

# THE SKEIN POLYNOMIAL OF CLOSED 3-BRAIDS

*This is a preprint. I would be grateful for any comments and corrections!*

A. Stoimenow<sup>\*†</sup>

Department of Mathematics,  
University of Toronto,  
Canada M5S 3G3

e-mail: alex@mpim-bonn.mpg.de,

WWW: <http://guests.mpim-bonn.mpg.de/alex>

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**Abstract.** Using the band representation of the 3-strand braid group, it is shown that the genus of 3-braid links can be read off their skein polynomial. Some applications are given, in particular a simple proof of Morton's conjectured inequality and a condition to decide that some polynomials, like the one of  $9_{49}$ , are not admitted by 3-braid links. Finally, alternating links of braid index 3 are classified.

## 1. Introduction and results

Braids are algebraic objects with a variety of applications. They were defined and studied by Artin [3, 4] and although the connection to knot theory was known by Alexander [1] and Markov [26], their importance in this context was not recognized until the mid 80's. On the topological side they were studied by Bennequin [5] in contact geometry and on the algebraic side used to discover the Jones polynomial [19]  $V$  and its generalization, the skein (HOMFLY) polynomial  $P$  [17, 37], and some relations between latter and braid representations were found, as the Morton–Franks–Williams inequality [28, 14] (henceforth called MWF).

We will use henceforth the variable convention of [28] for  $P$ :  $v^{-1}P(L_+) - vP(L_-) = zP(L_0)$ . Here as usual  $L_{\pm,0}$  denote links with diagrams equal except near one crossing, which is resp. positive, negative and smoothed out.

In [41], Rudolph studied the topology of knots and links via (braid closures of) representations of the braid groups  $B_n$  by embedded bands. These “band” representations have been algebraically studied recently in [8], and before in the 3-strand case by Xu. In [48], he considered the new generator  $\sigma_3 = \sigma_1^{-1}\sigma_2\sigma_1$  of  $B_3$  ( $\sigma_{1,2}$  denoting Artin's generators), with which  $B_3$  gains the representation

$$B_3 := \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_2\sigma_1 = \sigma_3\sigma_2 = \sigma_1\sigma_3 \rangle.$$

As in [48], we set  $\sigma_{i\pm 3} = \sigma_i$  (i.e., consider the subscript only mod 3) to avoid awkward notation, in which case the relations read  $\sigma_i\sigma_{i-1} = \sigma_{i+1}\sigma_i$ . We will also sometimes denote a word  $\sigma_{s_1}^{\varepsilon_1} \cdots \sigma_{s_k}^{\varepsilon_k}$  ( $\varepsilon_i = \pm 1$ ) by  $[l_1 \dots l_k]$ , where  $l_i = \varepsilon_i s_i$ .

Then for any representation of a braid  $\beta \in B_3$  as word in  $\sigma_{1,2,3}$  one obtains a Seifert surface of the closure  $\hat{\beta}$  of  $\beta$  by inserting disks for each braid strand and connecting them by half-twisted bands (the “embedded bands” in Rudolph's terminology) along each  $\sigma_i^{\pm 1}$ . We will call this construction *band algorithm*; it has obviously generalizations to higher braid groups. The surfaces thus obtained were studied by Bennequin, who showed in particular:

**Theorem 1** (Bennequin [5]) For every 3-braid link  $L$  there exists a band (algorithm) Seifert surface for  $L$  of maximal Euler characteristic  $\chi(L)$ .

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<sup>†</sup>On leave from: Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

Based on this, Xu gave an algorithm to obtain the shortest word representation of any  $\beta \in B_3$  (in  $\sigma_{1,2,3}$ ) and thus to calculate  $\chi(\hat{\beta})$ .

In this paper, we link his method with the skein polynomial and show

**Theorem 2** For every 3-braid link  $L$  we have  $\max \deg_z P(L) = 1 - \chi(L)$ , where  $\chi(L)$  is the maximal Euler characteristic of all Seifert surfaces for  $L$ .

Thus we can relate for any 3-braid *knot*  $K$  the two inequalities (latter coming from [28])

$$\max \deg \Delta(K) \leq g(K) = \frac{1}{2} \max \deg_z P(K) \leq \bar{g}(K),$$

where  $\Delta(K)$  is the Alexander polynomial [2],  $g(K)$  is the genus of  $K$  and  $\bar{g}(K)$  its canonical genus, i.e. the minimal genus of the canonical Seifert surfaces of all diagrams of  $K$  (see e.g. [44, 24]). There are examples, like the knot  $12_{2038}$  of [18] (a picture may be found in [43]), where the first (and very classical, see e.g. [38, exercise 10, p. 208]) inequality is not exact, thus making the relation between the genus and  $P$  even more surprising (for this example we have  $\max \deg \Delta = 2$  and  $g = 1/2 \max \deg_z P = 3$ ). Such examples do not occur among the homogeneous knots of Cromwell [12].

Here are some straightforward and useful consequences of theorem 2:

**Corollary 1** There is no non-trivial 3-braid link with the skein polynomial of the (1, 2, 3-component) unlink(s), and there are only finitely many 3-braid links with the same skein polynomial.

This in particular allows us to decide for a given  $P$  polynomial whether it is the polynomial of a 3-braid link. The general problem whether the MWF bound can be realized among knots of given  $P$  polynomial was raised by Birman in problem 10.1 of [29]. The negative answer was given in [45] by means of a computer example and a braid index inequality of Jones [20], but the problem for specific examples of polynomials, like those of  $9_{42}$  and  $9_{49}$  (see problem 10.2 of [29]), remained open. Now we can answer Birman's question negatively also for these two polynomials. By Bennequin's theorem all 3-braid knots of genus 2 can be easily written down (in fact all they have at most 10 crossings, as will follow from an inequality proved below in proposition 1, and hence are listed in the tables of [38]), and no one has such a polynomial. (We will later see that in fact for  $9_{49}$  there is an even faster method to conclude this, just looking at the polynomial.)

It also shows that, inspite of Birman's examples [6], later extended by Kanenobu [21], the failure of the skein polynomial to distinguish 3-braid links is limited, and series of the type of [22] do not exist among such links.

**Corollary 2** For any 3-braid link  $L$ , we have  $\min \deg_v P(L) \leq 1 - \chi(L)$ .

**Proof.** The identity [25, proposition 21] implies that  $\min \deg_v P(L) \leq \max \deg_z P(L)$  for any link  $L$ . (I'm grateful to H. Morton for pointing this out to me.)  $\square$

Corollary 2 is a recent result of Dasbach–Mangum [13], a special case of a long-standing problem of Morton [29]. Their proof is slightly less involved than ours of theorem 2, but the ' $\leq$ ' part of our equality follows rather easily (see lemma 1) so that we simplify the Dasbach–Mangum proof (in particular we will not need the Scharlemann–Thompson result [42]).

**Remark 1** In [28], Morton remarks that some knots with  $2g < \max \deg_z P$  exist. Since one of the 11 crossing knots with unit Alexander polynomial,  $11_{409}$  (denoted according to [18]), is such a knot (it has  $g = 2$  by [15, figure 5 below],  $\max \deg_z P = 6$  by [25, p. 111] and braid index 4, see [25, figure 26]), the ' $\leq$ ' inequality in theorem 2 is not true for 4-braids. The problem of an example of a knot with the contrary inequality  $2g > \max \deg_z P$  is less straightforward to solve, and an example was given only in [45]. I have not investigated whether 4-braid examples exist.

## 2. The proof of theorem 2

We start with the easy part of theorem 2, the inequality ‘ $\leq$ ’.

**Lemma 1** For every 3-braid link  $L$  we have  $\max \deg_z P(L) \leq 1 - \chi(L)$ .

**Proof.** Because of theorem 1, it suffices to prove that for  $L = \hat{\beta}$  we have

$$\max \deg_z P(L) \leq \text{len}(\beta) - 2, \quad (1)$$

where  $\text{len}(\beta)$  is the length of  $\beta$  as word in  $\sigma_{1,2,3}$ .

We proceed by induction on  $\text{len}(\beta)$  (outer induction) and for fixed value of  $\text{len}(\beta)$  on the crossing number  $c(\beta)$  of  $\beta$ , or equivalently by the number of letters  $\sigma_3^{\pm 1}$  (inner induction). In case  $\sigma_3^{\pm 1}$  does not occur or  $\text{len}(\beta) \leq 1$ , the inequality follows from [28], or is straightforward to verify. Otherwise consider a letter  $\sigma_3^{\pm 1}$  in  $\beta$ . If it is followed by another letter  $\sigma_3^{\pm 1}$ , then we apply the skein relation on one of the band crossings, use (outer) induction assumption on  $L_0$ , and switch the second band reverse to the first one, getting through by outer induction. Else  $\sigma_3^{\pm 1}$  is followed by  $\sigma_2^{\pm 1}$  or  $\sigma_1^{\pm 1}$ . Again applying the skein relation on one of the band crossings, we can switch it so that the subword of the 2 letters reduces to one of the same length (two), but without occurrence of  $\sigma_3^{\pm 1}$ , so we are done by the inner induction.  $\square$

**Remark 2** Not only lemma 1, but in fact also the inequality (1) (with ‘2’ replaced by ‘3’) is not true for 4-braids, since the above quoted knot  $11_{409}$  has a 7-band 4-braid representation:  $[(23-2)(1-2-1)2(2-3-2)(12-1)-32]$ . (Here – and only here! –  $\sigma_3 = [3] \in B_4$  does not refer to the element  $[-121] \in B_3$ .)

The use of  $\sigma_{1,2,3}$  and Bennequin’s result can be applied also for the following useful inequality.

**Proposition 1** Let  $c_3(K)$  be the minimal crossing number of a 3-braid representation of a 3-braid knot or link  $K$ . Then

$$c_3(K) \leq 5/3(3 - \chi(K)). \quad (2)$$

In particular if  $K$  is a knot

$$c_3(K) \leq 2 \left\lfloor \frac{5}{3}(g(K) + 1) \right\rfloor \quad \text{or} \quad g(K) \geq \frac{3}{10}c_3(K) - 1. \quad (3)$$

**Proof.** Consider a minimal length (in  $\sigma_{1,2,3}$ ) word representation for  $\beta \in B_3$  with  $\hat{\beta} = K$ , and among such word representations one of minimal crossing number (that is, minimal number of letters  $\sigma_3^{\pm 1}$ ), up to cyclic permutations. Every (maximal) subword  $\sigma_3^k$ ,  $k \in \mathbb{Z} \setminus \{0\}$  of  $\beta$  must be (cyclically) followed by  $\sigma_2^{-1}$  or  $\sigma_1$  and preceded by  $\sigma_2$  or  $\sigma_1^{-1}$ , otherwise the first or last copy of  $\sigma_3^{\pm 1}$  can be eliminated by a relation, preserving the word length of  $\beta$ . Since  $\sigma_3^k = \sigma_1^{-1}\sigma_2^k\sigma_1$ , from each subword  $\sigma_3^k$  only one copy of  $\sigma_3^{\pm 1}$  contributes three to the minimal crossing number of a  $(\sigma_{1,2}$ -word) representation of  $\beta$  (the others contribute 1), (2) follows from Bennequin’s result. For knots this is equivalent to the second inequality of (3), the first inequality follows by the remark that  $2 \mid c_3(K)$  for knots  $K$ .  $\square$

For  $g = 1$  we get from (3)  $c \leq 6$ , thus obtaining Xu’s list of  $3_1, 4_1$  and  $5_2$  as the only genus 1 3-braid knots. For  $g = 2$  we get  $c \leq 10$ . The knots are in fact  $3_1\#(!)3_1, 5_1, 6_2, 6_3, 7_3, 7_5, 8_{20}$  and  $8_{21}$  (compare the discussion after corollary 1). There are three 12 crossing knots of genus 3, and still one 16 crossing knot of genus 4, so that (3) is exact in these cases.

We obtain from theorem 2 as corollary:

**Corollary 3** If the crossing number  $c(K)$  of a knot  $K$  satisfies the inequality

$$c(K) > 2 \left\lfloor \frac{5}{6}(\max \deg_z P(K) + 2) \right\rfloor,$$

then  $K$  is not a closed 3-braid.

It can already be expected from the proof of lemma 1 that the inequality  $\max \deg_z P(L) \leq 1 - \chi(L)$  should be fairly sharp. However, by computer check it turned out to be sharp without any exception up to 18 bands, thus leading me to the investigation of theorem 2.

To carry out the rest of the proof of theorem 2, we need to recall some of the work in [48]. There a fast algorithm to get any  $\sigma_{1,2,3}$  word-representation of  $\beta \in B_3$  into one of minimal length (and thus to calculate  $\chi(\hat{\beta})$ ) is given.

We recall this algorithm as it will be important in the proof.

- (i) Move all  $\sigma_i^{-1}$  to the left using  $\sigma_i \sigma_j^{-1} = \sigma_{i+1}^{-1} \sigma_{j+1}$ . Thus  $\beta = L^{-1}R$  with  $L$  and  $R$  positive.
- (ii) As long as  $L$  or  $R$  contain some subword  $\sigma_{i+1} \sigma_i$ , this subword can be moved to their beginning, giving  $\beta = L^{-1}[21]^k R$  with  $L$  and  $R$  positive and non-decreasing.
- (iii) Applying  $[21]^{-1} \sigma_{i+1} = \sigma_i^{-1}$  and  $\sigma_i^{-1} [21] = \sigma_{i-1}$  and cyclic reductions, one of the 3 factors in  $L^{-1}[21]^k R$  can be eliminated.

Although this may not be evident from the algorithm, we remark that Bennquin's result implies that the minimal length of  $\beta \in B_3$  is conjugacy invariant, and thus whether a word can be reduced by Xu's algorithm or not is invariant under cyclic permutations of its letters.

**Proof of theorem 2.** Since we need to consider only one of two mirror images for  $L$ , we may assume at one point in our proof for every case that  $\beta \in B_3$  with  $\hat{\beta} = L$  has non-negative exponent sum  $e(\beta)$ .

By Xu's algorithm, each  $\beta \in B_3$  can be written in one of the two forms

- (A)  $[21]^k R$  or  $L^{-1}[21]^{-k}$  ( $k \geq 0$ ), or
- (B)  $L^{-1}R$ ,

where  $L$  and  $R$  are positive words with (cyclically) non-decreasing indices (i.e. each  $\sigma_i$  is followed by  $\sigma_i$  or  $\sigma_{i+1}$ ). Since the form (B) must be cyclically reduced, we may assume that  $L$  and  $R$  do not start or end with the same letter.

If  $\beta$  is of type (A) (we call this case "strongly quasipositive" conforming to Rudolph [39, 40]), then by the mirroring argument we may assume  $e(\beta) > 0$ , and then have from [28] and theorem 1

$$1 - \chi(\hat{\beta}) = e(\beta) - 2 \leq \min \deg_v P \leq \max \deg_z P, \quad (4)$$

and thus the reverse inequality to lemma 1.

Thus we need to consider only the case (B).

A fair part of our argument will go like this: We choose a band (crossing) in  $\beta = \beta_{\pm}$  and apply the skein relation at this crossing, expressing the polynomial of  $L = L_{\pm} = \hat{\beta}$  by those of  $L_0 = \hat{\beta}_0$  and  $L_{\mp} = \hat{\beta}_{\mp}$ . (Here  $\beta_0$  and  $\beta_{\mp}$  are obtained by deleting resp. reversing the band in  $\beta$  we consider.) Then we show that only one of  $\beta_0$  and  $\beta_{\mp}$  contributes to the coefficient  $[P(L)]_{z^{1-\chi(L)}}$  of  $z^{1-\chi(L)}$  in  $P(L)$ . Because of lemma 1 for this it suffices to show that the other one is not of Xu's minimal (word length) types (A) and (B). This way we lead back inductively the case of  $L$  to some simple cases.

We have

$$\beta = L^{-1}R \quad \text{with} \quad R = \prod_{j=0}^l \sigma_{i+j-1}^{k_j} \quad \text{and} \quad L = \prod_{j=0}^{l'} \sigma_{i'+j-1}^{k'_j}, \quad (5)$$

with  $i \neq i'$  and  $i+l \not\equiv i'+l' \pmod{3}$ .

The first application of the skein relation argument is that we can make induction on the  $k_j$  and  $k'_j$ , thus being left just with the cases where all  $k_j = k'_j = 1$ , in which case  $L$  and  $R$  get the simpler form

$$R = (\sigma_i \sigma_{i+1} \sigma_{i+2})^k \alpha \quad \text{and} \quad L = (\sigma_{i'} \sigma_{i'+1} \sigma_{i'+2})^{k'} \alpha' \quad (6)$$

with  $\alpha$  and  $\alpha'$  of length  $\leq 2$ . Again we can assume modulo mirror images, that  $R$  is not shorter than  $L$ , i.e.  $\beta = L^{-1}R$  with  $L$  and  $R$  as in (6) has  $e(\beta) \geq 0$  (it may originate from a braid in (5) with negative exponent sum!).

Now consider the case where  $6 \mid e(\beta)$  and use the representation theory of  $P$  on 3-braids (see [20]). Let  $\Delta = [121]$  be the square root of the center generator of  $B_3$  [10]. Define  $\beta^* = \Delta^{2/3e(\beta)}\beta^{-1}$  to be the *dual* of  $\beta$  (clearly  $\beta^{**} = \beta$ ). Then, as observed in [6, proof of proposition 2],  $\beta$  and  $\beta^*$  have the same polynomial, because they have the same (normalized) Burau trace and the same exponent sum. But for  $\beta$  as in (5) we have because of  $\Delta^2\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i-1}^{-1} = \sigma_i^3$  that

$$\beta^* = \gamma\sigma_i^{k_i}\sigma_j^{k_j}\gamma', \quad (7)$$

where  $\gamma$  and  $\gamma'$  have length at most 2,  $i \neq j$  and  $k_i k_j \leq 0$ , and are thus left with showing that for such words  $\beta^*$ ,  $\max \deg_z P(\hat{\beta}^*) = \text{len}(\beta^*) - 2$ . Again by the skein induction argument on the  $k_i$  and  $k_j$  this can be reduced to the cases where  $\beta^*$ , and hence  $\beta$ , have small crossing number, and they checked directly.

Now consider the case where  $e(\beta) \equiv 1, 2 \pmod{6}$ . We apply the skein relation at the rightmost letter/band in  $\beta$ . Then one of  $\beta_0$  or  $\beta_-$  have  $6 \mid e$ . It suffices to show that the other one is not minimal (and apply lemma 1). For this one checks that either  $\beta_0$  is not cyclically reduced (starts and ends with opposite letters), or that when permuting the rightmost (negative) letter of  $\beta_-$  to the left, the word  $L$  in (5) is not increasing (and hence  $\beta_-$  can be reduced by Xu's algorithm). For example for  $\beta = [-2 - 1 - 3 - 21231231]$  we get  $\beta_- \doteq [-1 - 2 - 1 - 3 - 2123123]$  (' $\doteq$ ' meaning equality up to conjugacy).

If  $e(\beta) \equiv 3, 4, 5 \pmod{6}$ , then apply the same argument at most 3 times, getting back to the  $6 \mid e$  case (except in the cases where  $R$ , and hence  $L$ , are short, and which can be checked directly). Since any  $6 \mid e$  word is reduced, and every pair  $(\beta_0, \beta_-)$  contains one reducible word, it will indeed be the  $6 \mid e$  braid wo which the argument recurs rather than some of its neighbors.

This completes the proof of theorem 2. □

**Remark 3** There is an alternative way to proceed with the proof after (6), namely to remark that in the application of the skein relation at every second stage it is  $\beta_0$  that is of Xu's form, and then to work by induction on the word length. Thus the representation theoretic argument can be avoided. However, the proof did not appear (to me) more elegant without it, and also, there are some insights which this argument explains better (in particular the cases of trivial Alexander polynomial, see question 2), so I consider it not inappropriate.

The representation theory also shows that  $P(\hat{\beta})$  for  $\beta \in B_3$  can be calculated in time  $O(c(\beta))$  (see [30]), and thus we have an even faster algorithm than the (quadratic) one of Xu to calculate  $\chi(\hat{\beta})$ .

**Corollary 4** For  $\beta \in B_3$ ,  $\chi(\hat{\beta})$  can be calculated in  $O(c(\beta))$  steps.

We will come back to this corollary at the end of the next section.

### 3. Further applications

We can even say a little more that theorem 2. Since throughout the proof,  $L_+$  inherited its maximal coefficient from  $L_-$  or  $L_0$  (up to multiplication with units in  $\mathbb{Z}[v, v^{-1}]$ ) and duality does not alter the polynomial, we see that in fact we can determine what maximal ( $z$ -)coefficients skein polynomials of 3-braid links can have by checking some simple cases. We have the following result (note that it somewhat depends on the convention for  $P$  chosen!):

**Theorem 3** Let  $L$  be a 3-braid link. Then  $[P(L)]_{z^{1-\chi(L)}}$  is up to units  $\pm v^k$ ,  $k \in \mathbb{Z}$  one of  $1 \pm v^2$  or 1, except for the 3 component unlink (where it is  $(1 - v^2)^2$ ). If  $L$  is a knot or 3 component link, then  $[P(L)]_{z^{1-\chi(L)}} \neq -v^k(1 + v^2)$ .

**Proof (sketch).** This is, as remarked, basically a repetition of the proof of theorem 2. In the strongly quasipositive case (which there could be dealt with immediately) the skein and duality arguments can be applied similarly, leaving us with a braid of the form  $(\sigma_2\sigma_1)^k\sigma_1^l\alpha'$  with  $\alpha'$  having small length. By skein argument induction,  $l$  can be reduced

to 1. For the first factor use now  $(\sigma_2\sigma_1)^3 = \sigma_2\sigma_1\sigma_2^2\sigma_1\sigma_2$ , and apply the skein argument on the ‘ $\sigma_2^2$ ’ in the middle until you get  $k$  small. The rest for the first statement is to compute the polynomial for some simple words.

To show that  $(1 - v^2)^2$  occurs only for the 3-component unlink, we need to verify this among the small words and to observe that the procedure of inductively simplifying the braid in the proof of theorem 2 at no stage gives the trivial (empty) word (for this  $i \neq j$  in (7) is needed).

For the second statement, consider  $\hat{\beta} = L$  for  $\beta$  of odd connectivity (i.e. even exponent sum) and the signs in the skein relations. It follows from the skein relation that expressing  $P(L_\varepsilon)$  for  $|\varepsilon| = 1$  by  $P(L_{-\varepsilon})$  and  $P(L_0)$ , latter’s coefficients are  $+1$  except for the one of  $P(L_0)$  when  $\varepsilon = -$ . Thus we need to take care only when we switch negative crossings. When reducing the  $k'_j$  in (5), we can maintain sign at the cost of leaving possibly one of them equal to 2. Denote by  $w$  the subword of  $L$  made up of this generator square. Then, a possible mirroring (to get in (6)  $R$  to be not shorter than  $L$ ) does not alter the sign of  $[P(L)]_{z^{1-\chi(L)}}$ . When mirroring puts  $w$  into  $R$ , the generator square has positive sign and can be reduced. Then, after going over to  $\beta^*$ , (except for the few small length cases where  $R$  is short) we switch negative crossings only when reducing the negative one of  $k_i$  and  $k_j$  in (7). Then we just choose to reduce it by steps of 2, thus preserving connectivity of the closure and sign of  $[P(L)]_{z^{1-\chi(L)}}$ . If  $w$  remains in  $L$  (in which case we don’t apply mirroring), reduce it as well, but then in  $\beta^*$  in (7) reduce the negative one of  $k_i$  and  $k_j$  first by one, thus canceling the negation. Checking some simple cases (just of odd connectivity) shows the result up to mirroring. Since mirroring does not negate  $[P(L)]_{z^{1-\chi(L)}}$  when the connectivity is odd (or equivalently  $2 \mid e(\beta)$ ), the result follows.  $\square$

We can say something on the cases where  $1 \pm v^2$  in the above theorem occurs as maximal coefficient. We rephrase this using the relation to the Conway polynomial  $\nabla(K)$  [11] and Alexander polynomial  $\Delta(K)$  (in the normalization  $\Delta(1) = 1$  and  $\Delta(t^{-1}) = \Delta(t)$ )

$$\Delta(t) = \nabla(t^{1/2} - t^{-1/2}) = P(1, t^{1/2} - t^{-1/2}). \quad (8)$$

**Proposition 2** If for a 3-braid link  $L$ ,  $\max \deg \nabla(L) < \max \deg_z P(L)$  or  $\max \text{cf} \nabla(K) = \pm 2$ , then  $L$  is (the closure of) a strongly quasipositive 3-braid.

**Proof.** Again check the small length cases and apply the previous type of induction.  $\square$

A small application of this is

**Corollary 5** Any homogeneous braid index 3 link  $L$  is fibered or positive.

**Proof.** Theorem 3 shows from (8) that  $\Delta(L)$  has leading coefficient  $\max \text{cf} \Delta = -1, 1$  or  $2$ , since  $2 \max \deg \Delta(L) = \max \deg_z P(L)$  by [12]. If the leading coefficient is  $\pm 1$ , then  $L$  is fibered (see [12, corollary 5.3]). Otherwise, it is strongly quasipositive, and lemma 1 and (4) imply  $\min \deg_v P(L) = \max \deg_z P(L)$ . Then apply [12, theorem 4].  $\square$

We will in the next section have to say much more about alternating links.

Some other worth remarking consequences follow now from the work of Rudolph [40]. For simplicity, call the Alexander polynomial  $\Delta(K)$  of a knot  $K$  *maximally monic*, if its leading coefficient is  $\pm 1$  (monicness) and its degree equal to  $g(K)$  (maximality). A classical result states that fibered knots have such Alexander polynomials. Here we obtain:

**Corollary 6** Any achiral or slice braid index 3 knot has maximally monic Alexander polynomial.

**Proof.** For slice knots this follows from proposition 2 and [40]. For achiral knots use theorem 3 and that  $[P(K)]_{z^{1-\chi(K)}}$  is self-conjugate.  $\square$

In the slice case neither maximality nor monicness need to hold for 4-braids, as show (slice) knots like  $8_8$  and the  $\Delta = 1$  11 crossing knot  $11_{409}$ . In the achiral case the situation is unclear since there may exist no achiral braid index 4 knot (see [45]).

In a similar way we get

**Corollary 7** There are only finitely many braid index 3 knots  $K$  of given unknotting number  $u(K)$ , whose Alexander polynomial is not maximally monic. (For unknotting number 1 this is just the knot  $5_2$ .)

**Proof.** Use that by [40],  $u(K) \geq g(K)$  for strongly quasipositive  $K$ . □

This is certainly not true, already for  $u(K) = 1$ , without the condition on the Alexander polynomial, an example being the rational knots with Conway notation  $(n11n)$ ,  $n \in \mathbb{N}$ .

A practical application of theorem 3 is related to corollary 4. In its formulation we meant the word “steps” w.r.t. the notion of complexity where an addition is regarded as an elementary operation no matter how large the summands are. Clearly, in practice they will grow with  $c(\beta)$ , though. Theorem 3 allows to circumvent this problem. We know that for knowing  $\max \deg_z P(v, z)$  it suffices to know  $P(v, z) \bmod 2$ . We have

$$P\left(\lambda, q - \frac{1}{q}\right) = X\left(q^2, \frac{\lambda^2}{q^2}\right),$$

where  $X$  is the Ocneanu-Jones parametrization of the skein polynomial described in [20], and by Ocneanu’s work for a 3-braid link  $L$  we know that  $X_L(\lambda, q) \cdot Q(q)$  for some monic polynomial  $Q$  (that is, with leading coefficient  $\pm 1$ ) is a linear combination (with coefficients being polynomials in  $\lambda, q$ ) of traces of (Bureau) matrices over  $\mathbb{Z}[q, q^{-1}]$ . Thus there is no problem calculating the polynomial via the Hecke algebra with coefficients reduced mod 2, and the problem of increasing magnitude of the numbers in the operations can be avoided.

Finally we remark that theorem 3 also gives another (and much more straightforward) way to see that  $9_{49}$  (and any other knot with such polynomial) is not a 3-braid knot. Unfortunately, this simple criterion does not always work, as shows the polynomial of  $9_{42}$ . This can always be decided as discussed after corollary 1, but for higher  $\max \deg_z P$  the process of generating the whole list of knots becomes tedious, so that our work here does not render obsolete examples like the one in [45].

#### 4. Alternating links of braid index 3

A final, and main, application of our method is to complete the description of alternating links of braid index 3.

**Theorem 4** Let  $L$  be an alternating braid index 3 link. Then (and only then)  $L$  is

- a) the connected sum of two  $(2, k)$ -torus links (with parallel orientation), or
- b) an alternating 3-braid link (i.e. the closure of an alternating 3-braid, including split unions of a  $(2, k)$ -torus link and an unknot and the 3 component unlink), or
- c) a pretzel link  $\mathcal{P}(1, p, q, r)$  with  $p, q, r \geq 1$  (oriented so that the twists corresponding to  $p, q, r$  are parallel).

From this theorem the classification of alternating links of braid index 3 is easy using [27]. For the alternating 3-braid links we need to figure out which (alternating) 3-braids admit a flype. They are all of the form  $[1^k - 21^l - 2^m]$  for  $k, l, m > 0$  (for special values of  $k, l, m$  the flypes may be trivial). See [7].

Murasugi [31] described the rational ones among them. Our result easily implies his.

**Corollary 8** If  $L$  is a rational link of braid index 3, then  $L$  has Conway notation  $(p, 1, 1, q)$  or  $(p, 2, q)$  for some  $p, q > 0$ .

**Proof.** Simply check which of the alternating diagrams of the above links are rational. □

**Proof of theorem 4.** We know from [12] that for an alternating link  $L$ ,  $\max \deg \nabla(L) = \max \deg_z P(L)$ , and thus for braid index  $b(L) \leq 3$  we have from proposition 2 that  $\max \text{cf} \nabla(L) \in \{-1, 1, 2\}$ .

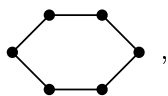
In case  $\max \text{cf} \nabla(L) = \pm 1$ ,  $L$  is fibered by [32] (or see [12, corollary 5.3]), and then by [31, theorem A(2)] any alternating diagram of  $L$  has  $b(L) \leq 3$  Seifert circles. This gives the cases a) and b). (The split cases are easy since  $b$

is additive under split union.) Since it is known from [33] that case a) includes all composite links of braid index 3, we may henceforth assume that  $L$  be prime, and also non-split.

Assume now that  $\max \text{cf} \nabla(L) = 2$ . We know from proposition 2 that  $L = \hat{\beta}$  with  $\beta \in B_3$  strongly quasipositive. By lemma 1 and (4) we have  $\min \text{deg}_v P(L) = \max \text{deg}_z P(L)$ , and then it follows from [12, theorem 4] (see also [36]) that any homogeneous (in particular, alternating) diagram of  $L$  is positive. Thus  $L$  has a special alternating diagram  $D$ .

For every such diagram  $D$  we consider the Seifert graph  $G(D)$ , with vertices corresponding to Seifert circles and edges to crossings (see [12, §1]).  $G$  is connected, planar (that is, in particular considered with its planar embedding) and bipartite, hence every cycle in  $G$  has even length (possibly 2, since  $G$  may have multiple edges). For every such  $G$  we can contrarily construct a special alternating diagram  $D(G)$  with  $G(D(G)) = G$  (which in general depends on the planar embedding of  $G$ ). It follows from [12] (see corollary 2.2 and the proof of theorem 5, p. 543) that if  $G'$  is obtained from  $G$  by deleting an edge, then  $\max \text{cf} \nabla(D(G')) \leq \max \text{cf} \nabla(D(G))$ . (Here deleting an edge  $e$  means deleting one single edge in a multiple one. If  $e$  is single and its deletion disconnects the graph, then if one of the 2 new components is a single vertex, this vertex is deleted as well, while if both components contain edges, the deletion of  $e$  is prohibited.)

If  $G$  is a cycle graph of length  $2k$  like

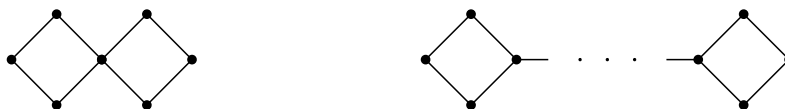


then  $D(G)$  depicts the  $(2, 2k)$ -torus link  $T_k$  with reverse orientation, and  $\nabla(T_k) = kz$ . Therefore, if  $\max \text{cf} \nabla(D(G)) = 2$ ,  $G$  cannot contain a cycle of length  $> 4$ .

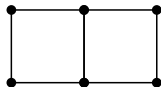
Since we excluded composite links, we may assume that in  $G = G(D)$  there is no vertex connected (by a possibly multiple edge) to only one single other vertex. Then we replace in  $G$  every multiple edge by a single one. We obtain a graph  $\hat{G}$  (called sometimes reduced Seifert graph), in which each vertex has valency  $\geq 2$  and there are no multiple edges. By [9, proposition 13.25] (see also [12, 46]),  $\max \text{cf} \nabla(D(\hat{G})) = 1$  iff  $\hat{G}$  is a tree, and in this case  $\hat{G}$  would have to be one single vertex, which is uninteresting.

Therefore,  $\max \text{cf} \nabla(D(\hat{G})) = 2$  and  $\hat{G}$  still contains a cycle. We know from  $G$  that any cycle in  $\hat{G}$  has length 4. We wish to show that there is only one such cycle.

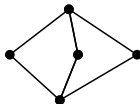
Assume there were two, call them  $C_1$  and  $C_2$ . If  $C_1$  and  $C_2$  have  $\leq 1$  vertex in common, then by deleting edges from  $\hat{G}$  we can obtain a graph  $\tilde{G}$  consisting of  $C_1$  and  $C_2$  joined by a (possibly trivial) path.



But  $D(\tilde{G}) = T_2 \# T_2$ , and  $\max \text{cf} \nabla(T_2 \# T_2) = 4$ . If  $C_1$  and  $C_2$  have two neighbored vertices in common, then  $G$  has a subgraph



and a cycle of length 6. Thus either  $C_1$  and  $C_2$  have 3 vertices in common or two vertices which are opposite (not neighbored). In both cases  $G$  contains the subgraph



This corresponds to the  $(2, 2, 2)$ -pretzel link (oriented so that the clasps are reverse) with  $\nabla = 3z^2$ .

Therefore,  $\hat{G}$  contains only one cycle (of length 4), and must be only this cycle. This shows that  $D$  is a diagram of the  $(p, q, r, s)$ -pretzel link (with parallel twists)  $\mathcal{P}(p, q, r, s)$ . Since for  $p = 1$  we have  $\mathcal{P}(1, q, r, s) = \hat{[1^q 2^s 3^r]}$ , it remains to show that  $L = \mathcal{P}(p, q, r, s)$  has braid index 4 for  $p, q, r, s > 1$ . Using MWF and  $\min \text{deg}_v P(L) = \max \text{deg}_z P(L) =$



$1 - \chi(L)$ , it suffices to show that  $\mu(L) := \max \deg_v P(L) = 7 - \chi(L)$ . For this, we verify it for  $p = q = r = s = 2$  and inductively use the skein relation, noticing that the signs of the  $z$ -coefficients of  $[P(L')]_{\nu^{\mu(L')}}_z$  are for  $L' = L_-$  and  $L' = L_0$  the same as for  $L' = \mathcal{P}(2, 2, 2, 2)$ , and thus their contributions to  $P(L_+)$  do not cancel.  $\square$

**Remark 4** K. Murasugi pointed out that an alternative proof of the conclusion  $L$  special alternating and  $\max \text{cf} \Delta_L = 2 \implies L = \mathcal{P}(p, q, r, s)$  was given in lemma 4.3 of [35].

By [23, 31, 47], each alternating 3-braid knot will have even crossing number. The theorem now shows:

**Corollary 9** Prime alternating braid index 3 knots, which are not closures of alternating 3-braids, have odd crossing number.

Also we have

**Corollary 10** Each alternating braid index 3 link is an alternating 3-braid link or is positive.

**Remark 5** The braid representations of 3-braid links were described in [7], but since braids have (at least so far) proved of little use in the study of combinatorial (diagrammatic) properties of their closures, the methods there are unlikely to approach such kind of results.

## 5. Problems

Here are some open questions one can ask. For example, one is the following question, suggested by computer experiment, in which braid index 3 knots of at most 16 crossings were identified in the tables of [18] and all were found to accord to the following conjectured rule (the same experiment pointed me to theorem 4).

**Question 1** Does any non-alternating braid index 3 knot have even crossing number?

Another problem which is possible to pursue by the methods of this paper, but which involves some technical difficulties is

**Question 2** Does for any 3-braid link  $K$  with  $\nabla(K) \neq 0$  hold  $\max \deg \nabla(K) \geq \max \deg_z P(K) - 2$ ? Are the only 3-braid links  $K$  with  $\nabla(K) = 0$  the split unions of  $(2, k)$ -torus links and an unknot and links of the form  $\hat{[123]^{2k}}$ ?

The proof should go similarly to theorem 2, but more care must be taken.

The origin of the investigations of this paper came from the attempt to compare the two estimates for  $\tilde{g}(K)$  given by  $g(K)$  and  $1/2 \max \deg_z P(K)$ . Now it was shown that they are equally good, but I do not know whether they are always sharp.

**Question 3** Is  $g(K) = \tilde{g}(K)$  for any 3-braid knot  $K$ ?

Since, as mentioned,  $2\tilde{g}(K) = \max \deg_z P(K)$  for  $K$  of  $\leq 12$  crossings, the answer is positive at least up to genus 3.

A final question concerns a possible generalization of Bennequin's result.

**Question 4** Does any knot have a minimal genus Seifert surface constructed by the band algorithm on a minimal strand representation?

It has been mentioned by Birman and Menasco that knots lacking the requested property should exist. However, I was unable to find (by computer) a concrete example. In [41], Rudolph showed that any Seifert surface is isotopic to some band algorithm surface (for a possibly non-minimal strand representation).

This question was subsequently answered in a joint paper with M. Hirasawa [16].

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