SQUARE NUMBERS AND POLYNOMIAL INVARIANTS OF ACHIRAL KNOTS

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Abstract. We describe the leading coefficients of the Alexander and skein polynomial of alternating achiral knots.
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1. Introduction

When the *Jones polynomial* V [J] appeared in 1984, one of its spectacular features was that it was often able to distinguish between a knot and its mirror image, or *obverse*, by taking different values on them. This property of V was a novelty, since its decades-old predecessor, the *Alexander polynomial* Δ [Al], is always equal for a knot and its mirror image. Nonetheless, Δ was known to satisfy certain properties for an *achiral*, or *amphicheiral* knot, a knot equivalent to its mirror image [Ha, HK]. The generalizations of the Jones polynomial, the *HOMFLY, or skein, polynomial* P [F&] and the *Kauffman polynomial* F [Ka], consequently also distinguish mirror images. More precisely, the V, P and F polynomial of an achiral knot has the special property of being *self-conjugate*, or reciprocal, that is, invariant when (one of) its variable(s) is replaced by its inverse.

In this paper we will consider the Alexander and skein polynomials of several large classes of achiral knots and prove some properties of their leading coefficients. We will primarily address achiral alternating knots, where such properties are closely related to Murasugi's $*$-product. We prove in particular that perfect squares are exactly the numbers occurring as leading coefficients $\max\operatorname{cf}\Delta$ of the Alexander polynomial of alternating achiral knots (corollary 5.12). This improves the previously known fact, shown by Murasugi and Przytycki [MP2, MP3], that these coefficients can not be primes. For strongly and negatively amphicheiral knots the same squareness result is obtained using [Ha, HK]. The theme of relations of achiral knots to square numbers was studied in detail in [St2], and the present paper is a continuation of this previous work. An independent proof of our result for alternating knots was obtained soon after our proof by C. Weber and C. V. Quach Hongler [QW]. (See the end of §3 for some explanation.)

We will illustrate our results with several examples. In particular we will solve negatively (see example 6.3) a related conjecture of Murasugi and Przytycki in [MP3], stating that the skein polynomial would always detect chirality whenever their primeness (chirality) condition on $\max\operatorname{cf}\Delta$ does.

Some open problems will be mentioned during the discussion. Particularly worth pointing to is a conjecture of Kawauchi (see remark 4.3), which will be seen to imply an extension of our results on the Alexander polynomial of alternating to general amphicheiral knots. A related interesting problem is discussed in [St4], namely whether all achiral knots have diagrams that can be transformed into their obverse by moves in S^2. We also omit here a detailed study of links (where also the additional technical issue of component orientation becomes relevant).

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2. Basic preliminaries

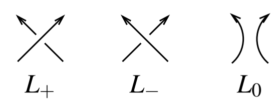
We begin by discussing the most basic concepts that will be employed in this paper. (A few more detailed definitions are given later.) We recall first the description of some link polynomials, and then general properties of knot diagrams. We will also use some abbreviations in the text: "w.r.t." stands for "with respect to", and "w.l.o.g." abbreviates "without loss of generality".

Knots and links are represented by *diagrams*. A knot (resp. link) diagram is understood as a smoothly immersed closed plane curve (resp. a collection of such curves) with transverse self-intersections (*crossings*), at which one of the intersecting plane curve segments is distinguished (over- and undercrossing). In the following knots and links, and their diagrams, will be assumed oriented, but sometimes orientation will be irrelevant. The *inverse* of a knot or link (diagram) is obtained by altering the orientation of *all* components.

The *Alexander polynomial* $\Delta_L(t)$ [Al] is an invariant of oriented knots and links L with values in $\mathbb{Z}[t,t^{-1}]$ (a Laurent polynomial in one variable t). Originally, one has a topological definition of Δ, which is accurate only up to multiplication with units in $\mathbb{Z}[t,t^{-1}]$ (see [Ro]). The fix of this ambiguity (normalization) is usually chosen so that $\Delta(t)=\Delta(1/t)$ and $\Delta(1)=1$. Later it was found that the so normalized polynomial can be alternatively characterized by the properties of having the value 1 on the unknot and satisfying the *"skein" relation*

$$\Delta(L_+)-\Delta(L_-) = (t^{1/2}-t^{-1/2})\,\Delta(L_0)\,. \tag{1}$$

Herein $L_{\pm,0}$ are three links with diagrams differing only near a crossing.



(2)

As above, we will often omit from the notation $\Delta_L(t)$ the link L or variable t (or both) if it is irrelevant or clear from the context, and we will also sometimes write $\Delta(L)$ for Δ_L. It is well-known that the the Alexander polynomial does not distinguish a knot K from its mirror image $!K$, that is, $\Delta_K(t)=\Delta_{!K}(t)$ for any knot K.

The *skein* or *HOMFLY polynomial* P [F&] is a Laurent polynomial in two variables l and m of oriented knots and links and can be specified by being equal to 1 on the unknot and the skein relation

$$l^{-1}\,P\big(L_+\big)+l\,P\big(L_-\big) = -m\,P\big(L_0\big)\,, \tag{3}$$

with $L_{\pm,0}$ as in (2). (This convention uses the variables of [LM], but differs from the convention there by the interchange of l and l^{-1}.) Here and below a similar explanation to the one for Δ applies on the use of $P(l,m)$ and $P_K=P(K)$ as abbreviations of $P_K(l,m)$.

It easily follows from (3) that P is a generalization of the Alexander polynomial. Consequently, there is a substitution formula (see [LM]), expressing the Alexander polynomial Δ, for the normalization so that $\Delta(t)=\Delta(1/t)$ and $\Delta(1)=1$, as a special case of the HOMFLY polynomial (i is here the complex unit):

$$\Delta(t) = P(i,i(t^{1/2}-t^{-1/2}))\,. \tag{4}$$

The skein polynomial satisfies the property $P_K(l,m)=P_{!K}(l^{-1},m)$. A similar identity holds for the Jones V [J] and Kauffman F [Ka] polynomials. Such relations mean that achiral knots K (i.e. knots with $K=!K$) have *self-conjugate* polynomials (in one of the variables). In contrast to Δ, this property is not automatic, and so allows for a chirality test. This test is efficient but not perfect. The popular example of a chiral knot K (i.e. a knot with $K\neq!K$) having self-conjugate polynomials is 9_{42} in the tables of [Ro, appendix].

Let $\mathcal{L}=\mathbb{Q}[t,t^{-1}]$ be the Laurent polynomial ring in one variable. For $X,Y\in\mathcal{L}$ we write $X\sim Y$ if X and Y differ by a multiplicative unit in $\mathbb{Z}[t^{\pm1}]$, that is, $X(t)=\pm t^nY(t)$ for some $n\in\mathbb{Z}$. For $Y\in\mathcal{L}$ and $a\in\mathbb{Z}$, let $[Y]_{t^a}=[Y]_a$ be the coefficient of t^a in Y. For $Y\neq0$, let $\mathcal{C}_Y=\{\,a\in\mathbb{Z}\,:\,[Y]_a\neq0\,\}$ and define

$$\min\deg Y=\min\,\mathcal{C}_Y\,,\quad \max\deg Y=\max\,\mathcal{C}_Y\,,\qquad\text{and}\qquad \operatorname{span}Y=\max\deg Y-\min\deg Y$$

to be the *minimal* and *maximal degree* and *span* (or *breadth*) of Y, respectively. We will denote the maximal degree, or *leading*, *coefficient* $[Y]_{\max\deg Y}$ of t in Y by $\max\operatorname{cf}Y$.

More generally, let $x_1,\dots,x_n$ be variables, and $Y \in \mathbb{Q}[x_1,\dots,x_n]$. Let $M=\{1,\dots,n\}$, $\Phi\subset M$, and $Q=\prod_{i\in\Phi} x_i^{k_i}$ for $k_i\in\mathbb{N}$ (not necessarily non-zero) be a monomial in $\{x_i\}_{i\in\Phi}$. Then we write $[Y]_Q$ for the coefficient of Q in Y, and understand it so that $[Y]_Q\in\mathbb{Q}[x_i]_{i\in\overline{\Phi}}$, with $\overline{\Phi}=M\setminus\Phi$. The maximal degree $\max\deg_{x_i} Y$ of Y w.r.t. the variable x_i is the largest power of x_i occurring in a monomial Q of Y (that is, a monomial Q in $x_1,\dots,x_n$ with $[Y]_Q\neq 0$ in $\mathbb{Q}$). The leading coefficient $\max\mathrm{cf}_{x_i} Y$ of Y w.r.t. x_i is defined by $[Y]_{x_i^{\max\deg_{x_i} Y}}\in\mathbb{Q}[x_j]_{j\in\overline{\{i\}}}$. For Laurent polynomials, we replace x_i by $x_i^{\pm 1}$ in the polynomial ring, and $k_i\in\mathbb{N}$ by $k_i\in\mathbb{Z}$ in Q.

Definition 2.1 The diagram on the right of figure 1 is called *connected sum* $A\#B$ of the diagrams A and B. If a diagram D can be represented as the connected sum of diagrams A and B, such that both A and B have at least one crossing, then D is called *disconnected*, or *composite*; otherwise it is called *connected*, or *prime*. Equivalently, a diagram is prime if any closed curve intersecting it in exactly two points does not contain a crossing in one of its interior or exterior.

Any diagram D can be written as $D_1\#D_2\#\cdots\#D_n$, so that all D_i are prime and have at least one crossing. Then D_i are called the *prime (or connected) components/factors* of D. A link L is prime if for any composite diagram $D_1\#D_2$ of L, one of $D_{1,2}$ is an unknot diagram.

Definition 2.2 A diagram D is *split* if there is a closed curve which does not intersect D, but which contains parts of D in both its interior and exterior.

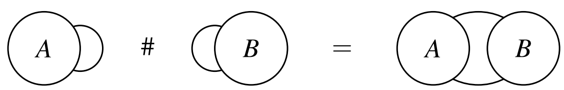


Figure 1

Definition 2.3 A crossing q in a link diagram D is called *nugatory* if there is a closed (smooth) plane curve γ intersecting D transversely in q and nowhere else. A diagram is called *reduced* if it has no nugatory crossings.

Theorem 2.4 ([Me]) An alternating reduced diagram of a link L is prime iff L is prime.

By replacing in a diagram D all fragments of the type $L_\pm$ in (2) by fragments of the type L_0, we obtain a collection of loops called *Seifert circles* of D. A Seifert circle is *separating* if it contains other Seifert circles in both its interior and exterior. A diagram with no separating Seifert circles is *special*. We call the crossings in the fragments of L_+ resp. L_- in (2) resp. *positive* and *negative*. A diagram is called positive or negative if all of its crossings are so. A diagram is *special alternating* if it is special and alternating. (For non-split special diagrams, alternating is equivalent to one of positive or negative.)

Definition 2.5 Let $c(D)$ be the *crossing number* of a link diagram D. Let $c(L)$ be the crossing number of a link L, which is the minimal crossing number of all diagrams D of L.

Theorem 2.6 ([Ka2, Mu2, Th]) Each alternating reduced diagram is of minimal crossing number (for the link it represents).

3. The leading coefficients of the Alexander and HOMFLY polynomial

In this section we formulate, and later partly explain, a very striking occurrence of (perfect) squares in connection with invariants of achiral knots, namely in the leading coefficients of their Alexander polynomial, and discuss a partial generalization of this property to the HOMFLY (skein) polynomial.

Definition 3.1 Let $g = g(K)$ be the *genus* of a knot K, the minimal genus of a *Seifert surface* S of K, which means an embedded compact oriented surface $S \subset \mathbb{R}^3$ with boundary $\partial S = K$ and no closed components. For a diagram D of K, the Seifert surface obtained by applying the Seifert algorithm on D is called *canonical Seifert surface* of D. The genus of this surface is called (canonical) genus $g(D)$ of D. This genus can be expressed as

$$g(D) = \frac{c(D) - s(D) + 1}{2},$$

with $s(D)$ being the number of Seifert circles of D (see [Ad, §4.3] or [Ro]). Call the minimal genus of a diagram D of a knot K the *canonical genus* of K.

Note that for the genus $g(K)$ of K arbitrary, *not* necessarily canonical, Seifert surfaces of K are taken into account. To underscore this, in [Mo], Morton showed the inequality

$$\max\deg_m P(D) \leq c(D) - s(D) + 1 \tag{5}$$

for any arbitrary link diagram D, and observed from it that there are knots K which do not possess a diagram D with $g(D) = g(K)$ (a fact that also implicitly follows from [Wh]). Such knots K are believed, though, to be somewhat exceptional. In fact, the main interest in the canonical genus comes from the study of diagrams D that satisfy the above equality. Of particular importance in this regard is the following classical fact:

Theorem 3.2 ([C, Mu3]) If K is an alternating knot with an alternating diagram D, then $g(D) = g(K)$. Also we have $\max\deg\Delta_K = g(K)$.

For an alternating knot K, the HOMFLY polynomial $P_K \in \mathbb{Z}[m^2, l^{\pm 2}]$ is known to have $\max\deg_m P_K = 2g(K)$, i.e.

$$P_K(l,m) = a_{2g}(l)m^{2g} + (\text{lower } m\text{-degree terms}), \tag{6}$$

with $a_{2g} \in \mathbb{Z}[l^2, l^{-2}]$ being a non-zero Laurent polynomial in l^2 (see [Cr]).

Some experimental and heuristical evidence (explained below) leads to two main questions we consider here. The first one concerns the Alexander polynomial.

Question 3.3 Is $\max\mathrm{cf}\,\Delta_K$ for an achiral knot K always a square up to sign, and if Δ is normalized so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$, is the sign $\mathrm{sgn}\,(\max\mathrm{cf}\,\Delta_K)$ always given by $(-1)^{\max\deg\Delta_K}$?

The question about Δ can be generalized to P. If K is achiral, then we have that $P_{\max} = \max\mathrm{cf}_m P$ (which equals a_{2g} in (6) if K is alternating) satisfies $P_{\max}(l^{-1}) = P_{\max}(l)$. However, another property of $P_{\max}$ becomes often apparent.

Question 3.4 Which achiral knots have $\max\mathrm{cf}_m P$ of the form $f(l^2)f(l^{-2})$ for some $f \in \mathbb{Z}[l]$?

We will be concerned with answering these questions for several classes of achiral knots, in particular for alternating knots. It appears convenient to summarize these classes for each question in one statement, but first we need some more terminology.

Recall that a knot K is *fibered* if $S^3 \setminus K$ fibers over S^1 (with fiber being a minimal genus Seifert surface for K).

The notion of a homogeneous diagram/link was introduced by Cromwell [Cr], in an attempt to extend certain results on positive and alternating links. It relies on the concept of (diagrammatic) Murasugi sum, which will be of central importance also here.

Definition 3.5 (see [Cr, §1]) The *Seifert picture* (union of Seifert circles) of a link diagram D separates the plane into regions. A non-empty part of D lying in some such region is called a *block*. Then we say that D is *homogeneous* if all blocks D_i of D are positive or negative (i.e. special alternating). A link is homogeneous if it has a homogeneous diagram.

In the terms of definition 3.5, a Seifert circle is separating if it bounds blocks on both sides, and a diagram is special if it has only one block. In particular, any of the blocks of D is special.

Since in positive/negative diagrams all blocks are positive/negative, such diagrams are homogeneous. Alternating diagrams are also homogeneous, this time so that blocks which are Murasugi-summed along a Seifert circle have opposite sign.

Definition 3.6 (see [MP] for example) The operation that reconstructs a diagram from its blocks by gluing them back along the separating Seifert circles is called (diagrammatic) $*$–*product* or *Murasugi sum*.

When considering knot orientation, then we distinguish among achiral knots between $+$achiral (or positive amphicheiral) and $-$achiral (or negative amphicheiral) ones, dependingly on whether the deformation into the mirror image preserves or reverses the orientation of the knot. (For links one has to attach a sign to each component, and take into account possible permutations of the components. Usually one calls the link $+$ or $-$achiral if the orientation of *all* components is preserved or reverted, *without* regard to their order, i.e. allowing one component to be mapped to the mirror image of another.)

Recall that a knot K is called *strongly achiral* if it admits an embedding into S^3 pointwise fixed by the (orientation-reversing) involution $(x,y,z) \mapsto (-x,-y,-z)$. Again dependingly on the effect of this involution on the orientation of the knot we distinguish between strongly $+$achiral and strongly $-$achiral knots.

Theorem 3.7 Let K be an $(+/-)$achiral knot. Then $\max\mathrm{cf}_m P_K$ is of the form $f(l^2)f(l^{-2})$ for some $f \in \mathbb{Z}[l]$, if

1) K is a fibered homogeneous knot,
2) K is a homogeneous knot of crossing number at most 16, or
3) K is an alternating knot.

From formula (4) it is straightforward to see that whenever the leading m-coefficient of P_K is of the above form, both the modulus and sign of $\max\mathrm{cf}\,\Delta_K$ are as specified in question 3.3. There are some more situations where these Alexander polynomial properties can be established. We list again all cases we know of, even if some of them are easy to prove.

Theorem 3.8 Let K be an achiral knot. Then $|\max\mathrm{cf}\,\Delta_K|$ is a square and $\mathrm{sgn}\,(\max\mathrm{cf}\,\Delta_K) = (-1)^{\max\deg\Delta_K}$, if

1) K is a fibered homogeneous knot,
2) K is a knot of crossing number at most 16,
3) K is an alternating knot,
4) K is strongly achiral, or
5) K is negative achiral.

Before the proofs of theorems 3.7 and 3.8 we give some motivational and historical comments.

Question 3.3 was the first one I came across, addressing special properties of the leading coefficients of the Alexander polynomial of achiral knots. This question came up when considering the formula

$$\max\mathrm{cf}\,\Delta_K = \pm 2^{-2g}\prod_{i=1}^{2g} a_i$$

for a rational (or 2-bridge) knot K given in its Conway notation $a_1 \dots a_{2g}$, with all $a_i \neq 0$ even and $g = g(K)$. If K is achiral, the sequence $(a_1,\dots,a_{2g})$ is palindromic, and so we have, up to the sign, the requested property. Then a verification of all prime knots in Thistlethwaite's tables [HT] up to 16 crossings failed to turn up an example answering the question negatively. (Note that the property for a composite knot will follow from that of its factors.) The rational knot argument led to question 3.4. Its diagrammatic, and not topological, origin suggested that the question may not have a positive answer in general, but at least on some (diagrammatically defined) nice knot classes, for example alternating knots. Some electronic search indeed found exceptions among non-alternating knots. Later the proof for alternating knots was found. While it subsumes the rational knot case, latter still deserves some attention by virtue of its generalization to Montesinos knots (see §7). Then the theorem of [HK] was used to prove the Alexander polynomial property for strongly achiral knots.

A last remark aims to clarify, from my perspective and knowledge, the relation of this work to [QW] and [St2]. The statements about alternating knots in theorems 3.7 and 3.8 were proved independently in [QW]. The present proof, apart from being simplified (using mutations instead of flypes), was obtained somewhat earlier. It was explained in summer 2001 to Prof. Murasugi and appeared in written form in the arXiv version v2 of [St2] dated 9/2/01. In summer 2002, Prof. Murasugi gave me a copy of the preprint of [QW]. The referee supporting the publication of [St2] asked to remove the section on theorems 3.7 and 3.8. I followed his advice, at that time unaware of the publication of [QW], which I found only shortly thereafter. Thus I present a separate exposition of this material here.

4. Non-alternating knots

We now collect the arguments that prove the statements of theorems 3.7 and 3.8 for non-alternating knots K. Some arguments are well-known, a matter of electronic verification, or follow algebraically from theorem 4.1. These parts are briefly discussed first. Our main result are the statements in the alternating case, which are proved subsequently. We will finish by mentioning some classes of independent interest that we identify as subclasses of the alternating knots.

We will use the following theorem on the Alexander polynomial of strongly achiral knots, due to Hartley and Kawauchi.

Theorem 4.1 ([HK]) If K is strongly negative amphicheiral, then $\Delta(t^2) \sim F(t)F(-t)$ for some $F \in \mathcal{L}$ with $F(-t) \sim F(t^{-1})$. If K is strongly positive amphicheiral, then $\Delta(t) = F(t)^2$ for some $F \in \mathcal{L}$ with $F(t) \sim F(t^{-1})$.

We require one standard number theory term.

Definition 4.2 The *p-adic valuation* of $s/t \in \mathbb{Q} \setminus \{0\}$ w.r.t. the prime p is given by

$$\mathrm{val}_p\left(\frac{s}{t}\right) := \max\{\, i \in \mathbb{N} : p^i \mid s \,\} - \max\{\, i \in \mathbb{N} : p^i \mid t \,\}.$$

It makes sense to set $\mathrm{val}_p(0) = \infty$. For $s,t \in \mathbb{Q}$, the properties

- $\mathrm{val}_p(-s) = \mathrm{val}_p(s)$,
- $\mathrm{val}_p(s \cdot t) = \mathrm{val}_p(s) + \mathrm{val}_p(t)$ and
- $\mathrm{val}_p(s+t) = \mathrm{val}_p(s)$ when $\mathrm{val}_p(s) < \mathrm{val}_p(t)$

are obvious and well-known.

Proof of theorems 3.7 and 3.8.

fibered knots. A class of knots for which the squareness property of $\max \mathrm{cf}\,\Delta$ is trivial are the fibered knots, since for them $\max \mathrm{cf}\,\Delta = \pm 1$. For fibered homogeneous (in particular, alternating) knots, the other properties also follow easily from known results, because by [Cr, corollaries 4.3 and 5.3] and [MP] we have for such knots that $\max \mathrm{cf}_m P = l^k$ for some $k \in 2\mathbb{Z}$, and then achirality shows $k = 0$.

strongly achiral knots. For strongly achiral knots we obtain the claims of theorem 3.8 from the results of [HK] stated in theorem 4.1. We start with the following observation.

For a prime p, the p-adic valuation of non-zero rationals is additive under multiplication. For a polynomial $F \in \mathcal{L} = \mathbb{Q}[t,t^{-1}]$, let its minimal p-adic valuation be the minimal p-adic valuation of all of its non-zero coefficients. An easy argument shows that, similarly to the case of rational numbers, the minimal p-adic valuation is additive under polynomial multiplication. It follows that if a non-zero polynomial in $\mathbb{Z}[t,t^{-1}]$ factors as $F\cdot\tilde F$ in $\mathbb{Q}[t,t^{-1}]$, and F has a coefficient with a negative p-adic valuation, then all coefficients of $\tilde F$ have positive p-adic valuation. However, in the way that $F(t)$ and $\tilde F(t)=\pm t^{\pm n}F(\pm t^{\pm 1})$ are related in theorem 4.1, this is impossible. Thus in the sequel we can assume w.l.o.g. that theorem 4.1 holds with $\mathcal{L}=\mathbb{Z}[t,t^{-1}]$ instead of $\mathcal{L}=\mathbb{Q}[t,t^{-1}]$.

The square property of $|\max\operatorname{cf}\Delta|$ is then obvious. To obtain the sign of $\max\operatorname{cf}\Delta$, consider first strongly positive amphicheiral knots. Since then $F(t)\sim F(t^{-1})$, and we have $\Delta(1)=1$, it follows that $F(1)=\pm 1$, and so $F(t)=+t^nF(t^{-1})$, rather than $F(t)=-t^nF(t^{-1})$. Moreover, since $F(-1)\equiv F(1)\bmod 2$, we see that $F(-1)\neq 0$, and n must be even. Then so is $\operatorname{span}F$. Hence $4\mid\operatorname{span}\Delta$, and $\max\operatorname{cf}\Delta=(\max\operatorname{cf}F)^2>0$, as desired.

Now consider the sign of $\max\operatorname{cf}\Delta$ for a strongly negative amphicheiral knot. To identify this sign, first normalize the polynomial F found by theorem 4.1 by some $+t^n$ so that $\min\deg F=-\max\deg F$. (Note that $\operatorname{span}\Delta=\operatorname{span}F$, which is even.) Then we have

$$F(t)=\pm F(-t^{-1}). \tag{7}$$

If we normalize Δ so that $\min\deg\Delta=-\max\deg\Delta$ and $\Delta(1)=1$, then the minimal and maximal degrees show that in

$$\Delta(t^2)\,=\,\pm t^nF(t)F(t^{-1})$$

we must have $n=0$, and the evaluation at $t=1$ shows that we must have the positive sign. Then, since $\Delta(1)=1$ and $\Delta(t)=\Delta(t^{-1})$, the absolute term $[\Delta(t)]_0$ of $\Delta(t)$ is odd. Thus the same is true for $\Delta(t^2)=F(t)F(t^{-1})$. But

$$\big[F(t)F(t^{-1})\big]_0\,=\sum_{i=\min\deg F}^{\max\deg F}[F(t)]_i^2\,,$$

and so from (7) we conclude that F must have non-zero absolute term. This determines the sign in (7) to be positive, and then $\max\operatorname{cf}F=\pm\min\operatorname{cf}F$ dependingly on the parity of $\max\deg F=\max\deg\Delta$.

negative amphicheiral knots. Hartley [Ha] has extended the result of [HK] for strongly negative amphicheiral knots to arbitrary negative amphicheiral knots. Thus the claim follows from the previous argument.

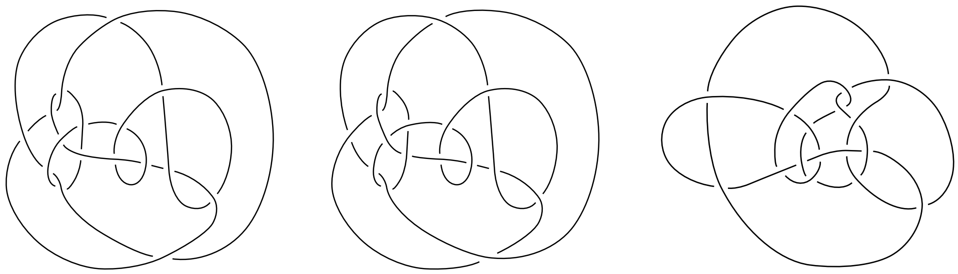
knots with at most 16 *crossings*. The positive answer to question 3.3 for these knots follows from some experimental results related to question 3.4. We examined the tables of prime knots of [HT]. Note that the verification of both the square and the sign property of $\max\operatorname{cf}\Delta$ reduce to prime knots. Prime factors of composite achiral knots are achiral or mutually mirrored in pairs. The sign property of the pairs is dealt with as in the argument for strongly +achiral knots above in the proof. Then both properties are preserved under multiplication of polynomials.

It is clear that an answer 'yes' to question 3.4 implies the same answer to question 3.3. A computer experiment found, though, that the answer to question 3.4 is not positive in general. The examples showing exceptional behaviour are not quite simple, and their location required the full extent of the tables presently available. Among ≤ 16 crossing knots, only three fail to have this property: $16_{1025717}$, $16_{1025725}$ and $16_{1371304}$. They are all +achiral and have $P=m^8(l^{-2}+3+l^2)+O(m^6)$. See figure 2. Since the Alexander polynomial of these knots has degree 4 and leading coefficient 1, they still conform to the properties requested in question 3.3. Also, the knots in figure 2 were found to be fibered (the first two by myself using the method of [Ga], and the third one by M. Hirasawa), showing that the homogeneity assumption in part 1) of theorem 3.7 is essential.

alternating knots. We defer this part to corollary 5.12. □

Remark 4.3 The property of the Alexander polynomial of a strongly negative amphicheiral knot proved in [HK] was conjectured by Kawauchi [Kw] to hold for arbitrary amphicheiral knots. This conjecture clearly implies a positive answer to question 3.3. Kawauchi's conjecture is true in particular for 2-bridge knots, since in [HK] he shows that all amphicheiral 2-bridge knots are strongly negative amphicheiral. Hartley's extension [Ha] of the result in [HK] confirmed the conjecture for general negative amphicheiral knots. (Note that Hartley also obtains a condition for positive amphicheiral knots, but it is too weak to address the conjecture or our question.) Along the proof of part 2) in theorem 3.8 Kawauchi's conjecture was also verified for all prime amphicheiral knots of ≤ 16 crossings.

Remark 4.4 Fibered homogeneous knots contain the homogeneous braid knots of [S], but also many more. For example, there are 15 fibered homogeneous prime 10 crossing knots, among them 12 alternating and 2 positive ones, which can be shown by the work of [Cr] and an easy computer check not to have homogeneous braid representations: 10_{60}, 10_{69}, 10_{73}, 10_{75}, 10_{78}, 10_{81}, 10_{89}, 10_{96}, 10_{105}, 10_{107}, 10_{110}, 10_{115}, 10_{154}, 10_{156} and 10_{161}.



$16_{1025717}$ $16_{1025725}$ $16_{1371304}$

Figure 2: Three $+$achiral knots with $\max \mathrm{cf}_m P$ not of the form $f(l^2)f(l^{-2})$.

Remark 4.5 It follows from our argument for strongly positive amphicheiral knots K that $\max\deg\Delta_K$ is even. Since $\max\deg\Delta$ is often equal to the genus $g(K)$ of K (for example if K is alternating of fibered), one can raise a natural question: are there strongly positive amphicheiral knots of odd genus? Such knots indeed exist. Among the tabulated knots, the simplest instances have no less than 16 crossings, and two examples I found are $16_{1227719}$ and $16_{1371180}$. They have 16 crossing diagrams obtained by Murasugi-summing two special pretzel diagrams $(+2,+2,-2,-2)$ along their valence-4-Seifert circles. These diagrams show that the knots are strongly positive amphicheiral. Their (genus 3) canonical surfaces can be proved to be of minimal genus using for example Gabai's work. For both knots $\max\deg\Delta = 2$.

5. Alternating knots

The remaining cases of theorems 3.7 and 3.8 are included in two more general theorems. The first one generalizes the result of [MP2], where it was shown that for an alternating amphicheiral knot the leading coefficient of the Alexander polynomial is not a prime. We require some definitions of properties that we will need to refer to below.

Definition 5.1 A *flype* is a move on a diagram shown in figure 3.

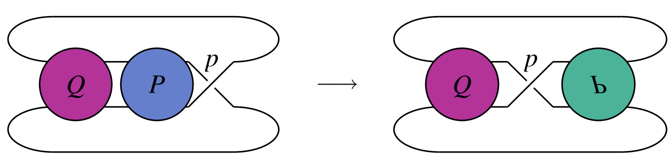


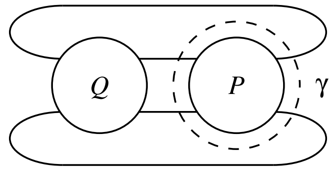
Figure 3: A flype near the crossing p

By the fundamental work of Menasco-Thistlethwaite, we have a proof of the Tait flyping conjecture.

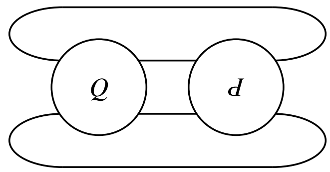
Theorem 5.2 ([MT]) For two alternating diagrams of the same prime alternating link, there is a sequence of flypes taking the one diagram into the other.

Herein we consider diagrams identical if they are transformable into each other by S^2-*moves*, meaning changes of the unbounded region, and *flip*. A flip is a rotation by 180° along some axis in the projection plane, or alternatively a combination of mirroring the diagram in the plane, and then changing all crossings. (A flip has the effect of seeing a diagram, projected on a sphere, from the other side of the sphere. It does not need to be realizable by a sequence of S^2-moves alone.)

Definition 5.3 Given a link diagram D and a closed curve γ intersecting D in exactly four points, γ defines a *tangle decomposition* of D.

$D \quad =$  γ (8)

Mutation is an operation introduced by Conway [Co]. It is performed by removing one of the tangles P in some tangle decomposition of D and replacing P by a tangle obtained from it by 180° rotation along the axis perpendicular to the projection plane, or horizontal or vertical in the projection plane. For example:



(To make orientations compatible, for oriented diagrams D possibly the orientation of either P or Q must be altered.) The curve γ is called the *Conway circle* for this mutation.

Remark 5.4 Note that a flype can be realized as a sequence of mutations.

Definition 5.5 As in definition 3.5, the Seifert picture of a link diagram D defines the blocks $D_1,\dots,D_n$ of D. These blocks may not be prime diagrams. We call the prime components $D_{i,j}$ of D_i *block prime components* of D. Then we say that D is *semi-homogeneous* if all block prime components are positive or negative. (That is, $D_{i,j}$ are special alternating, but, unlike homogeneity, possibly of different sign within the same block/for the same i.)

An example of a semi-homogeneous diagram which is not homogeneous will occur later as the (unique) minimal crossing diagram of the knot 14_{45601} in figure 5.

Remark 5.6 Almost all results of [Cr] extend to semi-homogeneity, but this point is not interesting to discuss. The new feature of semi-homogeneity which will turn out useful below is that, unlike homogeneity, it is invariant under flypes and mutations.

Theorem 5.7 Let a knot or link K have a semi-homogeneous diagram D which can be turned into its mirror image (possibly with opposite orientation) by a sequence of mutations and S^2-moves. Then $|\max\mathrm{cf}\,\Delta_K|$ is a square.

Remark 5.8 There are two ways of building a mirror image of a diagram: to mirror in the projection plane (plane mirroring) or to change all crossings (crossing mirroring). However, here distinguishing between the two mirrorings is irrelevant, since they differ by the flip mentioned after theorem 5.2, and such a flip is indeed a special type of mutation.

Proof of theorem 5.7. For a link diagram D, we denote by $\tilde{D}$ the mutation equivalence class of D, that is, the set of all diagrams that can be obtained from D by a sequence of mutations. We consider orientation reversal as a special type of mutation, so a diagram and its inverse belong to the same mutation equivalence class. Also, the skein and Alexander polynomial are invariant under mutation, so they are well-defined on a mutation equivalence class.

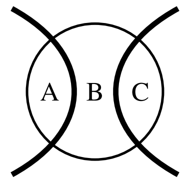
The treatment of split diagrams D easily reduces to the non-split ones, so assume below that D is non-split. Let D_i be the blocks of D, and $D_{i,1},\dots,D_{i,n_i}$ be the prime components of D_i. Note that all $D_{i,j}$ are positive or negative. They will have no nugatory crossings if D has neither.

Define

$$I(D) := \{\tilde{D}_{i,j}\}_{i=1,\dots,n,\,j=1,\dots,n_i}\,.$$

Here a set is to be understood with the order of its elements ignored, but with their multiplicity counted (i.e., $\{1,1,2,3\}=\{1,1,3,2\}\neq\{1,2,3\}$).

Now apply a mutation on D. The Seifert picture separates the Conway circle into 3 parts A, B and C.



Because the Conway circle intersects the Seifert picture only in 4 points, all parts A, B and C represent prime components of the blocks in D they belong to, or possibly connected sums of several such prime components.

Mutation then has the effect of applying mutation on B and interchanging and/or reversing A and C. Therefore, $\mathcal{I}(D) = \mathcal{I}(D')$ for any iterated mutant diagram D' of D.

If D has the property assumed in the theorem, then $\mathcal{I}(D) = \mathcal{I}(!D)$, or

$$\{\,\tilde{D}_{i,j}\,\}_{i=1,\dots,n,\,j=1,\dots,n_i} = \{\,\widetilde{!D_{i,j}}\,\}_{i=1,\dots,n,\,j=1,\dots,n_i}\,.$$

Let $\phi : \{\,\tilde{D}_{i,j}\,\} \to \{\,\tilde{D}_{i,j}\,\}$ be the bijection induced by $\tilde{D}_{i,j} \mapsto \widetilde{!D_{i,j}}$.

Since mutation preserves the writhe, ϕ has no fixpoints (unless some $D_{i,j}$ has no crossings, which leads to the previously excluded situation that D is split). Thus ϕ descends to a bijection

$$\phi : \{\,\tilde{D}_{i,j} \,:\, D_{i,j} \text{ positive}\,\} \to \{\,\tilde{D}_{i,j} \,:\, D_{i,j} \text{ negative}\,\}\,.$$

Then by the work of [Mu] we have that $\max\mathrm{cf}\,\Delta$ is multiplicative under $*$-product, and hence

$$\begin{aligned}
\max\mathrm{cf}\,\Delta_D &= \prod_{i,j} \max\mathrm{cf}\,\Delta_{D_{i,j}} \\
&= \prod_{i,j\,:\,D_{i,j}\text{ positive}} \max\mathrm{cf}\,\Delta_{D_{i,j}} \cdot \max\mathrm{cf}\,\Delta_{!D_{i,j}} \\
&= \prod_{i,j\,:\,D_{i,j}\text{ positive}} \max\mathrm{cf}\,\Delta_{D_{i,j}} \cdot \pm\max\mathrm{cf}\,\Delta_{D_{i,j}} \\
&= \pm\left(\prod_{i,j\,:\,D_{i,j}\text{ positive}} \max\mathrm{cf}\,\Delta_{D_{i,j}}\right)^2,
\end{aligned}$$

as desired. □

Definition 5.9 We say that a link diagram D is *P-maximal* if it makes Morton's inequality (5) exact (that is, this inequality becomes an equality).

In [Cr, corollary 4.1] it was shown that homogeneous diagrams are P-maximal, and by remark 5.6 so are semi-homogeneous ones. Many knots have P-maximal diagrams – beside homogeneous knots, for example all the knots in Rolfsen's tables [Ro, appendix], and also all 11 and 12 crossing knots tabulated in [HT]. However, some knots do not – in [St, fig. 9] we gave four examples of 15 crossings.

In [MP] it was shown that $\max\mathrm{cf}_m P$ is multiplicative under $*$-product of P-maximal diagrams. From this result we have now

Theorem 5.10 Under the same assumption as theorem 5.7 we have $\max\mathrm{cf}_m P(D) = f(l^2)f(l^{-2})$ for some $f \in \mathbb{Z}[l]$.

Proof. Using [MP] instead of [Mu], we obtain $\max\mathrm{cf}_m P(D) = \tilde{f}(l)\tilde{f}(l^{-1})$. Additionally, $\tilde{f}$ is up to units in $\mathbb{Z}[l^{\pm1}]$ a product of elements in $\mathbb{Z}[l^{\pm2}]$, since for any link diagram $D_{i,j}$ only even or only odd powers of l occur in $\max\mathrm{cf}_m P$. Then the result follows. □

Remark 5.11 Beware that simply using $X(l) = \tilde{f}(l)\tilde{f}(l^{-1}) \in \mathbb{Z}[l^{\pm 2}]$ does not suffice to conclude $X(l) = f(l^2)f(l^{-2})$. Take for example $\tilde{f}(l) = -l^{-1}+1+l$. Then $X(l) = -l^{-2}+3+l^2 \in \mathbb{Z}[l^{\pm 2}]$, but $X(\sqrt{l}) \in \mathbb{Z}[l^{\pm 1}]$ is irreducible. (In particular, theorem 1.2 (2) of [QW], even with signs ignored, is stated weaker than theorem 5.10.)

We complete now the proof of theorems 3.7 and 3.8.

Corollary 5.12 If K is an alternating achiral knot, then $|\max \mathrm{cf}\,\Delta_K|$ is a square and $\mathrm{sgn}(\max \mathrm{cf}\,\Delta_K) = (-1)^{\max\deg \Delta_K}$. Moreover, $\max \mathrm{cf}_m P_K = f(l^2)f(l^{-2})$.

Proof. The claim on $\max \mathrm{cf}_m P$ follows from the facts that any alternating diagram is homogeneous, theorem 5.2, and remark 5.4. The properties of Δ then follow from (4). □

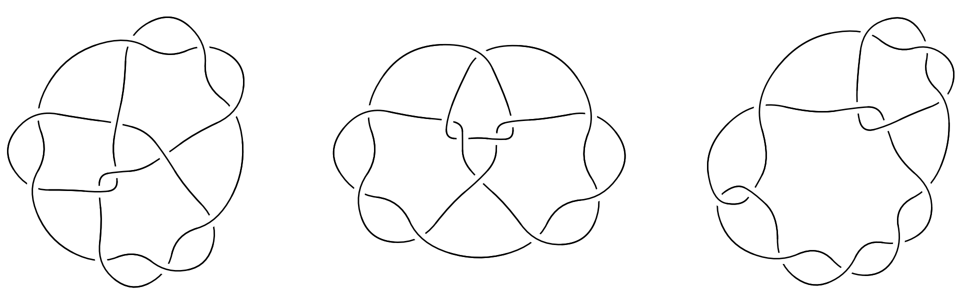
Note that with corollary 5.12 we obtain an exact condition what numbers occur as $|\max \mathrm{cf}\,\Delta_K|$ for an alternating achiral knot K, since for every perfect square finding a proper K is easy.

6. Some examples

One can visualize the squareness properies by some examples.

Example 6.1 Alternating knots of small genus are completely classified. For genus one this was done explicitly in [St3], although I was aware of, and motivated by, a related previous remark by Rudolph. The result there shows that the achiral alternating genus one knots are the rational knots (a,a), with a even (and so for example implies easily Corollary 1.4 of [QW]). The classification was continued for genus up to 4, but can unlikely be (technically) completed for genus 5.

Example 6.2 It is useful to see examples showing that theorems 5.7 and 5.10 can be applied as a chirality test for an alternating knot. For interesting examples it makes sense to consider only alternating knots with self-conjugate skein polynomial. The knots 12_{1171} and 12_{1205} in [HT], in figure 4, are the simplest ones that have $|\max \mathrm{cf}\,\Delta|$ a square, but



12_{1171} $\qquad$ 12_{1205} $\qquad$ 12_{669}

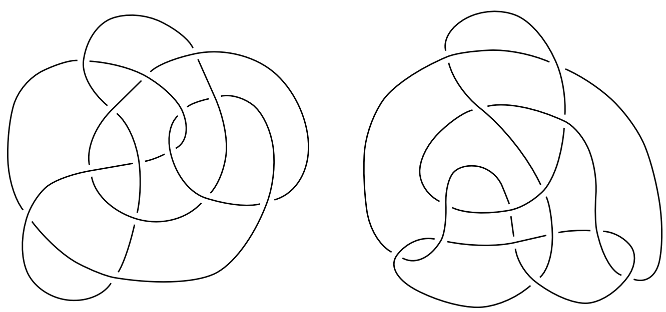
Figure 4

$\max \mathrm{cf}_m P \neq f(l^2)f(l^{-2})$. For all such knots up to 16 crossings where the test of $\max \mathrm{cf}_m P$ excludes achirality, so does already the (weaker) test for $\mathrm{sgn}(\max \mathrm{cf}\,\Delta)$, and also the Kauffman polynomial. Some knots, for example 16_{130184}, have zero signature, but the signature condition and the squareness test of $|\max \mathrm{cf}\,\Delta|$ combinedly also apply for all examples.

The existence of a sufficient variety of knots with self-conjugate skein polynomial leads also to the following related examples.

Example 6.3 Murasugi-Przytycki conjectured (for general knots) that if the leading coefficient of the Alexander polynomial is (up to sign) a prime then the skein polynomial is not self-conjugate; see conjecture 15.10 in [MP3]. This conjecture is wrong, even for alternating knots; 12_{669} (figure 4) is among the simplest counterexamples. It may be worth saying the following on such knots we found.

- We have no alternating counterexamples of zero signature, but have non-alternating examples of both zero (12_{1538}) or non-zero signature (12_{1850}).
- We have no alternating counterexamples of "right" sign $(-1)^{\max\deg\Delta}$ of $\max\mathrm{cf}\,\Delta$; but have non-alternating examples of right (12_{1538}) or wrong (16_{976307}) sign.
- We have no alternating counterexamples of self-conjugate Kauffman F polynomial (because such have zero signature), but have such non-alternating examples (14_{19544}). Even though for non-alternating knots self-conjugate F polynomial does not imply zero signature (cf. 9_{42}), all our examples of self-conjugate F are of zero signature.



14_{45317} 14_{45601} 14_{41330}

Figure 5

Although corollary 5.12 is the most interesting special case of the preceding theorems 5.7 and 5.10, latter give indeed more general statements. To illustrate this, we give the following example.

Example 6.4 The non-alternating achiral knots 14_{45317} and 14_{45601} in [HT], shown in figure 5, have unique minimal crossing (number) diagrams, which therefore must be transformable into their mirror images by S^2-moves and flip only. These diagrams are homogeneous and semi-homogeneous, respectively. Note that, unlike [HT] which identifies mirrored diagrams, here we consider diagrams equivalent only up to S^2-moves and flip. Uniqueness is then meant w.r.t. this equivalence. There are achiral knots with two minimal crossing diagrams, which are mutually mirrored (so that there is a unique minimal diagram up to mirroring), but not interconvertible by S^2-moves, flip and flypes. One such example is 14_{41330}.

7. Montesinos and other knots

We finish with a brief treatment of some further noteworthy special classes of knots, for which theorems 3.7 and 3.8 hold by means of the fact that all such achiral knots are alternating.

Montesinos knots can be described as the closure of a sequence of rational tangles, such that the tangles are glued along disjoint disks. See [BZ].

Proposition 7.1 Achiral knots in the following classes are alternating:

1) rational (or 2-bridge) knots,

2) 3-braid knots,

3) Montesinos knots.

Proof. It is well-known that 2-bridge knots are alternating. It follows also from [BM] that achiral 3-braid knots have alternating 3-braid representations. The case of Montesinos knots is recurred to that of 2-bridge knots by means of proposition 7.2 below. □

Proposition 7.2 An achiral Montesinos knot is 2-bridge.

This property seems suggestive, but I have not found a record of it in this form. In [BZ], a weaker claim was left as an exercise, with the restriction to odd length knots. With a different small restriction (nonellipticity), the result was given in [BZi, corollary 1.4]. However, the heavy machinery of Thurston's hyperbolization theorem is not needed, and a proof can be obtained (without restrictions) from the classification of Montesinos links.

Proof of proposition 7.2. Let

$$K = M(p_1/q_1,\dots,p_n/q_n\,;e) \tag{9}$$

be the representation with $0 < p_i/q_i < 1$. If $n \le 2$, then K is 2-bridge, so assume $n \ge 3$. Then the representation (9) is canonical up to cyclic permutation and possible reversal of the vector

$$\bar{\alpha} := (\alpha_1,\dots,\alpha_n) = (p_1/q_1,\dots,p_n/q_n)\,.$$

We call (9) then *normal form*. So comparison of normal forms of K and $!K$ shows that $e = -n/2$ (in particular n is even) and $\bar{\alpha}$ differs from $1-\bar{\alpha}$ by cyclic permutation and possible reversal. Let Γ be this dihedral transformation. If Γ has no reversal, then $\Gamma(\alpha_i) = \alpha_{(k+i) \bmod n}$ for some k, where '$\bmod n$' is taken with values between 1 and n. If now the map $i \mapsto (k+i) \bmod n$, acting iteratedly on $\{1,\dots,n\}$, has an orbit of odd order, then for all i in this orbit $\alpha_i = 1/2$. Since for a knot at most one of the α_i can be $1/2$, the orbit must be trivial. So $k = 0$, but then all $\alpha_i = 1/2$. Thus Γ has only even order orbits, and the α_i complete each other to 1 in pairs:

$$\{1,\dots,n\} = \bigcup_{j=1}^{n/2} A_j\,, \quad |A_j| = 2\,, \quad \sum_{i\in A_j} \alpha_i = 1\,.$$

If Γ contains a reversal, then $\Gamma(\alpha_i) = \alpha_{(k-i) \bmod n}$. If k is odd, then the map $i \mapsto (k-i) \bmod n$ has two fixpoints, so two $\alpha_i = 1/2$, which we excluded; otherwise this map is a fixpoint-free involution, and again the α_i complete each other to 1 in pairs.

So the α_i complete each other to 1 in pairs in both cases. Again, no $\alpha_i = 1/2$. So we can incorporate the $-n/2$ twists in e into the $n/2$-many $\alpha_i > 1/2$, and then we have a representation $K = M(\alpha'_1,\dots,\alpha'_n;0)$ with α'_i opposite in pairs:

$$\{1,\dots,n\} = \bigcup_{j=1}^{n/2} A_j\,, \quad |A_j| = 2\,, \quad \sum_{i\in A_j} \alpha'_i = 0\,.$$

Now mutation does not change the number of components, and we can permute the α'_i so that $\alpha'_i = -\alpha'_{n/2+i}$ for $i \le n/2$. Then the diagram of $M(\alpha'_1,\dots,\alpha'_n;0)$ has a tangle decomposition as in (8) with the two tangles $Q = (\alpha'_1,\dots,\alpha'_{n/2})$ and $P = (\alpha'_{n/2+1},\dots,\alpha'_n)$ being mirrored to each other, and hence having the same connectivity. Such a link cannot be a knot. □

Remark 7.3 In studying the polynomials of achiral Montesinos *links* some caution is needed. The classification of Montesinos links (and hence the examination of their achirality) is done in the unoriented sense, while our polynomial properties could extend to links only for oriented achirality (as in [QW]). Thus such an extension would require beside the normal form a careful analysis of component orientations.

Remark 7.4 With a bit more argument, one can refine proposition 7.2 to show that an achiral (in the unoriented sense) Montesinos link has even number of components.

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References

[Ad] C. C. Adams, *The knot book*, W. H. Freeman & Co., New York, 1994.

[Al] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), 275–306.

[BM] J. S. Birman and W. W. Menasco, *Studying knots via braids III: Classifying knots which are closed 3 braids*, Pacific J. Math. **161** (1993), 25–113.

[BZi] M. Boileau and B. Zimmermann, *Symmetries of nonelliptic Montesinos links*, Math. Ann. **277(3)** (1987), 563–584.

[BZ] G. Burde and H. Zieschang, *Knots*, de Gruyter, Berlin, 1986.

[Co] J. H. Conway, *On enumeration of knots and inks*, in "Computational Problems in abstract algebra" J. Leech, ed.), 329–358. Pergamon Press, 1969.

[Cr] P. R. Cromwell, *Homogeneous links*, J. London Math. Soc. (series 2) **39** (1989), 535–552.

[C] R. Crowell, *Genus of alternating link types*, Ann. of Math. **69(2)** (1959), 258–275.

[F&] P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–246.

[Ga] D. Gabai, *Detecting fibred links in S^3*, Comment. Math. Helv. **61(4)** (1986), 519–555.

[Ha] R. Hartley, *Invertible amphicheiral knots*, Math. Ann. **252(2)** (1979/80), 103–109.

[HK] ——— " ——— and A. Kawauchi, *Polynomials of amphicheiral knots*, Math. Ann. **243(1)** (1979), 63–70.

[HT] J. Hoste and M. Thistlethwaite, *KnotScape*, a knot polynomial calculation and table access program, available at `http://www.math.utk.edu/~morwen`.

[J] V. F. R. Jones, *A polynomial invariant of knots and links via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[Ka] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[Ka2] ——— " ——— , *State models and the Jones polynomial*, Topology **26** (1987), 395–407.

[Kw] A. Kawauchi, *H-cobordism I. The groups among three dimensional homology handles*, Osaka J. Math. **13(3)** (1976), 567–590.

[LM] W. B. R. Lickorish and K. C. Millett, *A polynomial invariant for oriented links*, Topology **26 (1)** (1987), 107–141.

[Me] W. W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23 (1)** (1986), 37–44.

[MT] ——— " ——— and M. B. Thistlethwaite, *The Tait flyping conjecture*, Bull. Amer. Math. Soc. **25 (2)** (1991), 403–412.

[Mo] H. R. Morton, *Seifert circles and knot polynomials*, Proc. Camb. Phil. Soc. **99** (1986), 107–109.

[Mu] K. Murasugi, *On a certain subgroup of the group of an alternating link*, Amer. J. Math. **85** (1963), 544–550.

[Mu2] ——— " ———, *Jones polynomial and classical conjectures in knot theory*, Topology **26** (1987), 187–194.

[Mu3] ——— " ———, *On the genus of the alternating knot*, J. Math. Soc. Japan **10** (1958), 94–105, 235–248.

[MP] ——— " ——— and J. Przytycki, *The skein polynomial of a planar star product of two links*, Math. Proc. Cambridge Philos. Soc. **106(2)** (1989), 273–276.

[MP2] ——— " ——— and ——— " ———, *Index of graphs and non-amphicheirality of alternating knots*, in "Progress in knot theory and related topics", Travaux en Cours **56** (1997), Hermann, Paris, 20–28.

[MP3] ——— " ——— and ——— " ———, *An index of a graph with applications to knot theory*, Mem. Amer. Math. Soc. **106 (508)** (1993).

[QW] C. V. Quach Hongler and C. Weber, *On the topological invariance of Murasugi special components of an alternating link*, Math. Proc. Cambridge Philos. Soc. **137(1)** (2004), 95–108.

[Ro] D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[S] J. R. Stallings, *Constructions of fibred knots and links*, "Algebraic and geometric topology" (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part **2**, 55–60.

[St] A. Stoimenow, *Knots of genus two*, preprint `math.GT/0303012`.

[St2] ——— " ———, *Square numbers, spanning trees and invariants of achiral knots*, `math.GT/0003172`, Comm. Anal. Geom. **13(3)** (2005), 591–631.

[St3] ——— " ———, *Knots of genus one*, Proc. Amer. Math. Soc. **129(7)** (2001), 2141–2156,

[St4] ——— " ———, *The existence of achiral knot diagrams*, proceedings of the Second East Asian School of Knots and Related Topics in Geometric Topology, Dailan, China, 2006, 15–26.

[Th] M. B. Thistlethwaite, *A spanning tree expansion for the Jones polynomial*, Topology **26** (1987), 297–309.

[Wh] W. C. Whitten, *Isotopy types of minimal knot spanning surfaces*, Topology **12** (1973), 373–380.