ON THE CROSSING NUMBER OF SEMIADEQUATE LINKS

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Abstract. The concept of adequacy of a link was introduced to determine its crossing number. We give crossing number estimates for the (much larger) class of semiadequate links, in particular improving the previously known estimates for positive links. We give also examples showing that semiadequacy may not be attained in minimal crossing number diagrams. Our approach uses the description of links of a given canonical genus, and is related to skein invariants, the signature, the new concordance invariants, and hyperbolic volume. It allows to settle the A/B-semiadequacy status of many examples, including all knots up to 10 crossings. As two applications, we describe generic alternating and positive knots of given genus, determining the asymptotical behaviour of the number of such knots, and classify Seifert fibered, and therewith also hyperbolic, Montesinos links.

Keywords: Kauffman polynomial, Jones polynomial, skein polynomial, crossing number, semiadequate link, positive link, hyperbolic volume, slice link
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Contents

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1. Introduction

The crossing number can be considered as the most fundamental invariant of a knot or link. It dates back to the early days when knot theory began with Tait's first knot tables. Yet this invariant is still in general very hard to compute. It is related to several long-standing problems, like additivity under connected sum and the Cable crossing number conjecture [FH]. The century-old problem of Tait about crossing numbers of alternating links was spectacularly solved in [Ka, Mu2, Th2]. Later on, Lickorish and Thistlethwaite [LT] introduced the concept of (semi)adequacy to help determining the crossing number of some other links. The concept was based on the study of Kauffman's bracket [Ka] for the Jones polynomial. At a later stage Thistlethwaite extended the study of (semi)adequacy to the Kauffman polynomial in [Th], and proved that adequate diagrams have minimal crossing number. (He thereby extended the result for alternating links, of which adequate links are a generalization.) Recently these properties were also related to Khovanov homology [Ko].

Semiadequacy seems to be much less understood. It follows from [Th] that semiadequate links are non-trivial, although a concrete crossing number estimate of this non-triviality has not been obtained. Beyond this non-triviality result, not much has been known until recently about semiadequate links in general. This might be due to the fact that they form a rather large class. Unlike adequate links, it completely contains other interesting classes like positive, Montesinos [LT], and 3-braid links [St12]. Semiadequacy also plays a central role in recent attempts [DL, St15] to understand some coefficients of the Jones polynomial. Based on this, it is a basic ingredient in the proof of existence of amphicheiral knots of almost all odd crossing numbers (relating to another old problem of Tait) [St12]. Thistlethwaite observed [Th, p. 290 top] that, although not all low crossing knots have adequate diagrams, "all diagrams of minimal crossing number of knots up to 11 crossings are semiadequate, as are all but a handful of diagrams of minimal crossing number of 12 crossing knots". This provided some evidence that
(a) any knot may have a semiadequate diagram, and
(b) that it may have one of minimal crossing number (assuming it has one at all, if we do not believe that (a) holds).
He gave an example of a knot, 12_{1750} on [Th, figure 2], for which it was unknown whether it has a semiadequate diagram.

Kauffman's bracket is a cornerstone in all study of semiadequacy so far, except [Th]. The aim of this paper is, on the theoretical side, to build on the work in [Th] and widen the scope of relations of this property. On the practical side, we will apply our method to many concrete examples.

First we will connect semiadequacy to skein invariants (the skein polynomial, and the Jones polynomial as a special case of the former, rather than as an equivalent of Kauffman's bracket), the signature and, for knots at least, its recent homological relatives τ, s introduced by Ozsváth–Szabó [OS] and Rasmussen [Ra]. Our main results show how such invariants can be used to estimate above the crossing number of a semiadequate diagram.

Theorem 1.1 Any semiadequate link has only finitely many semiadequate reduced diagrams, which can be effectively generated provided the link can be identified from a semiadequate diagram.

By variations of our arguments, we will be able to naturally relate our estimates also to the classical invariants of L, like unknotting (or unlinking) number, genus, braid index, or hyperbolic volume. We remark that other topological or diagrammatic properties, like sliceness and achirality, also give additional useful information in our approach.

Thistlethwaite's results are the starting point, but will be used only limitedly. A more important tool will be the estimates of Morton-Williams-Franks [Mo, FW] and Bennequin-Rudolph [Be, Ru] for what is called now the Thurston-Bennequin invariant. (A good discussion is given by Fuchs-Tabachnikov [FT].) Most substantially, however, we will use the generator description of canonical Seifert surfaces [St, St6, SV].

We will show how our method leads to efficient tests for (A/B)-semiadequacy. They allow us to decide many cases, including the example 12_{1750} in [Th] and all Rolfsen [Ro, appendix] knots. We have examples of various difficulty showing

Theorem 1.2 There are knots having neither A-semiadequate nor B-semiadequate diagrams.

The conditions in [LT, Th] on the Jones and Kauffman polynomial of semiadequate links also yield such examples (see i.p. §4.1), but they are scarce, since the resulting tests are not very efficient. Apart from doing much better in this regard, we will also disprove all the aforementioned evidence on realization of semiadequacy properties by minimal crossing diagrams. This is opposed to the intrinsic relationship between the combination of A- and B-semiadequacy to the crossing number.

Theorem 1.3 For any of the properties A-semiadequate, B-semiadequate and semiadequate, there exist knots with diagrams having this property, but no such diagrams of minimal crossing number.

Besides general semiadequate links, we will investigate in some detail two important subclasses, of alternating and positive links. It turns out that our estimates can handle these particular cases more efficiently. For these links we show that the difference between the crossing number of a reduced semiadequate diagram and the crossing number of the link can be controlled by the degrees of the Jones polynomial and the smooth 4-ball Euler characteristic. As an application of this fact we prove, building on and extending previous work with A. Vdovina [SV]:

Theorem 1.4 A generic (see definition 4.1) positive or alternating knot of a fixed genus higher than 1 is special alternating. Its special alternating diagram is its only positive (in particular, also its only alternating) diagram.

This result allows us to determine the asymptotical number of such links for growing crossing number. Another result is the easy (and precise) derivation of the classification of the Seifert fibered (and thus also the hyperbolic) Montesinos links. A further application to the Cable crossing number conjecture has been moved out to a separate paper for length reasons.

In [St12], the approach in [LT] focused on Kauffman's bracket is pursued, and many of the results there extended. For example, we prove that semiadequate links and Whitehead doubles of semiadequate knots have non-trivial Jones polynomial. The solution in [St12] of Tait's 120-year-old problem to determine the crossing numbers of amphicheiral knots uses heavily an invariant quantity of semiadequate diagrams, the atom number, which we work out in §3.1 here. So our paper can be considered also as a preparation to that solution.

M. Ozawa pointed to his paper [O2], where he extended the non-triviality result of semiadequate links to a larger class he defines (Theorem 1.15). Our theorems 1.1 and 1.2 (partly) address problems (1) and (3) in §4 of *op. cit.*

2. Preliminaries

We begin by introducing necessary notions, notations, and recalling earlier results that will be used throughout the paper. Most of these are well-known, but some are more specific, and introduced in our own previous work.

2.1. General and link notations

The cardinality of a (finite) set S is denoted by $\#S$ or $|S|$. For any $S \subset \mathbb{R}$, by $\sup S$ we denote the supremum of S, with the natural convention that $\sup \varnothing = -\infty$.

The *crossing number* $c(L)$ of a link L is the minimal number of the crossing number $c(D)$ of all diagrams of L.

The *mirror image* of K is denoted by $!K$, the *inverse* (oppositely oriented knot) by $-K$. By T_k for $k \in \mathbb{Z} \setminus \{-1,0,1\}$ we denote the $(2,k)$-torus knot or link (in latter case with parallel orientation), with the mirroring convention that T_k is positive for $k \geq 2$. Similarly let $\overline{T}_k$ for $k \in 2\mathbb{Z} \setminus \{0\}$ be the $(2,k)$-torus link with reverse orientation, again with the mirroring convention that $\overline{T}_k$ is positive for $k \geq 2$. Thus $T_2 = \overline{T}_2$ is the (positive) Hopf link and T_3 is the (positive or right-handed) trefoil.

Knots and links of ≤ 10 crossings will be denoted according to Rolfsen's tables [Ro, appendix] (including mirroring convention), and knots of ≥ 11 crossings according to Hoste and Thistlethwaite [HT]. We shift down by one the index of the last 4 Rolfsen knots, discarding the Perko duplication $10_{162} = !10_{161}$.

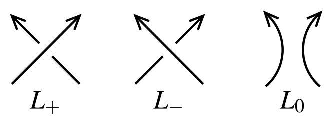
2.2. Polynomial invariants

The various link polynomials were introduced in the papers [F&, LM, Ka, J].

The *skein (HOMFLY) polynomial* P is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1}\,P\big(L_+\big) + l\,P\big(L_-\big) = -m\,P\big(L_0\big)\,. \tag{2.1}$$

It involves diagrams of three links

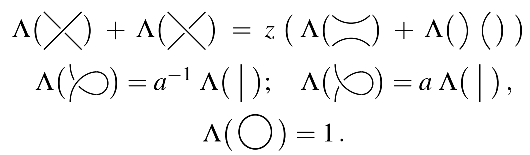


(2.2)

differing just at one crossing. The convention in (2.1) uses the variables of [LM], but differs by the interchange of l and l^{-1}. We call the three diagram fragments in (2.2) from left to right a *positive* crossing (of sign $+1$), a *negative* crossing (of sign -1) and a *smoothed out* crossing. We also say that the signs are meant *in the skein sense*. (We will briefly use a different, Kauffman, sign later.) The *writhe* $w(D)$ of a link diagram D is the sum of signs of all its crossings.

The *Kauffman polynomial* [Ka2] F is usually defined via a regular isotopy invariant $\Lambda(a,z)$ of unoriented links.

We use here a slightly different convention for the variables in F, differing from [Ka2, Th] by the interchange of a and a^{-1}. Thus in particular we have the relation $F(D)(a,z) = a^{w(D)}\Lambda(D)(a,z)$, where $w(D)$ is the writhe of D, and $\Lambda(D)$ is given in our convention by the properties



Note that for P and F there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity.

The *Jones polynomial* V is obtained from P and F by the substitutions (with i the complex unit)

$$V(t) \quad = \quad P(-it, i(t^{-1/2} - t^{1/2})) \quad = \quad F(-t^{3/4}, t^{1/4} + t^{-1/4})\,. \tag{2.3}$$

(See [LM] or [Ka2, §III] but mind our conventions.)

Below Δ is the *Alexander polynomial*. It is an invariant with values in $\mathbb{Z}[t,t^{-1}]$, and can be defined by being 1 on the unknot and a relation, with $L_{\pm,0}$ as in (2.2),

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\,\Delta(L_0)\,. \tag{2.4}$$

This relation is easily seen to be a special case of the skein polynomial relation. Consequently, there is the substitution formula (see [LM]),

$$\Delta(t) = P(i, i(t^{1/2} - t^{-1/2}))\,.$$

Let $[Y]_{t^a} = [Y]_a$ be the *coefficient* of t^a in a polynomial $Y \in \mathbb{Z}[t^{\pm 1}]$. Similarly one defines for $Y \in \mathbb{Z}[x_1,\dots,x_n]$ the coefficient $[Y]_X$ for some monomial X in the x_i. For a multi-variable polynomial the coefficient may be taken with respect only to some variables, and is a polynomial in the remaining variables, for example $[Y]_{x_1^k} \in \mathbb{Z}[x_2,\dots,x_n]$. (Thus it must be clear as a monomial in which variables X is meant. For example, for $X = x_1^k \in \mathbb{Z}[x_1]$, the coefficient $[Y]_X = [Y]_{x_1^k} \in \mathbb{Z}[x_2,\dots,x_n]$ is *not* the same as when regarding $X = x_1^k = x_1^k x_2^0 \in \mathbb{Z}[x_1,x_2]$ and taking $[Y]_X = [Y]_{x_1^k x_2^0} \in \mathbb{Z}[x_3,\dots,x_n]$.)

Let for $Y \in \mathbb{Z}[t^{\pm 1}]$

$$\min\deg Y = \min\{\, a \in \mathbb{Z} \,:\, [Y]_a \neq 0 \,\}\,,\;\; \max\deg Y = \max\{\, a \in \mathbb{Z} \,:\, [Y]_a \neq 0 \,\}\,,\;\; \operatorname{span} Y = \max\deg Y - \min\deg Y$$

be the *minimal* and *maximal degree* and *span* (or breadth) of Y, respectively. If Y is a multi-variable polynomial, one can define minimal and maximal degrees $\min\deg_{x_i} Y$ and $\max\deg_{x_i} Y$ with regard to a *single* variable x_i analogously. Finally, define the *leading coefficient* of x_i in Y to be

$$\max\mathrm{cf}_{x_i} Y \,:=\, [Y]_{x_i^{\max\deg_{x_i} Y}}\,.$$

In a similar way the *trailing coefficient* is

$$\min\mathrm{cf}_{x_i} Y \,:=\, [Y]_{x_i^{\min\deg_{x_i} Y}}\,.$$

2.3. Signature

The *signature* σ is a $\mathbb{Z}$-valued invariant of knots and links. Originally it was defined terms of Seifert matrices [Ro]. We have that $\sigma(L)$ has the opposite parity to the number of components of a link L, whenever the *determinant* $\det(L) = \big|\Delta_L(-1)\big| \neq 0$. This in particular always happens for L being a knot (since $\Delta_L(-1)$ is always odd in this case), so that σ takes only even values on knots. The most of the early work on the signature was done by Murasugi [Mu], who showed several properties of this invariant.

Then, with $L_{\pm,0}$ as in (2.2),

$$\sigma(L_+) - \sigma(L_-) \quad \in \quad \{0,1,2\} \tag{2.5}$$

$$\sigma(L_\pm) - \sigma(L_0) \quad \in \quad \{-1,0,1\}\,. \tag{2.6}$$

(Note: In the first property one can also have $\{0,-1,-2\}$ instead of $\{0,1,2\}$, since other authors, like Murasugi, take σ to be with opposite sign. Thus (2.5) not only defines a property, but also specifies our sign convention for σ.)

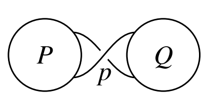
We have additivity of σ under split union (denoted by '$\sqcup$') and connected sum (denoted by '#')

$$\sigma(L_1\#L_2) \,=\, \sigma(L_1 \sqcup L_2) \,=\, \sigma(L_1) + \sigma(L_2)\,.$$

Also, $\sigma(!L) = -\sigma(L)$, where $!L$ is the mirror image of L.

2.4. Diagrams

A crossing p in a link diagram D is called *reducible* (or *nugatory*) if D can be represented in the form



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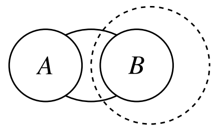
The diagram D is called reducible if it has a reducible crossing; otherwise it is called *reduced*.

A link diagram D is *composite*, if there is a closed curve γ intersecting (transversely) the curve of D in two points, such that both in- and exterior of γ contain crossings of D. That is, D has the form



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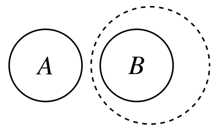
Otherwise D is *prime*.

The diagram is *split*, if there is a closed curve not intersecting it, but which contains parts of the diagram in both its in- and exterior:



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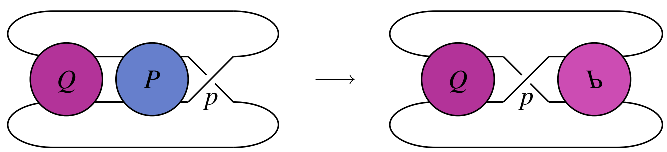
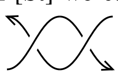
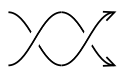
A link is split if it has a split diagram.

A *flype* is the move



Two crossings p and q of a *knot* diagram D are *linked*, iff their basepoints are passed in cyclic order $pqpq$ along the line of D (rather than $ppqq$).

Recall that in [St] we called two crossings $\sim$*-equivalent*, if after a sequence of flypes they can be made to form a *reverse clasp*  . Similarly we call (see [St6]) two crossings $\overset{\sim}{\ast}$*-equivalent*, if after a sequence of flypes they can be made to form a *parallel clasp*  . It is an exercise to check that both are equivalence relations. We observed (see [St, St6, STV]) that $\sim$- and $\overset{\sim}{\ast}$-equivalent crossings of a knot diagram are linked with the same set of other crossings in the diagram.

We call a crossing p connected to a Seifert circle s also *adjacent* or *attached* to s. If only negative crossings, in the skein sense explained below (2.1), are adjacent to a Seifert circle s, we call s *strongly negative*. Let $s_-(D)$ be the number of strongly negative Seifert circles of a diagram D.

In [St4] we called a diagram *k-almost positive*, if it has exactly k negative crossings (in the skein sense). A link L is k-almost positive, if it has a k-almost positive diagram, but no l-almost positive one for any $l < k$. We call a diagram or link *positive*, if it is 0-almost positive (see [Cr, O, Yo, Zu]), and *almost positive* if it is 1-almost positive [St8]. Similarly one defines *k-almost negative*, and in particular *almost negative* and *negative* links and diagrams to be the mirror images of their k-almost positive (or almost positive or positive) counterparts.

Adequacy consists of the combination of two weaker properties called jointly *semiadequacy*. They are defined as follows.

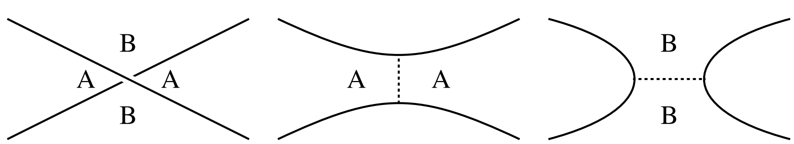


Figure 1: The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.

One defines for each crossing of a diagram an *A*- and *B-splitting* as in figure 1. One calls a diagram D *A-(semi) adequate* if the number of loops obtained after A-splicing all crossings of D is more than the number of loops obtained after A-splicing all crossings except one. Similarly one defines the property B-(semi) adequate. Then we set

$$\begin{aligned} \text{adequate} &= \text{A-semiadequate } \textit{and} \text{ B-semiadequate}\,, \\ \text{semiadequate} &= \text{A-semiadequate } \textit{or} \text{ B-semiadequate}\,, \\ \text{inadequate} &= \textit{neither} \text{ A-semiadequate } \textit{nor} \text{ B-semiadequate}\,. \end{aligned}$$

We call a link adequate resp. (A/B)-semiadequate if it has an adequate resp. (A/B)-semiadequate diagram.

An alternative way to understand A-semiadequacy is to keep the trace of the crossings after each splitting. Then we have each of the traces of the crossings joining two loops, obtained after the splittings. The property A-semiadequate means that, in the set of loops obtained by A-splitting all crossings, each crossing connects two different loops. We call this set of loops the *A-state* of the diagram.

Since in a positive diagram the A-splittings are the same as the splittings according to the skein relation, the loops obtained are just the Seifert circles. Then since any crossing connects different Seifert circles, it follows that positive diagrams (and hence also links) are A-semiadequate. (This was also previously observed, see e.g. [Fi].)

A standard result on the signature, which will be used, is that non-trivial positive links (in our convention for σ) have positive signature. The first proof (for knots, which however easily extends to links) is given in [CG]. Another proof follows from the work of Taniyama [Ta]. (See [St11, St9] for a more detailed discussion.) A significant improvement of this result for a general positive link is unfortunately not known. Such an improvement would also have impact on the applicability of some of our estimates (see §3.6).

2.5. Critical line polynomials

The relation between k-almost positivity and A-adequacy originates from Thistlethwaite's work, but was not substantially pursued. Let us explain the few easy known facts (see also Cromwell's book [Cr3]). We need the Kauffman polynomial F [Ka2], and some of its properties studied by Thistlethwaite.

As observed in [Th], the relation for Λ implies that the coefficient of $z^l a^m$ in $\Lambda(D)$, denoted by $[\Lambda(D)]_{z^l a^m}$, does not vanish only if $|m|+l \le c(D)$. The main contribution of [Th] was the study of the "critical line" polynomials

$$\phi_\pm(D)(z) = \sum_{i=0}^{c(D)} [\Lambda(D)]_{z^{c(D)-i}a^{\mp i}}\, z^{c(D)-i}\,. \tag{2.7}$$

Note the sign switch resulting from our different convention. We changed also the variable and exponent from t^i to $z^{c(D)-i}$. It was shown in [Th], in particular, that $\phi_\pm(D) \neq 0$ iff D is A/B-semiadequate (theorem 3 (i)), and that in this case all the coefficients are always non-negative (corollary 1.1 (i)).

Let us specify a slightly different definition of $\phi_\pm$. For a diagram D, let $X(D)$ resp. $x(D)$ be the maximal sum $l+m$ resp. minimal (as an integer, not in absolute value) difference $l-m$, such that $[F(D)(a,z)]_{a^l z^m}$ is a non-zero coefficient. Let $\phi_-(D)(z)$ be the sum over m of monomials $[F(D)(a,z)]_{a^l z^m} z^m$, such that $l+m = X(D)$. Similarly $\phi_+(D)(z)$ is the sum of the same sort of monomials, but for l,m with $l-m = x(D)$. For semiadequate diagrams this formulation is equivalent to (2.7); see p. 287, l. -14f and Theorem 3 in [Th]. These arguments imply also that when D is A-adequate, then $-x(D) = 2c_-(D)$ is the number of negative crossings of D. Similarly for a B-adequate diagram, $X(D) = 2c_+(D)$. (See below proposition 2.1.)

One can also write $x(L), X(L)$ for a link L, since x, X depend only on F, which is a link invariant. In that sense, we define the following quantity, that will be of major importance throughout the rest of the paper:

$$a(L) = -\frac{1}{2}x(L) = \frac{1}{2}\max\{\, m-l : [F(L)]_{z^m a^l} \neq 0\,\}\,. \tag{2.8}$$

Thus if the F polynomial of a knot K has the property that there are $l,\ m \in \mathbb{Z}$, such that

$$[F(K)]_{z^l a^m} < 0 \ \text{ and } \ m-l = \max\{\, m-l : [F(K)]_{z^l a^m} \neq 0\,\}\,, \tag{2.9}$$

then K cannot be A-semiadequate. Similarly, if there are l', m' with

$$[F(K)]_{z^{l'}a^{m'}} < 0 \ \text{ and } \ m'+l' = \max\{\, m'+l' \,:\, [F(K)]_{z^{l'}a^{m'}} \neq 0\,\}\,, \tag{2.10}$$

then K cannot be B-semiadequate.

Both conditions give a semiadequacy test, but it is not efficient in practice (see §4.1).

Proposition 2.1 A diagram D of L is A-semiadequate iff it has exactly $a(L)$ (defined in (2.8)) negative crossings (in the skein, not checkerboard, sense!). Moreover, for any (not necessarily semiadequate) link L there is no diagram with fewer negative crossings than $a(L)$.

Proof. Since $[\Lambda(D)]_{z^m a^l} \neq 0$ only for $m+|l| \leq c(D)$, we have $[F(D)]_{z^m a^l} \neq 0$ only for

$$m+w(D)-l \leq m+|l-w(D)| \leq c(D)\,.$$

Thus $w(D) \leq c(D)-(m-l)$, that is,

$$w(D) \,\leq\, c(D) + \min\{\, l-m \,:\, [F(D)]_{z^m a^l} \neq 0\,\}\,. \tag{2.11}$$

Then for a diagram D we have

$$\begin{aligned} D \text{ is } A\text{-semiadequate} \quad &\Longleftrightarrow \quad \exists \text{ some } m \text{ and } l \text{ with } m-l=c(D) \text{ and } [\Lambda(D)]_{z^m a^l} \neq 0 \\ &\Longleftrightarrow \quad \exists m-l = c(D)-w(D) \ :\ [F(D)]_{z^m a^l} \neq 0 \\ &\Longleftrightarrow \quad (2.11) \text{ is sharp.} \end{aligned}$$

This easily implies the claim. □

Proposition 2.1 can be rephrased thus (see also the remarks preceding proposition 3.1):

Corollary 2.1 For any k-almost positive link L we have $k \geq a(L)$. Moreover, L has an A-semiadequate diagram D if and only if $k = a(L)$, and then D has exactly k negative crossings. □

In particular, we have the following important special cases:

Corollary 2.2 If a link L is positive, then $a(L) = 0$. Moreover, a diagram of L is A-adequate exactly if it is positive. □

Corollary 2.3 An adequate achiral knot of $2k$ crossings is k-almost positive. □

This extends the result for achiral alternating knots and $k \leq 4$, whose proof in [St4] using the Jones polynomial is much more awkward. (Note that for alternating knots the corollary follows also from an inequality of Murasugi [Mu4]; see the proof of proposition 3.9.)

2.6. Topological invariants

An compact orientable spanning surface of a link L is called a *Seifert surface*. We will allow disconnected surfaces, but exclude closed components. A Seifert surface of L is called *canonical* if it is obtained by Seifert's algorithm applied on some diagram D of L.

The *canonical Euler characteristic* of a link diagram D is called the Euler characteristic of D's canonical Seifert surface. It satisfies

$$\chi(D) \,=\, -c(D)+s(D)\,,$$

where $c(D)$ is the number of crossings of D, and $s(D)$ the number of its Seifert circles. The *canonical genus* $g(D)$ is given by

$$g(D) \,=\, \frac{1}{2}\big(2-n(D)-\chi(D)\big)\,,$$

where $n(D)$ is the number of components of (the link represented by) D. The canonical Euler characteristic $\chi_c(L)$ and canonical genus $g_c(L)$ of a link L are the maximal canonical Euler characteristic resp. minimal canonical genus of any diagram D of L.

The (classical, or Seifert) *Euler characteristic* $\chi(L)$ resp. *genus* $g(L)$ of a link L is called the maximal Euler characteristic resp. minimal genus of all its Seifert surfaces (i.e. not necessarily those canonical for some diagram of L).

The (smooth) *slice or 4-ball genus or Euler characteristic* $g_s(L)$ and $\chi_s(L)$ are the maximal Euler characteristic resp. minimal genus of all smoothly properly embedded surfaces in B^4, whose boundary is L.

From their definition, we have the inequalities

$$\chi_s(L) \geq \chi(L) \geq \chi_c(L) \quad \text{and} \quad g_s(L) \leq g(L) \leq g_c(L)\,.$$

Bennequin's inequality [Be, theorem 3] , states that for any reduced diagram D of L,

$$-\chi(L) \geq w(D) - s(D)\,, \tag{2.12}$$

with $s(D)$ being the number of Seifert circles of D, and $w(D)$ its writhe. (This is rather an extension of Bennequin's formulation from braids to diagrams, see [Ru2, St2].) Later this inequality was improved, and its strongest version known presently (see [Ru, K]) is

$$1 - \chi_s(L) \geq w(D) - s(D) + 1 + 2s_-(D)\,, \tag{2.13}$$

with $s_-(D)$ being explained in §2.4.

The *unknotting (or unlinking) number* $u(L)$ of a link L is the minimal number of crossing changes needed to turn L into the trivial link (of the same number of components). A link is trivial if all its split components are unknots.

The *braid index* of L is the minimal number of strands of a braid, whose closure is L [Mo, FW].

2.7. Montesinos links

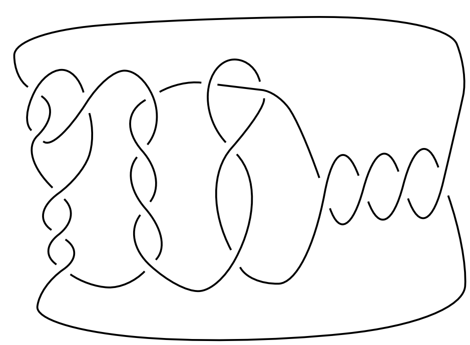


Figure 2: The Montesinos knot $M(3/11, -1/4, 2/5, 4)$ with Conway notation $(213, -4, 22, 40)$.

To avoid confusion, let us fix some terminology related to Montesinos links. The details on Conway's notation can be found in his original paper [Co], or for example in [Ad].

Let the *continued fraction* $[[s_1, \dots, s_r]]$ for integers s_i be defined inductively by $[[s]] = s$ and

$$[[s_1, \dots, s_{r-1}, s_r]] = s_r + \frac{1}{[[s_1, \dots, s_{r-1}]]}\,.$$

The rational tangle $T(p/q)$ is the one with Conway notation $c_1\ c_2\ \dots c_n$, when the c_i are chosen so that

$$[[c_1, c_2, c_3, \dots, c_n]] = \frac{p}{q}\,. \tag{2.14}$$

One can assume without loss of generality that $(p,q)=1$, and $0<q<|p|$. A *rational (or 2-bridge) link $S(p,q)$* is the closure of $T(p/q)$.

Montesinos links (see e.g. [BZ]) are generalizations of pretzel and rational links and special types of arborescent links. They are denoted in the form $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$, where e,p_i,q_i are integers, $(p_i,q_i)=1$ and $0<|q_i|<p_i$. Sometimes e is called the *integer part*, and the $\frac{q_i}{p_i}$ are called *fractional parts*. They both together form the *entries*. If $e=0$, it is omitted in the notation. Our convention follows [Oe] and *may differ* from other authors' by the sign of e and/or multiplicative inversion of the fractional parts. For example $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$ is denoted as $\mathfrak{m}(e;\frac{p_1}{q_1},\dots,\frac{p_n}{q_n})$ in [BZ, definition 12.28] and as $M(-e;(p_1,q_1),\dots,(p_n,q_n))$ and the tables of [Kw].

If all $|q_i|=1$, then the Montesinos link $M(\pm\frac{1}{p_1},\dots,\pm\frac{1}{p_n},e)$ is called a *pretzel link* $P(\pm p_1,\dots,\pm p_n,\varepsilon,\dots,\varepsilon)$, where $\varepsilon=\operatorname{sgn}(e)$, and there are $|e|$ copies of it. We also say it is a $(\pm p_1,\dots)$-pretzel link.

To visualize the Montesinos link from a notation, let p_i/q_i be continued fractions of rational tangles $c_{1,i}\dots c_{n_i,i}$ with $[[c_{1,i},c_{2,i},c_{3,i},\dots,c_{l_i,i}]]=\frac{p_i}{q_i}$. Then $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$ is the link that corresponds to the Conway notation

$$(c_{1,1}\dots c_{l_1,1}),(c_{1,2}\dots c_{l_2,2}),\dots,(c_{1,n}\dots c_{l_n,n}),e\,0\,. \tag{2.15}$$

An example is shown in figure 2.

An easy exercise shows that if $q_i>0$ resp. $q_i<0$, then

$$M(\dots,q_i/p_i,\dots,e)=M(\dots,(q_i\mp p_i)/p_i,\dots,e\pm 1)\,, \tag{2.16}$$

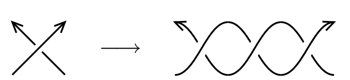
i.e. both forms represent the same link. In our convention the identity (2.16) can be read naturally in the sense that the sum of all entries is preserved. Theorem 12.29 in [BZ] asserts that the entry sum, together with the vector of the fractional parts, modulo $\mathbb{Z}$ and up to cyclic permutations and reversal, determine the isotopy class of a Montesinos link L. So the number n of fractional parts is an invariant of L; we call it the *length* of L. If the length $n<3$, an easy observation shows that the Montesinos link is in fact a rational link.

A minimal crossing diagram $D(L)$ of the form (2.15) for each Montesinos link L was described in [LT]. There are 2 cases. Either $e\neq 0$, and all $c_{i,j}$ in (2.15) are of the same sign as e (the Montesinos link is alternating), or $e=0$, $n\geq 3$, and all $c_{j,i}$ are of the same sign for given i, depending on the sign of q_i. This diagram is A resp. B-adequate, unless exactly one of the q_i is positive resp. negative. In particular, at least one of both applies, so the diagram is always semiadequate.

2.8. Genus generators

The description of canonical Seifert surfaces by generators was initiated by myself in [St] and independently by Brittenham [Br]. It was set forth in [St6], and later in [STV, SV]. (Some explanation of this work is given also in section 5.3 of Cromwell's recent book [Cr3].)

Now, we consider the move we call a $\bar{t}'_2$ *twist* or $\bar{t}'_2$ *move*. Up to mirroring this is given by



(2.17)

We call diagrams that cannot be reduced by flypes and inverses of the move (2.17) *generating* or *$\bar{t}'_2$-irreducible*. The importance of this concept comes from our previous work in [St]. Reduced knot diagrams of given genus g (with $1-\chi=2g$) decompose into finitely many equivalence classes under $\bar{t}'_2$ twists and their inverses. We call these collections of diagrams *series*. Since the number of series grows rapidly with the genus, practical classification is possible only up to genus 4 [St11]. In manageable form, the list of generators can be given for genus 1 and 2 [St, St6].

Theorem 2.1 ([St, St6]) A genus one knot diagram is (modulo crossing changes) a rational diagram $C(p,q)$ with $p,q>0$ even, or a pretzel diagram $P(p,q,r)$, with $p,q,r>0$ odd. That is, it can be obtained via $\bar{t}'_2$ moves and crossing changes from the alternating trefoil and figure eight knot diagram.

A prime genus two knot diagram can be obtained via $\bar{t}'_2$ moves and crossing changes from an alternating diagram of the following 24 knots: 5_1, 6_2, 6_3, 7_5, 7_6, 7_7, 8_{12}, 8_{14}, 8_{15}, 9_{23}, 9_{25}, 9_{38}, 9_{39}, 9_{41}, 10_{58}, 10_{97}, 10_{101}, 10_{120}, 11_{123}, 11_{148}, 11_{329}, 12_{1097}, 12_{1202}, and 13_{4233}.

The finiteness of generators, together with the Flyping theorem [MT], shows

Theorem 2.2 (see [St]) Let $a_{n,g}$ be the number of prime alternating knots K of genus $g(K)=g$ and crossing number $c(K)=n$. Then for $g\geq 1$

$$\sum_n a_{n,g}x^n \,=\, \frac{R_g(x)}{(x^{p_g}-1)^{d_g}}\,,$$

for some polynomial $R_g\in\mathbb{Z}[x]$, and $p_g,d_g\in\mathbb{N}$. Alternatively, this statement can be written also in the following form: there are numbers p_g (period), n_g (initial number of exceptions) and polynomials $P_{g,1},\dots,P_{g,p_g}\in\mathbb{Q}[n]$ with $a_{n,g}=P_{g,n \bmod p_g}(n)$ for $n\geq n_g$.

This was explained roughly in [St], and then in more detail in [SV], where we made effort to characterize the leading coefficient of these polynomials $P_{g,i}$. Even if the polynomials vary with a very large period p_g, the leading coefficients depend only on the parity of n. The degrees of all $P_{g,i}$ are independent on i, and $\max\deg P_{g,i}=c_g-1$, where c_g is the maximal number of $\sim$-equivalence classes of diagrams of canonical genus g. In [SV] it was later shown that $c_g=6g-3$. An exceptional case is $g=1$ and i even, when $\max\deg P_{g,i}=1$ instead of 2.
We will use an extension of the result $c_g=6g-3$ to link diagrams.

Theorem 2.3 In a connected link diagram D of canonical Euler characteristic $\chi(D)\leq 0$ there are at most

$$\begin{cases} -3\chi(D) & \text{if } \chi(D)<0 \\ 1 & \text{if } \chi(D)=0 \end{cases}$$

$\sim$-equivalence classes of crossings. If D is $\bar{t}'_2$-irreducible and has $n(D)$ link components, then

$$c(D)\,\leq\,\begin{cases} 4 & \text{if } \chi(D)=-1 \text{ and } n(D)=1\,, \\ 2 & \text{if } \chi(D)=0\,, \\ -6\chi(D) & \text{if } \chi(D)<0 \text{ and } n(D)=2-\chi(D)\,, \\ -5\chi(D)+n(D)-3 & \text{else}\,. \end{cases} \tag{2.18}$$

Combined with the examples in [SV], this settles the problem the maximal crossing number of a generator for knots.

Corollary 2.4 The maximal crossing number of a knot generator of genus $g\geq 2$ is $10g-7$.

Even albeit the inequalities in theorem 2.3 are optimal in general, they can be improved under additional conditions. One such condition is related to the signature $\sigma(L)$ of the underlying link L, in comparison to the maximal Euler characteristic $\chi(L)$ of an orientable spanning surface of L. It is well-known [Mu3] that $|\sigma(L)|\leq 1-\chi(L)$, at least when L does not bound disconnected surfaces of small genus, so for example for knots or (non-split) positive links. We can say something about the situation when $\sigma(L)\ll 1-\chi(L)$.

Lemma 2.1 Assume we have a diagram D of a link L with $\chi(D)<0$ and $c_-(D)$ negative crossings, and

$$k(D)\,:=\,\frac{1-\chi(D)-2c_-(D)-\sigma(D)}{2}\,\geq\,0\,.$$

Then D has at most $-3\chi(D)-\dfrac{3}{2}k(D)$ $\sim$-equivalence classes, and its underlying generating diagram has at most

$$-5\chi(D)+n(D)-2-\frac{1}{2}k(D)$$

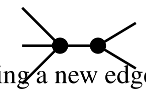
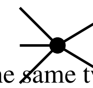
crossings.

Originally theorem 2.3 and lemma 2.1 were proved in this paper. Then we decided, though, to move out the material. This was done both for length reasons, and because, while the paper evolved, the method of proof became gradually unrelated to the rest of the exposition. The proof is now in the preprint [St11], which will likely become part of a monograph. Still the present place was referred to for the proof from elsewhere, and at least the reference in [St10] is published and cannot be altered. We apologize to have to forward the reader to [St11] at this point.

2.9. Graphs

A graph G will have for us possibly multiple edges (edges connecting the same two vertices). It will, though, usually have no loop edges (edges connecting one and the same vertex). By $V(G)$ we will denote the set of vertices of G, and by $E(G)$ the set of edges of G. (In the latter, each multiple edge will count as a set of single edges.) Similarly $v(G)$ and $e(G)$ will be the number of vertices and edges of G (counted with multiplicity), respectively.

For a graph, let the operation $\longrightarrow$ (adding a vertex of valence 2) be called *bisecting* and its inverse (removing such a vertex) *unbisecting* of an edge. We call a graph G' a *bisection* of a graph G, if G' is obtained from G by a sequence of edge bisections. We call a bisection G' *reduced*, if it has no adjacent vertices of valence 2 (that is, each edge of G is bisected at most once). Contrarily, if G' is a graph, its *unbisected graph* G is the graph with no valence-2-vertices, of which G' is a bisection.

Similarly, a *contraction* is the operation  $\longrightarrow$  , and a *decontraction* its inverse.

The *doubling* of an edge consists in adding a new edge connecting the same two vertices.

A graph is *n-connected*, if n is the minimal number of edges needed to remove from it to disconnect it. (Thus connected means 1-connected.) Such a collection of edges is called an *n-cut*. An edge forming a 1-cut is called an *isthmus*.

Hereby, when we delete an edge, we understand that a vertex it is incident to is *not* to be deleted too. In that sense, the set I_v of edges incident to a given vertex v always forms a cut – if we delete these edges, v gets an isolated component, so the graph is not connected anymore. A *cut vertex* is a vertex which disconnects a graph, when removed *together* with all its incident edges.

A graph G is *planar* if it is embeddable in the plane *and equipped* with a fixed such planar embedding. Observe that there is a natural bijection of edges between a planar graph G and its dual graph G^*; in that sense we can talk of the dual $e^* \in E(G^*)$ of an edge $e \in E(G)$. The operations doubling and bisection become dual to each other.

3. Crossing number estimates for semiadequate links

For many knots, like the one 12_{1750} quoted by Thistlethwaite, it is still difficult to decide by the previously known methods alone whether they have a semiadequate diagram. This motivates the following results, which will basically settle the problem in theory.

Now we can proceed to prove our main result in several qualitative versions. The Kauffman polynomial will always be necessary (at least in the proof), but the additional ingredients can be varied a lot. This way one can put into use many invariants known from standard knot theory. The number of possible ways in which the arguments can be combined is too large. We can therefore present only the most self-contained and simple cases, and explain how is to proceed otherwise. We regroup the results into five subsections, named after the invariants that come to use at each new stage.

Let us meet one other convention in advance. In some situations the estimates we will write down will not be integers. This can be corrected by using integer parts of certain subexpressions, or case distinctions, which would both make the form, however, less pleasant. Thus we (largely) avoid this option, even if it would sometimes slightly improve the inequalities. The results that follow are meant anyway in a theoretical purpose – to explain the interrelation between the various invariants, and how they can be applied to study semiadequacy. How to proceed in practice will be shown at the end of the paper.

3.1. Critical line polynomials

Thistlethwaite studied in [Th] the partial terms $\phi_\pm$ of the Kauffman polynomial $F(a,z)$. Our convention and notation differs from his at several places. To minimize confusion, we explain once more how to convert between both in the below arguments. First we conjugate Λ and F in a in comparison to his. To account for this, we interchange the polynomials $\phi_\pm$, and also replace therein each t^n by $z^{c(D)-n}$ The formulation of his result in (3.1) changes accordingly. Also, if we apply Thistlethwaite's results stated for $+$-adequate (his term for A-adequate) diagrams to B-adequate ones, we must interchange the role of G_+ with G_- (and $\bar{G}_+$ with $\bar{G}_-$). We do retain the same notion as his of positive and negative crossings in the *checkerboard (Kauffman) sense*. (Note that this signing of crossings is *different* from the one in (2.2). The checkerboard sign is used only here in §3.1.)

Let, as explained in [Th], G be the signed checkerboard graph of D (any of the two choices, equivalent up to duality, will do). Let G_- be obtained from G by deleting all $+$ edges and $\bar{G}_+$ be obtained by contracting all $-$ edges (compare p. 286 l. 2f in [Th]). Let bl denote the number of blocks of a graph, except isolated vertices (with no edges incident).

By Thistlethwaite [Th, corollary 1.1 (iii)], we have (with our conventions) that

$$c(D) - \max\deg_z \phi_\pm(D) \,=\, bl(G_\pm) + bl(\bar{G}_\mp)\,. \tag{3.1}$$

Proposition 3.1 Let D be a non-trivial reduced B-adequate diagram. Then $c(L) - 1 \geq \max\deg_z \phi_-(D) \,\geq\, c(D)/2$.

Proof. Using (3.1), we need to estimate the number of blocks of G_- and $\bar{G}_+$. By reducedness of D and by construction of G_- and $\bar{G}_+$ we can assume that G_- has no loop edges and $\bar{G}_+$ has no isthmuses. Since D is B-adequate, by proposition 2 (ii) in [Th], also $\bar{G}_+$ has no loop edges and G_- has no isthmuses. Then each block in G_- and $\bar{G}_+$ must consist of at least two edges, and the total number of edges in G_- and $\bar{G}_+$ is equal to the crossing number of D. This shows the second inequality we claim. The first inequality follows from [Ki] because $\max\deg\phi_\pm \leq \max\deg_z F$. (Note that [Ki] uses $Q = F(1,z)$, but the argument for F is the same.) □

Corollary 3.1 There are only finitely many semiadequate links with given Kauffman polynomial, or even $\max\deg_z F$. In particular, there are only finitely many 3-braid, positive, or Montesinos links.

Proof. The first claim of the corollary is clear; the second claim follows from the first, because these classes of links are semiadequate. For positive links this was explained above. For Montesinos links, it was known from [LT]. For 3-braid links, it was shown in [St12]. □

Remark 3.1 A still unsolved problem of Kanenobu (problem 1.91 (3) of [Kr]) is whether there is at all an infinite family of links with the same Kauffman polynomial.

The fact that the $(2,2n+1)$-torus knots have positive diagrams of $4n$ crossings shows at least that any constant $> 1/2$ cannot be expected, also for positive links. However, in the case of positive links, we can be more precise. Let $m(D)$ be the number of *Murasugi atoms* of a positive diagram D. These are the connected sum factors of Murasugi sum components of D, as called by Quach-Weber [QW].

Proposition 3.2 Let D be a positive diagram. Then $\max\deg_z \phi_+(D) \,=\, c(D) - m(D)$. In particular $\max\deg_z \phi_+(D) \geq c(D) - 1 + \chi(D)$.

This settles a conjecture made in [St3].

Corollary 3.2 Let D be a positive reduced non-trivial diagram of a link L. Then $c(L) \,\geq\, c(D) + \chi(D)$, and if equality holds, then L is (special) alternating.

Proof. Clearly $\max\deg_z \phi_+(D) \,\leq\, \max\deg_z F \,\leq\, c(D)$. If $c(D) = \max\deg_z F$, then D is a trivial unlink diagram, and if $c(D) - 1 = \max\deg_z F$, then D is alternating [Ki]. □

Proof of proposition 3.2. Consider the Murasugi sum decomposition of D along separating Seifert circles. It decomposes D into *diagram blocks*, as defined in [Cr, §1]. (We call them not just 'blocks' in order to emphasize the distinction from Thistlethwaite's 'graph blocks'.) Choose an even-odd type black-white coloring $\mathcal{B}$ of the diagram blocks of D, so that blocks neighbored along a separating Seifert circle have opposite color.

Choose then also a checkerboard coloring $\mathcal{A}$ of D. Then the $+$ edges in the checkerboard coloring $\mathcal{A}$ correspond to the crossings in the blocks of D with one of the colors, say white, in $\mathcal{B}$. So G_+ is a disjoint union of the checkerboard graphs of these blocks. Since the number of graph blocks in the checkerboard graph is equal to the number of connected sum factors, we identified $bl(G_+)$. To deal with $bl(\bar{G}_-)$, switch colors in $\mathcal{A}$. Then $\bar{G}_-$ is taken to G_+, and the positive crossings in G become the crossings of the diagram blocks colored by the other color (black) in $\mathcal{B}$. Using (3.1) we are done. □

Remark 3.2 We will use the fact that we have now a lower bound, which for fixed χ and number of components is of the form $c+\text{constant}$, to extend the main result of [STV] in section 4.

The proof of proposition 3.2 also shows immediately how to interpret $\max\deg_z\phi_+(D)$ in terms of the A-state of D.

Proposition 3.3 Decompose an A-adequate link diagram D along separating loops of its A-state into diagrams D_i in analogy to the Murasugi sum decomposition along separating Seifert circles. Let $m(D)$ be the total number of connected sum factors of all D_i. Then $\max\deg_z\phi_+(D) = c(D) - m(D)$. □

We call $m(D)$ the *atom number* of D. (It is straightforward that for a positive diagram it coincides with the Murasugi atom number, as we previously specified.) It will be used heavily in [St12] for the construction of odd crossing number amphicheiral knots.

While we discuss the work in [Th] only briefly in this paper, let us give also a description of $\min\deg_z\phi_+(D)$. Let $v(G)$ be the number of vertices of G and $p_1(G)$ the first Betti number of G (the minimal number of edges to remove to destroy all cycles in G).

Lemma 3.1 (see corollary 1.1 (iv) in [Th]) We have for an A-adequate diagram D,

$$\min\deg\phi_+(D) = c(D) - p_1(G_+) - v(\overline{G_-}) + 1\,, \tag{3.2}$$

and the minimal degree term in $\phi_+(D)$ has coefficient $+1$.

Now we use this to show the following

Proposition 3.4 Let D be A-adequate. Then $p_1(G_+) + v(\overline{G_-})$ equals the number $|A(D)|$ of loops in the A-state of D.

Proof. While there may be an intrinsic, but more elaborate, combinatorial argument, we show this much faster by appealing to (3.2) and some other observations.

If D is positive (i.e. all crossings are positive in the *skein* sense), $|A(D)| = s(D)$, so, using (3.2), the claim we like to show reads $\min\deg\phi_+(D) = 1-\chi(D)$. This now follows from the work of Yokota [Yo]. He shows that $\min\deg_a F(D) = 1-\chi(D)$, and up to signs and change of variables that $[F(D)]_{a^{1-\chi}} = [P(D)]_{l^{1-\chi}}$. The non-vanishing of the coefficient of $(lm)^{1-\chi}$ in $P(D)$ follows from $\min\deg_l P(D) = 1-\chi(D)$ and the identity [LM, proposition 21]. This finishes with the case that D is positive.

Now clearly altering the orientation of single components does not change the (validity of) the claim in the lemma, and we observed in [St12] that D admits a positive orientation iff its A-state graph $G(A) = G(A(D))$ is bipartite. We recall form [St12] that $G(A)$ has a vertex for each loop in $A(D)$, and an edge for each trace of a crossing. (This is *not* the same as the checkerboard graph G of D.)

Now for every diagram D, its graph $G(A)$ can be made bipartite by edge bisections – give in some arbitrary way a sign to each vertex, and bisect edges between vertices of equal sign. Such a bisection adds one loop in $A(D)$ (or vertex in $G(A)$). In D it displays as a turning a crossing into a clasp (in an appropriate way). Its effect on G is that either a negative edge is bisected, or a positive edge is doubled. Then from $p_1(G_+)$ and $v(\overline{G_-})$ one is preserved, and the other is augmented by one. More precisely, $v(\overline{G_-})$ is augmented if the edge in $A(D)$ is a trace of a checkerboard-negative crossing in D, or a negative edge e in G (which is bisected), and $p_1(G_+)$ is augmented if e is positive (and doubled). So the claim follows from the positive diagram case. □

Now, Kauffman's state model [Ka] relates the number of loops in $A(D)$ also to the Jones polynomial. We recall the number $a(L)$ defined in (2.8). Now by straightforward calculation, we obtain, using $w(D) = c(D) - 2a(D)$:

Corollary 3.3 If a link L is non-split and A-adequate, then we have

$$\min\deg V(L) = \frac{1}{2}\min\deg_z\phi_+(L) - \frac{3}{2}a(L)\,, \tag{3.3}$$

and the trailing coefficient of $V(L)$ is $(-1)^{\min\deg_z\phi_+(L)}$. □

In particular, one easily sees

Corollary 3.4 If L is non-split and alternating, then $\min\deg_z \phi_+(L) + \min\deg_z \phi_-(L) = c(L)$.

Proof. One applies (3.3) to L and $!L$. Then one notes that $\max\deg V(L) = -\min\deg V(!L)$, and that $a(L) + a(!L) = c(L)$ when L is adequate and that $\max\deg V(L) - \min\deg V(L) = c(L)$ when L is alternating. $\square$

While (3.3), is a simple yet efficient tool to rule out A/B-adequacy (see example 4.3), it is certainly not universal, and cannot account for most of the work that follows. We show next how to improve the inequality in proposition 3.1 under certain circumstances and in a variety of ways. We will then treat with the tools we develop several examples, including Thistlethwaite's knot 12_{1750}.

3.2. Kauffman polynomial and Euler characteristic

We start with a series of inequalities involving the Euler characteristic and Kauffman polynomial F. We need again the quantity defined in (2.8).

Theorem 3.1 Let L be an oriented non-split link of maximal Euler characteristic $\chi(L)$, and D be a reduced A-semiadequate diagram of L. If $L \neq \overline{T}_k$, $k \geq 2$, then

$$c(D) \leq \frac{17a(L) - 9\chi(L)}{2} + \max\deg_z F(L)\,. \tag{3.4}$$

The A-semiadequate reduced diagrams of $\overline{T}_k$, $k \geq 2$, are exactly their (only) alternating diagrams.

Remark 3.3 By [Th, corollary 3.2], if D is non-split, so is L. In the following we will pose the non-splitness condition for merely technical reasons. It can easily be dropped in most cases, or at least weakened to the exclusion of trivial (unknotted) split components.

Remark 3.4 Note that semiadequacy and crossing numbers of diagrams and links, and also $\max\deg_z F$, do not depend on the orientation of single components. Contrarily χ and $a(L)$ do (except for knots). Thus one has the freedom to alter, and possibly improve, the estimate (3.4) by proper choice of orientations.

Remark 3.5 It follows from the proof that if the quantity in (2.8) is negative, L cannot be A-semiadequate. It would be curious to find examples, where this happens, but at least among knots up to 16 crossings there are no such ones.

Proof of theorem 3.1. As noted in proposition 2.1, D has $a(L)$ negative crossings. Thus Bennequin's inequality (2.12) now reads

$$1 - \chi(L) \geq w(D) - s(D) + 1 = 1 - \chi(D) - 2a(L), \tag{3.5}$$

with $s(D)$ being the number of Seifert circles of D, and $\chi(D)$ the Euler characteristic of D's canonical Seifert surface. We have

$$1 - \chi(D) = c(D) - s(D) + 1\,. \tag{3.6}$$

From the improvement (2.13) and (3.6) we have

$$1 - \chi(L) \geq w(D) - s(D) + 1 + 2s_-(D) = 1 - \chi(D) - 2a(L) + 2s_-(D)\,. \tag{3.7}$$

(Note that we use here only χ instead of χ_s; we will consider χ_s separately later.)

Now, clearly any semiadequate diagram D will have in each $\sim$-equivalence class only crossings of the same (skein) sign. We call a $\sim$-equivalence class of D positive or negative depending on the sign of its crossings. By applying flypes, we can assume without loss of generality that the l crossings in a negative $\sim$-equivalence class of D bound $l - 1$ Seifert circles with only two adjacent crossings each (both negative).

Let $c_-(D)$ be the number of negative $\sim$-equivalence classes of D. (Clearly, $c_-(D) = 0$ for $a(L) = 0$, and $1 \le c_-(D) \le a(L)$ otherwise.) Then we have from (3.7),

$$1 - \chi(D) \le 1 - \chi(L) + 2c_-(D). \tag{3.8}$$

We know now from theorem 2.3 that D has at most

$$-3\chi(D) \,\le\, -3\chi(L) + 6c_-(D) \tag{3.9}$$

$\sim$-equivalence classes. This conclusion may be incorrect if $\chi(D) = 0$. But the estimate remains valid unless $-\chi(L) + 2c_-(D) = 0$. Then $\chi(D) = c_-(D) = a(L) = 0$. The property $\chi(D) = 0$ implies that D is a diagram of $L = \overline{T}_k$, and $a(L) = 0$ that an A-semiadequate diagram D of L is positive, hence in particular $k \ge 2$. Since a positive diagram has maximal χ, we obtain the exceptional case of the theorem. Otherwise, we can assume in the following that $-\chi(L) + 2c_-(D) > 0$, and hence that the quantity on the right of (3.9) is correct as an estimate for the number of $\sim$-equivalence classes of D.

Consider one of the series $\mathcal{D}$ of n-component link diagrams, differing by $\bar{t}_2'$ moves, in which D lies. If we have > 3 crossings in a positive $\sim$-equivalence class in D, we can reduce them by (possible flypes and) undoing $\bar{t}_2'$ twists, thereby preserving A-semiadequacy. Similarly, if we have > 2 crossings in a negative $\sim$-equivalence class in D, we can reduce them by reverse $\bar{t}_2'$ twists, too. This does not spoil A-semiadequacy either, since a $\bar{t}_2'$ twist applied at a negative crossing creates two crossings joining the same loops in the A-state as the original crossing.

Thus the undoing of $\bar{t}_2'$ twists leads to an A-semiadequate diagram D'' with ≤ 3 crossings in each positive $\sim$-equivalence class, and ≤ 2 crossings in each negative $\sim$-equivalence class. Also, the number of positive and negative $\sim$-equivalence classes of D'' is the same as in D. Thus the crossing number of D'' is at most

$$3 \cdot (-3\chi(D'')) - c_-(D'') \,=\, 3 \cdot (-3\chi(D)) - c_-(D) \,\le\, 3\big(-3\chi(L) + 6c_-(D)\big) - c_-(D) \,=\, 17c_-(D) - 9\chi(L). \tag{3.10}$$

Now, repeated applications of $\bar{t}_2'$ twists on D'' to obtain D preserve A-semiadequacy.

The effect of a $\bar{t}_2'$ twist on the checkerboard graph G is the replacement of an edge by three edges of the same (now checkerboard) sign. This is done either by bisecting the edge twice (adding on it two vertices of valence two), or by tripling it (replacing it by three edges connecting the same vertices). This procedure preserves the number of blocks of the graphs $G_\pm$ and $\overline{G_\pm}$ introduced in [Th], which are not isolated vertices. Thus by [Th, corollary 1.1(iii)], $c(\,.\,) - \max\deg_z \phi_+(\,.\,)$ remains constant under the $\bar{t}_2'$ moves taking D'' into D. That is, each $\bar{t}_2'$ move augments $\max\deg\phi_+$ by two. Since we know from proposition 3.1, that $\max\deg\phi_+(D'') \ge c(D'')/2$, and $\max\deg_z F(D) \ge \max\deg\phi_+(D)$, we have

$$c(D) - c(D'') \,=\, \max\deg\phi_+(D) - \max\deg\phi_+(D'') \,\le\, \max\deg_z F(D) - \frac{c(D'')}{2}\,.$$

So with (3.10) we find:

$$\begin{aligned} c(D) \quad &\le \quad \frac{c(D'')}{2} + \max\deg_z F(D) \\ &\le \quad \frac{17c_-(D) - 9\chi(L)}{2} + \max\deg_z F(D)\,. \end{aligned} \tag{3.11}$$

This implies the inequality we claimed, as $c_-(D) \le a(L)$. $\square$

Remark 3.6 As one can see from the proof of theorem 3.1, the coefficient 17 in (3.4) results from the s_- term in Rudolph's improvement (2.13) of the slice Bennequin inequality. In the previous form without s_- on the right of (2.13) we would have 18 instead. (Our method will lead to several rather unusual constants in the following estimates.)

As the exceptions $\overline{T}_k$ cause extra effort, in the following most formulas will hold *modulo this family* of links. If they are to be incorporated, they must be checked separately.

Since the inequality in theorem 3.1 may appear a bit involved, we can derive (at the price of efficiency) some simplified version of it.

Corollary 3.5 Let L be a non-split and non-trivial link of maximal Euler characteristic $\chi(L)\le 0$ and crossing number $c(L)$. If $L\ne\overline{T}_k$, $k\ge 2$, has an A-semiadequate reduced diagram D, then

$$c(D)\,\le\,\frac{17a(L)-9\chi(L)}{2}-1+c(L)\,,$$

with equality only if L is alternating.

Proof. Now, $\max\deg_z F(D)\le c(L)-1$ by [Ki]. If $\max\deg_z F(D)=c(L)-1$, then L would be alternating. □

3.3. Skein and Jones polynomial

There are two similar versions of theorem 3.1, using instead of the Euler characteristic the skein and Jones polynomial. For the skein polynomial, we have:

Theorem 3.2 Under the hypothesis of theorem 3.1, we have

$$c(D)\,\le\,\frac{18a(L)+9\min\deg_l P(L)-9}{2}+\max\deg_z F(L)\,,$$

except if $L=\overline{T}_k$ for $k\ge -2$. Then '-9' must be replaced by '-7'.

Proof. The proof is similar to that of theorem 3.1, so some parts are not repeated. However, the following important modifications are necessary. First, we replace the Rudolph-improved version of Bennequin's inequality (3.7) by its weaker form (3.5). Then the last inequality in the estimate

$$1-\chi(D)\,=\,c(D)-s(D)+1\,=\,w(D)-s(D)+1+2a(L)\,\le\,1-\chi(L)+2a(L)\,,\tag{3.12}$$

must be modified using an inequality of Morton [Mo] (and essentially also found by Williams-Franks [FW]). This inequality states

$$w(D)-s(D)+1\,\le\,\min\deg_l P(L)\,\max\deg_l P(L)\,\le\,w(D)+s(D)-1\,.\tag{3.13}$$

We obtain

$$1-\chi(D)\,=w(D)-s(D)+1+2a(L)\le\,\min\deg_l P(L)+2a(L)\,.\tag{3.14}$$

The inequality (3.9) modifies to

$$-3\chi(D)\,\le\,-3+3\min\deg_l P(L)+6a(L)\,.$$

The rest of the calculation follows as before, by replacing '$\chi(L)$' by '$1-\min\deg_l P(L)$', and '$c_-(D)$' by '$a(L)$', and omitting the terminal '$-c_-(D)$' in (3.10).

We must again verify the case $\chi(D)=-1+\min\deg_l P+2a=0$ separately, which leads to the exceptional $\overline{T}_k$. Then D'' has one more $\sim$-equivalence class, of an even number of crossings, which contributes with (3.11) one more crossing to the total estimate of $c(D)$. □

The proof shows

Corollary 3.6 If $\min\deg_l P(L)<1-\chi(L)-2a(L)$, then L is not A-semiadequate.

Proof. For a semiadequate, reduced diagram D of L we would have from (3.14) that $1-\chi(D)<1-\chi(L)$, which is clearly impossible. □

Let us give an application that shows how this simple condition can be useful in some cases.

Proposition 3.5 Let β be a positive 3-braid of $n>2$ crossings, and its closure link $L=\hat{\beta}$ not be the $(2,n-1)$-torus link. Then L cannot be projected with fewer than n positive crossings. Also, L is B-semiadequate (or adequate) if and only if β contains no $\sigma_1\sigma_2\sigma_1$ or $\sigma_1\sigma_2\sigma_1$ as a cyclic subword.

This shows in particular that $(3,n)$-torus links are not B-semiadequate for $n \geq 3$. We will use this proposition for the description of Seifert fibered Montesinos links in §4.3.

Proof. The link L is positive, with $\chi(L) = 3-n$, and by the result of [Yo], we have $\min\deg_a F(L) = 1-\chi(L) = n-2 > 0$ and that $[F(L)]_{z^{1-\chi}a^{1-\chi}} \neq 0$. Now, because of the identity $F(\sqrt{-1},z) = 1$, the polynomial $[F(L)]_{z^{1-\chi}}$ (in a) must have at least two monomials, so $\max\deg_a [F(L)]_{z^{1-\chi}} \geq 3-\chi = n$. It implies that $a(!L) \geq n-1$.

If $!L$ is $\leq n-1$-almost positive, then by corollary 2.1, we must have $a(!L) = n-1$, and that $!L$ is A-semiadequate. Now by [N2], when $L \neq T_{n-1}$, we have $\max\deg_m P(L) = 5-\chi(L)$. So we can apply corollary 3.6, obtaining a contradiction. This shows the first claim.

The second claim follows from the first and a few other results. The link L is clearly A-semiadequate. By corollary 2.1 it would be B-semiadequate if and only if $a(L) = n$. This condition is equivalent to the B-semiadequacy (and hence adequacy) of the diagram $\hat{\beta}$, which by [St12] in turn is equivalent to the stated property of β. □

Now we can replace the skein polynomial by the Jones polynomial, using an analogue of Morton's inequality proved in [St4].

Proposition 3.6 ([St4]) If D is a k-almost positive diagram (that is, one with exactly k negative crossings), then

$$\min\deg V(D) \geq \frac{1-\chi(D)}{2} + 1 - 2k$$

for $k > 0$.

For positive diagrams D (that is, $k = 0$), we have, as well-known previously (see [Zu, Fi]), $\min\deg V(D) = \frac{1-\chi(D)}{2}$, so that using V instead of χ does not give any new information; likewise with P and χ_s, since also $\min\deg_l P = 1-\chi = 1-\chi_s$ in this case (see [Cr, theorem 4] and [St2]).

Using proposition 3.6, we obtain an analogous inequality. (We exclude the case $a(L) = 0$ as in proposition 3.6 just for formal reasons; in this case one would have to replace the constant -14 by -4.5 if $L \neq \overline{T}_k$, $k \geq 2$, and by -3.5 otherwise. But positive links were treated satisfactorily with corollary 3.2, so it is no real contraint to exclude them when convenient.

Theorem 3.3 Under the hypothesis of theorem 3.1, but assuming that also $a(L) > 0$, we have

$$c(D) \leq 27a(L) - 14 + 9\min\deg V(L) + \max\deg_z F(L).$$

Proof. Instead of (3.12), we have now

$$1-\chi(D) \leq 2\min\deg V(L) + 4a(L) - 2$$

with a similar calculation. In (3.10) we use $c_-(D) \geq 1$ (because $a > 0$). Verifying the analogue of (3.9) would lead to exceptions if $\chi(D) = 2\min\deg V + 4a - 3 = 0$. The only $\overline{T}_k$ with $k < 0$ satisfying this condition is for $k = -2$. (The assumption $a > 0$ excludes $k > 0$.) For $\overline{T}_{-2}$ we see directly that the estimate we claim is weaker than the one of theorem 3.2. □

As for theorem 3.1, the choice of orientation of single components has influence on the estimate for links. Using the result of [LM2, Mo2] (following also from (2.3)), we can control this change for theorem 3.3. Let us denote by $lk(L_i, L_j)$ the linking number of two components of a link $L = L_1 \sqcup L_2 \sqcup \ldots \sqcup L_n$, and by $lk(L_i)$ the total linking number of component L_i, given by

$$lk(L_i) = \sum_{j \neq i} lk(L_i, L_j).$$

Then reversal of the orientation of L_i changes $\min\deg V(L)$ by $-3lk(L_i)$, while it changes $a(L)$ by $2lk(L_i)$. Thus the choice of orientation for which $\min\deg V(L)$, and hence the writhe of the diagram, is maximal (and $a(L)$ minimal) will give the best estimate (excluding positive diagrams). For theorem 3.1 and 3.2 the change of χ (resp. χ_s) and $\min\deg_l P$ cannot be reasonably controlled, so some trial and error may be appropriate.

The comparison of the efficiency of the estimates in theorems 3.1, 3.2 and 3.3 basically depends on the quantities

$$\alpha(L) = 1-\chi(L)\,,\quad \beta(L) = \min\deg_l P(L)\,,\quad \text{and}\quad \gamma(L) = 2\min\deg V(L) - 2\operatorname{sgn}(a(L)) + 2a(L)$$

as upper bounds for $w(D)-s(D)+1$. In general, β is the smallest among the three, thus making the estimate in theorem 3.2 the best one. It is easy to find examples when β is lower than α and γ; consider e.g. 4_1 and $!10_{132}$. These two examples show also that α and γ are independent (no one is always sharper than the other one), and apply also for $\alpha'(L) = 1-\chi_s(L)$ instead of α. However, to show that some of α and γ can become smaller than β is rather non-trivial. For α this was an old problem of Morton, resisting solution over 15 years, for which examples have been found only recently [St3]. For γ, as for Morton's problem, a computer check showed that no prime knot of ≤ 16 crossings satisfies $\gamma<\beta$.

Nonetheless, theorem 3.3 deserves consideration in its own right, even if just for theoretical purposes. Since F determines V (by (2.3) or [Ka2, §III]), we have with theorem 3.3 a new estimate entirely determined by the Kauffman polynomial. This implies again corollary 3.1 (for F rather than just $\max\deg_z F$ at least).

One can prove a yet different such estimate also from a weaker version of theorem 3.2, since

$$\min\deg_l P(L) \leq \min\deg_l [P(L)]_{m^{1-n(L)}} \tag{3.15}$$

and, up to change of variable, $[P(L)]_{m^{1-n(L)}} = [F(L)]_{z^{1-n(L)}}$ (see [L, proposition 4.7]). So we have

$$c(D) \leq \frac{18a(L) + 9\min\deg_a [F(L)]_{z^{1-n(L)}} - 9}{2} + \max\deg_z F(L)\,. \tag{3.16}$$

However, with the inequality (3.15) we may sometimes lose a lot, so that the Jones polynomial variant can do sensibly better. An example is the knot $!13_{2917}$, where $\min\deg_l P = 0$ but $\min\deg_l [P]_{m^0} = 6$. From theorem 3.3 we obtain that an A-semiadequate diagram with its Kauffman polynomial will have at most 43 crossings, while (3.16) gives only at most 52.

On the other hand side, the weak appearing estimate (3.16) can occasionally dominate the universal bound $c(D) \leq 2\max\deg_z F(L)$ in proposition 3.1. This is shown by the twist knots with $a=2$ and further examples like 15_{5399}. We do not know whether the estimate in theorem 3.3 can dominate proposition 3.1 (or even (3.16) simultaneously).

Next we will explain how the other invariants get into relation with almost everything dealt with so far.

3.4. Canonical, 4-ball genus, and new concordance invariants

Since Bennequin's inequality (2.13) applies to the 4-genus, one can incorporate several (smooth) 4-ball invariants into the picture. We will concentrate on the (smooth) 4-ball genus g_s and 4-ball Euler characteristic χ_s. The other invariants are less important and more auxiliarily defined, and their relation to χ_s is well-known. (A section of [St2] gives a brief survey.)

When using χ_s instead of χ, a bit more care is necessary since, unlike χ, even for non-trivial non-split links χ_s may be positive. Nevertheless, most of the results in §3.2 hold by simply replacing 'χ' by 'χ_s'. In particular, Theorem 3.1 generalizes without much difficulty, and we have:

Corollary 3.7 Under the same hypothesis as theorem 3.1,

$$c(D) \leq \frac{17a(L) - 9\chi_s(L)}{2} + \max\deg_z F(L)\,. \tag{3.17}$$

Proof. The only point in the to-be-modified proof of theorem 3.1 to remark is when verifying (3.9). Then we must consider the case that $\chi(D)=0$, so that $L=\overline{T}_k$, and $\chi(L)=\chi_s(L)=0$, implying again $c_-(D)=a(L)=0$, etc. $\square$

Corollary 3.5 also generalizes accordingly.

For knots one can obtain a relation to the more recent invariant τ defined by Ozsváth and Szabó [OS] using Floer homology, and also the invariant s of Rasmussen [Ra] and Bar-Natan [BN] (modulo the conjecture that Bar-Natan's definition is legitimate). The slice Bennequin inequality for knots was proved, in the original form of the estimate, for τ by Livingston [Lv], and for s (without being stated explicitly) by Rasmussen [Ra]. More recently, Kawamura [K2] showed the improvement with the $s_-(D)$ term:

$$s(L) \geq w(D) - s(D) + 2s_-(D) + \begin{cases} -1 & \text{if } D \text{ is negative} \\ 1 & \text{otherwise} \end{cases} . \tag{3.18}$$

On the left one can place also $2\tau(L)$ instead of $s(L)$. So we can obtain new inequalities involving τ and s.

Corollary 3.8 Under the assumption of theorem 3.1, and if L is a *knot*, we have

$$c(D) \leq \frac{17a(L) - 9 + 18\tau(L)}{2} + \max\deg_z F(L) \tag{3.19}$$

and

$$c(D) \leq \frac{17a(L) - 9 + 9s(L)}{2} + \max\deg_z F(L) . \tag{3.20}$$

Proof. We replace in theorem 3.1 and its proof $1 - \chi(L)$ by s or 2τ, using (3.18). The argument goes through, except in the case that D is negative. For a negative diagram, (3.8) becomes

$$1 - \chi(D) \leq 2\tau(L) + 2 + 2c_-(D) . \tag{3.21}$$

This case is handled by noting that in passing from (3.7) to (3.8), we ignore the (strongly negative) Seifert circles of valence at least 3. So in fact, instead of (3.21), we have

$$1 - \chi(D) \leq 2\tau(L) + 2 + 2c_-(D) - 2s_*(D) ,$$

where we write $s_*(D)$ for the number of strongly negative Seifert circles of valence at least 3. To remedy the modification of (3.8), we need just to exclude the case $s_*(D) = 0$. But the only (negative) diagrams D with $s_*(D) = 0$ are those of the $\overline{T}_k$ (for $k \leq 0$), which are not knot diagrams. □

As opposed to σ, the new "signatures" τ and s are more limitedly computable, and also lack proper definition for links. This makes the application of estimates like (3.19) or (3.20), and the practical handling of some examples (see in particular §4.4) more difficult. The advantage of τ and s is, though, that, unlike σ, they allow for a Bennequin type inequality, yet, unlike $1 - \chi_s = 2g_s$, admit negative values. So when τ can be calculated, and is negative, (3.19) gives a better result than theorem 3.1. One can propagate the treatment of τ and s at places below where χ_s occurs (and L is a knot), but we omit this discussion for space reasons.

We can then further replace the slice Euler characteristic by the *unknotting (or unlinking) number* $u(L)$ and number of components $n(L)$, as $-\chi_s(L) \leq 2u(L) - n(L)$:

$$c(D) \leq \frac{17a(L) + 18u(L) - 9n(L)}{2} + \max\deg_z F(L) .$$

Let us also say something more in a different special case. Often when the Bennequin-Rudolph inequality detects non-sliceness, the estimates can be improved by an additional argument.

Theorem 3.4 Assume L satisfies $a(L) < -2\chi_c(L)$. Then all previous estimates for $c(D)$ can be improved by adding $\frac{a(L)+2\chi_c(L)+1}{2}$.

Proof. This results from improving the estimate for the crossing number of D'' in the proof of theorem 3.1. In that proof D'' was chosen so that it transforms into D under $\bar{t}'_2$-moves, and so that it is A-semiadequate. If $a(L) < -\chi_c(L)$, we can choose D'' more economically (w.r.t. crossing number).

Assume w.l.o.g. that in D (and hence D'') $\sim$-equivalent crossings are neighbored equivalent. Assume first that D's generator has the maximal number $-3\chi(D)$ of $\sim$-equivalence classes. (Since for $\chi_c(L) = 0$ there is nothing to prove by remark 3.5, we can assume that $\chi(D) < 0$.) Then D has $-2\chi(D)$ *essential* Seifert circles, i.e. such not enclosed between two $\sim$-equivalent crossings. Call Seifert circles of latter type *intermediate*.

Consider the Seifert graph G of D, and sign the edges according to the (skein) sign of the crossings. Each valence-2-vertex of G is incident to two edges of the same sign, and corresponds to an intermediate Seifert circle of D. Obtain a (signed) graph G', by removing all these vertices by undoing bisections. The vertices of G' are ≥ 3-valent, and correspond to essential Seifert circles of D. The edges of G' correspond to $\sim$-equivalence classes of D. (Note that G' is still 2-connected, but may have loop edges. It may also have multiple edges, even with the different copies of the single edge differently signed.)

The loops in the A-state of D are obtained from the Seifert circles of D, in which intermediate Seifert circles of a negative $\sim$-equivalence class are removed, and the two essential Seifert circles connected (possibly via intermediate Seifert circles) by crossings in such a class are joined. We call the loops in the A-state of D thus obtained *essential*. Essential loops include essential Seifert circles, which have not been involved in a join of the described type. The *intermediate* loops are the intermediate Seifert circles of positive $\sim$-equivalence classes, which remain also unaffected by joins. So any loop in the A-state is either essential or intermediate.

Take a spanning tree T of G' containing the minimal number of positive edges. We claim now that any positive edge e of T corresponds to a positive $\sim$-equivalence class of D, connecting (possibly via intermediate loops) two different essential loops in the A-state. Otherwise, there must be a path in G' made of only negative edges, connecting the endpoints of e. This gives a spanning tree with fewer positive edges than T.

We assumed that D has the maximal number $-3\chi(D)$ of $\sim$-equivalence classes, so that G' has the maximal number $-2\chi(G') = -2\chi(G) = -2\chi(D)$ of vertices, i.e., is trivalent. Then T has $-2\chi(D)-1$ edges, and at most $c_-(D) \leq a(L)$ of them are negative. Thus G' has at least $-2\chi(D)-1-a(L) \geq -2\chi_c(L)-1-a(L)$ positive edges, whose $\sim$-equivalence class connects two different essential loops in the A-state. We call such classes *good*. The number of crossings in such a class can be reduced to ≤ 2 (and not as before ≤ 3) crossings, without spoiling A-semiadequacy.

Thus we save for each good $\sim$-equivalence class one crossing in $c(D'')$, and hence $1/2$ from the estimate on $c(D)$ by (3.11).

If now, the number of $\sim$-equivalence classes of D decreases by 1, so does the estimate for the number of positive good $\sim$-equivalence classes, and we save 1 crossing less. However, in this case the disappearing $\sim$-equivalence class saves at least one crossing, and so the estimate remains valid. $\square$

The inequality $a < -2\chi_c$ is satisfied by many non-slice (but also some slice) links. Of course, the statement remains true even if this condition fails (in particular also if $\chi_c(L) = 0$), only that the estimate becomes weaker. The improvement is particularly efficient when $a = 0$, but this case was treated well in §3.1. One can also vary on this theme. For example, we have:

Corollary 3.9 Under the hypothesis of theorem 3.1, we have

$$c(D) \leq \frac{19a(L)}{2} + \frac{9\min\deg_l P(L)}{2} - 3 - \max\deg_m P(L) + \max\deg_z F(L),$$

if $L \neq \overline{T}_k$ for $k \geq 2$ (and '$-5/2$' for '-3' otherwise).

Proof. This is obtained from theorem 3.4 using the Morton inequality $1-\chi_c \geq \max\deg_m P$ in [Mo] and theorem 3.2. $\square$

3.5. Hyperbolic volume

When encountering the canonical Euler characteristic, it is worth making a few remarks on the hyperbolic volume. Even if it is not the most useful invariant in our situation, it is still possible to extract some applicable information from it. Beside for semiadequacy, one can establish some properties of the volume for itself.

The hyperbolic volume was related to our context by a preprint of Brittenham [Br], in which he came up simultaneously and independently from us with the generator approach for the canonical genus. His motivation was to show that, in the case $n=1$ of knots, the supremum

$$v_{n,\chi} := \sup \{ \operatorname{vol}(L) : n(L)=n, \chi_c(L) = \chi \}$$

is finite, that is, the canonical genus bounds the hyperbolic volume of the knot. The generator estimates he obtained were better than the one I gave originally in [St], but were slightly improved in [St6] by referring to our work in [STV]. The volume bound was also improved, by Lackenby [La], and later by Ian Agol and Dylan Thurston (see the appendix to Lackenby's paper):

Theorem 3.5 (Agol and Thurston) Let $V_0 \approx 1.01494$ be the volume of the ideal tetrahedron. Then we have for any (non-trivial) diagram D of a link L the inequality

$$\operatorname{vol}(L) \le 10V_0\big(t(D)-1\big), \tag{3.22}$$

where $t(D)$ is the twist number of D, and with the convention that $\operatorname{vol}(L)=0$ if L is not hyperbolic.

Note that $t(D)$ is always less than or equal to the number of $\sim$-equivalence classes of D. Then, using this result and theorem 2.3, we can state Brittenham's result in a sharper and more general form. (We omit the obvious proof.)

Theorem 3.6 Any link L with $\chi_c(L)<0$ satisfies

$$\operatorname{vol}(L) \le -10V_0\big(3\chi_c(L)+1\big). \tag{3.23}$$ □

This inequality can be applied wherever χ_c occurs with a positive sign in the estimate. (Note that (3.23) does not hold for $\chi_c=0$, as the r.h.s. is negative, but its below consequences are easily checked for the $\overline{T}_k$.) Thus we have, for example,

Corollary 3.10 Under the same hypothesis as in theorem 3.1, and if $L \ne \overline{T}_k$, $k>0$, we have

$$c(D) \le \frac{18a(L)-9\chi(L)}{2} + \max\deg_z F(L) + \frac{1}{6} - \frac{\operatorname{vol}(L)}{30V_0}.$$

If $L=\overline{T}_k$, $k>0$, we replace $1/6$ by 1.

Proof. Combine theorems 3.1, 3.4 and 3.6. For $L=\overline{T}_k$, $k>0$, use the previously established fact that $c(D)=c(L)$. □

Since we have from [Br] that $\operatorname{vol}(L) \le 4V_0\,c(L)$, the volume term can improve the inequality only limitedly. Although Agol-Thurston show that their inequality (3.22) is (asymptotically) sharp in general, it does not seem so in our context. For example, it follows from the computation in [St6] that for canonical knot genus two (3.23) gets sharp up to a factor of about 1.3, but it is weaker in general. with a hypothetical improvement coming from having the Agol-Thurston's inequality sharp, the volume term will remain smaller in magnitude than the crossing number term. Thus, as mentioned, in practice it is not optimal to apply the volume, at least for our purpose.

More optimistically, under special conditions theorem 3.6 can be improved. For example, we have:

Proposition 3.7 In corollary 3.10 we have for a *rational* (or 2-bridge) link L

$$c(D) \le \frac{18a(L)-9\chi(L)}{2} + \max\deg_z F(L) + \frac{1}{2} - \frac{\operatorname{vol}(L)}{10V_0}, \tag{3.24}$$

if $L \ne \overline{T}_k$ for $k \ge 2$. For such L, replace $1/2$ above by 1.

Proof. If a link L has a diagram $\tilde{D}$, which is an iterated plumbing of Hopf bands (not necessarily with ± 1 full twist), then we know from the results of Gabai [Ga, Ga2] that $\chi(\tilde{D}) = \chi_c(L) = \chi(L)$, and the number of such bands is $1-\chi_c(L) = 1-\chi(L)$. Since any band gives a $\sim$-equivalence class, we have from (3.22)

$$\mathrm{vol}(L) \le -10V_0\chi(L)\,.$$

This inequality seems the best possible estimate that one obtains with the Brittenham-Lackenby-Agol-Thurston method of proof, at least as far the generator part of the argument goes. However, it is achieved for all links with a diagram $\tilde{D}$. This includes the rational links. Thus we obtain the stated result. (For $L = \overline{T}_k$, $k > 0$, we again adjust the estimate directly.) □

Remark 3.7 Note that there are many more links having such a diagram $\tilde{D}$, and hence for which (3.24) holds. For example consider Montesinos links $L = M(p_1/q_1,\dots,p_n/q_n;e)$ with even $e \ne 0$ when the p_i, q_i can be adjusted so that $|p_i| < q_i$ and all p_iq_i are even, and there is an appropriate orientation. (In fact they are a generalization of rational links.)

Again one has a (straightforward, and hence omitted) version for achiral and slice rational knots. One can also obtain further (more general) inequalities using theorem 3.4.

There is more to say on sliceness, but for this we must discuss two further major tools, the Jones polynomial and signature. The approach of genus generators has also further applications to the volume on its own, but since they are not directly related to semiadequacy, we defer them to a separate paper [St13].

3.6. Signature and sliceness

The appearance of the signature σ seems in some way closely related to Hirasawa's algorithm, as investigated in [St5, St10]. It is clear from theorem 3.1 that the largest economy can be achieved if one can properly bound $a(L)$. The easy inequality $a(L) \le c(L)$ is in general highly unsharp. In practice this fact causes no problem, since one can calculate $a(L)$ directly for any given L. If one wants to improve the inequality for $a(L)$ in theory, one can use properties of L. For example, using σ, we have:

Proposition 3.8 Assume that $\sigma(L) \le 1-\chi(L)-2a(L)$. Then the estimate in theorem 3.1 can for $L \ne \overline{T}_k$, $k \ge 2$ (under the same hypothesis) be improved to

$$c(D) \le \frac{43}{4}a(L) - \frac{27}{8}\chi(L) - \frac{9}{8} + \max\deg_z F(L) + \frac{9}{8}\sigma(L)\,.$$

Proof. We apply lemma 2.1, which is possible since for any diagram D of L, $1-\chi(D) \ge 1-\chi(L)$, and if D is A-semiadequate, $a(D) = a(L)$, so that

$$k(D) \ge k(L) := \frac{1-\chi(L)-2a(L)-\sigma(L)}{2} \ge 0\,.$$

Then the estimate for the crossing number of D'' in the proof of theorem 2.3 reduces by

$$3\cdot\left(\frac{3}{2}k(D)\right) \ge \frac{9}{2}k(L)\,,$$

and hence by one half of this quantity does the estimate for $c(D)$. Thus

$$c(D) \le \frac{17a(L)-9\chi(L)}{2} + \max\deg_z F(L) - \frac{9}{8}\big(1-\chi(L)-2a(L)-\sigma(L)\big)\,,$$

which is the inequality we claimed. □

Using lemma 2.1, one can make use of σ and χ also to improve the inequalities involving the skein and Jones polynomial, but this is now straightforward and not worth discussing about. However, for V there is a new way to bring σ, χ and $\min\deg V$ into interplay. This happens when one uses corollary 3.5 and a result of Murasugi.

Proposition 3.9 For a reduced A-semiadequate diagram D of a link L we have

$$c(D) \le \frac{19}{2}c(L) + \frac{17}{4}n(L) - \frac{17}{2}\max\deg V(L) - \frac{35}{4}\chi_s(L) - 1\,, \tag{3.25}$$

with equality only if L is alternating. If L is alternating, then also

$$c(D) \le c(L) + \frac{17}{4}n(L) - \frac{17}{2}\min\deg V(L) - \frac{35}{4}\chi_s(L) - 1\,. \tag{3.26}$$

Proof. From Theorem 13.3 of [Mu4] we have that if L is A-semiadequate,

$$a(L) \ge \frac{\sigma(L)}{2} - \min\deg V(L)\,, \tag{3.27}$$

with equality if L is alternating. (Keep in mind our convention for σ.) By applying Murasugi's estimate (3.27) on a minimal crossing diagram of $!L$ and using corollary 2.1, we find also

$$a(L) \le c(L) + \frac{\sigma(L)}{2} - \max\deg V(L)\,.$$

Using the 4-ball modification of corollary 3.5 (which holds by the remark in the beginning of §3.4), we obtain for a reduced A-semiadequate diagram D of L

$$c(D) \le \frac{19}{2}c(L) + \frac{17}{4}\sigma(L) - \frac{17}{2}\max\deg V(L) - \frac{9}{2}\chi_s(L) - 1\,. \tag{3.28}$$

Now apply the Tristram-Murasugi inequality $\sigma \le n - \chi_s$, and (3.25) follows. For L alternating use [Ka, Mu2, Th2] that $c(L) = \operatorname{span} V(L)$ to obtain (3.26). □

For a strongly slice link, $\sigma(L) = 0$ and $\chi_s(L) = n(L)$, so that the inequality (3.28) simplifies to

$$c(D) \le \frac{19}{2}c(L) - \frac{9}{2}n(L) - 1 - \frac{17}{2}\max\deg V(L)\,.$$

Since obviously $c(D) \ge c(L)$, we obtain in particular:

Corollary 3.11 If a strongly slice non-trivial non-split link of $n(L)$ components is A-semiadequate, then

$$\max\deg V(L) \le c(L) - \frac{2+9n(L)}{17}\,. \qquad \square$$

Note that strongly slice links of > 1 component have vanishing Alexander polynomial, in particular there are no alternating (non-split) such links. (There *are*, though, a plenty of slice alternating *knots*.) So again by the non-alternation argument in corollary 3.5, we can replace '2' by '4' for $n(L) > 1$. We see also that (3.26) becomes vacuous for strongly slice (alternating) links with $n(L) > 1$, but we obtain a new statement for *knots*.

Corollary 3.12 If a non-trivial alternating knot K is slice, then any A-semiadequate reduced diagram D of K satisfies

$$c(D) - c(K) \le -\frac{17}{2}\min\deg V(K) - \frac{11}{2}\,.$$

In particular, $\min\deg V(K) < 0$. Also, the number of A-semiadequate reduced diagrams of alternating slice knots K with given $\min\deg V(K)$ grows polynomially in $c(K)$.

Proof. The inequality we claim follows directly from the above proposition. As for the statement on the number of diagrams, we observe from Murasugi's (now sharp in)equality (3.27) and the slice Bennequin inequality that

$$-\min\deg V(K) = a(K) \ge g_c(K)\,.$$

So we must consider diagrams of canonical genus at most $-\min\deg V(K)$ and crossing number at most $c(K) - {}^{17}\!/_2 \min\deg V(K) - {}^{11}\!/_2$ (see §4.2). Then we can apply the polynomial growth property in theorem 2.2. (Precisely speaking, we must argue that it is not changed even after allowing flypes. This can be done as explained in [St11].) □

Remark 3.8 The condition $\min\deg V(K)$ to be fixed is clearly necessary. It is easy to construct otherwise exponentially growing diagram families of the same alternating slice knot. The easiest example is to take connected sums of square knots, and to turn some of the positive trefoils therein into their positive 4 crossing diagrams.

Remark 3.9 The property $\min\deg V(K) < 0$ can also be deduced from Murasugi's remark that his inequality (3.27) is sharp for alternating links and the property $\sigma = 0$ for a slice knot and $a > 0$ (since non-trivial positive links are not slice).

4. Applications and examples

4.1. Initial examples

We start with a brief account on theorems 1.3 and 1.2. These simplest examples do not formally rely on our new theoretical results. How they later come into practice is shown in §4.4 and 4.5.

A-adequate examples. For the property A-semiadequate an easy example follows from our previous work [St3]. The knot 11_{550} was found there to have a positive diagram, but no such diagram of minimal crossing number. We mentioned in [St3] that 11_{550} has a single 11 crossing diagram, which is almost positive (that is, has exactly one negative crossing). For the same reason why positive diagrams are A-semiadequate, almost positive diagrams are never A-semiadequate. Therefore, 11_{550} has no A-semiadequate diagrams of minimal crossing number. (The mirror image of 11_{550} applies then for the property B-semiadequate.) However, the 11 crossing diagram of 11_{550} is B-semiadequate, and thus it cannot serve as an example for the property semiadequate in theorem 1.3. On the other hand, this also shows that a knot may have diagrams which are A-semiadequate, and such which are B-semiadequate, but no diagrams which are simultaneously both. (Perko's knot 10_{161} is another such example.)

Inadequate examples. Let us next discuss some examples showing theorem 1.2. We remarked before that up to 10 crossings all knots are semiadequate. As Cromwell pointed out [Cr3, figure 9.12], the knot 11_{462} is inadequate (and thus a lowest crossing number example). Its inadequacy is exhibited by its Jones polynomial and the results in [LT]. The other inadequate 11 crossing knot is 11_{485} of figure 3, but we need the work in this paper to prove its inadequacy (see example 4.3). The claim of Thistlethwaite, quoted in the introduction, that all minimal crossing number diagrams of 11 crossing knots are semiadequate is thus, taken precisely, not correct due to these two knots (though it is true for all the other 11 crossing knots).

From the point of view of the methods of this paper, the theoretically (though not in crossing number) easiest examples rely merely on the tests (2.9) and (2.10). Thus we are seeking knots with polynomials which admit both pairs (l,m) and (l',m'). It already follows from our above remarks that such knots can not be of very low crossing number. It turned out that in fact the simplest examples have 15 crossings, and they are 36. (These examples were comparatively easy to spot by means of requiring merely the calculation of the Kauffman polynomial, even although this calculation can be time consuming.) Two of the knots are shown in figure 3. It is worth remarking that all 36 knots have leading z-coefficient $\max\mathrm{cf}_z F$ of the form $-a^k(1+a^2)$ for some $k \in \mathbb{Z}$.

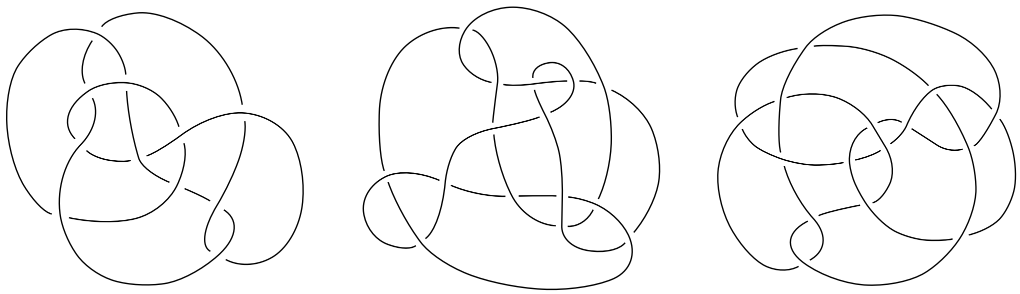
Semiadequate examples. Examples for the property semiadequate in theorem 1.3 were much harder to find. Consider the knot 15_{249826} in figure 4. The 15 crossing diagram on the left is its only minimal crossing number diagram, which is not semiadequate. However, the same knot has the 16 crossing diagram on the right, which is semiadequate. There are 6 further such knots of 15 crossings, only with 16 crossing semiadequate diagrams. Apparently they are the simplest examples of this type. They emerged from a calculation which was more substantial both theoretically and practically, but we skip details for space reasons.

4.2. Properties of generic alternating or positive knots of given genus

Given a class of knots, it is suggestive to ask how a generic representative of this class looks like. We give a precise meaning to 'generic' with the following definition:

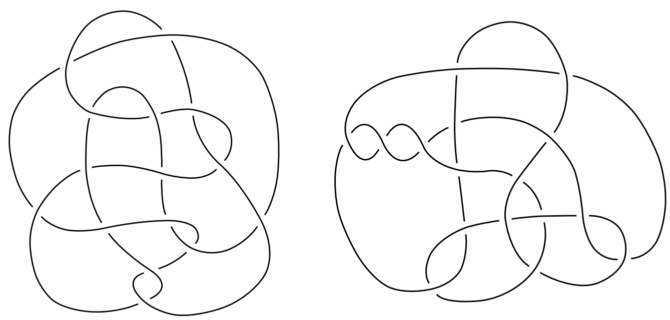
Definition 4.1 A subclass $\mathcal{C} \subset \mathcal{B}$ in a class $\mathcal{B}$ of links is called *asymptotically dense* or *generic*, if

$$\lim_{n\to\infty} \frac{\big|\{\,K \in \mathcal{C}\,:\,c(K)=n\,\}\big|}{\big|\{\,K \in \mathcal{B}\,:\,c(K)=n\,\}\big|} = 1\,.$$



11_{485} 15_{125441} 15_{216196}

Figure 3: Knots with no semiadequate diagrams.



15_{249826}

Figure 4: Two diagrams of the knot 15_{249826}. On the left its (only) minimal diagram, which is not semiadequate, and on the right a semiadequate 16 crossing diagram.

For example, in [Th3] Thistlethwaite showed that the non-alternating links are generic in the class of all (unoriented) links. Similarly, a result of [St7] implies that any generic subclass of the class of alternating links contains mutants.

In [SV] we showed that a generic alternating knot of some genus is *special alternating*, that is, its alternating diagram has only positive or only negative crossings. (It will be convenient to identify in the sequel a special alternating knot with its mirror image, and so to assume that all crossings are positive.) We will now apply the methods of [SV], together with the inequality for positive diagrams in the previous section, to extend this result. We describe precisely the situation for the class

$$\mathcal{B}_g := \{\ K \text{ alternating or positive knot}, g(K) = g\ \}.$$

Theorem 4.1 Define the class

$$\mathcal{C}_g := \{\ K \text{ special alternating knot, with a unique positive diagram and } g(K) = g\ \}.$$

Then $\mathcal{C}_g$ is asymptotically dense in $\mathcal{B}_g$ for $g > 1$.

Note that in particular the knots in $\mathcal{C}_g$ have also a unique alternating diagram (see [N]). We also remark, that for any diagram two of the properties special, positive and alternating imply the third.

Remark 4.1 One should note that, although theorem 4.1 shows that $\mathcal{C}_g$ contains a plenty of knots, it is difficult to show for any single particular example to belong to this class. We have a description of $\mathcal{C}_g$ only for $g = 1$, as it follows from [St] that $\mathcal{C}_1$ is made up of the pretzel knots $P(p,q,r)$ with $p,q,r \geq 3$ odd. For $g \geq 2$ we know from [St6] that $!10_{120} \in \mathcal{C}_2$, but further examples are not known.

Here and below '$a_i \sim_i b_i$' denotes $\lim_{i\to\infty} a_i/b_i = 1$. We omit the index of '$\sim$' if it is clear.

Proof of theorem 4.1. By [STV] a genus g knot diagram has $\le 6g-3$ $\sim$-equivalence classes. If it has exactly $6g-3$ $\sim$-equivalence classes, then by theorem 5.3 and theorem 5.7 of [SV] it is special alternating and (its alternating diagram) does not admit a flype. The number of generators is finite, and positive for both crossing number parities if $g > 1$.

Now we must argue for both crossing number parities separately. Since the argument is the same, we consider below only odd crossing number c.

Then we have

$$\#\{\, c \text{ crossing alternating genus } g \text{ diagrams of } n \sim\text{-equivalence classes} \,\} \sim_c A_{g,n} c^{n-1},$$

for some constant $A_{g,n}$.

Since $C_{g,o} := A_{g,6g-3} > 0$ for $g > 1$, as shown in [SV], it follows that special alternating knots with alternating diagrams of $6g-3$ $\sim$-equivalence classes are asymptotically dense in the subclass of alternating knots in $\mathcal{B}_g$. We also know that all such diagrams are special.

We now show that these knots are also asymptotically dense in the subclass of positive knots in $\mathcal{B}_g$.

For this we use corollary 3.2. We have for a knot K of genus $g(K)$ that $-\chi(K) = 2g(K)-1$. Thus the corollary implies that a positive reduced diagram of K has at most $c(K)+2g(K)-1$ crossings.

Alternating diagrams up to mirroring and positive diagrams correspond bijectively, and thus we have again

$$s_{g,n}(c) := \#\{\, c \text{ crossing positive genus } g \text{ diagrams of } n \sim\text{-equivalence classes} \,\} \sim_c A_{g,n} c^{n-1}.$$

Since by [SV] a non-special positive diagram has $< 6g-3$ $\sim$-equivalence classes, by proposition 3.2

$$\# \left\{ \begin{array}{c} \text{non-special positive diagrams } D \text{ of} \\ \text{a knot } K \text{ with } c(K)=c,\ g(K)=g \end{array} \right\} \le \sum_{c'=c}^{c+2g-1} s_{g,6g-4}(c') \sim_c O(c^{6g-5}). \tag{4.1}$$

(Here we must allow for c' of the opposite parity to c, but the estimate remains valid as written.) However,

$$\#\ \{\text{ special } c \text{ crossing diagrams with } 6g-3 \sim\text{-equivalence classes }\} \sim_c C_{g,o}\, c^{6g-4}.$$

Since by Theorem 5.7 of [SV] such diagrams do not admit flypes, we have

$$\# \left\{ \begin{array}{c} \text{knots } K \text{ with } c(K)=c,\ g(K)=g, \text{ and alternating} \\ \text{diagrams of } 6g-3 \sim\text{-equivalence classes} \end{array} \right\} \sim_c C_{g,o}\, c^{6g-4}. \tag{4.2}$$

Combining (4.1) and (4.2) shows that for genus $g > 1$ there are asymptotically fewer non-special positive diagrams representing positive knots of crossing number c, than special alternating knots of crossing number c with alternating diagrams of $6g-3$ $\sim$-equivalence classes. This argument implies that a generic positive knot (alike an alternating knot, as we already concluded above) is special alternating with an alternating diagram of $6g-3$ $\sim$-equivalence classes, as desired. But the same argument also implies that a generic special alternating knot with an alternating diagram of $6g-3$ $\sim$-equivalence classes has no non-special positive diagrams. Then any positive diagram of such a knot is special, hence special alternating, and, as argued, from Theorem 5.7 of [SV] we have that such a diagram is unique. □

Remark 4.2 The constant $C_{g,o}$, and its counterpart $C_{g,e}$ for even crossing number, also have an interpretation, as described in [SV] – they are, up to the factor $(6g-4)!$, equal to the number of maximal planar odd and even Wicks forms of genus g defined there.

As a consequence, we have the exact asymptotical behaviour of the number of positive knots of given genus.

Corollary 4.1 The number $c_{n,g}$ of positive genus g knots of crossing number n satisfies $c_{n,g} \sim_n C_{g,n \bmod 2}\, n^{6g-4}$ as $n \to \infty$ keeps the same parity. (Here in the index of C we identify $e = 0$, $o = 1$.)

Proof. We know from [SV] that the number of special alternating genus g knots of n crossings has the stated asymptotics. $\square$

Let $a_n \dot{\sim} b_n$ if there are constants $C_{0,1} > 0$ with $a_n \sim C_{n \bmod 2} b_n$ for $n \to \infty$ even/odd. Let furthermore $\lfloor x \rfloor$ be the largest integer not exceeding x, and $\lceil x \rceil = -\lfloor -x \rfloor$. Then one proves similarly using lemma 2.1 (and the integrality of several of the quantities occurring in its proof):

Proposition 4.1 The number $c_{n,\chi,\sigma}$ of non-split positive links of crossing number n, Euler characteristic $\chi < 0$ and signature σ satisfies for $n \to \infty$

$$c_{n,\chi,\sigma} \dot{\sim} O(n^{p(\chi,\sigma)}) \quad \text{with} \quad p(\chi,\sigma) = \left\lfloor -3\chi - 1 - \frac{3}{2}\left\lceil \frac{1-\chi-\sigma}{2} \right\rceil \right\rfloor .$$

Also $c_{n,\chi,1-\chi} \dot{\sim} n^{-3\chi-1}$.

Proof sketch. The first claim follows along similar lines as in the proof of theorem 4.1, while for the second claim (for $\sigma = 1-\chi$) one must argue like in the proof of theorem 2.2. We will, however, not repeat all the details. $\square$

This means in particular that at most $O(1/n^2)$ of the links of given χ have $\sigma < 1-\chi$. But such examples exist in multitude, and in fact it is not known, apart from its positivity, how small σ can become as compared to $1-\chi$. What we know now is that *generically* for fixed χ the value of σ is the maximal possible.

Remark 4.3 Similarly one can use lemma 2.1 for positive links with $\sigma < 1-\chi$ to improve theorem 3.6; we have for such a positive link L the inequality

$$\mathrm{vol}(L) \le 10 V_0\, p(\chi(L), \sigma(L)) .$$

4.3. Hyperbolicity of Montesinos links

We give some applications of our formulas to Montesinos links, which in particular imply the classification of Seifert fibered and non-hyperbolic ones. We resume the language of §2.7.

Let us note the following easy and useful application of proposition 3.3.

Corollary 4.2 Let $L = M(\frac{q_1}{p_1}, \dots, \frac{q_n}{p_n}, e)$ be a Montesinos link in the representation giving a minimal crossing diagram $D(L)$ of [LT] (as described at the end of §2.7). Then

$$\max\deg_z F(L) = c(D(L)) - 1 - \min(\#\{\, i : q_i < 0\,\}, \#\{\, i : q_i > 0\,\}) .$$

Proof. The inequality '$\ge$' is easy to see by applying proposition 3.3 to $\max\deg_z \phi_\pm(D(L))$. For the opposite, let us assume modulo mirroring that at least as many q_i are positive as negative. We can also assume that some q_i is negative; otherwise L is alternating, and the equality we claim is known from [Ki]. Permute the tangles $T_i = T(p_i/q_i)$ so that $q_i q_{i+1} < 0$ for as many $i = 1, \dots, n-1$ as possible. (This results in mutations, which do not change F.) Thus one can group the T_i into tangles U_j consisting of the sums of consecutive T_i for exactly one negative q_i and at least one positive q_i. Then apply Theorem 1.3 of [KSt] to the tangle decomposition that breaks $D(L)$ into the sums of U_j. $\square$

We start with a statement, which is more general, and *a priori* not related to Seifert fibered links.

Proposition 4.2 Let K be a fibered positive link with a minimal crossing positive diagram. Then K is not a Montesinos link, except the torus link $T(2,k)$ with $k > 0$ and the $(p,q,-2)$-pretzel link with $p,q > 0$.

Remark 4.4 The existence of a minimal (crossing) positive diagram, i.e. a diagram which realizes the crossing number of the link, and which is simultaneously positive, is a technical assumption. Presumably it can be removed using an extra argument based on [St15]. However, this argument appears involved, and would lead us too far aside. Such a diagram often exists, but not always, as shown in [St3, St12]. The related question about a minimal positive braid diagram of a positive braid link is open. It was known to exist (and hence so does a minimal positive diagram) from

[St14] for closed positive braids of at most 4 strings, and from [St3] for positive braid knots of up to 16 crossings. It also easily follows from [FW, Mu] for closed positive braids with a full twist, which includes torus links.

On the opposite side, Hirasawa and Murasugi [HM] determined the fiberedness for all Montesinos *knots*, but it appears too awkward to derive our statement (even for knots) from their work.

Proof of proposition 4.2. The case that K is an alternating (including rational) link, the statement is easy to see: using [St4] and that K is prime, we have only $T(2,k)$. So assume that K is non-alternating, and in particular its length $n\geq 3$.

Case 1. Assume the minimal crossing diagram D of K of [LT] is not A-adequate. This means that w.l.o.g. (applying a cyclic permutation) $q_1>0$ but $q_i<0$ for $i=2,\dots,n$. Then D is B-adequate, and by direct calculation of $m(D)$ for the B-state of D as for corollary 4.2 (or alternatively from [LT]), we have $\max\deg\phi_-(D)=c(D)-2=c(K)-2$, but by [Th] also $\phi_+(D)=0$.

Now looking at the coefficient of $z^{c(K)-2}$ in the identity $F(\sqrt{-1},z)\equiv 1$, and using [Th], we see that a positive (i.e. A-adequate) minimal diagram D' of K must have $\max\deg\phi_+(D')=c(K)-2=c(D')-2$. So by proposition 3.3, $m(D')=2$, and this easily implies that D' is a diagram of a closed positive 3-braid.

Now were D' not B-adequate, K would not be either by proposition 3.5. Thus D' is adequate, and hence so is K. But by [Th], all minimal crossing diagrams of an adequate link are adequate, in contradiction to our assumption D is not A-adequate.

Case 2. So we can assume that D is A-adequate (i.e. positive). Now we know from [St12, St15] that the second coefficient V_1 of the Jones polynomial of a fibered positive link is 0. We can apply to D the formula for V_1 of an A-adequate diagram given there. Then by a combinatorial observation we can conclude that D must be a $(p,q,-2)$-pretzel diagram, with $p,q>0$. $\square$

Note that the $(p,q,-2)$-pretzel link, with the positive orientation, is the closure of the 3-braid $\sigma_1\sigma_2^p\sigma_1\sigma_2^q$. If $p,q\geq 2$, this braid can always be written as $\delta^2\alpha$ with $\delta=\sigma_1\sigma_2\sigma_1$ and α positive. If some of p or q is 1, it reduces to a 2-braid.

It is for many problems useful to decide whether a link is hyperbolic. By Thurston and Jaco-Shalen-Johannson (JSJ) this amounts to deciding about essential tori and a Seifert fibration in the link complement. For Montesinos links, both were studied, but a proper account, at least on latter, seems lacking. Essential tori in Montesinos link complements were classified by Oertel [Oe] and (under a mild restriction) Boileau–Zimmermann [BZi, proposition 2.1]. Both papers then go on to treat hyperbolicity, but their accounts are incomplete and contain errors and inaccuracies, which convey a confusing impression. This motivated us to record the list of Seifert fibered and hyperbolic Montesinos links with some care. Our proof can be obtained by combining several results from this paper with the work in [BM, St12].

The following case is attributed in [Oe] to Bonahon-Siebenmann. Sadly, their monograph has not appeared in the 30 years since it was announced. (This contributes further to the aforementioned confusion. Update: there seems finally a completed version available, in [BS].) We understand that there is now also a proof of this corollary using Khovanov homology.

Corollary 4.3 The only torus links $T(m,n)$ among the Montesinos links are $T(2,n)$, $T(3,3)$, $T(3,4)$ and $T(3,5)$.

Proof. By applying proposition 4.2, we are left to deal only with $(p,q,-2)$-pretzel links $P(p,q,-2)$ on the Montesinos link side, and torus links $T(m,n)$ for $m\leq 3$. Now $q=1$ gives $T(2,n)$, $p=q=2$ gives $T(3,3)$, $p=q=3$ gives $T(3,4)$ and $p=3$, $q=5$ gives $T(3,5)$. It remains to to show that $T(3,n)$ for $n\geq 6$ do not occur.

Assume $P(-2,p,q)=T(3,n)$ for $n\geq 6$. We can also assume that $p,q\geq 2$. Note that by comparing crossing numbers we have $2n=p+q+2$, so it is enough to consider $p+q$ even. If both p,q are even, we have a 3-component link, so $3\mid n$. In that case one of the component linking numbers of $P(-2,p,q)$ is one, and looking at those for $T(3,n)$, we have $n=3$.

So consider the knot case, and assume p,q are odd and $3\nmid n$. Then $p,q\geq 3$ and $p+q=2n-2\geq 12$. To distinguish $P(-2,p,q)$ and $T(3,n)$ we can use the determinant $\det=|\Delta(-1)|=|V(-1)|$. Using the skein relation (2.4) for Δ it is easy to calculate that

$$\det(P(-2,p,q))=2(p+q)-pq\,, \tag{4.3}$$

up to sign. (This formula is valid also if some of p,q is even.) On the opposite hand-side (again up to sign),

$$\det(T(3,n)) = \begin{cases} 1 & 2 \nmid n \\ 3 & 2 \mid n \end{cases},$$

as can be found easily for example from Jones' formula for $V(T(p,q))$ in [J2]. An easy check shows that for $p+q = 2n-2$, $p,q \geq 3$ and $n \geq 7$, determinants cannot match (even up to sign), except in the cases we already had. □

Proposition 4.3 The Seifert fibered Montesinos links are, up to mirroring

- the torus links $T(m,n)$ for $m = 2$, or $m = 3$ and $3 \leq n \leq 5$, latter realized as $P(-2,2,2)$, $P(-2,3,3)$ and $P(-2,3,5)$,
- the $(4,3,-2)$-pretzel link,
- or the $(-2,2,p)$-pretzel links for $p \geq 3$.

Proof. The Seifert fibered links are given by Burde-Murasugi [BM]. They consist of a torus link $T(m,n)$, by possibly joining one or both cores of the solid tori complementary to the torus on which $T(m,n)$ lies. Such links can always be oriented so as to be closed positive braids. We will choose this orientation, which we have the freedom to do, since the property a link to be Montesinos does not depend on orientation. It is easy to see then that such links satisfy the hypothesis of proposition 4.2. This is true for $T(m,n)$ by choosing a positive m-string braid representation for $m \leq n$, and the addition of a/the core (unknotted) circle(s) preserves minimality of the crossing number by a linking number argument.

So the only Seifert fibered Montesinos links are to be sought among the $(-2,p,q)$-pretzel links. Such links have at most 3 components and braid index at most 3.

Case 1. The torus links (no unknots are added; case (c) in [BM]) follow from corollary 4.3.

Case 2. The Seifert fibered links with 2 circles added (case (a) in [BM]) have, except the Hopf link, at least 3 components. So for braid index reasons the third component must be unknotted, too. Looking at linking numbers we obtain the $(-2,2,p)$-pretzel links for even p. (For $p = 0$ we have the connected sum of two Hopf links, while a Montesinos link is prime, and $P(-2,2,2) = T(3,3)$.)

Case 3. Finally, consider the links with one circle added in case (b) in [BM].

Case 3.1. If (in the notation there) $|\beta| > 1$, there is a non-trivial torus knot component. Because of the braid index it must be $T(2,n)$ for some $n \geq 3$ odd. Thus L has only two components, $T(2,n)$ and an unknot O. The unknotted circle O is either the braid axis of σ_1^n, giving the 3-braid representation $\sigma_2\sigma_1^2\sigma_2\sigma_1^n$, or the core of σ_1^n, giving the 3-braid $(\sigma_1\sigma_2\sigma_1)^n$.

Case 3.1.1. The 3-braid representation $\sigma_2\sigma_1^2\sigma_2\sigma_1^p$ leads to the $(-2,2,p)$-pretzel links for odd p. (If $p = \pm 1$, then $P(-2,2,1) = T(2,4)$.)

Case 3.1.2. So consider $(\sigma_1\sigma_2\sigma_1)^n$, and the knotted component links the unknotted one n times. We may assume that n is odd, otherwise we have case 1. The links for odd n occur in [Tr]. They have determinant 2.

Two components mean that in $P(-2,p,q)$ exactly one of p,q is even. Say w.l.o.g. p is even, q is odd. Then the linking number of the two components is $p/2+1$, and the knotted component is a $(2,q)$-torus knot. From the Burde-Murasugi description it follows that $p = 2n-2$ and $q = n$.

In the case $p = 4$, $q = n = 3$ the pretzel link matches the Burde-Murasugi link. If $n \geq 5$, using (4.3), we see that the determinants are different.

Case 3.2. If $|\beta| = 1$, all components of the Burde-Murasugi links are unknotted and interchangeable. Moreover, they are three, because for fewer components the Burde-Murasugi description gives only $T(2,n)$ (with even n), while more components contradict the braid index 3. Then p,q are both even, and thus $P(-2,p,q)$ has a pair of components of linking number 1. By interchangeability, all linking numbers must be 1. Then $p = q = 2$, a link we already had. (Moreover, Burde-Murasugi's $\alpha = 1$, contradicting the assumption $\alpha > 1$ in the beginning of their case (b).)

With this the proof of proposition 4.3 is complete. □

Using the correct part of corollary 5 of [Oe], we can fix its incorrect part. (A graph manifold is one whose all pieces in the JSJ decomposition are Seifert fibered.)

Corollary 4.4 The non-hyperbolic Montesinos links are, up to mirroring, $T(2,n)$ and the pretzel links $P(2,2,-2,-2)$, $P(1,-3,-3,-3)$, $P(-2,4,4)$, $P(-2,2,p)$ for $p \geq 2$, and $P(-2,3,p)$ for $3 \leq p \leq 6$. All their complements are graph manifolds. □

4.4. More examples on deciding semiadequacy

A couple of examples, ordered by increasing complexity, should demonstrate how the methods of our paper can be used in practice to show that certain knots are not A-semiadequate. The first one is very simple.

Example 4.1 A consequence of corollary 2.1 is that if $a(L) = 1$, L is not A-semiadequate, even if no pair (l,m) with (2.9) exists. One such knot is $!10_{133}$. (A-semiadequacy is not ruled out by the Jones polynomial either.) A similar situation occurs if $a(L) = 0$, but we can show by other means that L is not positive, as for example for $!10_{145}$ (see [Cr, St6]).

Example 4.2 As a consequence of proposition 3.5, we see that $!8_{19}$, $!10_{124}$ and $!10_{139}$ are not A-semiadequate. The example $!9_{43}$ falls outside the scope of the proposition, but one can apply to it the (more general) corollary 3.6.

Example 4.3 The knots 9_{43}, 10_{126}, 10_{130}, 10_{132}, 10_{134}, 10_{143}, 10_{146}, 10_{147}, 10_{148} and 10_{150} can be ruled out from being B-adequate and 10_{156} from being A-adequate using (3.3). The same holds for 11_{485} (figure 3) and 12_{1750} (the knot of [Th, figure 2]). In some situations, like for $!12_{1409}$ and $!12_{1726}$, we have (3.3) and $\min \mathrm{cf} V = \pm 1$, but the sign is opposite to the specified one.

Note that (3.3) is a condition formulated enitrely in terms of the Kauffman polynomial; so any other (potential) knot with the Kauffman polynomial of one of the knots in example 4.3 will not be A/B-adequate either.

Clearly, the genus contradiction of corollary 3.6, as used in example 4.2, will apply only in few cases. For practical analysis, the structure of the set of knot diagrams of given (canonical) genus turns out useful. As an application of this classification of series $\mathcal{D}$ for genus one and two (see theorem 2.1) we give the next examples.

Example 4.4 The knots $!8_{20}$ and $!10_{140}$ have $a = 2$. Thus if they were A-semiadequate, then they would have diagrams with two negative crossings. Since $\min \deg_l P = 0$ for both knots, this diagram must have genus ≤ 2. To show that such a diagram does not exist, use that the knots are slice. A result of [St6] (§6 therein), a consequence of theorem 2.1, is that the only slice knots with 2-almost positive diagrams of genus ≤ 2 are the unknot and $!6_1$. Thus $!8_{20}$ and $!10_{140}$ are not A-semiadequate. Again their Jones polynomial does not rule out the A-semiadequacy. A similar argument applies to $!10_{130}$, with the modification that in [St6] we just used $\sigma = 0$ (see the remark at the end of §6 there).

In the following, most general type of example, the classification in theorem 2.1 is used to full extent, and the idea used in the proofs of §3 comes to practical application in most detail. We formulate the example as a proposition, since the argument is longer, and explains how to proceed in other cases.

Proposition 4.4 The knot $!10_{137}$ is not A-semiadequate.

Proof. We have $a(!10_{137}) = 3$, so an A-semiadequate diagram D of $!10_{137}$ would have 3 negative crossings. Since $\min \deg_l P(!10_{137}) = -2$, D must have genus at most 2, and hence exactly two, since $g(!10_{137}) = 2$. Thus by theorem 2.1, D lies in the series of one of the 24 knots listed therein. Note that, since $!10_{137}$ is prime, D cannot be composite. The factors of an A-semiadequate diagram are A-semiadequate, and the unknot has no non-trivial (reduced) A-semiadequate diagrams.

Consider the diagram $\tilde{D}$, obtained from D by reducing each $\sim$–equivalence class to at most two elements by undoing $\bar{t}'_2$ moves. Then $\tilde{D}$ is obtained from an alternating diagram of one of the above 24 knots by crossing changes and possible flypes. Moreover, $\tilde{D}$ has one or 3 negative crossings.

If $\tilde{D}$ has one negative crossing, then the 3 negative crossings of D lie in a tangle, which is the mirror image of the r.h.s. of (2.17). Thus we have $s_-(D) \geq 2$ Seifert circles of D bounding only negative crossings. Then using (2.13) and that $g(D) = 2$ and $w(D) = c(D) - 6$, we can estimate the slice genus by

$$g_s(!10_{137}) \geq \frac{w(D) - s(D) + 1}{2} + s_-(D) \,=\, \big(g(D) - 3\big) + 2 \,=\, 1\,,$$

and obtain a contradiction to the sliceness of $!10_{137}$ (see [Kw, table p. 276]).

Thus $\tilde{D}$ has 3 negative crossings. Since $\tilde{D}$ cannot have trivial clasps, these 3 negative crossings, and hence those of D, must be either in 3 $\sim$–equivalence classes of a single crossing, or in 2 $\sim$–equivalence classes of 1 and 2 (reverse clasp) elements resp. We obtain this way from the diagrams of the 24 knots up to flypes 386 possible diagrams $\tilde{D}$.

Since D is obtained from $\tilde{D}$ by $\bar{t}'_2$ moves at positive crossings, we have for the signature $0 = \sigma(D) \geq \sigma(\tilde{D})$. Thus we can discard all diagrams $\tilde{D}$ with positive signature. There remain 173 diagrams.

Note that the A-semiadequacy of D could have been spoiled in building $\tilde{D}$, because we may have reduced some $\sim$–equivalence class to a single element. To bring the semiadequacy condition into the game, consider the diagram $\hat{D}$, obtained from $\tilde{D}$ by applying $\bar{t}'_2$ moves on all $\sim$–equivalence classes in $\tilde{D}$ having a single (and) positive crossing.

There is a little difference between $\hat{D}$ and D'' from the proof of theorem 3.1. Both arise form $\tilde{D}$ by $\bar{t}'_2$ moves on $\sim$–equivalence classes of a single positive crossing. However, for D'' we apply $\bar{t}'_2$ moves only on *some*, while for $\hat{D}$ on *all* such classes. Thus $\hat{D}$ can be obtained from D'' by $\bar{t}'_2$ moves, and since D'' is A-semiadequate, $\hat{D}$ must be now, too.

Testing semiadequacy (I tested actually A- *or* B-semiadequacy), we obtain from the above 173 diagrams $\tilde{D}$ only 36 diagrams $\hat{D}$.

One finds by calculation of the Kauffman polynomial that for all such diagrams $\hat{D}$

$$\max\deg\phi_+(\hat{D}) \,=\, \max\big\{\, l \,:\, \min\deg_a[F(\hat{D})]_{z^l} - l \,=\, \min\{\, m - l \,:\, [F(\hat{D})]_{z^l a^m} \neq 0\,\}\,\big\} \,\geq\, c(\hat{D}) - 3\,. \qquad (4.4)$$

We know that a $\bar{t}'_2$ move preserves A-semiadequacy and $c(\,.\,) - \max\deg\phi_+$. Also, D and $\hat{D}$ are stably equivalent under $\bar{t}'_2$ moves (although one may not be convertible into the other directly). It follows that $c(D) - \max\deg\phi_+(D) \leq 3$. Since for $K = !10_{137}$ we have

$$\max\big\{\, l \,:\, \min\deg_a[F(K)]_{z^l} - l \,=\, \min\{\, m - l \,:\, [F(K)]_{z^l a^m} \neq 0\,\}\,\big\} \,=\, 8\,,$$

we obtain that D must have at most 11 crossings. Then it is not hard to check all genus two 10 and 11 crossing 3-almost positive diagrams. We find (e.g. by comparing the Jones polynomial) that $!10_{137}$ is not represented by such a diagram. Thus $!10_{137}$ is not A-semiadequate. $\square$

We could handle by similar arguments 11_{485} (see §4.1), and Thistlethwaite's original problem knot 12_{1750}. But this is now obsolete and superseeded in simplicity by example 4.3. (It was our initial proof before we found the test (3.3).)

It is worth remarking that for all examples given no pair (l,m) with (2.9) exists. Thus we see the increase in efficiency in our tests in comparison to Thistlethwaite's positivity condition on $\phi_\pm$.

4.5. Knots with up to 10 crossings

Using the above explained methods, and as an extension of the specific examples discussed above, the A/B-semiadequacy status of all knots up to 10 crossings could be determined. Note that all such composite knots are alternating, and alternating knots are adequate, so it is enough to focus on non-alternating (prime) knots. It is essential to specify mirroring, and we point out again that we use the convention in Rolfsen's tables [Ro, appendix]. We also shift down by one the index of the last 4 knots, discarding the Perko duplication $10_{162} = !10_{161}$. (So, the last knot of Rolfsen, 10_{166}, is for us 10_{165}.) For some (accidental?) reasons, the 50 not adequate knots in Rolfsen's tables are almost entirely (with only 4 exceptions) mirrored so as to be A- rather than B-adequate. Thus, the outcome is shortly summarized in the below statement.

Proposition 4.5 The following table indicates A/B-semiadequacy status of non-alternating prime knots up to 10 crossings (with Rolfsen's mirroring, and the Perko duplication removed):

adequate	A- and B-adequate but not adequate	B-adequate but not A-adequate	A-adequate but not B-adequate
10_{152} – 10_{154}	10_{161}	10_{155} – 10_{157}, 10_{165}	all others

Proof. The A/B-semiadequacy is directly checked in Rolfsen's diagrams, so we need to rule out the other semiadequacy. Let us mirror the knot so that it is A-semiadequate. The easiest test comes from V. Several knots are done by seeing that the minimal coefficient is not ± 1. One can also use corollary 3.3 (see example 4.3). The exclusion of the other knots is mainly an application of the methods of this paper, so we indicate only specific features that differ from the preceding examples.

First we determine $a(L)$ from the Kauffman polynomial, and then use the left inequality in (3.13) to bound the genus of an A-adequate diagram. It turns out that it always suffices to consider genus 2 or 3, where we have the list of generators. For $!10_{132}$, $!10_{150}$ and 10_{156} Morton's inequality is not sharp, and one should use either corollary 3.3, or the 2-cabled skein polynomial, as explained in [MS], to rule out $g(D) = 4$. For $!10_{132}$, one can also argue using the second Jones polynomial coefficient $V_1 = 0$. By [St12], if A-adequate, it would be positive, which it is clearly not (for example because $\sigma = 0$ or $a > 0$).

There are two further useful simplifications during the search for A-adequate diagrams D. Call two crossings *Seifert equivalent*, if they connect the same pair of Seifert circles. Now combinatorially using [MP] or geometrically using [Ga, Ga2], we see that whenever the diagram D is of minimal genus (i.e. $g(D) = g(K)$), there are no Seifert equivalent crossings in D of opposite sign.

Also, there is an additional condition, which we call *S condition*, resulting from an improvement of Morton's inequalities (3.13). This improvement is given in [MP2, Theorem 8.3] (modulo a correction which is required, as described in [St11]). It implies that when the left (resp. right) inequality (3.13) is exact (i.e. equality) for D, no negative (resp. positive) crossing in D forms a trivial (one-crossing) Seifert equivalence class. For negative crossings this implies that there are no $\sim$-equivalence classes of > 2 negative crossings in D. In particular D is obtained from D'' by positive $\bar{t}_2'$ twists only. One can establish the S condition also for some diagrams with inexact (3.13) by arguing with the 2-cabled polynomial.

When the S condition does not apply, and $a(L) > 2$, one must consider also the possibility of negative $\bar{t}_2'$ twists from D'' to D. Often, however, the use of σ, and the s and τ invariants can rule this case out, or restrict it. If $a(L)$ is large, then we switched positive crossings one by one and checked the signature at intermediate stages, to discard crossing switches that do not lower σ sufficiently.

The work took, using a computer and proper programming, a few minutes per knot. (Some knots can be handled simultaneously.) Note that we have also a crossing bound from proposition 3.1. But using only this bound and $a(L)$, i.e. the information that follows easily from [Th], would blow up the effort per knot from several minutes likely to several days. □

As a consequence, using corollary 2.1, we can determine for what k the knots up to 10 crossings are k-almost positive.

Corollary 4.5 Let K be a knot up to 10 crossings, and a minimal crossing diagram of K have k_+ positive crossings, and k_- negative crossings. Then K is k_--almost positive, and $!K$ is k_+-almost positive. The exception is $K = 10_{161}$, which is 9-almost positive, and $!10_{161}$, which is positive. □

5. Conclusions and problems

Since all steps in the proof of the inequality (3.4) are constructive, this allows – in theory – to decide whether a link is semiadequate or not: we can write down an explicit list of (semiadequate) diagrams among which it would have to be if it is semiadequate. As said, the crossing number inequalities are intended primarily in a theoretical spirit. In practice, by using the arguments in their proof, the list of potential diagrams becomes much smaller than simply the list of all diagrams up to the crossing number stated in the inequality. Nonetheless, this list may still be quite long even for some simple examples. In particular, the number of genus generators will quickly grow beyond our compiling capacity for genus > 4.

On the other hand, the condition of semiadequacy is much weaker than that of adequacy, and thus our estimates are much more generally applicable. A good illustration, roughly mentioned before, is that of the prime knots of 15 crossings given in [HT]. As opposed to the 85,263 alternating knots, among the diagrams, in which the 168,030 non-alternating knots are tabulated there, 148,720 are semiadequate, while only 15,669 are adequate. Thus, in practice, the concept of adequacy seems – unlike semiadequacy – to be only a slight generalization of alternation.

Since we noted that for an A-semiadequate link A-semiadequate diagrams are exactly those with the minimal number of negative crossings, one may ask for a possible extension of our main result:

Question 5.1 Does any oriented link have only finitely many reduced diagrams with the minimal number of negative (or positive) crossings?

We proved above that this is true if this minimal number is 0, what we already knew for knots from [St2]. For knots we settled in [St8] also the case that this minimal number is 1. The methods of these papers can, however, unlikely be pushed forward in the here desired direction.

Another problem is contained implicitly in the consideration of sliceness. We observed that the (slice) Bennequin inequality cannot distinguish between achiral and slice knots since it fails to react to (change of) orientation. One can ask how significant this failure can become.

Question 5.2 Are there achiral knots of arbitrarily large 4-genus?

A final question is whether the visual primeness results of Menasco [Me] (for alternating links) and Cromwell-Ozawa [Cr2, O] (for positive links) can be extended.

Question 5.3 Is a prime semiadequate diagram always representing a prime link?

This is true for knot diagrams up to 16 crossings. It is a partial case of problem (2) in §4 of [O2].

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References

[Ad] C. C. Adams, *Das Knotenbuch*, Spektrum Akademischer Verlag, Berlin, 1995 (*The knot book*, W. H. Freeman & Co., New York, 1994).

[BN] D. Bar-Natan, *On Khovanov's Categorification of the Jones Polynomial*, Algebraic and Geometric Topology **2** (2002), 337–370.

[Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Soc. Math. de France, Astérisque **107-108** (1983), 87–161.

[BZi] M. Boileau and B. Zimmermann, *Symmetries of nonelliptic Montesinos links*, Math. Ann. **277(3)** (1987), 563–584.

[BS] F. Bonahon and LĊ. Siebenmann, *New Geometric Splittings of Classical Knots and the Classification and Symmetries of Arborescent Knots*, monograph available on `http://www-rcf.usc.edu/~fbonahon/Research/Preprints/Preprints.html`

[Br] M. Brittenham, *Bounding canonical genus bounds volume*, preprint (1998), available at `http://www.math.unl.edu/~mbritten/personal/pprdescr.html`.

[BM] G. Burde and K. Murasugi, *Links and Seifert fiber spaces*, Duke Math. J. **37** (1970), 89–93.

[BZ] ——— " ——— and H. Zieschang, *Knots*, de Gruyter, Berlin, 1986.

[CG] T. D. Cochran and Robert E. Gompf, *Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and Property P*, Topology **27(4)** (1988), 495–512.

[Co] J. H. Conway, *On enumeration of knots and links*, in "Computational Problems in abstract algebra" (J. Leech, ed.), 329-358, Pergamon Press, 1969.

[Cr] P. R. Cromwell, *Homogeneous links*, J. London Math. Soc. (series 2) **39** (1989), 535–552.

[Cr2] ——— " ———, *Positive braids are visually prime*, Proc. London Math. Soc. **67** (1993), 384–424.

[Cr3] ——— " ———, *Knots and Links*, Cambridge University Press, 2004.

[DL] O. Dasbach and X.-S. Lin, *A volume-ish theorem for the Jones polynomial of alternating knots*, `math.GT/0403448`, Pacific J. Math. **231(2)** (2007), 279–291.

[Fi] T. Fiedler, *On the degree of the Jones polynomial*, Topology **30** (1991), 1–8.

[FW] J. Franks and R. F. Williams, *Braids and the Jones-Conway polynomial*, Trans. Amer. Math. Soc. **303** (1987), 97–108.

[FH] M. H. Freedman and Z. H. He, *Divergence-free fields: energy and asymptotic crossing number*, Ann. of Math. **134(1)** (1991), 189–229.

[F&] P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–246.

[FT] D. Fuchs and S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36(5)** (1997), 1025–1053.

[Ga] D. Gabai, *The Murasugi sum is a natural geometric operation*, Low-dimensional topology (San Francisco, Calif., 1981), Contemp. Math. **20**, 131–143, Amer. Math. Soc., Providence, RI, 1983.

[Ga2] ——— " ———, *The Murasugi sum is a natural geometric operation II*, Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), 93–100, Contemp. Math. **44**, Amer. Math. Soc., Providence, RI, 1985.

[HM] M. Hirasawa and K. Murasugi, *Genera and fibredness of Montesinos knots*, Pacific J. Math. **225(1)** (2006), 53–83.

[HT] J. Hoste and M. Thistlethwaite, *KnotScape*, a knot polynomial calculation program, available at `http://www.math.utk.edu/~morwen`.

[J] V. F. R. Jones, *A polynomial invariant of knots and links via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[J2] ——— " ———, *Hecke algebra representations of of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.

[Ka] L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.

[Ka2] ——— " ———, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[K] T. Kawamura, *On unknotting numbers and four-dimensional clasp numbers of links*, Proc. Amer. Math. Soc. **130(1)** (2002), 243–252.

[K2] ——— " ———, *The Rasmussen invariants and the sharper slice-Bennequin inequality on knots*, Topology **46(1)** (2007), 29–38.

[Kw] A. Kawauchi, *A survey of Knot Theory*, Birkhäuser, Basel-Boston-Berlin, 1996.

[Ko] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101(3)** (2000), 359–426.

[Ki] M. E. Kidwell, *On the degree of the Brandt-Lickorish-Millett-Ho polynomial of a link*, Proc. Amer. Math. Soc. **100(4)** (1987), 755–762.

[KSt] ——— " ——— and T. Stanford, *On the z-degree of the Kauffman polynomial of a tangle decomposition*, "Knots, braids, and mapping class groups", papers dedicated to Joan S. Birman (New York, 1998), AMS/IP Stud. Adv. Math. **24**, Amer. Math. Soc., Providence, RI, 2001, 85–93. See also `math/9903055`.

[Kr] R. Kirby (ed.), *Problems of low-dimensional topology*, book available on `http://math.berkeley.edu/~kirby`.

[La] M. Lackenby, *The volume of hyperbolic alternating link complements*, with an appendix by I. Agol and D. Thurston, Proc. London Math. Soc. **88(1)** (2004), 204–224.

[L] W. B. R. Lickorish, *Polynomials for links*, Bull. London Math. Soc. **20** (1988), 558–588.

[LM] ——— " ——— and K. C. Millett, *A polynomial invariant for oriented links*, Topology **26 (1)** (1987), 107–141.

[LM2] ——— " ——— and ——— " ———, *The reversing result for the Jones polynomial*, Pacific J. Math. **124(1)** (1986), 173–176.

[LT] ——— " ——— and M. B. Thistlethwaite, *Some links with non-trivial polynomials and their crossing numbers*, Comment. Math. Helv. **63** (1988), 527–539.

[Lv] C. Livingston, *Computations of the Ozsvath-Szabo knot concordance invariant*, `math.GT/0311036`, Geom. Topol. **8** (2004), 735–742.

[Me] W. W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23 (1)** (1986), 37–44.

[MT] ——— ” ——— and M. B. Thistlethwaite, *The Tait flyping conjecture*, Bull. Amer. Math. Soc. **25 (2)** (1991), 403–412.

[Mo] H. R. Morton, *Seifert circles and knot polynomials*, Proc. Camb. Phil. Soc. **99** (1986), 107–109.

[Mo2] ——— ” ——— , *The Jones polynomial for unoriented links*, Quart. J. Math. Oxford Ser. 2 **37(145)** (1986), 55–60.

[MS] ——— ” ——— and H. Short, *The 2-variable polynomial of cable knots*, Math. Proc. Camb. Philos. Soc. **101** (1987), 267–278.

[Mu] K. Murasugi, *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326 (1)** (1991), 237–260.

[Mu2] ——— ” ———, *Jones polynomial and classical conjectures in knot theory*, Topology **26** (1987), 187–194.

[Mu3] ——— ” ———, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.

[Mu4] ——— ” ———, *On invariants of graphs with applications to knot theory*, Trans. Amer. Math. Soc. **314** (1989), 1–49.

[MP] ——— ” ——— and J. Przytycki, *The skein polynomial of a planar star product of two links*, Math. Proc. Cambridge Philos. Soc. **106(2)** (1989), 273–276.

[MP2] ——— ” ——— and ——— ” ———, *An index of a graph with applications to knot theory*, Mem. Amer. Math. Soc. **106 (508)** (1993).

[N] T. Nakamura, *Positive alternating links are positively alternating*, J. Knot Theory Ramifications **9(1)** (2000), 107–112.

[N2] ——— ” ———, *Notes on the braid index of closed positive braids*, Topology Appl. **135(1-3)** (2004), 13–31.

[Oe] U. Oertel, *Closed incompressible surfaces in complements of star links*, Pacific J. Math. **111(1)** (1984), 209–230.

[O] M. Ozawa, *Closed incompressible surfaces in complements of positive knots*, Comment. Math. Helv. **77** (2002), 235–243.

[O2] ——— ” ———, *Essential state surfaces for knots and links*, preprint `math.GT/0609166v5`.

[OS] P. Ozsváth and Z. Szabó, *Knot Floer homology and the four-ball genus*, Geometry & Topology **7** (2003), 615–639.

[QW] C. V. Quach Hongler and C. Weber, *On the topological invariance of Murasugi special components of an alternating link*, Math. Proc. Cambridge Philos. Soc. **137(1)** (2004), 95–108.

[Ra] J. Rasmussen, *Khovanov homology and the slice genus*, preprint `math.GT/0402131`.

[Ro] D. Rolfsen, *Knots and links*, Publish or Parish, 1976.

[Ru] L. Rudolph, *Positive links are strongly quasipositive*, Geometry and Topology Monographs **2** (1999), Proceedings of the Kirbyfest, 555–562. See also `http://www.maths.warwick.ac.uk/gt/GTMon2/paper25.abs.html`.

[Ru2] ——— ” ———, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. **29** (1993), 51–59.

[St] A. Stoimenow, *Knots of genus one*, Proc. Amer. Math. Soc. **129(7)** (2001), 2141–2156.

[St2] ——— ” ———, *Positive knots, closed braids and the Jones polynomial*, `math/9805078`, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **2(2)** (2003), 237–285.

[St3] ——— ” ———, *On the crossing number of positive knots and braids and braid index criteria of Jones and Morton-Williams-Franks*, Trans. Amer. Math. Soc. **354(10)** (2002), 3927–3954.

[St4] ——— ” ———, *On some restrictions to the values of the Jones polynomial*, Indiana Univ. Math. J. **54(2)** (2005), 557–574.

[St5] ——— ” ———, *Bennequin's inequality and the positivity of the signature*, Trans. Amer. Math. Soc. **360(10)** (2008), 5173–5199.

[St6] ——— ” ———, *Knots of genus two*, `math.GT/0303012`, Fund. Math. **200(1)** (2008), 1–67.

[St7] ——— ” ———, *On the number of links and link polynomials*, Quart. J. Math. Oxford **55(1)** (2004), 87–98.

[St8] ——— ” ———, *Gauß sums on almost positive knots*, `math.GT/9803073`, Compositio Mathematica **140(1)** (2004), 228–254.

[St9] ——— ” ———, *The signature of 2-almost positive knots*, J. Of Knot Theory and Its Ram. **9(6)** (2000), 813–845.

[St10] ——— ” ———, *Genus generators and the positivity of the signature*, Alg. Geom. Top. **6** (2006), 2351–2393.

[St11] ——— ” ———, *Diagram genus, generators and applications*, preprint.

[St12] ——— ” ———, *Coefficients and non-triviality of the Jones polynomial*, `math.GT/0606255`. A part is accepted by J. Reine Angew. Math. (Crelle's J.)

[St13] ——— ” ———, *On dual triangulations of surfaces*, to appear.

[St14] ——— " ———, *Properties of closed 3-braids*, preprint `math/0606435`.

[St15] ——— " ———, *On polynomials and surfaces of variously positive links*, `math.GT/0202226`, Jour. Europ. Math. Soc. **7(4)** (2005), 477–509.

[STV] ——— " ———, V. Tchernov and A. Vdovina, *The canonical genus of a classical and virtual knot*, Geometriae Dedicata **95(1)** (2002), 215–225.

[SV] ——— " ——— and A. Vdovina, *Counting alternating knots by genus*, Math. Ann. **333** (2005), 1–27.

[Ta] K. Taniyama, *A partial order of knots*, Tokyo J. Math. **12(1)** (1989), 205–229.

[Th] M. B. Thistlethwaite, *On the Kauffman polynomial of an adequate link*, Invent. Math. **93(2)** (1988), 285–296.

[Th2] ——— " ———, *A spanning tree expansion for the Jones polynomial*, Topology **26** (1987), 297–309.

[Th3] ——— " ———, *On the structure and scarcity of alternating links and tangles*, J. Knot Theory Ramifications **7(7)** (1998), 981–1004.

[Tr] P. Traczyk, 3-*braids with proportional Jones polynomials*, Kobe J. Math. **15(2)** (1998), 187–190.

[Yo] Y. Yokota, *Polynomial invariants of positive links*, Topology **31(4)** (1992), 805–811.

[Zu] L. Zulli, *The rank of the trip matrix of a positive knot diagram*, J. Knot Theory Ramif. **6 (2)** (1997), 299–301.