether 'planar' should mean 'planarly embeddable' or 'planarly embedded'. But by Whitney's theorem [W], a planar embedding of a 3-connected 3-valent graph, if existent, is unique, so that our sloppiness is justified. Note that the uniqueness of the planar embedding of G holds only up to the change of the infinite region. We will assume in the sequel that this freedom is always given for planar embeddings, so that they are regarded the same as spherical embeddings.

We remark that $G \longmapsto L_G$ is injective, as one can argue by considering the linking numbers of the components of L_G.

The links L_G arise by elaborating on Brittenham's observation. Brittenham did not restrict himself to 3-connected and 3-valent graphs; these properties emerged more clearly from our work in [SV], as we will explain below. The L_G are also similar to the links occurring in recent work by van der Veen [vdV], which we will extensively use in a moment.

Let

$$v_\chi := \sup \{ \operatorname{vol}(L) : \chi_c(L) = \chi \} = \max_n v_{n,\chi}, \tag{21}$$

with the maximum taken over all $1 \le n \le 2-\chi$ with $n+\chi$ even. Note that the upper bound for n follows from the fact that a canonical Seifert surface of a non-split diagram is connected, and split links are non-hyperbolic. We will assume throughout the rest of the paper that only pairs (n,χ) satisfying these conditions are considered. Moreover, we will assume $\chi < 0$, since the values $v_0 = v_1 = 0$ are of little interest.

One first application is a description of v_χ in terms of the L_G. The proof will be given below.

Theorem 4.1 For $\chi < 0$,

$$v_\chi = \max_{\chi(G)=\chi} \operatorname{vol}(L_G) = \sup \{ \operatorname{vol}(L) : \chi(L) = \chi,\ L \text{ special alternating} \},$$

with the maximum taken over 3-connected 3-valent planar graphs G, considered up to isotopy in the sphere.

For special values of (n,χ) we can say even more, in that how the supremum is approximated from links of given crossing or component number, i.e. on the relation between the $v_{n,\chi}$. The discussion will be done below. There we will also explain the special status of the case $n = 2-\chi$. The main problem is to understand the difference between even and odd crossing number, which was already strongly apparent in [St, SV]. Here we will focus only on the case $n = 1$ of knots.

Proof of theorem 4.1. A consequence of the special diagram algorithm of [Hr] is that the series of special generators contain diagrams of the links of all the other series. Then from [SV] we know that the series of the maximal generators contain the series of all special generators. We also know that maximal generators are those whose unbisected Seifert graph is trivalent and 3-connected. (These arguments were applied in [SV] only for knots, but this restriction is not relevant for them.)

To attain the maximal volume, by W. Thurston's hyperbolic surgery theorem, one needs to let the number of twists in each $\sim$-equivalence class to go to ∞, *regardless* of their sign. Thus one can choose a positive sign everywhere, so that the diagrams become special alternating. By a result of Adams [Ad], one can (ignoring orientations) change the parity of the number of crossings in each $\sim$-equivalence class, without changing the volume of the augmented alternating link. (This circumstance is also explained by van der Veen's (un)zipping move in [vdV], which we will recall below.) By this result $\mathrm{vol}(L_G)$ is the volume of the augmented alternating link of any generator diagram with given G. The rest of the argument is as before. $\square$

4.2. Some volume inequalities

Thus one gains interest in the volumes of the links L_G. The generator estimates Brittenham obtained were better than the one I gave originally in [St], but were slightly improved in [St4] by referring to our work in [STV]. The volume bound was also improved, by Lackenby [La], and later by Agol and D. Thurston (see the appendix to Lackenby's paper). Lackenby's reverse lower bound for the volume of alternating links was also improved, by Agol, Storm and W. Thurston [AST].

For the following let

$$V_4 \approx 1.01494\,, \quad \text{the volume of the regular ideal hyperbolic tetrahedron}\,, \tag{22}$$

$$V_8 \approx 3.66386\,, \quad \text{the volume of the regular ideal hyperbolic octahedron}\,. \tag{23}$$

Let per convention $\mathrm{vol}(L)=0$ if a link L is not hyperbolic. The inequalities (into which work of Agol, Lackenby, Storm, D. Thurston and W. Thurston – below acronymed with their initials – goes in) can then be stated as follows:

Theorem 4.2 (LASTT) We have

$$\frac{V_8}{2}\big(t(D)-2\big) \le \mathrm{vol}(L) \le 10V_4\big(t(D)-1\big)\,. \tag{24}$$

Here for the left inequality D is an alternating diagram of a link L, for the right inequality an arbitrary non-trivial diagram of L, and $t(D)$ is the twist number of D.

Before continuing, we make a few short remarks that help putting the present work into a broader perspective. There have been so far at least two situations, in which the hyperbolic volume exhibits a relation to a quantum algebra structure.

The first, and (theoretically) most important one, is Kashaev's conjecture [Ks]. As it was later put in [MM], it asserts that the volume can be determined as a limit of unity root values of colored Jones polynomials. The practical use is unclear. Numerically, there are much more efficient ways to calculate the volume. (Below a program of J. Weeks, included in KnotScape, was used on the links L'_G in (39).) However, the conjecture is lent fundamental theoretical importance, and much work has been done it evaluating the colored Jones limit. In this context we became aware of a recent paper by van der Veen [vdV], which turned out of direct relevance.

Another more direct correspondence was observed by Dunfield [Df]. He found that, for low crossing numbers, the volume surprisingly well approximates a logarithm of the determinant alternating knots. Khovanov suggested that such a correspondence may extend to non-alternating knots (and links) if instead of the determinant we take the total degree of his generalization of the Jones polynomial [Ko]. For alternating links, Dunfield's conjecture was proved in [St6] using the above LASTT inequalities. More exactly, we showed for any non-trivial non-split alternating link L the inequality $\det(L) \ge 2\cdot C^{\mathrm{vol}(L)}$, for a constant $C>1$. Probably, however, a correspondence between determinant and volume is only approximate, and not exact, as explained in [St6]. We will propose a new relation in §5.2.

We will use the LASTT inequalities also here (see proposition 7.1), but in view of van der Veen's work, subsequently, a result better fitting into our context turned out to be one due to Atkinson (see [At, Theorem 2.4 and Theorem 3.2]). He proved various bounds on the volume of hyperbolic polyhedra. The 1-skeleton of an ideal $\pi/2$–equiangular polyhedron is a four-valent graph on S^2. Consequently it admits a *checkerboard coloring*, i.e., a black-white coloring of its regions specified by the property that regions sharing an edge should have opposite colors. Let $\mathcal{B}$ be the black and $\mathcal{W}$ be the white faces of such a coloring. (The sets $\mathcal{B}$ and $\mathcal{W}$ are determined up to interchange.)

Theorem 4.3 (Atkinson) If $\mathcal{P}$ is an ideal $\pi/2$–equiangular polyhedron with N vertices, and $|\mathcal{B}| \ge |\mathcal{W}|$, then

$$(N-|\mathcal{W}|)\cdot\frac{V_8}{2} \le \mathrm{vol}(\mathcal{P}) \le (N-4)\cdot\frac{V_8}{2}.$$

Then, using this result and theorem 3.2, we can state a volume bound in a sharper and more general form.

Corollary 4.1 For $\chi < 0$, we have

$$V_8(-6\chi-2) \le v_\chi \le V_8(-7\chi-4). \qquad \square$$

This is a direct consequence of theorem 4.1 and the following estimate, which is implied by theorems 4.3 and 3.2 (as explained below in §4.3).

Corollary 4.2 Let L_G be the link of a graph with $\chi(G)=\chi$. Then

$$V_8(-6\chi-2) \le \mathrm{vol}(L_G) \le V_8(-7\chi-4). \tag{25}$$

It is evident that the lower bound in theorem 4.3 can be realized by infinitely many N, by taking $\mathcal{P}$ to be composed of octahedra. These $\mathcal{P}$ correspond to links which were studied by van der Veen in [vdV]. These links will be of less interest to us, since we are looking to maximize volume, but van der Veen's construction will be very relevant, and will be discussed below. In the argument for theorem 1.2 in §4.3, we will also see a natural role of the lower bound in (25).

As for Atkinson's upper bound in theorem 4.3, Atkinson states that it is asymptotically best possible, in the sense that there is a sequence $\mathcal{P}_N$ with

$$\lim_{N\to\infty}\frac{\mathrm{vol}(\mathcal{P}_N)}{N} = \frac{V_8}{2}. \tag{26}$$

However, we are in a special situation, in that in the checkerboard coloring all (say) black faces are triangles (see remark 4.1 and corollary 4.4). In that case, a better bound follows from Agol-Thurston's proof of the right inequality in (24) in the appendix to [La].

Proposition 4.1 (Agol-Thurston) Let $\mathcal{P}$ be an ideal $\pi/2$–equiangular polyhedron with a checkerboard coloring where all (say) black faces are triangles, and let T be the number of such triangles. Then

$$\mathrm{vol}(\mathcal{P}) \le \frac{5}{2}(T-2)\cdot V_4. \tag{27}$$

This estimate is asymptotically best possible, in the sense that there is a sequence $\mathcal{P}_{[T]}$ with

$$\lim_{T\to\infty}\frac{\mathrm{vol}(\mathcal{P}_{[T]})}{T} = \frac{5V_4}{2}. \tag{28}$$

To see the (slight) improvement, compare the r.h.s. of (26) and (28) using the numerical values (22) and (23). (In our context $N = v(M(G)) = e(G)$ and $T = v(G)$, so that $N = 3T/2$.)

Note that Agol-Thurston never explicitly write the inequality (27) (and its asymptotic sharpness), but all argument is there in their proof. (I am grateful to J. Purcell for clarifying this with me.) They argue that the 1-skeleton of the volume-maximizing polyhedron $\mathcal{P}_{[T]}$ for $T\to\infty$ "approximates" the David-star tessellation of the plane.

Using this and lemma 4.1, along with its preceding explanation, we obtain the following. (Proof will appear in §4.3.)

Corollary 4.3

$$\mathrm{vol}(L_G) \le v_\chi \le -4\chi V_8 - 10\chi V_4 - 10V_4, \tag{29}$$

asymptotically best possible, i.e., there is a sequence L_{G_χ} for $\chi = \chi(G_\chi)\to-\infty$ with

$$\lim_{\chi\to-\infty}\frac{\mathrm{vol}(L_{G_\chi})}{-\chi} = 4V_8 + 10V_4.$$

4.3. Decomposing graph and link complements

For the proof of corollaries 4.2 and 4.3 it is necessary to gain some knowledge of the hyperbolic structure of the complements of L_G. We will now follow van der Veen's exposition in [vdV, §4], which clarifies most of what we will be able to understand (although certain aspects have been known before, in particular from [Ad]). I am indebted to Roland van der Veen for his extensive clarification, and reference to theorem 4.3.

The first step is to gain a proper understanding of the complement of a trivalent graph G as a hyperbolic manifold. The stipulation is made in [vdV, Definition 14].

The idea is to consider a neighborhood $N(G)$ of G, made up of 3-balls with geodesic 2-spheres as boundary around each vertex of G, and geodesic cylinders around each edge of G. The complement of $N(G)$ becomes a hyperbolic manifold with a rigid hyperbolic structure. In this sense we speak of the *graph volume* $\mathrm{vol}(G)$ of G.

When G is planar (and it will always be so for us) and $G \neq \theta$, the plane in which G lies cuts $S^3 \setminus N(G)$ into two ideal hyperbolic $\pi/2$-angled polyhedra Γ. Such polyhedra

$$\Gamma = \Gamma_{M(G)}$$

are determined by their 1-skeleton, which is the *median graph* $M(G)$ of G. The median graph $M(G)$ of G is a planar 4-valent graph obtained as follows: vertices of $M(G)$ are edges of G, and an edge connects v_1 and v_2 in $M(G)$ if and only if v_1 and v_2 are incident to the same vertex in G. One has to check that $M(G)$ for a 3-connected $G \neq \theta$ satisfies Andreev's conditions given in [At, theorem 2.1]; for this see remark 4.1 below. Thus

$$\mathrm{vol}(G) = 2\,\mathrm{vol}(\Gamma_{M(G)})\,. \tag{30}$$

In particular, when $G = \tau$, we obtain two regular octahedra, and

$$\mathrm{vol}(\tau) = 2V_8\,. \tag{31}$$

We can set

$$\mathrm{vol}(\theta) = 0\,. \tag{32}$$

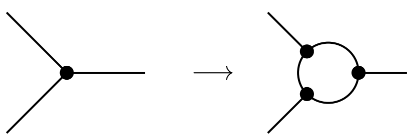
As another illustration, the 1-skeleton of Agol-Thurston's asymptotic volume-maximizer $\mathcal{P}_{[T]}$ for proposition 4.1 is $M(G)$ for the "bee-comb" (hexagonal) lattice G. The intuition behind conjecture 4.1 and remark 4.3 is very much compatible with this insight.

Lemma 4.1 $\mathrm{vol}(L_G) = 2v(G)\cdot V_8 + \mathrm{vol}(G)$

Proof. The 1-unzipping move in [vdV, figure 4] is essentially drawn in (20), but the understanding is slightly different. (An example to see the difference are L'_τ and L_τ in (39).) Now we undo a 1-unzipping move for every fragment of L_G shown on the right of (20), and use that it does not change volume (see proof of lemma 3 in [vdV]). We obtain

$$\mathrm{vol}(L_G) = \mathrm{vol}(G')\,,$$

where G' is obtained from G by doing van der Veen's *triangle move*



(33)

at each vertex. The change of volume under this operation is explained by the triangle move in proof of lemma 3 of [vdV], induction step. The outcome is that (33) augments the volume by $2V_8$ for each application.

More particularly, use (31) in lemma 4.3 below. Note that the case $G = \theta$ fits well, since (35) is compatible with (32). In that case, our $G' = \tau\#\tau$, so we apply (33) only once on a pair of octahedra. The result follows. □

An immediate consequence of this lemma, together with the said before it, is:

Corollary 4.4

$$v_\chi = -4\chi V_8 + 2\max_\Gamma \mathrm{vol}(\Gamma)\,,$$

where the maximum is taken over Γ being a regular hyperbolic polyhedron whose net (1-skeleton) is the median graph of a planar 3-connected cubic graph G of $\chi(G) = \chi$. □

Remark 4.1 Note that the net of Γ can also be characterized by being a planar 4-valent cyclically 5-connected graph, in whose checkerboard coloring all (say) black faces are triangles. (Cyclically 5-connected means that every 4-cut consists of the edges incident from a vertex, and there is no smaller cut; compare with remark 4.3.)

Proof of corollary 4.2. Just combine corollary 4.4 with theorem 4.3. For the lower bound use the checkerboard coloring of the net where the black faces are the triangles for each vertex of G. We have then $v(G) = -2\chi$, and in theorem 4.3, $N = e(G) = -3\chi$ and $|\mathcal{W}| = 2-\chi$. □

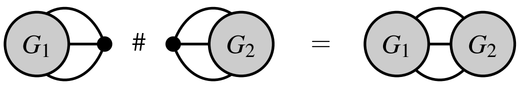
Proof of corollary 4.3. Use corollary 4.4 and that in (27) the number of (black) triangles

$$T = v(G) = -2\chi\,. \tag{34}$$

For asymptotic optimality, set G_χ so that $\mathcal{P}_{[T]} = \Gamma_{M(G_\chi)}$ for (28) (with (34)). □

It will become of some relevance that the graph volume is additive w.r.t. a simple operation we define now. This operation will be of fundamental importance for the paper and will continue occurring in different contexts[2].

Definition 4.1 The *composition* '#' of two cubic graphs is defined by



Note that this operation is highly ambiguous. It depends not only on the choice of vertices in G_1 and G_2, but also on the (mutual) cyclic ordering of their incident edges. It will be thus relevant to specify whether some way or all ways of performing the operation for two given graphs is considered. Usually all ways will be meant, but there is a crucial exception, in lemma 6.1.

Also note that planarity is not needed to define '#'; we will stick to planar graphs for the time being, but there are results for non-planar graphs as well (§5.4).

When $G_2 = \theta$, then $G_1\#\theta = G_1$. When $G_2 = \tau$, then composition is given by the triangle move (33). Since τ is symmetric, the result depends only on the choice of vertex v of G_1; we write for it $G_1\#_v\tau$.

The following is rather easy to see.

Lemma 4.2 If G_1 and G_2 are planar 3-valent graphs and are 3-connected, then $G_1\#G_2$ is (planar and) 3-connected, regardless of how the composition is performed. □

Lemma 4.3 If G_1 and G_2 are planar 3-valent 3-connected graphs, then

$$\mathrm{vol}(G_1\#G_2) = \mathrm{vol}(G_1) + \mathrm{vol}(G_2)\,, \tag{35}$$

for whatever way of performing composition. Consequently

$$\mathrm{vol}(L_{G_1\#G_2}) = \mathrm{vol}(L_{G_1}) + \mathrm{vol}(L_{G_2}) - 4V_8\,.$$

[2] It is called '⊕' in [Rs], for example.

Proof. The second equality is a consequence of the first and lemma 4.1. The first equality is explained by a straightforward generalization of the triangle move argument in the proof of lemma 3 of [vdV], as follows.

For a triangle move the augmentation of volume is explained by [vdV, Figure 13], showing that in each half-space an extra octahedron can be glued into the sphere of v_1 in $N(G_1)$ to obtain a simplicial decomposition of $S^3 \setminus N(G_1 \#_{v_1} \tau)$.

This explains that case $G_2 = \tau$. For arbitrary G_2, the octahedra in the preceding argument can be replaced by the ideal polyhedra which compose $S^3 \setminus N(G_2)$. □

Remark 4.2 Note in particular that when G is an iterated composition of the tetrahedral graph τ with itself, then L_G makes the left inequality in (25) exact. V.d. Veen was interested in these links, for which he proves the Volume conjecture, but in our context they thus represent the 'worst' way of augmenting volume.

The operation of composition gains some geometric flavor with lemma 4.3. This will motivate us to treat it in the sequel.

Proof of theorem 1.2. When changing the iterated composing graph in remark 4.2 to an arbitrary (3-valent 3-connected planar) graph, one sees as an easy consequence of lemma 4.3 and corollary 4.4 that there exists the 'stable volume-χ ratio' (2).

Of course the additive term $-2V_8$ in the denominators in (2) is not relevant for the limit. But it is for the supremum, which originates from a certain superadditivity property; that also lends a natural role to the lower bound in corollary 4.1.

An application of corollary 4.3 is the identification of the constant (2) as given in (3). This explains theorem 1.2. □

Note that corollary 4.1 would just imply $1 \leq \delta \leq {}^7\!/_6$. The computation related to conjecture 4.1 below can approximate experimentally this stable volume ratio δ from below. (See the last column of table 2.)

4.4. Number of link components

An immediate problem when studying L_G, in particular with the focus on knots, is: which L_G are relevant in taking the limit for volumes of alternating links L (with $\chi(L) = \chi(G)$) of a given number of components? In other words, for which G do links of what number of components occur as thickenings?

We bring the number of link components into the picture with the following result. (See §2.2 for the usage of '%'.)

Proposition 4.2 For $\chi < 0$,

$$v_\chi = v_{2-\chi,\chi} \geq v_{-\chi,\chi} \geq \dots \geq v_{\chi\%2,\chi}\,.$$

Proof of proposition 4.2. An easy modification of the proof of theorem 4.1 shows that for $v_{n,\chi}$ a similar formula involving the $\mathrm{vol}(L_G)$ holds, only that the maximum is taken over G, which have n-component markings $L = L_{G,O}$ for some O. Now it is to show that one can arbitrarily augment the number of components of L for fixed G by varying O.

For this use that there is another even-odd edge coloring of G, namely all edges even, giving rise to a diagram of the maximal number $n = 2-\chi$ of components. To conclude the claim for the other n, note that the change of orientation of any Seifert circle changes the number of components by 0 or ± 2. □

Then we can show that v_χ can be approximated by links of very special type.

Definition 4.2 Fix $\chi < 0$ and $1 \leq n \leq 2-\chi$ with $n+\chi$ even. We say that $P \subset \mathbb{N}$ is (n,χ)-*good*, if one of the following 3 conditions holds:

1) P is infinite and $n < 2-\chi$ and $\chi < -1$,

2) P contains arbitrarily large even numbers (i.e. $P \cap 2\mathbb{N}$ is infinite) and $n = 2-\chi$, or

3) P contains arbitrarily large odd numbers (i.e. $P \cap 2\mathbb{N}+1$ is infinite) and $n = -\chi = 1$.

We will give a proof of the following further-going statement in §8.2. This theorem is an application of (and was, in fact, original motivation for) our main result theorem 1.1.

Theorem 4.4 Fix $\chi < 0$ and $1 \le n \le 2-\chi$ with $n+\chi$ even, and let $P \subset \mathbb{N}$. Then the following 3 conditions are equivalent:

1)
$$v_{n,\chi} = \sup \{ \,\mathrm{vol}(L) : n(L) = n, \chi(L) = \chi, c(L) \in P, \ L \text{ special alternating} \,\}, \tag{36}$$

2)
$$v_{n,\chi} = \sup \{ \,\mathrm{vol}(L) : n(L) = n, \chi(L) = \chi, c(L) \in P, \ L \text{ alternating} \,\}, \tag{37}$$

3) P is (n,χ)-good.

An appealing improvement of proposition 4.2 is that in fact all inequalities therein are equalities.

Here comes in the statement of theorem 1.3. What this says is that (some of) the graph(s) G maximizing $\mathrm{vol}(L_G)$ in theorem 4.1 has a 1- or 2-component marking. (By proposition 4.2, augmenting the number of components is never a problem.) This appears quite plausible, since most graphs, as mentioned, have such markings. For knot markings exceptions start at $\chi = -9$; see table 4 or [St10]. So in particular we see theorem 1.3 to be true for odd $\chi \ge -7$.

However, what will be said in the proof of theorem 7.3 at least very strongly cautions that deciding the existence of knot markings is likely NP-hard. The complexity of this problem is also manifested in the fact that the full argument for theorem 1.3 is far longer than the above observations. A substantial part of it was moved out to [St10]. See theorem 6.2 in §6.

In relation, here is a more specific related conjecture, which is motivated by lemma 4.3 and the computation in the next subsection. (For the connection, see also remark 6.1 below.)

Now we consider *cyclically 4-connected* graphs. This property turns out to be of considerable importance in the following, so we give a formal (though not entirely standard) definition.

Definition 4.3 We call a 3-cut of a 3-valent graph G to be *essential* if it does not consist of the three edges incident from (i.e., the star of) a vertex.

For 3-valent graphs G the common definition of *cyclic 4-connectedness* can be paraphrased to require that G is 3-connected and has no essential 3-cuts. That is, if some ≤ 3 edges disconnect G, then they are the 3 edges incident to some vertex of G. To save space, we will often write "c4c" for "cyclically 4-connected".

In this way c4c means "prime" with respect to composition: G is c4c iff whenever $G = G_1\#G_2$, one of G_1 or G_2 is θ.

Lemma 4.4 In a 3-connected *planar* 3-valent graph, cut curves of essential 3-cuts can be made disjoint. Any two such cuts cannot have more than one edge in common. □

We recorded the above straightforward observation for later reference. There are ways to deal with cuts in non-planar graphs as well, but they meet added technical difficulties (e.g., fixing a spatial embedding and moving separating surfaces; see §5.4). The following is worth keeping track of, though.

Lemma 4.5 In a 3-connected 3-valent graph $G \ne \theta$ the decomposition

$$G = G_1\#\cdots\#G_k \tag{38}$$

for c4c graphs $G_i \ne \theta$ is unique up to permuting factors. Write $k = k(G)$. Also, $k-1$ is the number of essential 3-cuts in G. □

One must note that this uniqueness carries some slight imprecision about which vertices the composition is performed at. Thus if (38) and $G = G_1'\#\ldots\#G_{k'}'$, then we claim that $k = k'$ and $(G_1,\ldots,G_k)$ is a permutation of $(G_1',\ldots,G_k')$, though the vector alone is not sufficient to reconstruct G via (38). In fact, this ambiguity will be quite important at some point (see the proof of theorem 1.3 in §6).

Conjecture 4.1 A (planar 3-valent 3-connected) graph G with $\chi(G) = \chi$ and $\mathrm{vol}(L_G) = v_\chi$ is cyclically 4-connected.

There are two versions depending on whether 'A' should mean 'some' (weak version) or 'every' (strong version). In either way, the property seems not at all easy to verify. At least we can confirm:

Proposition 4.3 Conjecture 4.1 (with 'every') is true for $\chi \geq -21$.

The reason we present this statement here is in part in order to emphasize how the insight we gained from [vdV] simplifies the concrete computation of the maximal volume. Later we will see that theorem 1.3 can be reduced, almost without computation, to a purely combinatorial property (see theorem 6.2).

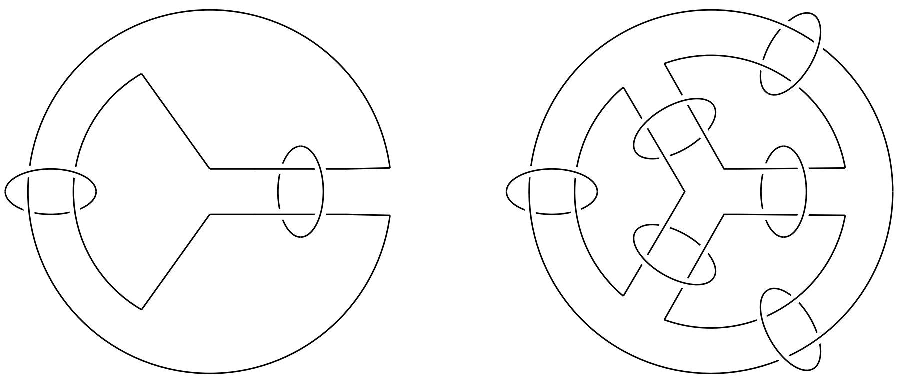
Proof. Lemma 4.3 explains that it is necessary to know $\mathrm{vol}(L_{G'})$ for planar c4c G' with $\chi(G') \geq \chi(G)$ in order to determine $\mathrm{vol}(L_G)$ for all cubic 3-connected planar G. This is the first enormous simplification of the calculation of v_χ in corollary 4.4.

Now, we have J.Weeks' software, as part of Hoste and Thistlethwaite's [HT], which can be used to calculate, in principle, $\mathrm{vol}(L_G)$. However, L_G tends to get quickly very complicated, which slows down the computation considerably. Much of this complexity is clearly redundant, since through (30) and lemma 4.1, we know that $\mathrm{vol}(L_G)$ can be obtained from $\mathrm{vol}(G)$. But unfortunately we had no tool available to calculate polyhedral volume directly.

Here, we can use v.d. Veen's ideas, and obtain $\mathrm{vol}(G)$ – and thus $\mathrm{vol}(L_G)$ – from a link much simpler than L_G. Namely, $\mathrm{vol}(G)$ can be represented as the volume of a link L_G' obtained when G is 1-unzipped along a *perfect matching*.

A perfect matching of G is a collection S of edges such that for every vertex of G there is exactly one incident edge in S. There has been considerable graph-theoretic interest in these matchings; see [CS, EK+]. For us it is only relevant that they always exist for the G we consider. This is known as Petersen's theorem. It is worth remarking here that the existence of these matchings for a planar 3-connected graph follows from Tait's theorem (presented in [BN2, §3]) and the 4CT.

It is then clear that (using a program I wrote) one perfect matching can be easily found for each G. The below diagrams show L_G and L_G' for the tetrahedral graph $G = \tau$.



(39)

L_τ' $\qquad$ L_τ

Note that the links L_G' so obtained have at most[3] one third of the crossing number of L_G, which makes their treatment using J. Weeks's software far easier.

[3] While the diagram of L_G is easily seen to be of minimal crossing number, the one for L_G' is not necessarily so: in (39), for example, L_τ' are the Borromean rings, so that the displayed diagram can be simplified by two more crossings.

With these two improvements, computations were possible until $\chi = -21$. We postpone computational data to table 2 in subsection 5.3, but notice already here that this computation allows us to approximate from below the value of δ in (3).

The entries of table 2 with a question mark are based on the prediction that the graph is in fact cyclically 5-connected (*c5c*; see following remark). Note that if this is true, the calculation would further drastically simplify, and might be possible for several more χ (if this is desired).

The gradually improving ratio in the last column of table 2 shows that conjecture 4.1 (in the strong version) is true up to, and including, the first row with a question mark, which is $\chi = -21$. □

Remark 4.3 As far as computation was possible, there is always a unique graph maximizing the volume. The intuition that this graph is "as complicated as possible" is also reflected in the expectation that, whenever such graphs exist, the graph is in fact cyclically 5-connected (c5c). This property means, for 3-valent graphs, c4c and that every 4-cut leaves a single edge as one of the connected components of its complement. (Compare with remark 4.1 in the case of 4-valent graphs.) In particular there is no 4-gonal face. Such graphs occur for $\chi = -10$ (the dodecahedral graph) and $\chi \le -12$. In this range, the prediction that a c5c graph maximizes volume is confirmed until (but *not* including) $\chi = -21$.

5. The sl_N weight system polynomial

5.1. Calculation via markings

It is easy to see after our work that theorem 3.3 can be extended to alternating links of any n and χ. Here it is convenient, as in [SV], to consider links up to mirroring but *with component orientation*.

Then again one has a period of polynomials, whose coefficients in degree $-3\chi-1$, call them $C_{n,\chi,e}$ and $C_{n,\chi,o}$, depend only on the parity of the crossing number. Some of them may now be 0, but this happens only in exceptional cases we will classify below.

Keep in mind that because of the said in (10) we ignore reversal of orientation of all components of the (alternating) link altogether (which immaterializes orientation of knots), and because of (41) we identify mirror images when counting links. This stipulation is implicit in the quantites $C_{n,\chi,*}$. (In the calculations given in §5.3, knot orientation has also been ignored.)

Compare to the definition of $C_{g,o} = C_{1,1-2g,o}$ and $C_{g,e} = C_{1,1-2g,e}$ in (13). To generalize this, let $a_{n,\chi,c}$ be the number of prime alternating links of $\chi(L) = \chi$, $c(L) = c$ and $n(L) = n$ components (again up to mirroring and simultaneous reversal of orientation of *all* components). Then one can express (with a normalization similar to (13), as explained in (14))

$$C_{n,\chi,e} = 2^{-3\chi-1}(-1-3\chi)!\lim_{c\to\infty}\frac{a_{n,\chi,2c}}{(2c)^{-3\chi-1}}, \quad C_{n,\chi,o} = 2^{-3\chi-1}(-1-3\chi)!\lim_{c\to\infty}\frac{a_{n,\chi,2c+1}}{(2c+1)^{-3\chi-1}}. \tag{40}$$

To simplify our language, let us generalize the definition of maximality of §3.1 (occurring also in [SV]).

Definition 5.1 Call a generator of given n and χ *maximal*, if it has -3χ $\sim$-equivalence classes. Call a generator *even* or *odd* if its crossing number is even or odd. Let G be the 3-valent graph obtained from a maximal generator D. Then vertices of G correspond to Seifert circles of D of valence 3. Orient cyclically the edges around the vertices of G according to the orientation of the corresponding Seifert circles in D.

The problem to control even and odd crossing number generators with the maximal number of $\sim$-equivalence classes is also related to the markings in (18). We need to equip ourselves with some more terminology.

Definition 5.2 Let $D_{G,O}$ be the special alternating diagram corresponding to a 3-connected 3-valent planar graph G with choice of vertex orientation O (= orientation of the corresponding Seifert circles = even-odd edge coloring). We call O also a *marking* of G. Let $L_{G,O}$ be the link represented by $D_{G,O}$. We call O an *n-component marking* (or *knot marking* for $n = 1$), if $n(L_{G,O}) = n$.

To make the combinatorics more explicit, notice that a maximal generator diagram D can be explained through its Seifert graph

$$G' = \Gamma(D)\,.$$

This graph is a reduced bipartite bisection of a planar 3-connected trivalent graph G. This means that G' can be specified by a $+/-$ marking of the vertices of G, where edges in G are bisected iff they connect equally marked vertices.

Obviously G' has only vertices of valence 2 and 3. This means that it cannot have a cut vertex, unless it has an isthmus (an edge whose deletion disconnects G'). Then D would have a reducible crossing.

By the said at the end of §2.4, one sees thus that a maximal generator D is special and prime. (See also [SV], where the knot case was discussed, but the constraint to knots is not relevant for this reasoning.) Since D is alternating, there are two choices: positively and negatively special alternating. We can fix this ambiguity by, say,

$$\textit{always considering the positively special alternating diagram}\,. \tag{41}$$

Thus $G' = \Gamma(D)$, with its planar embedding, determines D (see the end of §2.4). Furthermore, the argument for knots based on [MT] (see above (14)) extends to show that D does not admit a flype, and thus it is the only alternating diagram representing its link.

Notice that the convention (41) is that in $G' = \Gamma(D)$ all *edges* are labelled $+$, in definition 2.1. This labelling just reflects that D is special alternating, and has nothing to do with the $\pm$ labelling of the vertices of G in a marking O. Since it brings no new information, but only leads to confusion with O, the labelling of edges in $\Gamma(D)$ will henceforth be ignored throughout.

Definition 5.3 For a graph G and marking O, let O_v be O with the marking of a vertex $v \in V(G)$ reversed. Note that we always have $n(D_{G,O_v}) - n(D_{G,O}) \in \{0, \pm 2\}$. We call v *good* (in O) if $n(D_{G,O_v}) = n(D_{G,O})$, and *bad* otherwise.

An important result about good vertices, which will also make its appearance later, was originally proved in [BV], but is given here in a reformulated version using the work of [SV].

Lemma 5.1 (Bacher-Vdovina [BV, proposition 2.1]) If $n(D_{G,O}) = 1$ (i.e. O is a knot marking), then O has exactly $-1-\chi(G)$ good vertices (and exactly $1-\chi(G)$ bad ones).

This fact is remarkable, because it means that the number of good vertices is independent on O, and dependent on G only through $\chi(G)$. Our work will contain noteworthy implications of this circumstance. Nothing like this holds for a general marking (with more components); see also the end of §7.3.

Remark 5.1 Bacher-Vdovina call good vertices positive and bad vertices negative. Their signing thus has nothing to do with the one we introduced in definition 3.1. In fact, the dependence between the sign of a vertex (according to our definition 3.1) and its good/bad status is very difficult to understand.

Let $W_N(G)$ be the sl_N-weight system polynomial of a 3-valent (unmarked) graph G, as specified in (18). It is useful to assume that G is (at least) 2-connected, because of the below trivial lemma.

Lemma 5.2 If G has split components G_i, then $W_N(G) = \prod_i W_N(G_i)$, and if G is connected but not 2-connected, then $W_N(G) = 0$. □

The following properties of $W_N(G)$ are well-known.

Proposition 5.1

a) All coefficients of $W_N(G)$ are even.

b) All powers of N in $W_N(G)$ are positive, do no exceed $2-\chi(G)$, and have the same parity as $\chi(G)$.

c) $W_N(G)$ vanishes at $N=1$.

d) $[W_N(G)]_{2-\chi(G)}$ is the number of planar embeddings of G on an oriented sphere if G is planar, and 0 otherwise. In particular, for a 3-connected graph G this number is 2 if G is planar, and 0 otherwise.

e) If G is planar, then $|W_2(G)|$ is $2^{v(G)/2-2}$ times the number of 4-colorings of the planar map whose borders are given by (some planar embedding of) G.

f) $W_N(G)$ satisfies the IHX relation of [BN].

A few remarks on the origin (rather than formal proofs) of these properties seem appropriate:

1) Properties a)-c) are obvious from the definition. (The evenness property a) follows from the complementary marking.) These properties can jointly be written as

$$W_N(G) \in 2N^{\chi\%2}(N^2-1)\mathbb{Z}_{\frac{-\chi\%2-\chi}{2}}[N^2],$$

with $\mathbb{Z}_k[x]$ being the degree $\leq k$-part of $\mathbb{Z}[x]$.

2) Properties d) and e) are (mainly) established in [BN2], which gives Bar-Natan's remarkable reformulation of the 4-Color-Theorem:

$$[W_N(G)]_{2-\chi(G)} \neq 0 \Longrightarrow W_2(G) \neq 0\,.$$

The sign of $[W_N]_{2-\chi}$ was not considered in [BN2], but this sign is positive by our convention to use the planar embedding of G (since the $+$-marking O with $O(v)=+$ for all $v \in V(G)$ is obviously spherical). That all planar embeddings give the same sign, and the property for 3-connected graphs in d), are consequences of Whitney's theorem (see the introduction and the explanation below (18)).

3) Property f) was known from [BN].

Apart from their introduction in [BN] and subsequent appearance in [BN2], these polynomials seem to have been little studied, and Bar-Natan's results d)-f) are the most substantial known facts about them.

We separated the even and odd parts of W_N because this way they are of more interest to us, as they are related to the enumeration of alternating links of given number of components and genus.

Then the explanation below definition 5.2 (and with the counting conventions clarified at the beginning of this subsection) shows the following.

Proposition 5.2

$$\left[W_{N,+}\left(\sum_{\chi(G)=\chi} G\right)\right]_n \geq 2C_{n,\chi,e}\,, \qquad \left[W_{N,-}\left(\sum_{\chi(G)=\chi} G\right)\right]_n \geq 2C_{n,\chi,o}\,, \tag{42}$$

where the sums on the left go over all 3-connected 3-valent planar graphs G of given number of vertices (equivalent up to isotopy on the sphere). □

The factor 2 comes from taking the complementary marking (reverse all signs of all vertices). The reason to have only inequality is the different handling of symmetries. When G admits symmetries, then for different colorings O the $L_{G,O}$ may be the same, although the colorings are counted multiply in $W_{N,*}$. If one can efficiently bound the number of symmetries, we have for small χ in both inequalities of proposition 5.2 '$\approx$' instead of '$\geq$', and hence a Lie-algebraic approximation of $C_{n,\chi,e} - C_{n,\chi,o}$. In theorem 7.4 we can do this for $n=1$. On the opposite side, it would be interesting whether $W_{N,-}$ and $W_{N,+}$ have themselves some more specific meaning in the Lie/Vassiliev theory context.

5.2. A relation to the volume

We mentioned (below theorem 4.2) two situations, in which the hyperbolic volume exhibits a relation to a quantum algebra structure: the Volume conjecture, and Dunfield's inequalities. We now elaborate on another dependence, which is more restricted, but more explicit. It was observed experimentally during our computations. We will try to explain how far it can be understood.

The planar ones among the graphs G, for odd $\chi(G)$, occurred in our previous discussion of enumeration of alternating knots. It motivated a more detailed study of their weight systems. In particular, it is curious to see how well the coefficients of $W_{N,-}+W_{N,+}$ approximate the total number of generators, and $W_N = W_{N,-}-W_{N,+}$ the difference between those of even and odd crossing number. Since we noticed that the error in these approximations results from graphs with symmetries, it is interesting to observe that its relative contribution decreases as the genus goes up. Similarly, one is interested in the volumes of the links L_G, in order to calculate v_χ. We already explained that both have a relation to our generator approach.

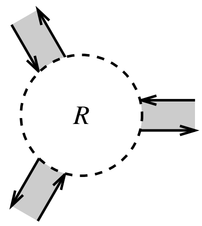
However, there is a very different similarity between $W_N(G)$ and $\mathrm{vol}(L_G)$, which seems to stem from their behaviour under graph composition. For the volume we saw this in lemma (4.3). For $W_N(G)$, this is the subject of the following easy lemma. (Its non-planar case will be needed later.)

Lemma 5.3 If G_1 and G_2 are (not necessarily planar) 3-valent graphs, then

$$W_N(G_1\#G_2) = \frac{W_N(G_1)W_N(G_2)}{2N(N^2-1)},$$

regardless of how composition is performed.

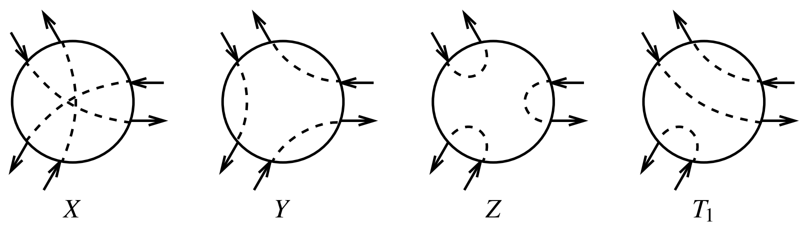
Proof. We fix the vertex v_1 of G_1 at which composition is performed. Then we regard each thickening $T(G_1)$ of G_1 (which is always orientable) in a complement of a neighborhood ball B of v_1, in such a way that $T\cap\partial B$ are three segments exposing the same side of $T(G_1)$ on the top (this may require to half-twist a band inside and outside of B).



(43)

This determines the orientation of the strands which enter $R = T(G_1)\setminus B$.

Depending on the orientation of the strands there are six ways in which in- and outputs can be connected (for a moment disregarding closed components).



(44)

The connections X, Y and Z are rotationally symmetric, and T_1 gives rise to the other two patterns T_2 and T_3 by $\pm 2\pi/3$ rotation.

Now consider the restricted weight system $W_N(G_1,v_1)$, defined as follows. It is a $\mathbb{Z}[N]$-linear combination of X, Y, Z and T_i. We consider $2^{\#v(G_1)-1}$ markings of the vertices of G_1 outside v_1.

The coefficient

$$W_{P,1} := [W_N(G_1,v_1)]_P \qquad (45)$$

of each pattern $P \in \{X,Y,Z,T_1,T_2,T_3\}$ in $W_N(G,v_1)$ is obtained by adding a monomial $\pm N^k$ for each thickening $T(G_1)$ with connectivity of $T(G_1)\setminus B$ matching P and k closed components (within $S^3\setminus B$). The sign is given by the parity of negative markings of the thickening (which does not take v_1 into account).

Now observe that when all markings (outside v_1) are reversed, the patterns Z and T_i do not change connectivity, and X and Y are exchanged, while the sign is reversed (since $\#v(G)-1$ is odd for a cubic graph G). It follows that

$$W_Z = W_{T_i} = 0, \quad \text{and} \quad W_X = -W_Y\,.$$

This gives

$$W_N(G_1) = (N-N^3)(W_{X,1}-W_{Y,1}) + N^3 W_{Z,1} \;=\; 2(N-N^3)W_{X,1}\,.$$

Similarly (with $W_{P,2}$ defined analogously to (45))

$$W_N(G_2) = 2(N-N^3)W_{X,2}\,,$$

while

$$W_N(G_1\#G_2) = 2(N^3-N)\,W_{X,1}\,W_{X,2}\,.$$

This proves the claim. □

Remark 5.2 There is some deeper insight into this type of behaviour gained by Vogel [Vo]. He introduced an algebra Λ acting on trivalent graphs by what we called composition. Then he showed that Λ induces (multiplicative) characters on the level of Lie-algebraic weight systems (not only the one for sl_N). I was pointed out that Vogel's work (Corollary 4.6 in [Vo]) gives a, similarly easy, alternative proof of Lemma 5.3. However, our argument will be relevant later (see the proofs of lemma 6.1 and, as will become relevant at a separate place, of theorem 7.3).

Lemma 5.3 motivates the following definition.

Definition 5.4 Let for a cubic graph G, the *reduced sl_N-polynomial* $\widetilde{W}_N(G)$ be

$$\widetilde{W}_N(G) \;=\; \frac{W_N(G)}{2N^3-2N}\,.$$

Call a polynomial in $\mathbb{Z}[N]$ a *basic polynomial* if it occurs as reduced sl_N-polynomial of a c4c planar cubic graph G.

Thus an analogy between W_N and $\mathrm{vol}(L_G)$ emerges. This raises questions about their possible relationship. In particular, it is legitimate to ask (how well) does one determine the other. (The first hint to such a relationship was given by the computation for odd $\chi \geq -7$, summarized in table 3 below.)

It is suggestive to restrict oneself to c4c graphs. Such graphs can be rendered, for given χ, by the program `plantri` of Brinkmann and McKay [BM], which was of fundamental assistance in all following calculations. Some computation was able to answer the above question easily (negatively) at least in one direction.

Example 5.1 There exist two c4c graphs of $\chi = -9$ with different W_N but equal $\mathrm{vol}(L_G)$ (at least up to 10^{-10}). Thus $\mathrm{vol}(L_G)$ does not determine $W_N(G)$.

However, the same computation temporarily added intrigue to the reverse direction. No counterexample could be observed, despite that coincidences of W_N are by no means sporadic.

Example 5.2 Among the 313 c4c graphs with $\chi = -11$, there are 114 coincidences of W_N (i.e. only 199 different values occur), but for all such cases $\mathrm{vol}(L_G)$ is equal (at least up to 10 significant digits). Similarly occurs with $\chi = -12$, where there are 617 coincidences of W_N among 1357 c4c graphs, and $\chi = -13$ with 3242 coincidences in 6244 graphs.

This justifies to highlight the question.

Question 5.1 Does $W_N(G)$ determine $\mathrm{vol}(G)$ (or equivalently, $\mathrm{vol}(L_G)$)?

We see from composition that in some way the multiplicative structure of W_N seems to correspond to additive contributions to $\mathrm{vol}(G)$. In other words, the volume turns out to be a type of "logarithm" of the weight system polynomial. Based on some further verification, we were able to conclude the following regarding question 5.1.

Proposition 5.3 The answer to question 5.1 is positive for $\chi \geq -14$, but negative for $\chi = -15$.

Proof. It is a consequence of lemmas 4.3 and 5.3, that if question 5.1 can be answered positively for c4c graphs G (up to given $-\chi$), then a factorization of $\widetilde{W}_N(G)$ into basic polynomials determines $\mathrm{vol}(G)$ for arbitrary (3-connected planar cubic) graphs G (up to that $-\chi$).

To the set of basic polynomials $P_i(N) = \widetilde{W}_N(G) \in \mathbb{Z}[N]$ for c4c graphs G, a positive answer to question 5.1 will yield corresponding real numbers $c_i = \mathrm{vol}(L_G) - 4V_8 > 0$. (For V_8 recall (23).) Then for every trivalent planar 3-connected graph G a factorization

$$\widetilde{W}_N(G) = \prod_{j=1}^{k} P_{i_j}(N) \tag{46}$$

(with i_j not necessarily different) will give

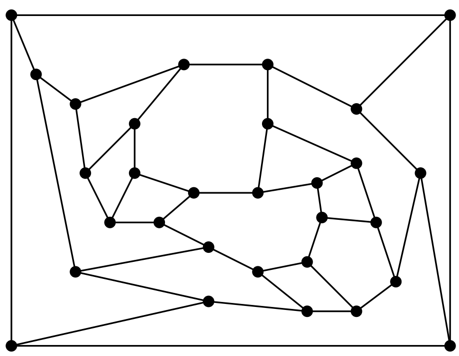
$$\mathrm{vol}(L_G) = 4V_8 + \sum_{j=1}^{k} c_{i_j}\,. \tag{47}$$

To examine question 5.1, we focused on c4c graphs G, and extended the calculation in example 5.2. In that example we determined W_N based on the original definition. For $\chi < -13$, this option becomes too time consuming to calculate W_N for all c4c graphs.

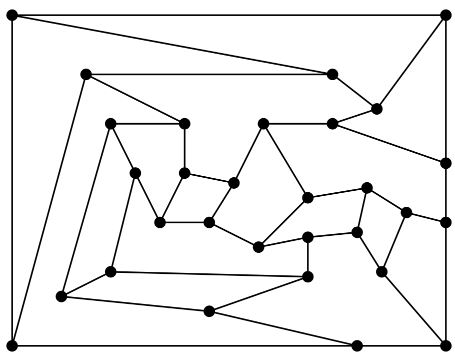
Thus we improved our method for calculating W_N by using an idea similar to lemma 4.3. For a c4c graph G, we seek 4- and 5-cuts in G so that the two parts G is split into have a large number of vertices. Then we calculate the parts of W_N on either side and combine them. This improved performance considerably: for $\chi = -13$ from 3 weeks for the old method to 5 minutes. The limit of computability reached conveniently $\chi = -17$, and with great effort $\chi = -18$.

First, we verified that equal W_N implies equal volume for c4c graphs. Then, from the W_N polynomials obtained, we reduced by $W_N(\theta) = 2N^3 - 2N$ and removed all coincidences. The list so obtained contains the basic polynomials. (Note that the degree of a basic polynomial is $-1-\chi(G)$.) Then we checked that this set does not yield a polynomial of degree ≤ 13 which factorizes as in (46) as a product of basic polynomials in two different ways (up to order of factors).

This method gave a positive answer to question 5.1 for $\chi \geq -14$. But the calculation of c4c graphs showed some counterexamples for $\chi = -15$. These are anything but trivial, and thus it is justified to give one pair explicitly:



$\mathrm{vol}(G) \approx 120.7043405$



$\mathrm{vol}(G') \approx 120.7043733$

g	$W^+_{\text{total}} + W^-_{\text{total}}$	$W^+_{\text{total}} - W^-_{\text{total}} = W_{\text{total}}$
1	$2N(1+N^2)$	$2N(-1+N^2)$
2	$2N(6+9N^2+N^4)$	$2N^3(-1+N^2)$
3	$2N(672+1644N^2+239N^4+5N^6)$	$2N^3(-28+23N^2+5N^4)$
4	$2N(71680+259444N^2+74375N^4+4051N^6+50N^8)$	$2N^3(-236-205N^2+391N^4+50N^6)$

Table 1

$$W_N(G) = W_N(G') = 10496N^3 - 1536N^5 - 9760N^7 - 7100N^9 + 6672N^{11} + 1156N^{13} + 70N^{15} + 2N^{17} \qquad \square$$

Despite that thus the answer to question 5.1, in this form, is negative in general, the mystery around it is not lifted. Examining the c4c graphs with $\chi \geq -18$ (see the second paragraph below (47)) has shown almost all among thousands of pairs of c4c graphs G, G' with $W_N(G) = W_N(G')$ to have equal volumes. Even in non-coincidences, W_N predicts the volume with unusual accuracy: when $W_N(G) = W_N(G')$, the relative difference

$$\frac{|\,\text{vol}(G) - \text{vol}(G')\,|}{\text{vol}(G)} < 6 \cdot 10^{-7}.$$

We have not pursued related questions, for example, whether the pair $(W_{N,+}, W_{N,-})$ (*cf.* §3.2) determines the volume. Such relationships seem too speculative as long not more than lemmas 4.3 and 5.3 is brought to light.

Remark 5.3 Lemma 5.3 is far from explaining in full the multiplicative structure of $W_N(G)$. The calculation in example 5.2 with planar c4c graphs G for $\chi \geq -13$ has shown most $\widetilde{W}_N(G)$ to be irreducible. For the few others, no conclusive patterns in factoring behavior could be observed. This raises a more specific question: can a polynomial $\widetilde{W}_N(G)$ factor as in (46) in two different ways into basic polynomials (up to order of factors)? Using the method outlined for proposition 5.3, I was able to check that multiple factorizations do not occur for $\max\deg \widetilde{W}_N \leq 17$ (i.e., $\chi \geq -18$).

5.3. Some computations of volume and sl_N-polynomials

The total even and odd parts of W_N, summed over all planar cubic 3-connected graphs G for odd $\chi \geq -7$, are given in table 1.

Note that the leading coefficients in the parentheses (i.e. with the factor 2 removed) always give the number of graphs in the second row of table 4.

The coefficients of $W^+ + W^-$ approximate from above the number of maximal generators of given number of components (analogously to the remark for knots in the proof of theorem 7.4). For knots the approximation is given by the ratio of the linear term in the second column of table 1 vs. the number in the fourth last row of table 4. For genus 4 the two numbers still differ by a factor of almost 1.7, but these approximations will improve when g increases and symmetries fade away (see the proof of theorem 7.4).

The next table, table 2, shows the computation of maximal volume, including the ratio of the maximal volume to the bound in (3).

Passing, by Adams' result [Ad], from generators to graphs in calculating v_χ saves an enormous amount of work. (For $\chi = -7$ it reduces the work by a factor of > 800.) The subsequent enormous simplification was achieved by the insight in §4.3. In particular, it allowed us to restrict ourselves to cyclically 4-connected graphs G when $v_{\chi'}$ for $\chi' > \chi$ are known (their number is given in the second column). Moreover, for such graphs we could determine $\text{vol}(L_G)$ by computing the volume of a link much simpler than L_G. (See the proof of proposition 4.3.) The last column gives the resulting approximations of δ defined (2) and given in (3) (see §4.3). This approximation seems slow in particular because of the constant $-10V_4$ on the right of (29).

χ	# G	max. vol. $v_{\chi\%2,\chi} = v_\chi \approx$	$\frac{v_\chi}{(-2-6\chi)V_8} \approx$
-1	0	$4V_8$	1
-2	1	$10V_8$	1
-3	0	$16V_8$	1
-4	1	82.7139821	1.02616
-5	1	105.8287878	1.03159
-6	2	129.3489143	1.03835
-7	4	153.3818722	1.04659
-8	10	177.4119910	1.05265
-9	25	201.2753427	1.05645
-10	87	226.3130252	1.06498
-11	313	249.8559926	1.06554
-12	1357	274.4419691	1.07007
-13	6244	298.6574449	1.07256
-14	30926	323.0102454	1.07514
-15	158428	347.2260495	1.07694
-16	836749	371.5783354	1.07891
-17	4504607	395.9131479	1.08059
-18	24649284	420.3944823	1.08246
-19	136610879	444.7966230	1.08394
-20	765598927	469.2471319	1.08538
-21	4332047595	493.7021266(?)	1.08669(?)
-22	?	518.1952668?	1.08796?

Table 2: Volume computation results. The second column gives the number of c4c graphs, which were examined in determining the maximal volume. The calculation for $\chi = -21$ was already rather hard and less reliable, being plagued by diverse technical difficulties. It was heuristically confirmed by verifying c5c graphs only (see remark 4.3). This (simplified) computation gave also the suggested value for $\chi = -22$ (and can be used to obtain heuristic values for more χ).

g	$W_N(G)$	$\mathrm{vol}(L_G) \approx$	$\#G$
1	$2N(-1+N^2)$	14.65544950684	1
2	$2N^3(-1+N^2)$	58.62179802734	1
3	$2N^5(-1+N^2)$	102.5881465478	3
	$2N^3(-12+11N^2+N^4)$	104.6971563678	1
	$2N^3(-16+15N^2+N^4)$	105.8287878445	1
4	$2N^7(-1+N^2)$	146.5544950684	24
	$2N^5(-12+11N^2+N^4)$	148.6635048883	11
	$2N^5(-16+15N^2+N^4)$	149.7951363650	7
	$2N^3(-48+16N^2+31N^4+N^6)$	150.5135355577	1
	$2N^3(-64+63N^4+N^6)$	151.0133385172	1
	$2N^3(-1+N^2)(12+N^2)^2$	150.7725147082	1
	$2N^3(24-38N^2+13N^4+N^6)$	151.3320885599	2
	$2N^3(-40+18N^2+21N^4+N^6)$	152.2447643764	1
	$2N^3(-16+15N^4+N^6)$	152.6890862227	1
	$2N^3(28-39N^2+10N^4+N^6)$	153.3818721750	1

Table 3

The following table 3 shows the weight system polynomials and volumes (with frequency) that occurred for the graphs of given genus $g=(1-\chi)/2\le 4$. (Our focus on knots led us to consider only odd χ.) It initially hinted to the possible Weight system-Volume-relation (question 5.1).

Remark 5.4

a) A *cut algorithm* was used to optimize the (exponential) cost of computing $W_N(G)$ for planar G, so that it is efficient to process enough practical examples. The algorithm tries to find a cut in G which is simultaneously small and divides G into close to half the number of vertices on either side of the cut circle. The cut circles were sought using the planar embedding rendered by the program of Brinkmann and McKay [BM]. This cut algorithm had to be substantially modified for non-planar G (see §5.4).

b) We have seen many non-isomorphic graphs giving rise to the same volumes. (This is explained partly by lemma 4.3, but by far not entirely, as shows example 5.2.) Thus the volume tells little about the injectivity of $G\longmapsto L_G$, even although it usually distinguishes better between links than many other invariants. Contrarily, for example the five links for $\chi=-5$ are distinguished by the Jones polynomial taken up to units in $\mathbb{Z}[t,t^{-1}]$ (although the leading and trailing coefficients still show coincidences). Note that, because of the lack of natural orientation on L_G, and the high crossing number, calculations for the other polynomials or for smaller χ make little sense, or are not worthwhile.

c) Even if not always, for many graphs more small degree terms in N vanish in W_N (beside the linear one). Graphs with high $\min\deg_N W_N$ occur more frequently than such with small $\min\deg_N W_N$, and have smaller $\mathrm{vol}(L_G)$. This is mostly clarified by the behaviour of vol and W_N under composition with simple graphs – like the tetrahedral graph (see remark 4.2). We see that the smallest volume we obtain for each genus g is $(12g-8)V_8$.

d) A feature of the algorithm used to generate the tables was that the graphs G were ordered (increasingly) by the number of their spanning trees. In this ordering graphs with the same W_N always appeared consecutively. The largest groups of graphs with equal W_N occur towards the beginning of the list (i.e. have few spanning trees), and are those with high $\min\deg_N W_N$ and small $\mathrm{vol}(L_G)$. On the opposite end, it appears that the G maximizing $\mathrm{vol}(L_G)$, and hence relevant for v_χ, is the one with the most spanning trees. (Compare with the remark on [Df] below theorem 4.2.)

5.4. Non-planar graphs and further study

Obviously, the sl_N polynomial is defined for non-planar graphs G as well, where our geometric connection to volume breaks down. Nevertheless, I have invested very extensive effort to make W_N practically computable for non-planar graphs, including implementing and optimizing the cut algorithm. (For non-planar graphs, one cannot identify a cut through a cut circle, so some adjustments were necessary.)

We will give output of this study of the sl_N polynomial in more detail at a separate place. The following is a simple but remarkable consequence of the Bacher-Vdovina lemma 5.1, which cannot be left unmentioned, though.

Theorem 5.1 If $G \neq \theta$, then the N-linear term in $W_N(G)$ is trivial.

Among some further new properties, we will find that W_N exhibits graphs as Hamiltonian. While Bar-Natan proved that the polynomial obstructs to planarity, we now know that (by itself) W_N does not determine planarity.

6. Existence and non-existence of minimal markings

In the sequel we return to the question considered in §4.4 on the number of components occurring for (markings of) a graph G. Pre-eminently, this entails the question when the minimal possible number of components occurs. Let us thus say something about when G has a 1- or 2-component marking. To simplify language, we define as follows.

Definition 6.1 Let G be a planar 3-valent graph. We call a marking of G *minimal*, if it is a knot marking for $\chi(G)$ odd, and a 2-component marking for $\chi(G)$ even.

Theorem 6.1 If $\chi(G)$ is odd, then G admits a knot marking iff it can be obtained recursively from θ by a sequence of (graphic) α, β and γ transformations, as defined in [V] (see figure 1).

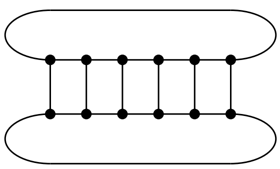
If $\chi(G)$ is even, then G admits a 2-component marking iff some application of the move (53) gives a graph G' (of odd χ) admitting a knot marking.

Proof. The first claim is a consequence of the work of [BV]. For the second part note that the existence of a 2-component marking implies a knot extension (53), while the smoothing out of a crossing in any knot marking of the r.h.s. of (53) gives a 2-component marking of the l.h.s. □

The problem which graphs admit knot markings (or Wicks forms), although having a recursive solution [V], seems too difficult to solve explicitly. A generic graph would likely have a knot marking, but graphs without knot markings exist. In fact we have the following, which is shown in [St10]:

Proposition 6.1 ([St10]) There exist 3-connected 3-valent planar graphs of both parities of χ with the minimal component number $\min\deg_N W_{N,\pm}$ of a marking arbitrarily large.

The explicit examples given in [St10] look as follows. Consider the 'ladder' graph A_n (or 'wheel' [BGRT]; depending on the embedding) of n stairs (or spokes). The representative example for $n = 6$ is



.

We build a graph B_n out of A_n by the local replacement (33) (which is composition with the tetrahedral graph τ) at each vertex. The graph B_n is planar, and 3-connected if $n \geq 3$. We argued that

$$\lim_{n\to\infty} \min\deg_N W_{N,\pm}(B_n) = \infty\,.$$

In the opposite direction, it is worth asking for a simple property of a graph that ensures a knot marking. The B_n suggested at first to exclude 3-cycles (i.e. consider graphs of *girth* ≥ 4). However, the move (33) can be generalized by adding edges inside the loop on the right, as long as the number of faces within the loop remains odd. (The argument uses theorem 6.1 and is a bit longer, so we omit it.) This easily implies that girth ≥ 4 is not sufficient. Since B_n (and many of its generalizations using the modified moves (33)) are Hamiltonian, if A_n is so, this property is also insufficient.

Then the insight we gained in §4.3 turned our attention to cyclic 4-connectivity. The following lemma, which was already used there, motivates this further. It explains that examples like B_n arise because we performed composition in an 'inappropriate' way.

Lemma 6.1 If G_1 and G_2 are planar 3-valent graphs and both have a minimal marking, then *there is a way* to perform the composition $G_1\#G_2$ so that $G_1\#G_2$ has a minimal marking.

Proof. Let us fix minimal markings O_i of G_i and consider fixed vertices $v_i \in v(G_i)$. We will see how they must be chosen.

Let us, as in (43), cut the thickening $T = T(G_i,O_i)$ of G_i corresponding to the marking O_i at a ball B around v_i so that $\partial B_i \cap T(G_i,O_i)$ are three segments exposing the same side of $T(G_i,O_i)$ on the top.

Let us also fix that if $\chi(G_i)$ is even, we choose v_i so as near v_i to have both components of ∂T. This means that (also for odd χ) there is no closed loop of ∂T in $T\setminus B$, and thus the connectivity is as one of the patterns in (44).

By reinstalling v_i, we see that patterns T_i correspond to two components of ∂T, which means that $\chi(G_i)$ is even, while X, Y and Z to one component, which occurs for odd $\chi(G_i)$. Moreover, for X and Y the vertex v_i is bad, while for Z it is good (*cf.* Definition 5.3.)

If both $\chi(G_i)$ are even, one compose T_1 with some of its rotated versions to give a knot marking of $G_1\#G_2$.

If, say, $\chi(G_1)$ is odd and $\chi(G_2)$ is even, then choose any vertex v_i in G_i. The composition of any T_i with X, Y and Z will give two components.

Thus assume both $\chi(G_i)$ are odd. It is important now that bad vertices exist by lemma 5.1. Our stipulation is that we choose v_1 to be bad. Note that we have the freedom of exchanging patterns X and Y by reversing the markings of all crossings outside v_i. With this freedom it is possible to combine X and Y with one of X, Y or Z to obtain one component. □

Then we can again consider cyclically 4-connected graphs (see above conjecture 4.1). Extensive computation led us to conjecture the following statement.

Theorem 6.2 ([St10]) Every cyclically 4-connected planar 3-valent graph has a minimal marking.

This purely combinatorial property gains some geometric motivation from conjecture 4.1, and is needed for theorem 1.3 (which we will prove in an instant).

Although "only" combinatorial, theorem 6.2 is not straightforward to prove. It is unclear how to find a knot marking directly, and in an inductive construction one must account for the fact that cyclic 4-connectedness is not inherited under undoing (any of the possible) Vdovina transformations in theorem 6.1. (Examples are the wheel graphs.)

These phenomena suggest why the determination of the minimal component number of a marking, and in particular finding good *explicit* criteria for a minimal marking, is difficult.

For theorem 6.2 we use a different set of transformations (though not disjoint; γ is included). It cannot be ascertained to generate all graphs, but it is sufficient for the ones we are interested in, and it makes a recursive work with the c4c property possible. This still requires some work, and the argument, which is purely combinatorial, is somewhat long. The details are thus deferred to the separate paper [St10].

Proof of theorem 1.3. For theorem 1.3 it is necessary to prove that a (planar cubic 3-connected) graph G maximizing $\mathrm{vol}(L_G)$ has a minimal marking. Now write (38) for c4c graphs $G_i \neq \theta$ (and $\chi(G_i) > \chi(G)$ unless $n = 1$). By theorem 6.2 all G_i have minimal markings. By lemma 6.1 one can 'restructure' the composition $G' = G_1\#\ldots\#G_k$ in such a way that G' has a minimal marking. And by lemma 4.3 we have $\mathrm{vol}(L_{G'}) = \mathrm{vol}(L_G)$. □

Remark 6.1 Note that conjecture 4.1 says that the "rearrangement" procedure in the preceding proof (and lemma 6.1) would be unnecessary.

7. Maximal even and odd generators

7.1. The non-planar case

We use the terminology of §5, in particular definitions 5.1 and 5.3.

If even and odd maximal generators exist, then we know from (the obvious generalization) of the arguments of [SV] that $C_{n,\chi,e}$ and $C_{n,\chi,o}$ are both non-zero, and hence

$$0 < \frac{C_{n,\chi,e}}{C_{n,\chi,o}} = \lim_{c\to\infty} \frac{\#\{\, L : c(L) = 2c,\ \chi(L) = \chi,\ n(L) = n, L \text{ alternating} \,\}}{\#\{\, L : c(L) = 2c+1,\ \chi(L) = \chi,\ n(L) = n, L \text{ alternating} \,\}} < \infty \tag{48}$$

(and that, in particular, the limit exists).

Now we can classify the cases in which even or odd maximal generators exist.

Theorem 7.1 If $n < 2-\chi$ and $(n,\chi) \neq (1,-1)$, then even and odd maximal generators exist. Contrarily, there exist only odd maximal generators for $(n,\chi) = (1,-1)$, and only even maximal generators for $n = 2-\chi$ (and $n > 1$).

Most of the theorem we can prove now. We use later in §7.2 the work of Bar-Natan on the Four color theorem [BN2] to complete the proof of the missing cases. Our approach, which will be pursued further to derive the main result, will base on the notion of a good vertex.

In the case of knot markings, for which we have the description in terms of maximal Wicks forms, we observed in [SV] that a good vertex is exactly the one, for which in property 3) of the description of Wicks forms in §3 the subword $c^{\pm 1}a^{\pm 1}$ occurs (in cyclic order) *after* $b^{\pm 1}c^{\pm 1}$ and *before* $a^{\pm 1}b^{\pm 1}$.

Proof of theorem 7.1. Let us first consider the exceptions. The situation for $(n,\chi) = (1,-1)$ is known from the classification of [St]. We can now deal with $n = 2-\chi$ only for $n \leq 4$. The argument will be extended to complete the proof for the other n with corollary 7.1.

The exceptional pairs $(n,\chi) = (2,0)$ and $(3,-1)$ are recognized as follows. From the same result of [St] it follows that the only canonical surfaces for $n = 2$ and $\chi = 0$ are those of the $\overline{T}_k$, so that odd generators do not exist. The same phenomenon occurs for $n = 3$ and $\chi = -1$, since one can check from theorem 3.1 that the only canonical surfaces are of the special (p,q,r)-pretzel diagrams with p,q,r even. To see this, apply twice

(49)

on a generator diagram. By theorem 3.2 the resulting knot diagram has ≤ 8 crossings. Then check which knot generators of genus two and ≤ 8 crossings have a pair of parallel clasps, such that smoothing out a crossing in each gives a 3-component link diagram. In the terminology of the end of §2.3, such a crossing pair is "unlinked". (Note contrarily that in each parallel clasp the two crossings themselves are linked.) The only such generator is 8_{15}, which comes from the pretzel diagram. With a little more effort one can also deal with $n = 4$ and $\chi = -2$. By theorem 3.2 a generator has at most 12 crossings, so that we need to seek (alternating) genus 3 knot generator diagrams of even crossing number ≤ 14, with 3 pairwise unlinked parallel clasps. The computer shows that here are no such diagrams. (For odd crossing number, there are 6 generators, of crossing number 11, 13, and 15, and all they are special.)

Now assume $n < 2-\chi$ and $\chi < -1$.

Consider first $n = 1$. By the work of [SV], the unbisected Seifert graph G of a maximal generator (diagram) is (planar) trivalent and 3-connected. Moreover, it is easy to see from [BV] (see also theorem 6.1 below) that for some 3-connected 3-valent planar graph G, there are knot generators, whose unbisected Seifert graph is G. To find even and odd such generators, use the existence of good vertices in maximal Wicks forms of genus > 1, a consequence of lemma 5.1. This deals easily with $n = 1$, and thus assume now that $n > 1$.

Consider next $n = -\chi$. The sought generators can be given explicitly. To construct an odd generator take a coloring of any G with exactly one vertex v of orientation opposite to the others. For an even generator take a pair of adjacent vertices $v_{1,2}$ of orientation opposite to the others. The corresponding link diagram has one component for each region, except for the four regions bounded by the two vertices; their loops are connected in pairs. The pair of regions bounded by the edge between $v_{1,2}$ is always distinct, and the other pair is too, unless $G = \theta$ and $\chi = -1$, which we excluded.

Now for fixed n, construct even and odd generators inductively over $-\chi$. Start with the odd generator we just constructed for $n = -\chi$. It contains a trefoil component. The Seifert circle of v is adjacent to all its 3 crossings. Between the passage of these 3 crossings the trefoil goes around the regions bounded by v. If one ignores the remaining components of the link, one can think of this part of the trefoil going along a fictive distant Seifert circle, so that we have the graph $G = \theta$. Our restriction is that we cannot alter the knot near the Seifert circle corresponding to one of the two vertices of G.

Since the edges $e_{1,2,3}$ incident to v are passed by the trefoil in both directions, we can apply a γ-construction of [V], adding two vertices on e_1 and one on e_2 (and one in the interior of the region bounded by $e_{1,2}$; see figure 1). This procedure shows that one can find orientations of the 4 new vertices such that the trefoil transforms into a new (alternating) knot (and not 3-component link). This knot has genus ≥ 2. Thus by lemma 5.1 its graph has at least two good vertices, so that at least one is different from the fictive vertex inherited from G (which must remain fixed). Changing the orientation of this good vertex alters the parity of the crossing number, but preserves the property the component to be a knot, and hence also preserves the total number of components of the link. Then $C_{n,\chi,e}$ and $C_{n,\chi,o}$ are both non-zero, and inequality (48) is clear.

The proof is complete up to the cases $n = 2-\chi > 4$, which will be settled in corollary 7.1. □

The following small application shows how well the maximal volume can be approximated by the volume of (sufficiently general but) particular links.

Proposition 7.1 If $n < 2-\chi$ and $\chi < -1$, then for $c \geq -5\chi + n - 4$, we have

$$v_{n,\chi} \leq \frac{14\chi+8}{3\chi+2} \max \{ \operatorname{vol}(L) : c(L) = c,\ \chi(L) = \chi,\ n(L) = n, L \text{ special alternating} \}. \tag{50}$$

Proof. This inequality follows from theorem 7.1. By theorem 3.2 the maximum is built over (the volume of) a link L with a maximal generator diagram. The estimate (50) follows from the lower bound in theorem 4.2 with the upper bound taken from corollary 4.1, as we explain.

We have that the twist number $t(D)$ is always less than or equal to the number of $\sim$-equivalence classes of D, which is $-3\chi(D)$.

It is equal to that number if no $\overset{\sim}{*}$-equivalent crossings exist, i.e. crossings which up to flype form a parallel clasp, as on the r.h.s. of the transformations in (49). (For the precise definition see, e.g., [St4].) Formally, this is true only if the diagram D is twist reduced (in a sense specified in [La]). But the property in theorem 4.2 the diagram to be twist reduced can always be achieved by flypes, and in fact we know from [SV] that flypes do not occur, so D is twist reduced.

Thus for applying theorem 4.2, it is enough to see that D has no $\overset{\sim}{*}$-equivalent crossings. Since (49) augments the number of $\sim$-equivalence classes at most by two, from theorem 3.2 we would have otherwise that D has at most $-3\chi - 1$ $\sim$-equivalence classes.

With this remark, we have

$$\frac{V_8}{2}(-3\chi-2) \leq \text{vol}(L) \leq v_{n,\chi} \leq V_8(-7\chi-4),$$

which implies the stated inequality. □

One can use (29), obtaining a constant in (50) slightly better, but clumsier, involving V_4 etc. A good quality of this constant cannot be achieved anyway. It is difficult to improve (50) for $c \to \infty$ because the convergence in Thurston's theorem is at best asymptotically understood (see [NZ]).

7.2. Planar surfaces

The sl_N weight system construction was applied by Bar-Natan in [BN2] to establish a relation to the Four color theorem. We can use the argument of Bar-Natan in the proof of proposition 1.2 of [BN2] also in our situation (referring to, but not repeating in full Bar-Natan's terminology). We will, though, later lift the restriction of 3-valency on the graphs, as the notion of a thickening extends straightforwardly to higher valence.

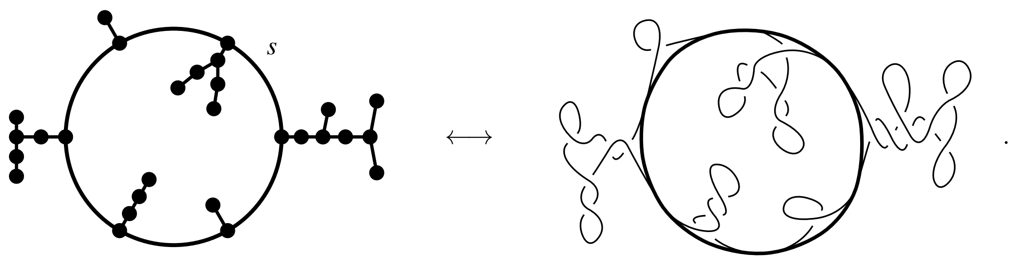
Lemma 7.1 Consider a 2-connected 3-valent graph G. Assume p_1 and p_2 are planar embeddings of G, and $O_{1,2} = O_{p_{1,2}}$ the canonical orientations of G corresponding to $p_{1,2}$. Clearly for any $v \in V(G)$, we have $O_1(v) = \pm O_2(v)$ (for one of the two signs). We claim that the number of v's for which the minus sign occurs is even, and so is the number of edges e_0 between vertices v_1 and v_2 with $O_1(v_1) = O_2(v_1)$ and $O_1(v_2) = -O_2(v_2)$.

Proof. By Whitney's theorem $p_{1,2}$ are interconvertible by a sequence of flips (see §1 or figure 2 of [BN2]). Each flip changes the canonical orientation corresponding to the embedding at an even number of vertices, and adds or removes exactly two edges e_0, namely the pair disconnecting the flipped part (the smiling face in Bar-Natan's drawing) from the rest of the graph. Since for $p_1 = p_2$ no v's and e_0's occur, we are done. □

Theorem 7.2 Let D be a prime reduced link diagram with planar canonical Seifert surface. Then D is special and of even crossing number, and all its components are unknotted.

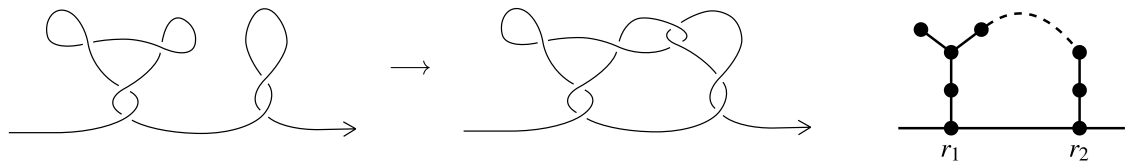
Proof. Let D have n components, and consequently Euler characteristic $\chi = 2-n$. As before, by moves (49), one can obtain from D a knot diagram D' of genus $n-1$ with $n-1$ pairwise unlinked parallel clasps. Then resolving these clasps (changing a crossing and eliminating both crossings by a Reidemeister II move) gives a genus 0 knot diagram D''.

Such a diagram is obtained from the 0 crossing diagram by Reidemeister I moves. Any special (Murasugi sum) component of D'' is described by its Seifert graph, which is a tree. Assume now s is a separating Seifert circle in D. Then so it is in D' and D''. Consider the (Seifert graph) trees of the two special components of D'' separated by s. These trees are rooted at the vertex corresponding to s. One can refine them by unsmashing s into a circle, and attaching the branches of the trees at basepoints along s from the inside and outside depending on the cyclic order of the crossings adjacent to s in D'' (the edges to the former root vertex in the Seifert graph):



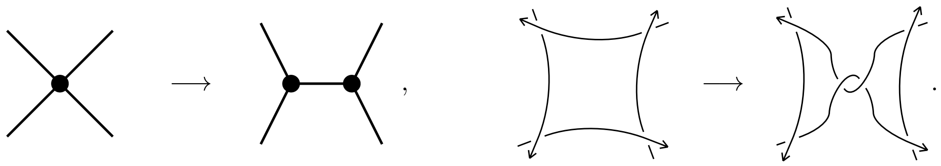
D' is then obtained by creating parallel clasps between the Seifert circles corresponding to certain vertices in these trees. In particular, to avoid nugatory crossings in D', all vertices of valence 1 (except the basepoints) must be involved in these clasps.

Since there must be non-nugatory crossings inside and outside of s in D', there must also be a pair among the $n-1$ parallel clasps obtained from (49), one lying inside and the other lying outside of s. Then consider for each such pair of clasps the two pairs $r_{1,2}$ and $q_{1,2}$ of basepoints of the trees inside and outside of s, whose vertices are connected by the clasps.



(It is possible that $r_1 = r_2$ or $q_1 = q_2$. If s is involved in some clasp, then one of the trees connected by a dotted line is a single vertex.) If D' is prime, then at least one set of $r_{1,2}$ and $q_{1,2}$ must lie in cyclic order $r_1q_1r_2q_2$ along s. Then the corresponding pair of clasps is linked, a contradiction. Thus D', and hence D is special.

Let G be the graph obtained from the Seifert graph G' of D by unbisecting (deleting vertices of valence 2). This graph G comes with a particular planar embedding we call p_1. Also, G is 2-connected as D is reduced, and it has no cut vertex[4] as D is special and prime. If G has a $\geq$ 4-valent vertex, one can apply a decontraction, which on diagrams means the separation of a Seifert circle into three by addition of a reverse clasp.


(51)

This argument shows that it suffices to consider the case that G is trivalent. Clearly any decontraction preserves 2-connectedness and lack of cut vertices, if the initial graph has these two properties.

We define a choice of orientation O of the vertices v of G by $O(v) = O_{p_1}(v)$ or $O(v) = -O_{p_1}(v)$ depending on the orientation of the Seifert circle corresponding to v in D. We assign to O a thickening T_O of G by putting a plus or minus sign in each vertex of G depending on which one of the two previous conditions holds, and then applying the construction of [BN2]. Then the planarity of the canonical Seifert surface of D is equivalent to the planarity of T_O. The thickening gives as in Bar-Natan's situation rise to a spherical embedding p_2 of G by gluing disks into the boundary components of T_O. Then p_2 is chosen so that $O = O_{p_2}$. The parity of the crossing number $c(D)$ of D is this of the number of odd edges in G, which are the edges e_0 in the previous lemma. Thus $c(D)$ is even.

From the description of D we obtain it is easy to see that all components of D are unknotted. However, there is a more direct argument. If such a component had a self-crossing, then its smoothing would augment both n and χ, which is impossible if $n+\chi = 2$. □

Corollary 7.1 There exist no odd generators for $n = 2-\chi$. □

This completes the proof of theorem 7.1.

Corollary 7.2 Let L be an alternating or positive link with a planar Seifert surface. Then L is special alternating and of even crossing number.

Proof. Consider first that L is non-split and prime. Then L possesses no disconnected Seifert surfaces. For positive links this follows directly from the linking number, and for alternating links because a boundary link (with more than one component) has vanishing Alexander polynomial (see theorem on bottom of page 196 in [Ro]). Then any maximal Euler characteristic Seifert surface of L is planar (and connected), and one such surface is realized as canonical Seifert surface for an alternating or positive diagram D of L. Then argue over the split components and prime factors of L separately, and use [Me, O]. □

[4]Note that Bar-Natan does not care about cut vertices, as for trivalent graphs 2-connectedness implies that they do not exist. However, we consider also higher valence.

Remark 7.1 The proof of theorem 7.2 also gives a very precise description of the diagrams in question. They are obtained from their unbisected Seifert graph by thickening (vertices are replaced by discs and edges by strips), and half-twisting the strips an even number of times each. As these diagrams are special, alternating and positive links can be classified from [MT]. It is easy to see the effect of a flype on such a diagram – it may half-twist an odd number of times a strip of an edge in a 2-cut of the graph.

7.3. Asymptotical behaviour of maximal knot generators

Our interest in the current investigation comes to a large part also from [SV], where the asymptotic enumeration of alternating links of given Euler characteristic (40) was considered in the case of knots ($n = 1$); see (13).

Theorem 7.3 For $* \in \{e, o\}$, the below limit exists, and

$$400 \le \lim_{g \to \infty} \sqrt[g]{C_{g,*}} < 1423\,.$$

However, due to space limitation, we move this study out to a separate place, and only give a very brief indication.

The proof of existence of this limit relies on the Bacher-Vdovina lemma 5.1 and the "combination" procedure of markings (lemma 5.3), which we saw closely related to Vogel's character for the sl_N invariant (see remark 5.2). The improved, in comparison to [SV], upper bound is obtained by estimating the number of knot markings. This problem is related to vertex arboricity of graphs. We also confirm that $C_{g,e}$ and $C_{g,o}$ are "exponentially close".

Theorem 7.4 For $g \to \infty$,

$$\frac{|C_{g,o} - C_{g,e}|}{C_{g,o}} \longrightarrow 0$$

exponentially fast.

It is an open problem whether $C_{g,o} > C_{g,e}$ for $g \ne 2$. We also emphasize that lemma 5.1 underlies the proof of theorem 7.4 as well, so that (while likely true) we cannot prove this property for multiple link components, i.e., for the $C_{n,\chi,*}$.

8. Proof of main result

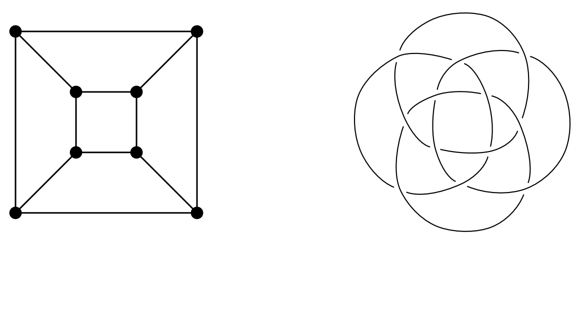
8.1. The self-crossing criterion

Convention. In this and the next section we will assume that all graphs are simple and 2-connected. We will lift the simpleness condition *only* in the proof of theorems 1.1 and 8.1 (second part), and corollary 8.1. Furthermore, in [SV] we considered only 3-connected graphs, but this restriction is not essential here. We will thus still talk of Wicks forms, meaning maximal planar Wicks forms, but only with the requirement their graphs to be planar, simple and 2-connected, and not necessarily 3-connected.

We now turn to prove the following theorem, giving a condition when a good vertex exists in a general marking. Although we will need only one direction, it is possible, and more valuable, to have the criterion proven complete.

Theorem 8.1 Assume G has $\chi(G) < -1$, is 3-valent, 2-connected, planar, but *not* necessarily simple. A marking O of G has a good vertex if and only if $D_{G,O}$ has a component with a self-crossing.

Beware that some markings have no component with a self-crossing. We showed that this is always true if $n = 2 - \chi$. However, there are more examples, one of them coming from the bipartite marking of the cube net.



(52)

Theorem 8.1 requires the most substantial effort, and thus we devote this separate section to its proof. Several steps within this proof will be singled out as separate lemmas to make the argument more tractable. We also separate the proof in two parts, first dealing with simple graphs G.

Proof of theorem 8.1 for simple G**.** First assume $D_{G,O}$ has a self-crossing. We describe a way to construct out of O a knot marking O' of a (simple) trivalent graph G', very similar to the construction in the proof of theorem 7.2.

Let $D_0 = D = D_{G,O}$, $i = 0$ and $G_0 = G$.

For $D_i = D_{G_i,O_i}$, as long as $n_i = n(D_i) > 1$, repeat the following step.

Since $n_i > 1$, then there exist two segments in the boundary of a non-Seifert circle region E of D_i, belonging to different components of L_i. Call these segments s_1 and s_2.

Let $v_{1,2}$ be the Seifert circles of D_i containing s_1 and s_2. If some v_i is trivalent, apply a $\bar{t}'_2$-move at one of the crossings bounding s_i. One of the two newly created Seifert circles of valence 2 is bounded from the side of E by (an edge belonging to) the same component of the link as s_i. Thus after possibly such move(s), assume the v_i are two-valent. Then connect s_1 and s_2 by a band of one or two half-twists (depending on whether their orientation w.r.t. E is the opposite or equal). See figure 2. The new diagram D_{i+1} comes from a marking O_{i+1} of a new graph G_{i+1} obtained

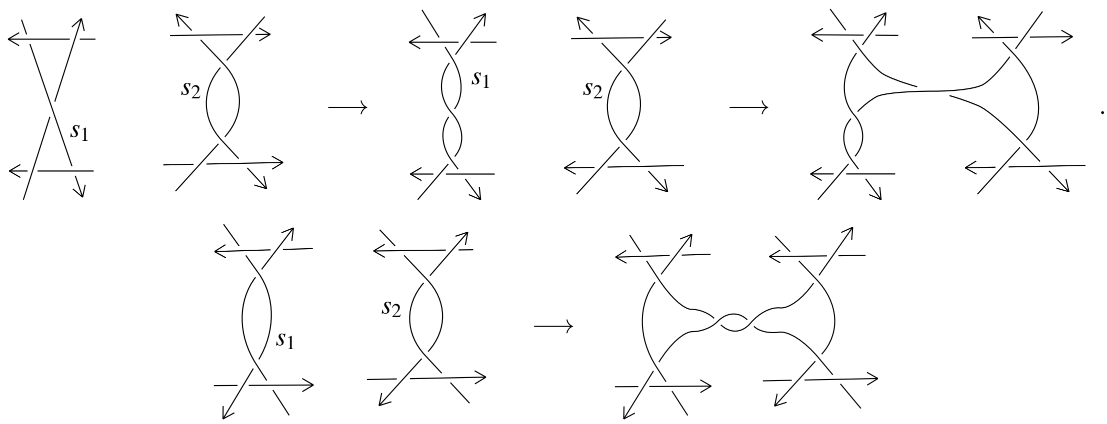
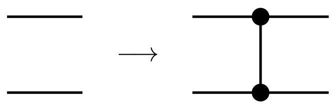


Figure 2

from G_i by separating one of its faces


, (53)

(O_{i+1} coincides with O_i outside the two new vertices added.) We have $n(D_{G_{i+1},O_{i+1}}) = n(D_{G_i,O_i}) - 1$. Clearly G_{i+1} is still planar and 2-connected if G_i is so. Note also that this procedure does not affect the edges in the original diagram D along its 3-valent Seifert circles.

The newly added edge (*not* the two previously bisected) in G_{i+1} is being *marked*, and we call its two ends *new*. The vertices recursively inherited from G_0 we call *old*. The edges inherited from G_0 we call *unmarked*. If at some point the move (53) bisects an unmarked edge to install a new vertex, the two resulting edges are still unmarked (unlike the third one added to that vertex).

By iterating this step, we obtain the desired knot marking (G',O'). We call (G',O') a *knot extension* of (G,O).

Note that the choice of (G',O') is highly non-unique. We need, and will henceforth work only with, one of a more specific nature.

Lemma 8.1 One can choose the moves in figure 2 so that in (53) always two *distinct* and (previously) *unmarked* edges are bisected and connected (by a new marked edge).

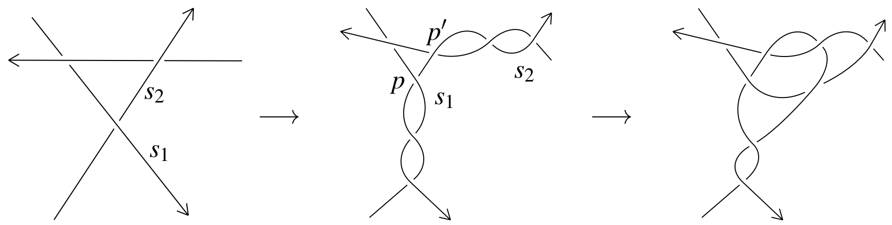
The lemma means that simpleness of G_{i+1} is also preserved, and that any vertex of G_{i+1} is not incident to more than one marked edge.

Proof. Let the link graph H of D_0 be with vertices given by the components of $D = D_0$ and edges between vertices of components with a mixed crossing. Take a spanning tree of T of H. It suffices to show that one can connect pairs of components of D corresponding to edges in T, when fixing a root of T and working with edges in T of decreasing distance from the root.

We do this in a specific way. Take a common crossing p of two components (given by a fixed edge in T) in D_0. Let v be a Seifert circle adjacent to p in D_0.

We can assume that v can be chosen at least trivalent. Otherwise one always takes the opposite crossing connected to one of the Seifert circles of valence 2, noting that it involves the same components. (If only 2-valent Seifert circles exist, we have $\chi = 0$, which is excluded.)

Now apply the following move:



(54)

(the vertical $\bar{t}_2'$ twist is only necessary if s_1 bounds a ≥ 3-valent Seifert circle).

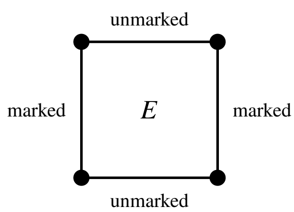
Let p' be one of the two crossings adjacent to v, which are neighbored to p along v. At p' apply a $\bar{t}_2'$-move, and so do at p, if it is not in a clasp. One has then two Seifert circles of valence 2 bounded from the side of the non-Seifert circle region E by segments of the two components intersecting at p. Then apply the previous band-connecting (as in figure 2).

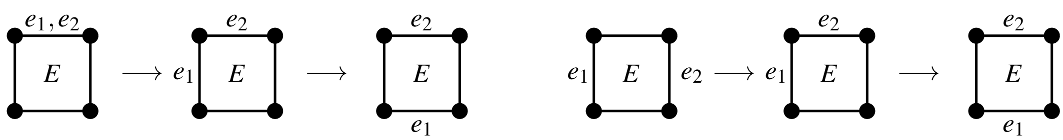
Now we iterate this by induction over i. We must now argue that, for fixed i, we are flexible enough to avoid $e_{1,2}$ being the same edge or one being marked.

The modification (54) shows that one can change e_j to a neighboring edge e_j' in the boundary ∂E of the non-Seifert circle region E. If e_j is marked, then its crossing(s) in D_i belong to the same component, and thus e_j can be changed to any of its either neighbors in ∂E.

Note that, since G_i is simple by induction, E has at least 3 edges in its boundary. (We assumed that $G_0 \neq \theta$.)

This way one can achieve that $e_1' \neq e_2'$, and also that e_i' are still unmarked. The following shows the changes needed in the situation that ∂E has 4 edges and two are marked. (Note that by construction two marked edges are never neighbored.)

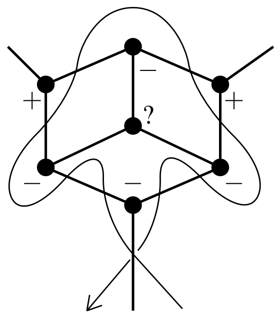




Notice that since s_1 and s_2 belong to different components, one cannot have $e_1 = e_2$ and being marked.

Thus we can ascertain that G_{i+1} has no multiple edge either. □

Now consider the self-crossing x in D_0. We can by minimality assume that one of the two arcs of the component separated by x has no further self-crossings. This means that it is a loop L of D_0 going around a cycle C in G_0, entering and exiting through the third edge incident to some vertex v in C.



Smooth out in D_0 the self-crossing x, also undoing a possible $\sim$-equivalent crossing y to x by a Reidemeister I move, and $\bar{t}'_2$-reducing. Let D''_0 be the diagram obtained this way. Remove L in D''_0, obtaining a diagram D'_0 with one separating Seifert circle. This Seifert circle is homotopic in a neighborhood of C to L with the reverse orientation, and will be denoted by $-L$ accordingly. Note that the vertices of C in D_0 correspond bijectively to crossings attached to $-L$. The vertices of C are connected by their third incident edges from inside and outside C depending on their sign in O_0, and in the same way, in which their corresponding crossings are adjacent from inside and outside to $-L$.

Lemma 8.2 We can apply the above procedure of knot extension to D'_0, such that any crossing adjacent to $-L$ is not affected, and $-L$ is affected only by attaching crossings to it.

Proof. To see this, we first argue that D'_0 is connected.

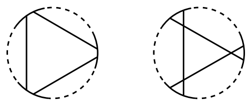
Assume s is a closed curve not intersecting D'_0 anywhere. Then s intersects in D''_0 only L. Assume by general position that these intersections occur outside crossings and are transversal. Let c be an edge of L intersected by s. (An edge is here understood as part of a component of a link diagram between two consecutive crossings.) This edge c bounds a Seifert circle r on L, which is ≤ 3-valent, i.e. consists of at most 3 edges. Since all the edges in r different from c belong to components intersecting L, and L has no self-crossings, these edges do not belong to L. Since s does not intersect other components, s must leave the interior of r by intersecting again (and only) c. Then s can be homotoped off c, without creating new crossings. By repeating this, we can homotope s off L. Since the Seifert graph G of D_0 was assumed 2-connected, the one of D''_0 is still connected, and then so is D''_0 itself. Then s has empty interior or exterior. Since the homotopy off L in D''_0 is trivial in D'_0 (where L is removed), we have that D'_0 is connected.

This means that in D''_0 we can always avoid L, if we want to connect all remaining components among each other. Now apply the move in (54). A $\bar{t}'_2$ move at a crossing on L means attaching two further crossings to $-L$, and since we do not band-connect L to any other component, the claim follows. □

Then we can reinstall L and undo the smoothing of x and possibly y. When recovering x and y one needs to remark why this can be done so that none of the constructed bands gets intersected. But note that the move (54) involves only edges with a common corner in a face of the Seifert graph. Thus intersection with a band is not a problem when possibly some $\bar{t}'_2$ moves are applied to the crossings adjacent to the Seifert circles x or y connected.

This shows that we can obtain a knot extension (G', O') of (G_0, O_0), such that the loop L is not affected. This means, in G' this loop passes a cycle C' of unmarked edges, obtained from those in C by possible bisections. (Wherever the edges in C have been bisected by (53), the third edge outside C incident to these vertices is sometimes marked.)

Consider the Wicks form v of the marking (G', O'). In the sequel it will be helpful to work with the Gauß diagram version of this Wicks form. It is obtained by putting the letters of the word v cyclically along a circle, and connecting by a chord a letter with its inverse. (This Gauß diagram recovers the word up to cyclic permutations, naming of letters, and interchange of a letter with its inverse, which are all irrelevant ambiguities.) This Gauß diagram has the following property: one can group its chords in triples of one of the forms



type I type II

, (55)

(dashed lines mean that other chord endpoints may lie there), such that each chord participates in two such triples. In both types of (55), the existence of two of the chords implies the existence of the third.

As we noted in [SV], such triples of chords correspond to the crossings adjacent to a Seifert circle of $D_{G',O'}$. Hereby type I corresponds to a (Seifert circle of) a good vertex of O' and type II to a bad vertex.

The Gauß diagram v carries a collection of $n-1$ distinguished chords. They come from the crossings of the bands added my the moves in figure 2. (The crossings added by possible $\bar{t}_2'$-moves prior to this band-connecting are not relevant to us.) We call these chords *marked*. Since the smoothing out of these $n-1$ crossings gives an n-component diagram $D_{G,O}$, these chords are pairwise non-intersecting (or unlinked in the terminology of the end of §2.3). This is equivalent to saying that for each pair of such chords, their letters appear cyclically as $\cdots a^{\pm 1} \cdots a^{\pm 1} \cdots b^{\pm 1} \cdots b^{\pm 1}$ in v, and not as $\cdots a^{\pm 1} \cdots b^{\pm 1} \cdots a^{\pm 1} \cdots b^{\pm 1}$.

Now the self-crossing x of $D_{G,O}$ gives one more letter (or chord) in v unlinked with all marked letters/chords. It is unlinked, since we chose the knot extension not to affect L. More precisely, on one of the segments of the circle of the Gauß diagram divided by x (namely the segment corresponding to L), there are no endpoints of marked chords.

We will need the following lemma, which we will prove later.

Lemma 8.3 A good vertex of $D_{G,O}$ corresponds exactly to one of the following types of vertices of v in $D_{G',O'}$ (made of triples of unmarked chords).

1) a good vertex of v with ≤ 2 of the chords intersected by marked chords, or

2) a bad vertex of v with exactly 2 of the chords intersected by marked chords.

(Note that in both types it is impossible that exactly one chord intersects a marked chord.)

Take a chord x enclosing no marked chords on one of its segments. Call this segment e. Now by descent among such x choose one, such that e does not contain both endpoints of a chord. Thus all chords on e end outside e, and they are all unmarked. Let $c_1, \dots, c_n$ denote the endpoints of chords along e labelled in clockwise order (the basepoints of x not included). Let $d_1, \dots, d_n$ be the chords they belong to, and let c_i' be the basepoint of d_i lying on the segment outside e, which we denote by e'. The third chords in the triples in (55), found to a pair of chords d_i, d_{i+1} with neighbored basepoints in e, are denoted d_i'. Some d_i' may be marked, some not. See figure 3.

Lemma 8.4 We can find a good vertex of $D_{G,O}$ if there is a chord c on e intersecting a marked chord.

Proof. Let p be the first (in clockwise order along e) endpoint of such a chord c, and q be the chord with the endpoint directly before p (in that order; possibly $q = x$). Since q does not intersect a marked chord (by assumption, or because we know x does not), we have that in the triple of c and q in (55) one or two chords intersect a marked chord. We excluded former option, and to latter option lemma 8.3 applies. □

Lemma 8.5 We can find a good vertex of $D_{G,O}$, unless there is a $1 \leq k < n$ such that, going clockwise along e', the c_i' lie in order $c_k' c_{k-1}' \cdots c_1' c_n' c_{n-1}' \cdots c_{k+1}'$.

Proof. If some c_{i+1}' lies before c_i' along e' in clockwise order (i.e. d_{i+1} and d_i are not linked), then d_i, d_{i+1} and the third chord d_i' in (55) form a triple of chords, two of which do not intersect, and do not intersect a marked chord (otherwise we are done by lemma 8.4). If d_i' is not marked, then we have type I with no chord intersecting a marked chord, and lemma 8.3 applies.

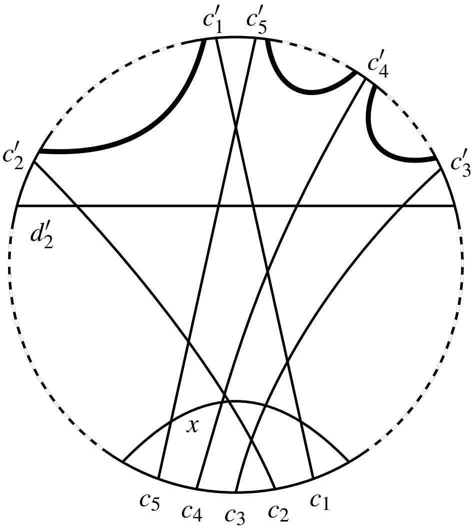


Figure 3: The chords occurring in the proof of lemma 8.5, with $n = 5$ and $k = 2$. The thick displayed chords are marked, and the dashed circle segments may contain endpoints of other chords.

Thus assume that d'_i is marked. Then by assumption d'_i does not intersect any $d_{i'}$, and since its basepoints are neighbored to c'_i and c'_{i+1}, there cannot be another $c'_{i'}$ lying between c'_i and c'_{i+1}.

Inductive iteration of this argument shows that if c'_j lies before c'_i along e' in clockwise order, with $j > i$, then the only $c'_{i'}$ between c'_j and c'_i are those with $i < i' < j$.

Now, notice, however, that x, d_1 and d_n form a type II triple, hence c'_1 and c'_n are neighbored on e', with c'_n following c'_1.

By the above argument the set I_1 of indices of $c'_{i'}$ lying on e' before c'_1, and the set set I_2 of indices of $c'_{i'}$ lying on e' after c'_n are ordered decreasingly in clockwise order along e'. Also there is no k with $k+1 \in I_1$ and $k \in I_2$. Otherwise a marked chord d'_k would intersect one of d_1 or d_n, in contradiction to our assumption, or if d'_k is not marked, we would be done with (d_k, d'_k, d_{k+1}). It follows then that each of $I_{1,2}$ is either empty, or an interval of the form $[2,k]$ resp. $[k+1, n-1]$. (Figure 3 shows the position of the chords in an example.) □

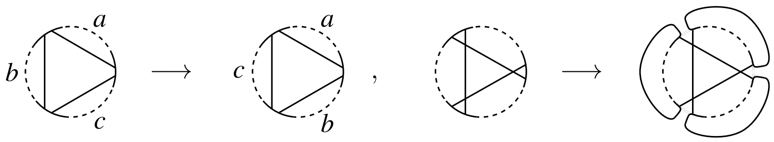
Now consider the graph G'. The d_i correspond to edges on C'. We know from the proof of the previous lemma that if c'_{i+1} lies on e' before c'_i, then d_i, d_{i+1} and d'_i correspond to a vertex on C' in G', incident to a marked edge (the edge of d'_i). The lemma implies that all vertices on C' are of such type except two: the ones incident to x and (the edge of) d'_k. But then undoing all moves (53), we find that in the original graph G, the cycle C has length (at most) 2, a contradiction to the simpleness (and 2-connectedness) of G.

Thus we are done finding a good vertex of $D_{G,O}$.

The opposite direction follows directly from lemma 8.3, which we prove below. It ensures that if $D_{G,O}$ has a good vertex, v has a chord x unlinked with any marked chord. When smoothing out the crossings of the marked chords in $D_{G',O'}$, we find that x comes from a self-crossing of one of the components of $D_{G,O}$.

This completes the proof of theorem 8.1 for simple G. □

Proof of lemma 8.3. For the proof it is essential to understand how the reversal of sign of some vertex affects $D_{G',O'}$. This is depicted in (56) (dashed lines mean that other chord endpoints may lie there).



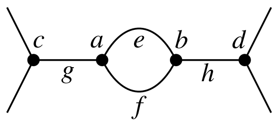
(56)

Assume in the sequel that w is an old vertex of $D_{G',O'}$, that is, all its incident edges are unmarked.

First assume the vertex w is good in $D_{G',O'}$. Let a, b, c be the segments of the circle in the Gauß diagram into which the chords x, y, z of the incident edges to w separate it. These chords form a triple of type I in (55). The effect of reversing the orientation of w is then that two of the segments a, b, c are interchanged (keeping the orientation of the circle segment). If now for two of the 3 pairs (x,y), (x,z), (y,z) there are (unlinked) marked chords intersecting such a pair, then after interchange of a and b (say) they would become linked. Then the smoothing out of the $n-1$ crossings of the marked chords, does not any longer give an n-component diagram, and w is bad in $D_{G,O}$. Contrarily, if only one of the three pairs has chords intersecting it, one of x, y, z, say x, is not intersected by any marked chord. Then the switch of two of the segments just translates the marked chords ending on a, which does not create a linked pair of marked chords. Then smoothing out of the $n-1$ crossings gives an n-component diagram, and w is good in $D_{G,O}$.

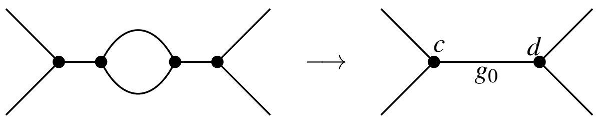
Now assume the vertex w is bad in $D_{G',O'}$. The switch of w in $D_{G',O'}$ has on the Gauß diagram of the Wicks form the result of splitting the circle into 3 components (or the word into 3 subwords), made of the segments a, b and c, without that the order of basepoints is further manipulated. Since we gained two new components, w is good if and only if we decrease the number of components under smoothing out the $n-1$ marked chords (rather than augmenting it) exactly once. It is easy to see that this happens exactly if two of the loops of a, b, c are connected by chords among each other, but the third one is not to any of them. □

Proof of theorem 8.1 for non-simple G. Now let G have a multiple edge. We use induction on $v(G)$ and the proof for the simple graphs. Let a and b be two vertices of G connected by a double edge (e,f). Let g, h be the third edges incident to a, b, and c, d the other two vertices adjacent to a, b. Since $\chi \neq -1$, we have $g \neq h$ and $b \neq c \neq d \neq a$, and G looks like



Case 1. If a and b have opposite sign in O, then c and d are both good in O, and $D_{G,O}$ has self-crossings (along g and h).

Case 2. If a and b have equal sign in O, then $D_{G,O}$ consists (up to homotopy) of a loop S along e and f, and the diagram $D_{G',O'}$. Hereby, to obtain G' from G, apply moves of the form



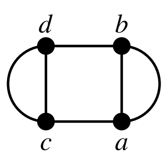
, (57)

and O' is the restriction of O to $v(G') = v(G) \setminus \{a,b\}$, forgetting the marking of a and b. First assume $\chi(G') = 1+\chi(G) < -1$ to apply induction.

Case 2.1. Consider now the situation that the two segments crossing on g_0 in $D_{G',O'}$ (or on g and h in $D_{G,O}$) belong to different components. Then $D_{G,O}$ has a self-crossing if and only if $D_{G',O'}$ has such. Also, a and b are bad in O. Moreover, a vertex v in $v(G') \subset v(G)$ is good in O' iff it is in O, since the orientation switch at v does not affect S. Thus $D_{G',O'}$ has a good vertex iff $D_{G,O}$ has one.

Case 2.2. If the two components crossing on g_0 are the same, $D_{G',O'}$, and hence $D_{G,O}$, have self-crossings. By induction, O' has a good vertex, and then so has O.

Case 2.3. This argument fails only if $G' = \theta$. Then consider the graph prior to the last application of (57), which is

$G =$ 

. (58)

If (a,b) or (c,d) have opposite sign in O, then case 1 (which does not use induction) applies. Up to simultaneous switch of all signs in O, we need to consider (a,b) positive and (c,d) negative, and all 4 vertices positive. It is easy to check directly that both markings have 4 components and no self-crossings, and have no good vertex. □

8.2. The existence of good vertices

Applying theorem 8.1, we can do now the main work for the proof of our main result.

Corollary 8.1 (a) If $\chi < -1$ and $1 \leq n < 2-\chi$, then each G having an n-component marking has an n'-component marking with a good vertex for each $n \leq n' \leq -\chi$ of the same parity as n.

(b) If G has a 1- or 2-component marking, then each such marking has a good vertex.

The example (52) quoted above (with $n = -\chi = 4$) shows that for $n > 2$ not every marking has a good vertex in general, even if $n < 2-\chi$. (I do not know the status of $n = 3$.)

Proof. Case 1. The case $n = 1$ in (b) follows directly from lemma 5.1.

Case 2. To establish the property for $n = 2$ in (b), first consider:

Case 2.1. $\chi = -2$. There are two graphs to deal with. The first one, being 2-connected, is (58). We know from case 1 of the proof of theorem 8.1 for non-simple G what is a marking with a good vertex for this graph. The second one is 3-connected (depicted in [BN2]), and hence gives rise to maximal generators. One can check that this graph has two different 2-component markings, which are interconvertible by a vertex switch. (These generators are the, properly oriented, links 9^2_{40} and 8^2_{14} in the tables of [Ro, appendix]. One can also alternatively check similarly to the said below (49) that 9^2_{40} and 8^2_{14} are the only generators of 6 $\sim$-equivalence classes.)

Case 2.2. Assume now $\chi < -2$ is even and G has a 2-component marking. One can bisect as in (53) two edges of G bounding a common region, and connect the two new vertices by an edge e, obtaining a new 3-valent planar graph G_1. If we do this properly (see the construction of a knot extension in §8.1), we find a knot generator realizing G_1, and in it smoothing out a crossing in the $\sim$-equivalence class corresponding to e (possibly subsequently undoing $\bar{t}'_2$ moves), we obtain a 2-component marking of G.

Since now $\chi < -2$, the knot marking of G_1 corresponds to a knot diagram of genus ≥ 3. By lemma 5.1, such markings have ≥ 4 good vertices, in particular at least two to which e is not incident. Then changing the orientation of the Seifert circle corresponding to one of these good vertices preserves the coloring of e in G_1, and hence alters the parity of the (2-component) marking of G.

Case 3. Now let $n \geq 3$ in both cases (a) and (b). For any G, we can find a face E of the planar embedding with at most 5 vertices. Take a marking O of G with $n(D_{G,O}) \leq n$ components. One can then, by switch of at most one vertex $q \in \partial E$, achieve that all the vertices of E are equally marked, except exactly one. Call this vertex v.

Case 3.1. $n(D_{G,O}) \geq n(D_{G,O_q})$ or switch of q is not necessary. Then the (possibly) new marking has $\leq n$ components. This marking has a self-crossing adjacent to the Seifert circle of v (of a loop going around ∂E). Now successively switch all vertices outside ∂E to have sign opposite to v. This preserves the self-crossing near v. The number of components changes at most by ± 2 every time, and at the end we arrive to a marking, in which v is the only vertex of its sign. Such a marking occurred in the proof of theorem 7.1, and we know that it has $-\chi$ components. Thus every number n' of components between n and $-\chi$ (of the proper parity) must have been attained at some stage.

Case 3.2. Switch of q is necessary and $n(D_{G,O_q}) = n(D_{G,O}) + 2$. It is easy to see that then the 3 segments bounding the Seifert circle of q in $D_{G,O}$ belong to the same component. Then $D_{G,O}$ has itself a self-crossing. Then switch one by one the vertices outside ∂E to a sign opposite to q, but *do not* switch q or a possible other vertex $v \neq q$ on ∂E of the same O-sign as q. At the end we arrive at a marking with at most two vertices of opposite sign to the others. It has at least $-\chi-2$ components. By the previous argument we cover every number of components between n and $-\chi-2$. To deal with $-\chi$ components, note that it follows from the proof of theorem 7.1. The argument therein in this case made no assumption on G, except that $\chi(G) < -1$. □

Proof of theorem 1.1. It suffices to show that for every G there are even and odd (maximal) generators of n components. For this we use corollary 8.1.

By assumption G has an n-component marking. (Take a neighborhood of the embedding of G on S'.) Then G has by corollary 8.1 an even and odd n'-component marking, and the two embeddings $p_{1,2}$ are obtained by gluing disks into the boundary components of the thickened surface. □

Corollary 8.2 If $\chi(G) < -1$, then $\min\deg_N W_{N,+}(G) = \min\deg_N W_{N,-}(G)$, and we will write for this magnitude in the sequel $\min\deg_N W_{N,\pm}(G)$. □

Let us mention in passing by that there is a possible generalization of theorem 1.1, to which, however, our method does not apply.

Question 8.1 Is theorem 1.1 true, if instead of planarity of G we demand only embeddability on a surface of smaller genus than S?

Proof of theorem 4.4. It follows from Thurston's hyperbolic surgery theorem that with any finite P, (36) will not hold, even if we remove any other restriction on L (except, of course, that $\chi_c(L) = \chi$). The property (36) certainly does not hold for $(n,\chi) = (1,-1)$. From the classification of knot diagrams of genus one we conclude that for $(n,\chi) = (1,-1)$ we need to compare the volume of the $(2,-2,2,-2,2,-2)$-pretzel link to that of the Borromean rings. The latter has smaller volume. The exceptional pairs with $n = 2-\chi$ were explained by the proof of theorem 7.1 (and corollary 7.1), as with the convention $\sup\varnothing = -\infty$, the property (36) fails trivially.

Otherwise, we apply the proof of proposition 4.2 and theorem 4.1. Then we restrict ourselves only to maximal generators, which are special, and whose unbisected Seifert graph is 3-valent (and planar).

Use Thurston's hyperbolic surgery theorem. Then the claim amounts to saying that the right hand-sides of (36) coincide for $P = 2\mathbb{N}$ and $P = 2\mathbb{N}+1$. For both parities one obtains the volume of an augmented alternating link. By the discussed result of Adams [Ad], one can (ignoring orientations) change the parity of the number of crossings in each $\sim$-equivalence class, without changing the volume of this augmented alternating link. (See the proof of Lackenby's original weaker version of theorem 4.2 in [La].) This volume then depends only on the unbisected Seifert graph G. It is realized by the limit link L_G of theorem 4.1. We know from [SV] that only 3-connected (planar) 3-valent graphs G are relevant. We consider now (one of) these G maximizing $\mathrm{vol}(L_G)$ among all G having an n-component marking (see the argument in the proof of proposition 4.2). Then it suffices to show that this G has an even and odd marking with n components, which in turn follows from corollary 8.1. □

8.3. Unlinked sets in Wicks forms

In this subsection we briefly mention a reformulation of our work in §8.1 and 8.2 to, and a problem on, Wicks forms. The following definition appeals to the end of §2.3.

Definition 8.1 Fix a maximal Wicks form w of genus $g > 0$. We call two letters a, b of w *unlinked* if w is of the form $w_1a^{\pm1}w_2a^{\mp1}w_3b^{\pm1}w_4b^{\mp1}$ for certain subwords w_i, or a cyclic permutation thereof. A set X of letters of w is *unlinked* if all letters in X are pairwise unlinked. X is *maximal unlinked* if it is unlinked and not a proper subset of another unlinked set of letters of w. Let $u(w)$ be the *minimal* size of a maximal unlinked set of letters in w.

We will continue allowing the graph of the Wicks form to be 2-connected, If it is 3-connected, we call the Wicks form so and write '3C'; we use '2C' for 2-connected *but not* 3-connected. Below we (obviously) assume that $n+\chi$ is even.

Theorem 8.2 If any maximal planar Wicks form w of genus $(n-\chi)/2$ for $\chi < -1$ has $u(w) \geq n$ in the sense of definition 8.1, then any n-component marking of a planar 2-connected 3-valent graph G with $\chi(G) = \chi$ has a good vertex.

The idea behind this theorem also lies behind the proof of theorem 8.1 and therewith theorem 1.1, and the quantity $u(w)$ in definition 8.1.

Proof of theorem 8.2. Take any n-component marking $D = D_{G,O}$ with $\chi(G) = \chi$, and consider then the Wicks form corresponding to a knot extension (see below (53)) of D. By assumption, it has a non-marked chord unlinked with any of the $n-1$ marked chords. Thus $D_{G,O}$ has a self-crossing, and so by theorem 8.1 a good vertex. □

It follows from the proof of theorem 8.2 that Wicks forms corresponding to knot extensions of planar markings have

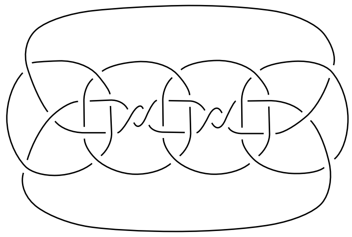
$$u(w) \leq g(w)\,. \tag{59}$$

But while $u(w)$ is minimization-defined, how small it is in general appears not clear. Here is what can be checked in low genus.

Example 8.1 It is easy to see that any maximal (not necessarily planar) Wicks form w of genus $g(w) > 1$ has $u(w) \geq 2$. A verification of Vdovina's list of genus 3 forms shows that all 158 planar 3C ones (*cf.* (19) and the fourth column of table 4) have $u(w) = 3$. But $u(w) = 2$ occurs for several non-planar 3C ones, and some planar 2C ones admit up to $u(w) = 6$. Most planar 3C w with $g(w) = 4$ (compiled as described in [St8], and also in §9 below) have $u(w) = 4$, but some have $u(w) = 3$ (*cf.* example 8.2).

So far experimentation was not able to refute the possibility that (59) holds at least for planar 3C forms. But we are cautioned that strict inequality is possible. In fact, (52) hints how to obtain w with comparatively low $u(w)$ more systematically. Such examples arise from markings with no self-crossings and with $n < 2 - \chi$.

Example 8.2 For every even genus $g = 2m$, there is a maximal planar 3C form w with $g(w) = 2m$ and $u(w) = m + 1$. Some of these forms for $m = 2$ can be constructed from example (52) by knot extension. For $m > 2$ consider the following generalization of (52), given for $m = 4$:



This gives (after applying several decontractions (51), and knot extension) forms w of genus $g = 2m$ with $u = m + 1$. By smoothing out a pair of crossings and doing decontractions with some care, one can modify these examples to obtain $g(w) = 2m - 1$ and $u(w) = m + 1$. In particular we see that $g(w) - u(w)$ can be arbitrarily large (even for planar 3C forms).

Naturally a few problems can be formulated.

Conjecture 8.1 For each n there is a g_n such that every maximal planar Wicks form w of genus $g(w) \geq g_n$ has $u(w) \geq n$.

Thus conjecture 8.1 is qualitatively the most optimistic. (Quantitatively one can after example 8.2 at best expect g_n to be linear in n.) It implies:

Conjecture 8.2 For any n, there are finitely many n component markings O of 2-connected graphs G, whose $D_{G,O}$ have no self-crossings.

9. Tables of maximal generators

It was possible to compile the maximal knot generators for genus 4, and at least to count them for genus 5 and 6. (The determination of all genus 4 generators, which includes a slightly different way to generate the maximal ones, is explained in [St8].) For the determination of maximal generators I used the program of Brinkmann and McKay [BM] to generate all 3-connected 3-valent planar graphs G. Then, using MATHEMATICA™ [Wo], I calculated the automorphisms of these graphs. We know that there is a bijection between maximal knot generators, their diagrams $D = D_{G,O}$, the Seifert graphs G' of these diagrams, and the marked unbisected Seifert graphs (G, O). Thus isomorphic

G' give rise to the same G and two markings transformable by an automorphism of G. Then to count maximal generators (with orientation ignored), one needs to count knot markings up to automorphisms of G. They were generated by a C++-program.

Table 4 summarizes the basic features of maximal knot generators. There, contrarily to our previous discussion, generators are considered *without* orientation.

The following comments are appropriate:

a) The number of graphs is small in comparison to the number of generators.

b) Trivalent 3-connected planar graphs (second row) have been enumerated for small number of vertices (see [BF], or [Sl, sequence A000109] for an extensive list of references, but beware that here only odd $\chi = 1-2g$ is considered).

c) The number of genus-3 maximal generators, 158, equals the one that appears in table (19), by the correspondence established in [SV].

d) A difference between even and odd generators always exists, but indeed seems to remain small (see theorem 7.4 above). It appears that there are always a few more odd generators than even ones.

The values in the table are given only for $0 \le \tilde{c} \le 4g-4$ (see its caption), which are bounds known *a priori* (*cf.* corollary 3.1). It seemed reasonable to believe that for each g, the set of $\tilde{c}$ realized is an interval. The zeros for $\tilde{c}=0$ and $2 \le g \le 5$ were explained in [St4], and it was shown that they terminate for $g \ge 6$. However, the zero for $\tilde{c}=1$ and $g=6$ was surprising. A subsequent check showed that there are no $\tilde{c}=1$ markings also for $g=7,8$. This led us to think about a proof, which we show below. Still, apart from this "gap" (for $g \ge 6$), we do not know of others (see question 9.1).

Theorem 9.1 There exist no maximal generators with $\tilde{c}=1$.

Proof. Assume there is such a generator. By taking the dual of the Seifert graph, and removing the double edge that corresponds to the unique non-trivial $\sim$-equivalence class, we see that the claim is equivalent to the following proposition, which we prove instead. □

(Note that the property the marking to give a knot will not be used in the below proof, so that the theorem applies to maximal link generators as well.)

Proposition 9.1 There exists no triangulation of the square (in the sense of [Tu]) only with even valence vertices.

Proof. Assume there exists such a triangulation G. We call the 4 vertices and edges of the square *external*, and the others *internal*. (Note that an internal vertex is incident to internal edges only.) All vertices have even valence, and so it is at least 4, except at most two of the external vertices, with valence 2. (This means that we do allow the infinite face to bound with a triangular face in two edges or two vertices; otherwise two triangular faces intersect non-trivially only in a vertex or an edge.) If we write $\#f$ for the number of faces (*without* the infinite one, the exterior of the square), $\#v$ for the number of vertices and $\#e$ for the number of edges, then we have

$$3\#f+4 = 2\#e,$$

and since

$$\#e = \frac{1}{2}\sum_{v\in V(G)} \mathrm{val}(v),$$

we find

$$\#f = \frac{2\#e-4}{3} = \frac{1}{3}\left(\sum_{v\in V(G)} \mathrm{val}(v) - 4\right).$$

So

$$\#v = -\#f+\#e+1 = \frac{1}{3}\left(4-\sum_{v\in V(G)} \mathrm{val}(v)\right)+\frac{1}{2}\left(\sum_{v\in V(G)} \mathrm{val}(v)\right)+1 = \frac{1}{6}\left(\sum_{v\in V(G)} \mathrm{val}(v)\right)+\frac{7}{3},$$

g		1	2	3	4	5	6
# G		1	1	5	50	1249(1)	49566(13)
	0	1	0	0	0	0	2
	1		0	0	0	0	0
	2		0	2	9	64	597
	3		1	3	36	376	4830
	4		1	12	205	3715	60042
	5			25	876	19951	418384
	6			47	2328	82285	2303377
	7			46	4882	267826	10183590
	8			23	8272	693131	35774314
	9				10236	1420434	103394088
$\tilde{c}$	10				9024	2357415	249500203
	11				5094	3184724	503417940
	12				1332	3412980	851355100
	13					2785919	1208059960
	14					1647144	1431786505
	15					628162	1397296172
	16					114194	1095013342
	17						664381258
	18						294362136
	19						84781034
	20						11809498
# max. gen.		1	2	158	42294	16618320	7943902372
# max. even gen.		0	1	74	21124	8307392	3971937256
# max. odd gen.		1	1	84	21170	8310928	3971965116
$\frac{\text{odd}}{\text{even}} \approx$		∞	1	1.13514	1.00218	1.00043	1.00001

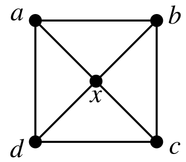
Table 4: The number of maximal knot generators up to genus $g \leq 6$, tabulated by $\tilde{c} = c - (6g - 3)$, which is the difference of the crossing number and the number of $\sim$-equivalence classes. The distinction between even and odd generators is of course still done according to c and not $\tilde{c}$. (The latter was only used to save space; its maximal possible value is $\tilde{c} = 4g - 4$.) The second row gives the number of planar 3-valent 3-connected graphs, and in parentheses the number of such (if any) without knot markings.

and this implies that there are at least 5 vertices of valence 4. (To see this, use that, except in one trivial case, there is at most one external vertex of valence 2.)

We work by induction on the number of vertices. We show that if G exists, then there exists also a simpler (i.e. with fewer vertices) such triangulation G'. For this we apply an appropriate local transformation on the graph. Then we obtain a contradiction by induction assumption.

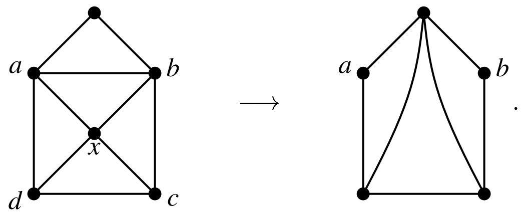
To start the induction, we check directly that G does not exist for at most 3 internal vertices. (All transformations we apply reduce the number of internal vertices by at most 3.)

Since G has at least 5 vertices of valence 4, there is at least one such internal one, call it x.



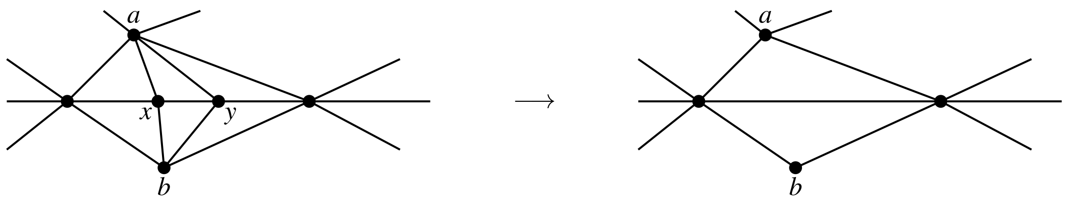
We call a vertex *reducible* if it is of valence ≥ 6 *or* an external vertex. The meaning of this is that its valence can be decreased by 2 by a transformation, without spoiling the property the graph to be a triangulation. The other vertices are called *irreducible*.

Assume first that *all* of a,b,c,d are reducible. It is easy to see that at most one of the edges ab,bc,cd,da is external (otherwise there is an external vertex of valence 3). So we can assume w.l.o.g. that ab is internal. Thus it has an opposite triangle. Then we can apply the move



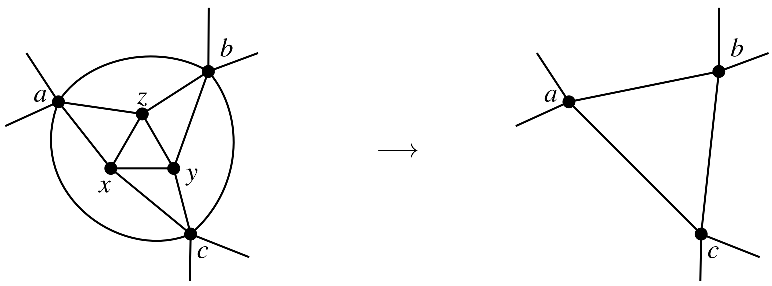
Even valence is preserved, a vertex removed, and since a,b are reducible, this is still a triangulation.

So assume next we have two adjacent irreducible vertices x,y. Then the move



reduces the triangulation, unless one of a and b is irreducible. (They cannot be both irreducible, because we would have two faces touching in two points.)

So it remains to check the case that there is a face bounded by 3 irreducible vertices x,y,z. Since their valence is 4, we find a subgraph of the sort shown in the following diagram on the left.



(60)

Now all of a, b and c are *a fortiori* reducible (otherwise we have a pair of faces bounding in two points, and G is not a triangulation). In other words, they are external vertices, or the edges going outside the picture on the left of (60) indeed exist (at least two per vertex). Thus the move shown above is always applicable, and reduces the triangulation.

With this the inductive proof is complete. □

For knot markings of a specific graph G the phenomenon of non-connected set of values for $\tilde{c}$ occurs already for $g=4$.

While it is clear why $\tilde{c}=0$ markings are scarce (G must be bipartite), one can look at the other extreme. We proved in [SV] that (for some graph G) there are knot markings with $\tilde{c}=4g-4$ and $\tilde{c}=4g-5$ (if $g>1$). One could then ask if $\tilde{c}=4g-4$ markings exist for each G (admitting knot markings at all). But 17 examples giving a negative answer occur for $g=6$.

Such examples show that the situation with crossing numbers is at least not obvious. So far we have not confirmed the cases $2\le\tilde{c}\le 4g-6$.

Question 9.1 For which $(g,\tilde{c})$ do knot markings exist?

Note also that the slight prevalence of odd generators as compared to even ones seems to persist (compare the remark below theorem 7.4). It can be easily noted in relation to theorem 5.1 and proposition 5.2 that their numbers are equal when automorphisms of G are not taken into account (and $g>1$). This means that odd knot markings are slightly more likely to inherit automorphisms of their underlying graphs.

10. Non-orientable surfaces

We conclude this exposition with a remark on cellular graph embeddings on non-orientable surfaces.

Note first that for such surfaces, of course, there is no issue of vertex orientation, and so theorem 1.1 does not make sense. Contrarily, it is well possible to define a flip (1) without requirement of surface orientation. Unfortunately our method, which decisively uses vertex orientation, cannot be used to examine moves on non-orientable surfaces. However, the following statement can easily be proved.

Theorem 10.1 Any planar connected graph G (*not* necessarily 3-valent or 2-connected) is cellularly embeddable on a non-orientable compact surface S, provided $\chi(G)<\chi(S)<2$.

This means that the obvious homological restriction is the only one in the non-orientable case. Thus the situation is very different to orientable surfaces, as we saw by the examples given in §6.

Proof. We give a sketch only.

First note that S are classified as connected sums of an orientable surface with the projective plane ($\chi=1$) or Klein bottle ($\chi=0$), so in particular are determined by $\chi(S)\le 1$.

We now apply a similar construction as for markings, only that now even-odd edge colorings (and the corresponding bisections of G) can be chosen freely, and do not come from vertex orientations. The surfaces so obtained are called *checkerboard surfaces*. They are non-orientable unless the bisections G' of G are bipartite graphs (and we are in the situation studied before). Then one obtains cellular embeddings of G on non-orientable S by gluing disks into the boundary components of the thickening.

To construct non-orientable thickenings with the proper number of (boundary) components, we apply induction on $v(G)+e(G)$. Vertices of valence ≤ 2 can be eliminated, so we assume G is ≥ 3-valent. However, we allow loop edges in G.

If G has a single vertex and loop, the required thickening is a Moebius strip.

Then we need to show that the property of possessing non-orientable thickenings with the proper number of components is inherited under edge decontraction and loop (edge) addition (and subsequent elimination of vertices of valence ≤ 2). In the case of edge decontraction (χ is preserved) simply always color the new edge even. In the case of loop edge addition (χ decreases by 1), we start with a non-orientable thickening, and need to show that we need either preserve or augment by one the number of components, keeping the thickening non-orientable. Since a loop addition on the thickening adds a band, and addition of a band to a non-orientable (thickening) surface does not make it orientable, latter condition is no problem. Neither is former, by adjusting the parity of half-twists of the band. □

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