REALIZING ALEXANDER POLYNOMIALS BY HYPERBOLIC LINKS

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Abstract. We realize a given (monic) Alexander polynomial by a (fibered) hyperbolic arborescent knot and link of any number of components, and by infinitely many such links of at least 4 components. As a consequence, a Mahler measure minimizing polynomial, if it exists, is realized as the Alexander polynomial of a fibered hyperbolic link of at least 2 components. For given polynomial, we give also an upper bound on the minimal hyperbolic volume of knots/links, and contrarily, construct knots of arbitrarily large volume, which are arborescent, or have given free genus at least 2.

Keywords: Alexander polynomial, genus, free genus, slice genus, fibered link, hyperbolic volume, Mahler measure.
AMS subject classification: 57M25 (primary), 57M12, 57M50 (secondary).

1. Introduction

The Alexander polynomial Δ [Al] has been the object of diverse investigation in low-dimensional topology due to its relations and importance for a variety of topics ranging from representation theory [J] to the geometry of 3- and 4-manifolds (e.g. [FS]). Its topological understanding has led for some time to the insight which values this invariant takes on knots [Le]. A similar result was proved briefly later for fibered knots [Bu].

More recently, the Alexander polynomial has been studied with regard to hyperbolic knots (e.g. Kalfagianni [Kf], Silver and Whitten [SWh]). The motivation for this comes at part from the desire to exhibit connections between the hyperbolic volume $\mathrm{vol}(L)$ of (the complement in S^3 of) a link L and the polynomial invariants of L, and to understand what geometric complexity is measured by these link polynomials. (The s.c. Volume conjecture is now a pre-eminent such problem.)

In this paper, we will offer some new realization constructions for the Alexander polynomial. Of central focus will be thereby the hyperbolicity of knots. On the one hand, we will (easily) have an upper estimate on what is the smallest volume of a hyperbolic knot with given Alexander polynomial. On the opposite hand, we will also address the question how to make our constructions yield knots of large volume.

For the most part (though not throughout) our knots will enjoy another well-known property, little studied in this context, namely being arborescent (or algebraic in Conway's sense [Co]). Furthermore, we are interested (following the work of Nakamura [Na]) in obtaining canonical or free Seifert surfaces for the knots, which are of (minimal) genus equalling the degree of Δ, and which in the case of monic polynomials Δ (i.e. such with leading coefficient ±1) are fiber surfaces. We now briefly introduce our constructions.

First we realize each polynomial by a certain arborescent knot (see Theorem 3.1). This yields an upper bound on the minimal volume of a knot with given polynomial, which depends only (linearly) on the degree of the polynomial. Apart from hyperbolicity, we obtain the mentioned minimal genus and fiber properties for canonical surfaces of these knots.

This is a preprint. I would be grateful for any comments and corrections.
*Financial support by the BK21 Project is acknowledged.

Later we show how to augment hyperbolic volume. We have two different constructions. The first one (Theorem 8.1) simultaneously augments the slice genus. The second construction (Theorem 8.2) extends a result of Brittenham [Br2]. It yields knots of arbitrarily large volume with given free genus at least 2, with the additional feature that we can again specify the Alexander polynomial. (We set forth this approach in [St7], bringing Vassiliev invariants into relation.)

A main theme will be to consider also various questions for links. The realization result is extended first to links of two (Theorem 4.1), and then of more components (Theorem 5.1). The hyperbolicity proof is, unlike for knots, more involved, and requires the main effort. It uses heavily the results of Oertel [Oe] and Wu [Wu]. A motivation was that for fibered links of given polynomial not even primeness issues seem to have ever been settled (and for more than 2 components, not even candidates for prime links have been available). Another motivation, and now application (Corollary 4.1), is to confirm a claim of Silver and Williams. We prove that a polynomial of minimal (positive) Mahler measure, if it exists, is realized as the Alexander polynomial of a fibered hyperbolic 2-component link (see Remark 4.1).

Later we succeed in partially extending the construction to obtain infinite families of links. An analogue of the infinite realizability result of Morton [Mo] for fibered knots is shown for (arborescent) links of ≥ 4 components (Proposition 7.1), even for canonical fiber surfaces (for which it is known not to hold in some other cases [St4]). Table 1 at the end of the paper summarizes these (in the context of some previous related) results.

We will use several methods, including Seifert matrices and skein relations (for realizing Alexander polynomials), tangle surgeries and Stallings twists (for generating infinite families of links), some cut-and-paste arguments (for showing hyperbolicity), and results of Gabai [Ga2, Ga3] based on his sutured manifold theory [Ga] (to prove fibering).

We mentioned the works in a similar, but somewhat different, spirit made recently by Kalfagianni [Kf], Nakamura [Na], and Silver and Whitten [SWh]. Most properties studied there can be obtained from our work, too (except for the knot group homomorphism in [SWh]; see remarks 8.4, 8.1 and 3.3). If one is mainly interested in Alexander polynomials and large volume (but not in genera, fibering and arborescency), there are generalizations in a further direction [Fr], using Kawauchi's imitation theory.

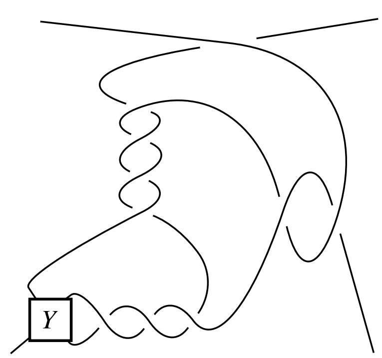
On another related (but not further pursued here) venue, I proved a conjecture of Dunfield [Df]. It relates the determinant, which can be expressed by the value $\Delta(-1)$, and volume of *alternating* links. In particular, the determinant has an exponential lower bound in terms of the volume. This way we have a different relation between $\Delta(L)$ and $\mathrm{vol}(L)$ when L is alternating.

2. Some preliminaries

2.1. Conway notation and Montesinos links

Definition 2.1 A *tangle* Y is a set of two arcs and possible circles (*closed components*) properly embedded in a ball $B(Y)$. Tangles are considered up to homeomorphisms of $B(Y)$ that keep fixed its boundary $\partial B(Y)$. Two tangles are *equivalent* (in the sense of [Wu]), if they are transformed by a homeomorphism of their ball that preserves (but does not necessarily fix) the 4 punctures of the boundary.

The effect of such a homeomorphism on the tangle is a *rational transformation*



(1)

Figure 1 shows the elementary tangles, tangle operations and notation, mainly leaning on Conway [Co]. A *clasp* is one of the elementary tangles ± 2 and its rotations. For two tangles Y_1 and Y_2 we write $Y_1 + Y_2$ for the *tangle sum*. This is a tangle obtained by identifying the NE end of Y_1 with the NW end of Y_2, and the SE end of Y_1 with the SW end of Y_2. The *closure* of a tangle Y is a link obtained by identifying the NE end of Y with its NW end, and the SE end with the SW end. The closure of $Y_1 + Y_2$ is called *join* $Y_1 \cup Y_2$ of Y_1 and Y_2.

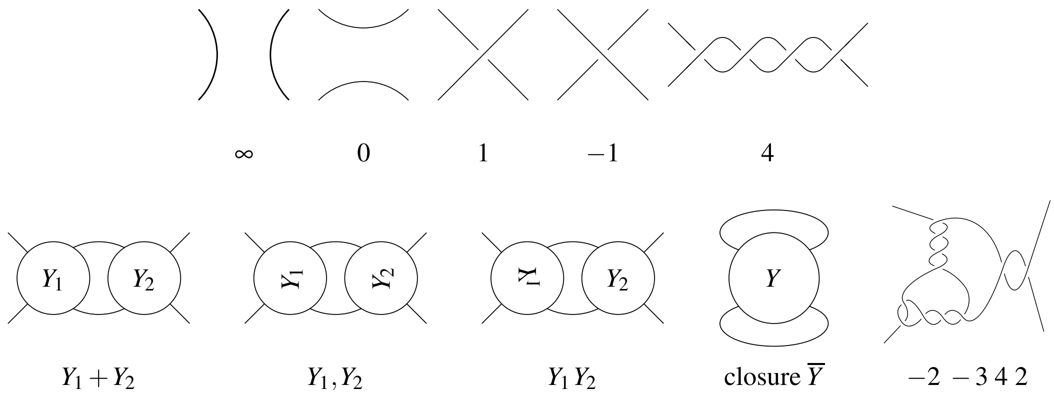


Figure 1: Conway's primitive tangles and operations with them.

Definition 2.2 A link diagram is *arborescent*, if it can be obtained from the tangles in figure 1 by the operations shown therein. An alternative description is as follows. Take a one crossing (unknot) diagram. Repeat replacing some (single) crossing by a clasp (of any orientation or sign). The diagrams obtained this way are exactly the arborescent diagrams. In Conway's [Co] terminology, these are diagrams with Conway polyhedron 1^*. A link is said to be arborescent if it admits an arborescent diagram.

A graph G is *series parallel*, if it can be obtained from •——• by repeated edge bisections and doublings. Such graphs correspond to arborescent link diagrams via the checkerboard graph construction (see [Ka, Mi, Th] for example).

Definition 2.3 A *rational* tangle diagram is the one that can be obtained from the primitive Conway tangle diagrams by iterated left-associative product in the way displayed in figure 1. (A simple but typical example of is shown in the figure.)

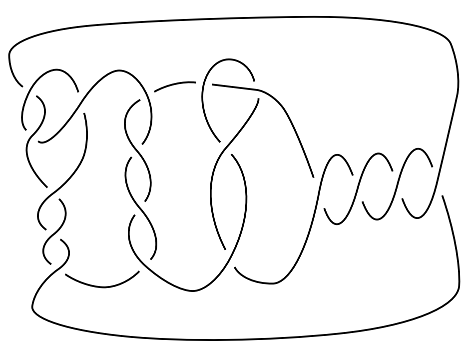


Figure 2: The Montesinos knot with Conway notation $(213, -4, 22, 40)$.

Let the *continued* (or *iterated*) *fraction* $[[s_1, \ldots, s_r]]$ for integers s_i be defined inductively by $[[s]] = s$ and

$$[[s_1, \ldots, s_{r-1}, s_r]] = s_r + \frac{1}{[[s_1, \ldots, s_{r-1}]]}\,.$$

The rational tangle $T(p/q)$ is the one with Conway notation $c_1\ c_2\ \dots c_n$, when the c_i are chosen so that

$$[[c_1, c_2, c_3, \dots, c_n]] = \frac{p}{q}\,. \tag{2}$$

One can assume without loss of generality that $(p,q)=1$, and $0<q<|p|$. A *rational (or 2-bridge) link* $S(p,q)$ is the closure of $T(p/q)$.

Montesinos links (see e.g. [BZ]) are generalizations of pretzel and rational links and special types of arborescent links. They are denoted in the form $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$, where e,p_i,q_i are integers, $(p_i,q_i)=1$ and $0<|q_i|<p_i$. Sometimes e is called the *integer part*, and the $\frac{q_i}{p_i}$ are called *fractional parts*. They both together form the *entries*. If $e=0$, it is omitted in the notation.

If all $|q_i|=1$, then the Montesinos link $M(\pm\frac{1}{p_1},\dots,\pm\frac{1}{p_n},e)$ is called a *pretzel link*, of type $(\pm p_1,\dots,\pm p_n,\varepsilon,\dots,\varepsilon)$, where $\varepsilon=\operatorname{sgn}(e)$, and there are $|e|$ copies of it.

To visualize the Montesinos link from a notation, let p_i/q_i be continued fractions of rational tangles $c_{1,i}\dots c_{n_i,i}$ with $[[c_{1,i},c_{2,i},c_{3,i},\dots,c_{l_i,i}]]=\frac{p_i}{q_i}$. Then $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$ is the link that corresponds to the Conway notation

$$(c_{1,1}\dots c_{l_1,1}),(c_{1,2}\dots c_{l_2,2}),\dots,(c_{1,n}\dots c_{l_n,n}),e\,0\,. \tag{3}$$

The defining convention is that all $q_i>0$ and if $p_i<0$, then the tangle is composed so as to give a non-alternating sum with a tangle with $p_{i\pm1}>0$. This defines the diagram up to mirroring. We sometimes denote the *Montesinos tangle* with Conway notation (3) in the same way as its closure link.

An easy exercise shows that if $q_i>0$ resp. $q_i<0$, then

$$M(\dots,q_i/p_i,\dots,e) = M(\dots,(q_i\mp p_i)/p_i,\dots,e\pm1)\,, \tag{4}$$

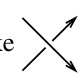
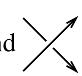
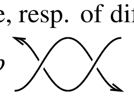
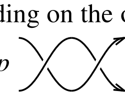
i.e. both forms represent the same link (up to mirroring).

Note that our notation *may differ* from other authors' by the sign of e and/or multiplicative inversion of the fractional parts. For example $M(\frac{q_1}{p_1},\dots,\frac{q_n}{p_n},e)$ is denoted as $\mathfrak{m}(e;\frac{p_1}{q_1},\dots,\frac{p_n}{q_n})$ in [BZ, definition 12.28] and as $M(-e;(p_1,q_1),\dots,(p_n,q_n))$ and the tables of [Kw].

Our convention chosen here appears more natural – the identity (4) preserves the sum of all entries, and an integer entry can be formally regarded as a fractional part. Theorem 12.29 in [BZ] asserts that the entry sum, together with the vector of the fractional parts, modulo $\mathbb{Z}$ and up to cyclic permutations and reversal, determine the isotopy class of a Montesinos link L. So the number n of fractional parts is an invariant of L; we call it the *length* of L.

If the length $n<3$, an easy observation shows that the Montesinos link is in fact a rational link. Then we could write rational links as Montesinos links of length 1. For example, $M(1)=M(\infty)$ is the unknot, and $M(0)$ is the 2-component unlink, while $M(2/5)=M(5/2)$ is the figure-8 knot. This simplification is not right, though, for Montesinos tangles with $n=2$. Thus we keep (and will need) the length-2 notation for tangles.

2.2. Diagrams and geometric invariants

Definition 2.4 A crossing in an oriented diagram looking like  is called *positive*, and  is a *negative* crossing. This dichotomy is called also *(skein) sign*. In an oriented diagram a clasp is called *positive*, *negative* or *trivial*, if both crossings are positive/negative, resp. of different sign. Depending on the orientation of the involved strands we distinguish between a *reverse clasp*  and a *parallel clasp* . So a clasp is reverse if it contains a full Seifert circle, and parallel otherwise. (We refer to [Li, Ro] for the notion of a Seifert circle.)

For the later explanations, we must introduce the notion of twist equivalence of crossings. The version of this relation we present here follows its variants studied in [St2, St3].

Definition 2.5 We say two crossings p and q of a diagram D to be *$\sim$-equivalent*, resp. *$\stackrel{\sim}{*}$-equivalent*, if up to flypes they form a reverse resp. parallel clasp. We remarked in [St2] that $\sim$ and $\stackrel{\sim}{*}$ are equivalence relations. We write $t_\sim(D)$ for the number of or $\sim$-equivalence classes of crossings in D. Set $t_\sim(K)$, the *reverse twist number* of a knot or link K, to be the minimum of $t_\sim(D)$ taken over all diagrams D of K.

In [St2] we noticed also that if $p \sim q$ and $p \stackrel{\sim}{*} r$, then $p = q$ or $p = r$. (There is the, not further troubling however, exception that D is the 2-crossing Hopf link diagram, or has such a diagram occurring as a connected sum factor.) So the relation $(p \sim q \vee p \stackrel{\sim}{*} q)$ is also an equivalence relation. We call this relation *twist equivalence*. Thus two crossings are twist equivalent if up to flypes they form a clasp. We will often call twist equivalence classes of crossings in a diagram simply *twists*. (Some twists may consist of a single crossing.) Let $t(D)$ denote the *twist number* of a diagram D, which is the number of its twists. The twist number $t(K)$ of a knot or link K is the minimal twist number of any diagram D of K. Clearly $t(D) \le t_\sim(D)$ and $t(K) \le t_\sim(K)$.

With this terminology, we can state the following inequality we need:

Theorem 2.1 ([La]) For a non-trivial diagram D of a link L, we have $10V_0\big(t(D)-1\big) \ge \mathrm{vol}(L)$, where $V_0 = \mathrm{vol}(4_1)/2 \approx 1.01494$ is the volume of the ideal tetrahedron.

Such an inequality, with the constant 10 replaced by 16, follows from well-known facts about hyperbolic volume (see for example the explanation of [Br]). Lackenby [La] (whose main merit is a lower volume bound for alternating links) repeated this observation, and Agol-Thurston found, in the appendix to Lackenby's paper, the optimal constant 10, which is used below for a better estimate.

Remark 2.1 Our notion of twist equivalence is slightly more relaxed than what was called this way in [La], the difference being that there flypes were not allowed. We call Lackenby's equivalence here *strong twist equivalence*. However, it was repeatedly observed that by flypes all twist equivalent crossings can be made strongly twist equivalent, which Lackenby formulated as the existence of *twist reduced* diagrams. Thus, assuming that the diagram is twist reduced, we can work with twist equivalence in our sense as with twist equivalence in Lackenby's sense (or strong twist equivalence in our sense).

A diagram is *special* if no Seifert circle contains other Seifert circles in both interor and exterior.

Definition 2.6 A *Seifert surface* S for an oriented link L is a compact oriented surface bounding L. A Seifert surface is *free* if its complement is a handlebody. It is *canonical*, if it is obtained by Seifert's algorithm from some diagram of L. A *slice surface* is a surface properly embedded in B^4 whose boundary is $L \subset S^3$. We denote by $g(L)$, $g_c(L)$, $g_f(L)$ and $g_s(L)$ the Seifert, canonical, free and smooth slice genus of L. These are the minimal genera of a (canonical/free) Seifert or slice surface of L, resp. For a link L we write $\chi(L)$, $\chi_c(L)$ and $\chi_s(L)$ for the analogous Euler characteristics (we will not need χ_f).

Seifert's algorithm is explained, for example, in [Ro]. We will use also some of the detailed discussion given to it in [St2, St3].

A canonical Seifert surface is free, and any Seifert surface is a slice surface. Thus $g_s(K) \le g(K) \le g_f(K) \le g_c(K)$ for any knot K. By $u(K)$ we denote the *unknotting number* of K. Then it is known that $g_s(K) \le u(K)$.

For a link L, let $n(L)$ be the *number of components* of L. Then $\chi_{[s/c]}(L) = 2 - n(L) - 2g_{[s/c]}(L)$.

2.3. The Alexander-Conway polynomial

Definition 2.7 Below it will be often convenient to work with the *Conway polynomial* $\nabla(z)$. It is given by the value 1 on the unknot and the *skein relation*

$$\nabla(D_+) - \nabla(D_-) = z\nabla(D_0)\,. \tag{5}$$

Here $D_\pm$ are diagrams differing only at one crossing, which is positive/negative, and D_0 is obtained by smoothing out this crossing. The Conway polynomial is equivalent to the (1-variable[1]) Alexander polynomial Δ by the change of variable:

$$\nabla(t^{1/2}-t^{-1/2})=\Delta(t)\,. \tag{6}$$

For that reason we will feel free to exchange one polynomial for the other whenever we deem it convenient. For knots $\nabla\in 1+z^2\mathbb{Z}[z^2]$ and for n-component links (with $n>1$) we have $\nabla\in z^{n-1}\mathbb{Z}[z^2]$. We call such ∇ and the corresponding Δ *admissible* polynomials. Each admissible polynomial is indeed realized by some knot or link.

There is another description for Δ. Given a Seifert surface S of genus $n=g(S)$ for a knot K, one associates to it a *Seifert matrix* V (a $2n\times 2n$ matrix of integer coefficients), and we have

$$\Delta(t)=t^{-n}\det(V-tV^T)\,,$$

where V^T is the transposed of V. This is described in [Ro], for example.

A direct understanding of the relation between the skein-theoretic and Seifert-matrix-related properties of Δ is still a major mystery in knot theory. Solving it may shed light on a topological meaning of the newer polynomials. To the contrary, the long-term lack of such a meaning justifies the pessimism in expecting the desired relation. Nonetheless, both descriptions of Δ offer two independent ways of keeping control on it, and we will successfully combine them in some of the below constructions.

We remark also that ∇ (and Δ) is symmetric resp. antisymmetric w.r.t. taking the mirror image, depending on the odd resp. even parity of the number of components. This means in particular that amphicheiral links of an even number of components have vanishing polynomial. (Here amphicheirality means that an isotopy to the mirror image is to preserve or reverse the orientation of *all* components *simultaneously*, while it is allowed components to be permuted.)

Definition 2.8 Let $[X]_{t^a}=[X]_a$ be the coefficient of t^a in a polynomial $X\in\mathbb{Z}[t^{\pm1}]$. For $X\neq 0$, let $C_X=\{\,a\in\mathbb{Z}:[X]_a\neq 0\,\}$ and

$$\min\deg X=\min\,C_X\,,\quad \max\deg X=\max\,C_X\,,\qquad\text{and}\qquad \operatorname{span}X=\max\deg X-\min\deg X$$

be the *minimal* and *maximal degree* and *span* (or breadth) of X, respectively. The *leading coefficient* $[X]_*$ of X is defined to be $[X]_{\max\deg X}$. If this coefficient is ±1, we call X *monic*.

A link in S^3 is *fibered* if its complement is a surface bundle over S^1. By a classical theorem of Neuwirth-Stallings, the fiber is then a minimal genus Seifert surface, and such a Seifert surface is unique. The operations *Hopf (de)plumbing* and *Stallings twist* are described, for example, in Harer [Ha]. (A Stallings twist is a ±1 surgery along an unknot in the complement of the fiber surface, which can be isotoped into the fiber.) Harer showed that every fiber surface in S^3 can be constructed from a disk by a sequence of these operations. Besides, there is Gabai's geometric work to detect (non-)fiberedness [Ga4]. We call a fibered link L *canonically fibered* if its fiber surface can be obtained by Seifert's algorithm on some diagram of L.

It is known that $\max\deg\Delta(K)\leq g(K)$ for any knot K, and similarly $2\max\deg\Delta(L)\leq 1-\chi(L)$ for any link L. The Alexander polynomial of a *fibered* link L satisfies $2\max\deg\Delta(L)=1-\chi(L)$ and $[\Delta]_*=\pm1$ (see [Ro]).

By $\lfloor x\rfloor$ we will mean the greatest integer not greater than x, and $\lceil x\rceil$ denotes the smallest integer not smaller than x.

3. Small volume knots

In this section we will consider the problem how one can estimate the volume of a hyperbolic knot in terms of the Alexander polynomial. Simultaneously, we will try to estimate the various genera (and for links, Euler characteristics). For instance, it makes sense to ask

Question 3.1 What is the minimal twist number, or the minimal volume of a hyperbolic knot, with given Alexander polynomial?

[1] In this paper Alexander polynomials are always understood to be the 1-variable versions.

As the Alexander polynomial provides upper bounds on the crossing number of *alternating* knots [C], it certainly does so for the twist number (and volume). Dunfield's correspondence mentioned in the introduction is a sharper version of this easy observation. There exist also, for arbitrary knots, lower bounds on the twist number from the Alexander polynomial, as we prove in joint work with Dan Silver and Susan Williams [SSW].

Note that one must exclude non-hyperbolic knots if we consider the volume in question 3.1. Otherwise take a knot K realizing Δ. Then a satellite around K with an unknotted pattern of algebraic degree 1, but geometric degree > 1, has the same Alexander polynomial.

The following result gives some information on question 3.1.

Theorem 3.1 Assume $\Delta \in \mathbb{Z}[t^{\pm 1}]$ satisfies let $\Delta(t) = \Delta(1/t)$, $\Delta(1) = 1$, and let $\max\deg\Delta = d$. Then there is an arborescent knot K with the following properties.

1. We have $\Delta(K) = \Delta$, $u(K) \le 1$, and $t_\sim(K) \le 4d - 1$ if $d > 0$.
2. A Seifert surface S of genus d for K is obtained as a canonical surface of a special arborescent diagram of K. In particular $g(K) = g_c(K) = d$, so S is of minimal genus.
3. If Δ is monic, then S is a fiber surface.
4. If Δ is not the unknot or trefoil polynomial, then K is hyperbolic, and

$$0 < \mathrm{vol}(K) \le 10V_0(4d-3)\,. \tag{7}$$

Remark 3.1 By a result of Hirasawa [H], a canonical surface from some diagram D of a link L is always canonical w.r.t. a special diagram D' of L. However, the procedure he uses to turn D into D' does not preserve arborescency (of the diagram).

Remark 3.2 It follows from [Ko, St2] that another knot of $g_c, u \le 1$ cannot have the Alexander polynomial of the unknot or trefoil. Contrarily, if we waive on $u \le 1$ (and on fibering, and $g_c(K) = 0$ for $\Delta = 1$), then there is an infinity of pretzel knots (p,q,r) for p,q,r odd with such polynomials.

Example 3.1 Among trivial polynomial knots, the two 11 crossing knots are arborescent, of unknotting number one, and have vol ≈ 11.2. The smallest volume knot with trivial polynomial I found is the $(-3,5,7)$-pretzel knot, where vol ≈ 8.5, but it is not of unknotting number one.

The knot 13_{5111} of [HT] is arborescent, has $u(K) = 1$ and the trefoil polynomial, and vol ≈ 11.3. The smallest volume knot I found with this polynomial is 13_{8541} with vol ≈ 7.8, but it is (apparently) not arborescent nor of unknotting number one.

There have been several other previous constructions of (fibered) knots (and links) with given (monic) polynomial, for example [Bu, Kn, Le, Mo, Q, Sa]. The new main features here are the volume estimate and arborescency and to somewhat smaller extent genus minimality of the canonical surface.

Remark 3.3 A triggering point for the present work was Nakamura's study of braidzel surfaces [Na3]. Using these, he showed in [Na] that one can choose K in part 1 of Theorem 3.1, so that it has braidzel genus n (and unknotting number one), by realizing a Seifert matrix in [Se]. But these braidzel surfaces are unlikely canonical. Then, simultaneously to this writing, he used a Seifert matrix of Tsutsumi and Yamada [TY] (see the below proof), to find braidzel surfaces isotopic to canonical surfaces of $4d-1$ twists [Na2]. (I was pointed to this matrix also by him; previously I used the one he gave in [Na] with a weaker outcome.) Thus he gives a method that combines all our properties except hyperbolicity and arborescency.

A different construction, producing (arguably always) hyperbolic knots, is due to Fujii [Fu]. His knots have tunnel number one, and are 3-bridge, but are unlikely arborescent, and do not (at least in an obvious way) realize the canonical genus by the degree of Δ. His diagrams have unbounded twist number even for fixed degree, and a similar volume bound using Thurston's surgery theorem appears possible, but more elaborate and likely less economical than ours.

After finishing this work, we learned that the same knots were considered by H. Murakami in [Mu]. We will nonetheless go beyond the reproduction of his result (which he uses with a different motivation from ours) that these knots have the proper Alexander polynomial.

Proof of Theorem 3.1. *parts 1 and 2.* Let $\nabla(z)$ be the Conway version of Δ, and

$$\nabla(z) = 1 - a_1 z^2 + a_2 z^4 - a_3 z^6 + \ldots + (-1)^n a_d z^{2d} \in \mathbb{Z}[z^2], \tag{8}$$

for integers $a_1, \ldots, a_d$, so $a_i = (-1)^i [\nabla]_{2i}$. By Tsutsumi and Yamada [TY], it suffices to realize the matrices V_n (shown for $d = 2, 4$, with omitted entries understood to be zero, and with the obvious generalization to arbitrary d)

$$V_2 = \left(\begin{array}{cc|cc} -1 & -1 & & \\ 0 & a_1 & 1 & \\ \hline & 1 & 0 & -1 \\ & & 0 & a_2 \end{array} \right), \quad V_4 = \left(\begin{array}{cc|cc|cc|cc} -1 & -1 & & & & & & \\ 0 & a_1 & 1 & & & & & \\ \hline & 1 & 0 & -1 & & & & \\ & & 0 & a_2 & 1 & & & \\ \hline & & & 1 & 0 & -1 & & \\ & & & & 0 & a_3 & 1 & \\ \hline & & & & & 1 & 0 & -1 \\ & & & & & & 0 & a_4 \end{array} \right) \tag{9}$$

as Seifert matrices of canonical surfaces. Then $\Delta(t) = t^{-n} \det(V_n - t V_n^T)$.

The solution is given by a sequence of graphs. We display the first three in figure 3; the series is continuable in the obvious way. The example for $d = a_1 = 3$, $a_2 = a_3 = -2$ is shown as a knot diagram on the left side below.

One obtains the surfaces from the graphs as follows. Each vertex corresponds to a Seifert circle of valence ≥ 3. (The valence of a Seifert circle is the number of crossings attached to it.) Each edge with label x corresponds to a band of $|x|$ reverse half-turns of (skein) sign $\operatorname{sgn}(x)$, enclosing $|x| - 1$ valence-2-Seifert circles in between.

To obtain the Seifert matrix, for each of the bounded regions of the complement of the graph, choose a loop going around the boundary. The rows of V_n (from top to bottom) and columns (from left to right) correspond to loops ordered alphabetically by the letter in their region. The orientation is coherently chosen, so two loops pass along a common edge (twisted band) in opposite direction. If the label of an inner edge is odd (always -1), the loops are intertwined. Let them intersect once on one of the neighbored Seifert circles, so as to reinstall their position. Otherwise loops do not intersect.

The graphs are series parallel (as defined in §2) so the knots are arborescent. Unknotting number one is visualized by drawing the knot diagram. Resolving the parallel clasp * (the double edge labeled -1 in the graph) gives an unknotting crossing change.

part 3. Assume Δ is monic. We show that S can be constructed from a genus one fiber surface S' by Hopf plumbings and Stallings twists. To that vein, we apply them in reverse order and reduce S to S'.

Deplumbing a Hopf band, one resolves one of the crossings in the clasp * in the diagram on figure 3. A Hopf (de)plumbing preserves the fiber property by [Ga2, Ga3]. By a Stallings twist, one cancels the other crossing, together with the twist of $2a_1 - 1$. Then one removes the Hopf link as connected sum factor (the clasp **) by deplumbing another Hopf band. By iterating this procedure, one reduces K to a diagram of a negative clasp and a twist of $+3$ or -1 (since Δ is monic). This is the fiber surface S' of the trefoil or figure-eight knot.

part 4. By the work of Hatcher and Thurston, we must argue that the knots are not satellite, composite or torus knots. It is known from [Oe, Wu2] that arborescent knot complements are atoroidal, so there is no satellite or composite arborescent knot. Arborescent torus knots are classified in the monograph of Bonahon-Siebenmann [BS], which is only told to exist. However, we can use a published argument. In our case also $u(K) = 1$, and only the trefoil is a torus knot of unknotting number one. This probably first follows from the signature formulas of torus knots [GLM, Hi], or more directly from the subsequent result of [KM]. So we have hyperbolic knots K except the trefoil and unknot.

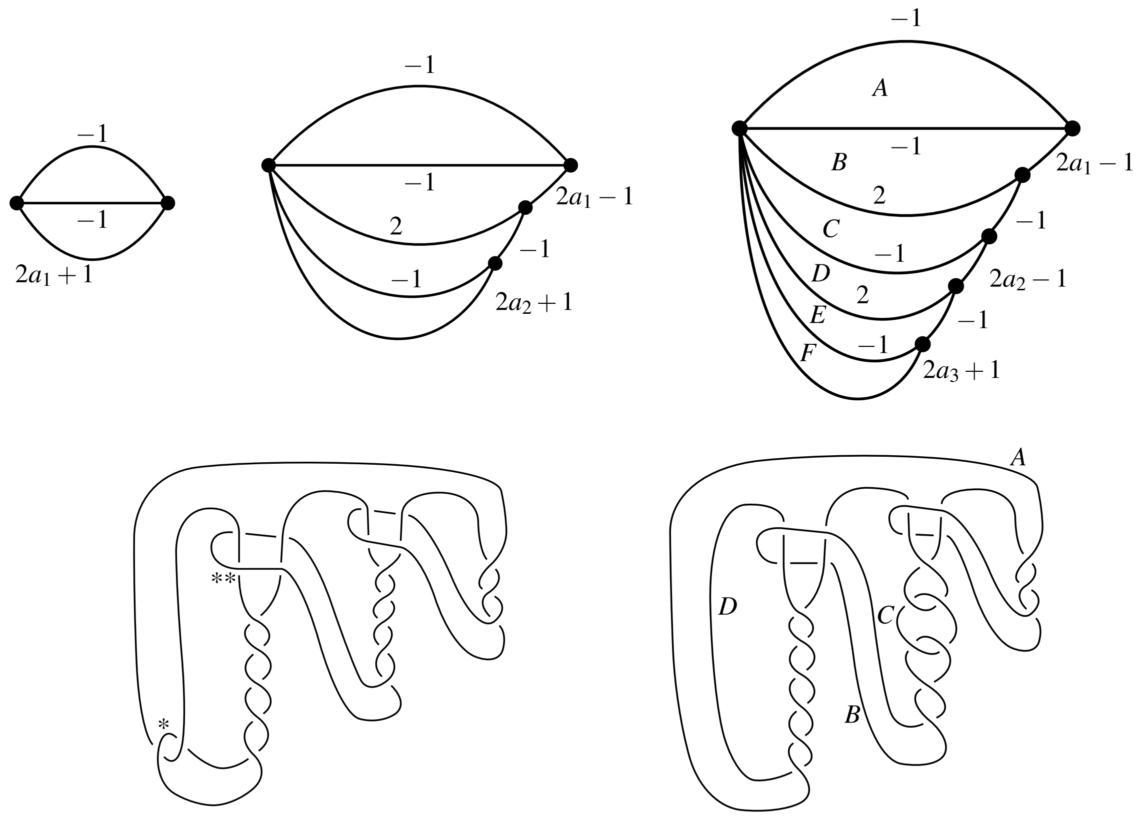


Figure 3

We have a diagram D with $4d-1$ $\sim$-equivalence classes, with two of them (of a single crossing each; the boundary of region A) forming the parallel clasp $*$, so $4d-2$ twist equivalence classes. Then applying Theorem 2.1, we have the stated volume estimate. $\square$

Remark 3.4 For an infinite series of knots, we can apply tangle surgery (see below), at the cost of slightly increasing the twist number. (However, it is not evident how to preserve fiberedness; see the remarks in §7.)

4. Two component links

With some more work, we can obtain a result of almost the same stature as Theorem 3.1 for links of two components.

Theorem 4.1 Any admissible Alexander polynomial of a 2-component link is realized by an arborescent link L with $d = 2\max\deg\Delta = 1-\chi_c(L)$, which can be chosen to have the following further properties.

1. If Δ is monic, then L is additionally fibered.
2. If $d > 1$ (that is, $\nabla(z) \neq kz$, $k \in \mathbb{Z}$), then L is hyperbolic, and

$$0 < \operatorname{vol}(L) \leq 20V_0(d-1)\,. \tag{10}$$

Remark 4.1 Silver and Williams were interested in proving, that if Lehmer's question on the existence of a Mahler measure minimizing polynomial f has an affirmative answer, then f can be chosen to be the Alexander polynomial of a fibered hyperbolic knot or 2-component link. They claimed this in a preliminary (arXiv v1) version of [SW], but there was an error in their reasoning (as has been noted in the revision). The provision of a correction motivated the study of two component links here. However, this correction requires some work, as a "pre-prepared" argument, like in the case of knots, does not seem available.

Theorem 4.1, beside confirming their claim, shows a bit more. While it is of course more interesting if one can exclude the 2-component links (or relatedly, to understand the significance of the condition $\Delta(1) = 1$ in Lehmer's question), once links come in, our theorem first eliminates the (need of) knots. We will see later, with Theorem 5.1, that we can choose the number of link components arbitrarily (as long as above 1).

Corollary 4.1 A polynomial of minimal Mahler measure (if such exists) is realized as the Alexander polynomial of a fibered hyperbolic arborescent 2-component link. $\square$

However, second, we see that, from the point of view of mere realizability, there is nothing special to Lehmer's (or any other monic reciprocal) polynomial. This should caution in seeking a topological meaning behind Lehmer's question along these lines.

Proof of Theorem 4.1. To obtain a link L of two components with given ∇, smooth out the unknotting crossing in the knot found for $1+z\nabla$ in the proof of Theorem 3.1. Observe that on the surface this is a Hopf deplumbing, so that fiberedness is preserved for monic polynomials. The Conway polynomial is $a_1z - a_2z^3 + a_3z^5 - \dots$, with the a_i as in (8).

The inequality (10) is clear once we show hyperbolicity. For this we assume that $a_1 \notin \{1,2,3\}$. Otherwise, realize $-\Delta$, and take the mirror image.

We show below in Lemma 4.2 that L is atoroidal if $a_1 \neq 1$. Atoroidality settled, hyperbolicity follows from Hatcher-Thurston once Seifert fibred link complements are excluded. Links with Seifert fibred complements are determined by Burde and Murasugi [BM]. It follows from their work that all components of such links are (possibly unknotted) torus knots. Excluding the case of $d = 1$, giving the $(2,.)$-torus links, in our examples we have an (obviously) unknotted component O, and a further component K. Now note that the knot K is of the form that is obtained by our previous construction in Theorem 3.1. By that construction,

$$\nabla_K \neq 1\,, \tag{11}$$

so K is knotted. Also, by the proof of part 4 of that theorem, K is hyperbolic (and in particular not a torus knot), unless it is a trefoil. If K is a trefoil, the proof in [BM] shows that a 2-component link of an unknot and a trefoil occurs only in their case (b). A look at the argument there shows that we must have $a_1 = lk(K,O) \in \{\pm 2, \pm 3\}$. This leaves only 4 links; they can be specified (up to component orientation) as the closures of the 3-braids $\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{2-2a_1}$. A check with Jeff Weeks' software SnapPea [We], available as a part of [HT], shows that for $a_1 = -2,-3$ the links are hyperbolic (while for $a_1 = 2,3$ they are not, which explains the other initial restriction). $\square$

Definition 4.1 In the following a *twist of* x for $x \in \mathbb{Z}$ is understood to mean a twist of $|x|$ crossings of (skein) sign $\operatorname{sgn}(x)$. We call $|x|$ the *length* of the twist. A twist is *reverse* or *parallel* if the crossings it contains are $\sim$ or $\tilde{*}$-equivalent resp., according to definition 2.5. (A twist of a single crossing is simultaneously both reverse and parallel.)

In order to avoid that the 2-component link is a connected sum with a Hopf link factor, we need $a_1 \neq 1$. First, we prove

Lemma 4.1 The above constructed link L is prime if $a_1 \neq 1$.

Proof. An easy "proof" is a routine application of the technique in [KL], but here is another proof (with a fully different argument, and worth dropping the quotes).

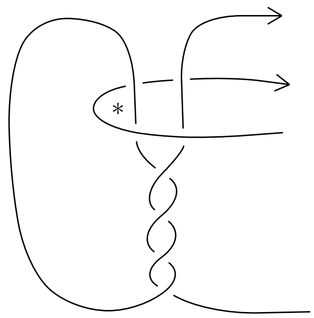
Since we assume $d > 1$, our link L consists of an (obviously) unknotted component O, and another component K. We observed that K actually is of the form that was constructed in Theorem 3.1. Then we have (11), so in particular K is knotted. Moreover

$$\max\deg\nabla_K = \max\deg\nabla_L - 1 \quad \text{and} \quad [\nabla_K]_* = \pm[\nabla_L]_*\,. \tag{12}$$

We also have $u(K) = 1$, so that K is prime by [Sc]. Hence the only possible way that L is composite is that $L = K\#L'$, where L' is a link of two unknotted components. Because of (12) we have $\nabla_{L'} = \pm z$. By additivity of the genus under connected sum, L' must bound an annulus, and then, since its both components are unknotted, L' must be a Hopf link. Now

$$a_1 = lk(K,O) = [\nabla_L]_z = \pm[\nabla_K]_{z^0} = \pm 1\,.$$

Since we excluded $a_1 = 1$, the sign is negative, and so L' is a negative Hopf link. Let $\widetilde{L}$ be the link obtained from L by reversing the orientation of O.



(13)

Then we must have

$$\nabla_{\widetilde{L}} = -\nabla_L = +z\nabla_K\,. \tag{14}$$

To show that this is not the case, we calculate $\nabla_{\widetilde{L}}$. Apply the skein relation (5) at the clasp $*$. (In $\widetilde{L}$ the orientation is so that the clasp is negative and parallel.)

$$\nabla(D_-) = \nabla(D_+) - z\nabla(D_0)\,.$$

Then D_+ depicts the connected sum of a parallel $(2,4)$-torus link with K, so $\nabla(D_+) = (2z+z^3)\nabla_K$. The diagram D_0 depicts a knot K', which is obtained from K by reversing the sign of the crossings in the unknotting (parallel) clasp.

If ∇_i are the polynomials of links L_i with diagrams equal except at one spot, where a parallel twist of i positive crossings is inserted, then by the skein relation

$$\nabla_4 = \nabla_2 + z\nabla_3 = \nabla_2 + z\nabla_1 + z^2\nabla_2 = \nabla_2 + \nabla_2 - \nabla_0 + z^2\nabla_2 = (2+z^2)\nabla_2 - \nabla_0\,.$$

So $\nabla(D_0) = \nabla(K') = z^2 + 2 - \nabla_K$. Then using (11), we have

$$\nabla_{\widetilde{L}} = \nabla_{D_-} = (2z+z^3)\nabla_K - 2z - z^3 + z\nabla_K \neq z\nabla_K\,,$$

with the desired contradiction to (14). $\square$

Lemma 4.2 The link L is atoroidal if $a_1 \neq 1$.

For the proof we require some cut-and-paste arguments. We lean closely on the work of Wu [Wu]. Let us fix some notation and terminology first. All manifolds are assumed in general position, so intersections are transversal. We use the formalism of tangle operations in figure 1 (see also the related explanation in and after Definition 2.1).

Writing again by $B(Y)$ the ball in which a tangle Y lives, we denote by $B(Y)\setminus Y = X(Y)$ the *tangle space* of Y. (This is a 3-manifold with a genus two surface as boundary; see [Wu].) By $E(L) = S^3 \setminus L$ we denote the complement of the link L.

We call a disk properly embedded in $X(Y)$ *separating* if both balls in its complement contain parts of Y. We call a tangle Y *prime* [KL] if it has no separating disk and every sphere in $B(Y)$ intersecting Y in two points bounds a ball in $B(Y)$ intersecting Y in an unknotted arc.

Proof of Lemma 4.2. If $d = 3$, then we have the Montesinos link $M(-\frac{2a_2}{4a_2+1}, \frac{1}{2}, \frac{1}{2a_1-2})$. Atoroidality follows then from [Oe]. Our form is not among those given in corollary 5 there (see in particular the proof of the corollary[2]).

[2]but beware that the hyperbolicity argument – which we do not require – contains an error; see the remarks at the end of §5 below.

Let now $d \geq 5$. In our situation, $L = Y_1 \cup Y_2$ is a 2-component link, and for integers $k \neq 0$, and m odd we can write in the notation of figure 1

$$Y_1 = (U\ 1\ 1, -2)\ m, \quad \text{and} \quad Y_2 = (2k, -2)\ 1\ 1 \ \ (= R[2k, -2; 1] + 1 \text{ in the notation of [Wu]}). \tag{15}$$

(U is a, possibly rational, arborescent tangle; Y_2 is the tangle in (13).) So Y_2 has an unknotted closed component O, but Y_1 has none. Let K be the other, knotted, component of L. It is easily verified using [KL] that Y_i are prime.

So now assume $T \subset E(L)$ is an essential (i.e., incompressible and not boundary parallel) torus. T bounds a solid torus S we call also *interior* $\operatorname{int} T$. If T bounds two solid tori, T is unknotted. Then choose one solid torus to be S. Let $R = S^3 \setminus S$ be the other complementary *region*, which we call also *exterior* $\operatorname{ext} T$. Let $B_i = B(Y_i)$ be the balls in which Y_i are contained (with $B_1 \cup B_2 = S^3$), $X_i = X(Y_i)$ be the tangle spaces and $P = \partial X_1 \cap \partial X_2$ their common boundary, a 4-punctured sphere $C = \partial B_i$. We call T *separating* if both regions of $S^3 \setminus T$ contain one component of L each.

Sublemma 4.1 Let $F \subset T$ be a circle, and assume F bounds a disk D in one of the complementary regions of T, and D is not parallel to T. Then $|D \cap L| \geq 2$.

Proof. An empty intersection is clearly out because T is incompressible. Assume $|D \cap L| = 1$. We produce a contradiction in cases by assuming that some meridional disk D of T intersects L in one point. (We choose the interior S of T to contain D.)

Case 1. T is knotted.

Case 1.1. If T is separating, the component M of L in $S = \operatorname{int} T$ is composite (and T is a swallow torus) or satellite, or T is ∂-parallel to M. Now neither of the components of L is a composite or satellite knot (see proof of Theorem 3.1, part 4), and T is essential, so we have a contradiction to all options.

Case 1.2. If T is not separating, then L is the connected sum of the knot type of T with some 2-component link (obtained by reembedding unknottedly $S = \operatorname{int} T$). This contradicts Lemma 4.1.

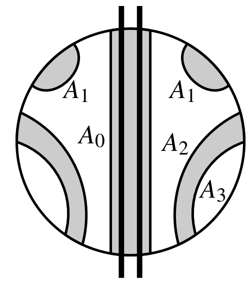
Case 2. So now consider the case T is unknotted. Then T must be separating (otherwise it compresses in its exterior). But then if T is not ∂-parallel, then L is the connected sum of the component of L in S with a satellite of the Hopf link (with a pattern that keeps the core of S). This again contradicts Lemma 4.1. □

We consider $T \cap X_i$. All disks therein can be removed (possibly after further components of $T \cap C$ are done so, i.e. components of $T \cap X_i$ are moved to X_{3-i}), because both Y_i have no separating disks. Thus (for Euler characteristic reasons) $T \cap X_i$ can be assumed to be a collection of annuli.

Sublemma 4.2 We can achieve by isotopy and proper choice of T that $T \cap X_i$ is either empty, the whole T, or a single annulus. Moreover, the intersection of an annulus $T \cap X_i$ with C is a pair of circles, each circle bounding a disk in $C \setminus T$ that contains exactly two of the 4 punctures $C \cap L$ of P.

For the proof let us fix a bit more language. Assume a torus T intersects a ball X so that an annulus A is a connected component of $X \cap T$. Assume also the two circles in ∂A are not contractible in T. (We will soon argue that this is always the case.) One can only place two unlinked unknotted not contractible loops on a torus, if they are two meridians, or two longitudes and the torus is unknotted. Since meridians (resp. longitudes) bound a disk only in the interior (resp. exterior) of a solid torus, we can choose one (and only one) of the complementary regions Y of T as the interior of T so that the loops ∂A collapse in Y.

We then choose one of the two regions Y' of $X \setminus A$ so that $Y' \cap C$ is a pair of disks (rather than an annulus). By Sublemma 4.1, both disks intersect L in exactly 2 of the punctures each. (T may enter into Y', so that not necessarily $Y' = X \cap Y$.) We call $Y' = \operatorname{int} A$ the interior of A, and the exterior of A is then obvious. Then Y' is a cylinder. We call A *(un)knotted* if the core of Y', or alternatively the intersection of a longitude of T with A, is a(n un)knotted arc in X. Similarly T is *(un)knotted (in X)* if $X \cap T = A$ and A is (un)knotted. With the same meaning we use this term when $X = X(Y)$ is a tangle space and A is disjoint from the tangle Y. (Then $\operatorname{int} A \cap Y \neq \varnothing$ in general, and knottedness of an arc is understood as w.r.t. the ball $B(Y) = X \cup Y$.) Note that T is unknotted in a ball (but not tangle space) X if and only if A is boundary parallel to X.



We introduce a relation $\succ$ among annuli of the considered type, saying for two such annuli A, A' that $A \succ A'$, if $A \subset \text{ext} A'$. It is easy to see that this defines a partial order. (Beware, though, that this is *not* equivalent to $\text{int} A \supset A'$, and this latter condition is not reflexive.) A maximal element in $\succ$ is called an *outermost* annulus.

Consider the example diagram on the right. It shows a view of $B(Y)$ from an equatorial section. The tangle Y is depicted by the thicker lines; the thinner lines indicate C and ∂T. The gray regions belong to $\text{int} T$. Then $A_1 \succ A_0$ and $A_3 \succ A_2 \succ A_0$, but A_1 does not compare to $A_{2,3}$. However, $A_1 \subset \text{int} A_2$ and also $A_2 \subset \text{int} A_1$ (and the same is true for A_3 instead of A_2). The outermost annuli are $A_{1,3}$.

Proof of Sublemma 4.2. There is easily seen to be no separating disk of Y_i in X_i, so one can remove from B_i all disks from $T \cap X_i$, together with any other parts of T in B_i that lie on one side of such disks. Then $T \cap X_i$ consists only of annuli. (They are finitely many by compactness.)

If one of the circles in $T \cap C$ bounding an annulus A of $T \cap X_i$ is contractible in T, then A is contained in a disk D that is isotopable into the exterior of T and not intersecting L. Since T is incompressible, the disk D, and hence A, is parallel to T, and so A can likewise be removed from $T \cap X_i$. So we can assume that both circles in ∂A are not contractible in T. So we have the situation, and terminology available, discussed before the proof.

Now we would like to rule out the possibility of several annuli in $T \cap X_i$. For this assume w.l.o.g. that among all essential tori T of L, ours is chosen so that $T \cap P$ has the fewest number of components (circles).

By the above argument, each annulus in $T \cap X_i$ bounds in P a pair of meridional disks (with respect to one of the complementary solid tori if T is unknotted). In particular, all the circles in $T \cap P$ are meridians of $\partial S = T$, w.r.t. the interior $S = \text{int} T$ of T, or a proper choice of interior if T is unknotted. (Because a longitude and meridian always intersect, the choice of S cannot be different for different circles in $T \cap P$.)

By Sublemma 4.1, each circle of $T \cap P = T \cap C$ which bounds a disk in C disjoint from $T \cap C$ (let us call such circles *innermost*) intersects ≥ 2 of the punctures $L \cap C$ of P. There are clearly at least two circles in $T \cap C$, and hence there are also at least two innermost. Since P has four punctures, we see that there must be exactly two innermost circles, each bounding a disk in C intersecting L in exactly two punctures. Then $S \cap P$ is a collection of two twice-punctured disks, and unpunctured annuli. Next we show that we can get disposed of the annuli in $S \cap P$.

Let A be an annulus of $S \cap P$. Then A forms a torus $T_{1,2}$ with each of the two annuli that ∂A cuts T into. The T_i inherit meridians from T, and their interior is defined again as the region where meridians collapse. Then $\text{ext} T_i$ is determined also, $\text{ext} T = \text{ext} T_1 \cup \text{ext} T_2$ and $A = \text{ext} T_1 \cap \text{ext} T_2$. We claim that at least one of $T_{1,2}$ is essential. Since A can be pushed into either X_1 or X_2, we have then a contradiction to the above minimizing choice of T.

First, $T_{1,2}$ do not compress in their interior, because T does not. If some T_j (is unknotted and) compresses in its exterior, then all components of L contained in $\text{ext} T_j$ lie within a ball contained in $\text{ext} T_j$. If there are such components, L is split, and otherwise, T is isotopic to T_{3-j}, and subsequently A can be removed.

If some T_j were ∂-parallel to a component of L in its interior then T would also be (and T and T_j would be isotopic). Finally, at least one of $T_{1,2}$ is not ∂-parallel in its exterior. If both were such, then because of $\text{ext} T_j \subset \text{ext} T$, we would have both two components of L in the exterior of T, in contradiction to $S \cap L \neq \varnothing$.

With this argument we showed that any annulus in $S \cap C$ (that comes from a pair of nested annuli in $T \cap X_i$) can be removed by isotopy. Thus we can achieve that $S \cap C$ consists only of disks. Also, by Sublemma 4.1, we argued that there is only one pair of disks, so we have only one annulus in $T \cap X_i$, and complete the proof of Sublemma 4.2. $\square$

We consider the two options for $T \cap C$ from Sublemma 4.2.

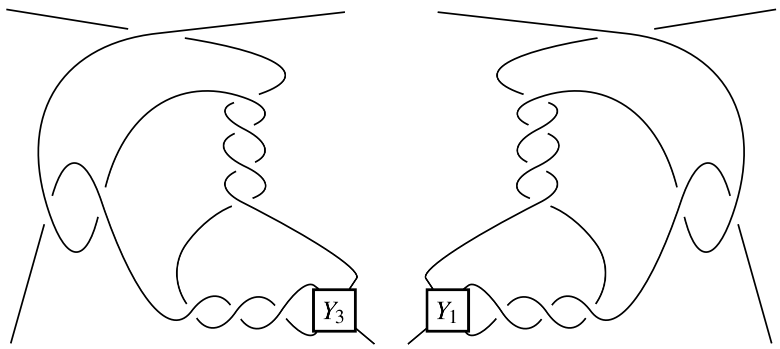
Case 1. $T \cap C \neq \varnothing$, that is, both $T \cap X_i$ are annuli.

Sublemma 4.3 T is unknotted in X_1.

Proof. Assume that T is knotted in X_1. Then $T \cap X_1$ is not parallel to the boundary of a string of Y_1. (Otherwise, the intersection of T with P is a pair of circles, each circle has only one, and not two as assumed, of the 4 punctures.) If $d \geq 7$, then U in (15) is not a rational tangle, and then Y_1 is not among the tangles in Theorem 4.9(a-d) of [Wu]. This theorem says then that T is simple, so excludes such an annulus $T \cap X_1$.

If $d = 5$, then Y_1 is equivalent (in the sense of definition 2.1) to a Montesinos tangle $M(1/2, p/q)$ with q odd. To obtain a contradiction in this case, assume w.l.o.g. $Y_1 = M(1/2, p/q)$. Let Y_3 be a prime tangle such that $L' = Y_3 \cup Y_1$ is a prime link of ≥ 2 components. Let $A' \subset B(Y_3)$ be an unknotted annulus identifying both circles of $T \cap P$ such that it contains Y_3 in its interior. Consider the torus T' in $X(L')$ obtained by gluing A' and $A = T \cap X_1$. So T' is knotted. Let S' be its interior. Then if T' is ∂-parallel, it must be ∂-parallel to a single link component in S'. But since L' has several components, and S' contains all of L', this is impossible. Since T' is knotted, if it is compressible, then a compressing disk must be meridional. Such a disk can be moved completely into either X_1 or X_3, using that $Y_{1,3}$ have no separating disks. But both is excluded, since $Y_{1,3}$ are prime and $P \cap L'$ is non-empty. Therefore, T' is essential, and L' is toroidal.

So any prime link $L' = Y_3 \cup Y_1$ of ≥ 2 components is toroidal. To see that this is not so, take $Y_3 = Y_1$. Since we do not know which pairs of punctures the two circles of $T \cap P$ enclose, to glue the two annuli A, A' properly, we may need to place them in a favorable position, e.g. so that their boundary circles are in a 'standard' shape. To achieve this for A, we apply a rational transformation (1) on Y_1, and then we use the inverse transformation on Y_3 before placing A' to be parallel to $\partial B(Y_3)$.



However, in all cases these modifications can be carried out so that L' becomes a Montesinos link of length 4. (This observation will be required and implicitly applied again in some of the below arguments.) Corollary 5 of [Oe] shows that such links are atoroidal except if $p/q \neq \pm 1/2$, which is clearly not the case here (because q is odd). $\square$

But now recall that Y_1 has no closed component. Then by Sublemma 4.2, all of Y_1 lies in the interior of T, i.e. in $S \cap B_1$. Since T is unknotted, it must be then ∂-parallel to C, and can be removed from X_1. Thus it suffices to deal with the next case.

Case 2. $T \cap C = \varnothing$. So T lies in some X_i. In our situation Y_1, Y_2 are, if not simple, up to equivalence $M(1/2, p/q)$. So it suffices that we study the case $Y_1 = M(1/2, p/q)$ (with q even or odd, that is, with or without closed component) and assume $T \subset X_1$.

We obtain by inclusion a torus T in the exterior of the link $L' = Y_1 \cup Y_3$ for any tangle Y_3. Again we want to obtain a contradiction from this by choosing Y_3 well and using Oertel. Assume Y_3 is prime and L' is non-split.

We claim that this torus $T \subset X_1$ is not compressible in $E(L')$. To see this, assume T were compressible. First note that if T separates components of Y_1 in X_1, it would too in L', in contradiction to the non-splitness of L'. So T separates no components in X_1. Then the only way in which T would be compressible in $E(L')$ but incompressible in $E(L)$ is that T is knotted, and $X_1 \supset \operatorname{ext} T$.

Let D be a compressing disk of T in $E(L')$. This disk may penetrate into $X_3 = X(Y_3)$. But since Y_3 was chosen prime, X_3 has no separating disks, and so D can be moved out of X_3, and into X_1. So T would compress in X_1 too, a contradiction.

With this we assure that $T \subset X_1$ is incompressible in $E(L')$. So it is essential, unless it is boundary parallel. It is not boundary parallel to a closed component of Y_1, because it is essential in $E(L)$, and so also in X_1. So T can only be boundary parallel in its region containing B_3. This can be avoided for example by choosing Y_3 to have a closed component.

Therefore, all $L' = Y_1 \cup Y_3$ where Y_3 has a closed component must be toroidal. This is easily disproved by choosing Y_3 well (so that L' is a Montesinos link) and using Oertel.

Since we obtained contradictions in all cases, we conclude that there is no T, and Lemma 4.2 is proved. $\square$

5. Links of more components

Now we derive some consequences and generalizations for links of more components. (In §5 we use consistently $n = n(L)$ for the number of components of a link L and $g = g(L)$ for its genus. The cases $n(L) \leq 2$ were discussed before, so assume throughout $n \geq 3$.)

The first theorem deals with fiberedness. Kanenobu [Kn] extended the realization of monic polynomials to fibered links. However, his construction, which seems the only one known, uses connected sum with Hopf links. Thus, for more than two components, surprisingly, the simple question to find a prime fibered link appears open (even for $n = 2$, Kanenobu's links are not proved to be prime). The theorem removes this shortcoming, with a more specific statement.

Theorem 5.1 Let ∇ be an admissible (as in definition 2.7) monic Conway polynomial of an n-component link, $n \geq 3$. Then, except for $n = 3$, $g(L) = 0$ and $\nabla = +z^2$, there exists a prime arborescent fibered link L with $\nabla_L = \nabla$, such that the fiber of L is a canonical surface obtained from a special arborescent diagram of L. Unless $n = 3$, $g(L) = 0$, and $\nabla = -z^2$, the link L is hyperbolic, and

$$\mathrm{vol}\,(L) \,\leq\, 10V_0 \cdot \left\{ \begin{array}{cc} 2\max\deg\nabla - n & \text{if } g(L) > 0 \\ n & \text{if } g(L) = 0 \end{array} \right. .$$

The following object will be useful for the primeness and hyperbolicity arguments.

Definition 5.1 Define the *linking graph* $G(L)$ of L by putting a vertex for each component of L and connecting vertices of components with non-zero linking number. Optionally, we may label an edge by the linking number.

Proof of Theorem 5.1. Let first $g(L) > 0$. We deal with the case $n = 3$ first. Consider the 2 component link L' found in Theorem 4.1 for $\nabla' = \nabla/z + z$. Assume the (reverse) clasp ** in the left diagram of figure 3 is negative, by possibly mirroring L' (mirroring preserves ∇ for even number of components). Recall that L' is obtained from a knot as on the left of figure 3 by smoothing out one crossing in its parallel clasp *.

Call the replacement of a crossing with a parallel clasp a *clasping*, and give it a sign as for the crossings involved:

(16)

Then apply a positive clasping at a crossing among those corresponding to the edge labeled $2a_2 - 1$ in figure 3. (If these crossings are negative, create a trivial clasp by a Reidemeister II move in advance.) We claim that the resulting 3-component link L is what we sought.

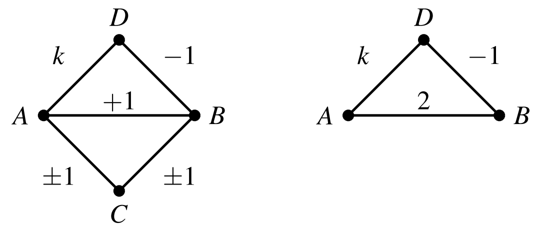
The Conway polynomial is easily checked using the skein relation (5) at the crossing created by the clasping. In that case D_+ depicts L, D_- depicts a $(2,-2,k)$-pretzel link (k even), and D_0 depicts L'. By the proper choice of ∇', we see that $\nabla_L = \nabla$.

The fibering is also easy, since a clasping results in a Hopf plumbing on the canonical surface. By [Ga2, Ga3], the fiber property is invariant under a Hopf plumbing.

It remains to see primeness. This can be shown again from the arborescency using [KL], but there is a more elementary argument. Note that all components of L are unknotted and have pairwise non-zero linking number. (Here the proper choice of signs of clasps is helpful.) Thus if we had $L = L_1 \# L_2$, former property excludes the option that some of $L_{1,2}$ is a knot, and latter property excludes the option that both are 2-component links.

For $n > 3$ we can use induction. Again we apply claspings (either sign may do) at some of the crossings of $2a_2 - 1$ (possibly creating new crossings by Reidemeister II moves). The link on the bottom right of figure 3 is a typical example (for $n = 4$). Again the check of ∇ is easy; $D_\pm$ depicts L, D_0 depicts a connected sum of a $(2,-2,k)$-pretzel link with Hopf links, and $D_\mp$ depicts a link of the sort constructed for $n-1$. The skein relation of ∇ again easily allows to adjust the polynomial of L properly.

To see primeness, use again that all components of L are unknotted. So if $L = L_1 \# L_2$, then both of $L_{1,2}$ are links. Then the linking graph $G(L)$ of L must have a cut vertex v (i.e. it must become disconnected when removing v and its incident edges). However, for our L this is easily seen not to be the case. Here $G(L)$ consists of a chain connecting all vertices, with an additional edge between two vertices of distance 2 in the chain. So L is prime. Let us display the graphs for 3 and 4 components, also for future reference. They look like (up to reversing sign in all linking numbers)

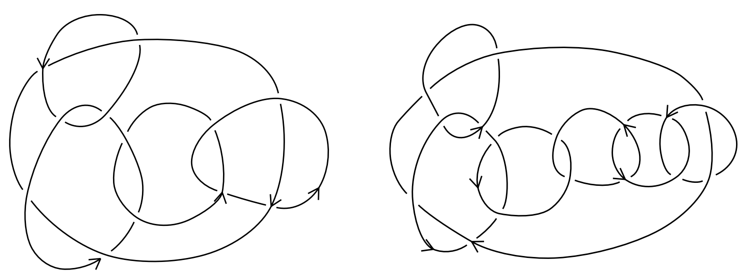


(17)

Here the component designation for $n = 4$ is as in figure 3. Note that, since the diagram is special, D and A have with B a linking of opposite sign.

For $n = 3$ we let C identify with A under undoing one of the claspings (16) in the $n = 4$ case. As occurred in the primeness argument, we can have also $lk(A,B) = 0$. We will need this case only once (at the end of the proof of Lemma 5.1), and otherwise stick with $lk(A,B) = 2$.

Our construction yields links with all desired properties (except hyperbolicity, which we treat below) whenever $g(L) > 0$. Finally, turn to the case $g(L) = 0$. We use the pretzel links of type I in Gabai's theorem 6.7 in [Ga4]. The links in case 1 (B), (C) there realize the stated polynomials. For even number of components, case (C) applies, and we get both possible polynomials $\pm z^{n-1}$ by mirroring (which changes sign of ∇). For odd number of components we have the pretzel links in case (B). To see that their polynomial is $(-1)^{\lfloor n/2 \rfloor} z^{n-1}$, one can use, for example, the formula of Hosokawa-Hoste [Ht]. For $n = 5, 7, \dots$ and $\nabla = (-1)^{\lceil n/2 \rceil} z^{n-1}$ we found, with the help of some computation, the sequence of links with Conway notation $(2,2,-2)(2,-2,2,\dots,-2,2)$, the first two of which look like:



(18)

(The orientation of components is so that all clasps are reverse.) The fibering of these examples can be confirmed by the disk (product) decomposition of Gabai [Ga4], and the proper ∇ using [Ht].

We postpone the hyperbolicity proof to lemmas 5.1 and 5.2. The volume estimate is again easy from Theorem 2.1. $\square$

Remark 5.1 The following observations indicate how one can (or can not) modify or extend Theorem 5.1.

1) For $n = 3$ the only diagrams with canonical surfaces of genus 0 are the (p,q,r)-pretzel diagrams, p,q,r even. Then Theorem 6.7 Case (1) of Gabai [Ga4] shows that there is no prime link for $\nabla = +z^2$, even with a canonical fiber surface from an arbitrary diagram.

2) The algebraic topologist considers Δ usually up to units in $\mathbb{Z}[t^{\pm 1}]$, in opposition to treating Δ as the equivalent (6) of ∇. In that weaker sense the exceptional links (18) in our proof could be avoided. For knots the ambiguity of Δ is not essential, because the condition $\Delta(1) = 1$ allows one to recover the stricter form. Note, though, that for links of more than one component, we lose the information of a sign in the up-to-units version.

3) The exception $n = 3, g = 0$ also disappears for the strict $\Delta \neq 0$ if we waive on fiberedness (and then also on monic polynomials) and demand $2 \max\deg \Delta = 1 - \chi$ instead. The corresponding statement follows just by an obvious modification of the proof we gave. (For genus 0 one can easily adjust infinitely many pretzel links to give the proper polynomial.)

If we like to keep small 4-genus, we have

Corollary 5.1 For any admissible Alexander polynomial Δ of a link, there exists an arborescent link L with $\Delta(L)=\Delta$, $\max\deg\Delta = 1-\chi_c(L)$ and $\chi_s(L)\geq -1$. Moreover, L can be chosen to be fibered if Δ is monic.

Remark 5.2 Clearly for an n-component link, $\chi_s \leq n$, but even below this bound, one cannot augment χ_s unrestrictedly, since it is related to (the vanishing of) certain linking numbers, which in turn have impact on the low-degree terms in ∇. (In particular $\chi_s = n$ means strongly slice, which implies that $\nabla = 0$.)

Proof. For one component, $u(K)\leq 1$ implies $\chi_s(K)\geq -1$. For a link of two components take the link constructed for Theorem 4.1. Observe that this link bounds a ribbon annulus, so $\chi_s \geq 0$. For $n\geq 3$ components, we can always achieve that $\chi_s \geq -1$ for the links L in Theorem 5.1, by varying the sign of claspings (16) with the parity of n. $\square$

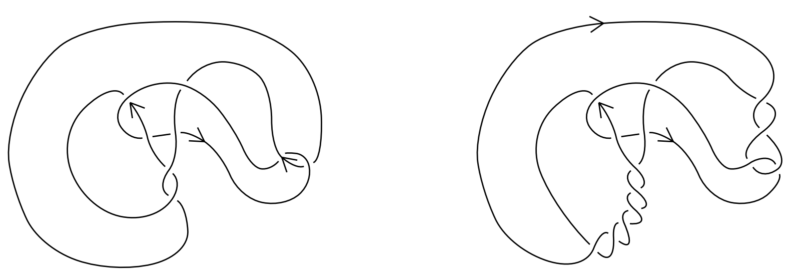
Lemma 5.1 The link L of Theorem 5.1 is choosable to have a complement which is not Seifert fibered, unless $n=3$, $g(L)=0$, and $\nabla = -z^2$.

Proof. Consider first $g(L)>0$. We use again the description in [BM]. Since $n\geq 3$, all components are unknotted, we have only the types shown in figures 2 (type (a)) and 3 (type (b)) therein. Now all these links have the following property: there is a component M having the same linking number with all the others, up to sign. Looking at $G(L)$ for our links L, we see that only $n\leq 4$ components come in question.

So for $n=4$, M can be only one of A or B (see (17)). However, the next property of Burde-Murasugi's links is that all components different from M have mutually the same linking number. This immediately rules out also $n=4$.

Now for $n=3$, M can be only D and $k=\pm 1$. In type (b) of Burde-Murasugi, the distinguished component M has linking number $\pm\alpha$ with all the other components, and in that case it was assumed that $\alpha>1$, so this option is ruled out. It remains their type (a). For these links, looking at Figure 2 of [BM] with $m=3$, and taking care of linking numbers, we see that we have the $(2,-2,4)$-pretzel link, oriented so as to be the closure of the 3-braids $\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{\pm 4}$, but for σ_1^{-4} one component involving these crossings must be reversed. Latter case gives a link of genus 0, so consider only former, i.e. with σ_1^4 in the braid.

The Conway polynomial of this link is $\nabla = -3z^2 - z^4$. The link L (up to mirroring) obtained from our construction with such polynomial is shown on the left of (19). It has the linking graph on the right of (17) for $k=-1$. It turns out that SnapPea reports this link non-hyperbolic, so apparently it is the Burde-Murasugi link.



(19)

However, now recall that we had some option in the construction of L. First we can change the sign of the clasp * in (13), which here leads to a composite link. Next, though, we can change the sign of the clasping (16). This leads to another link with the same polynomial, given on the right of (19). SnapPea reports it to be hyperbolic, with which the case $g(L)>0$ is finished.

The links L of genus 0 are dealt with by the same argument. Again by linking numbers we are down to 4 components (in particular all those links of (18) are done). For 4 components, the linking graph of a pretzel is a cycle of length 4, so this case is out too, and for $n=3$ we arrive at the additional exception we had to make – the $(2,-2,r)$-pretzel links are indeed Seifert fibered. $\square$

Lemma 5.2 The link L of Theorem 5.1 is atoroidal.

Proof of Lemma 5.2. Let us focus on $g(L)>0$. We adapt the proof, as far as possible, from lemma 4.2, and use the notation from there. The tangle decomposition of L in (15) modifies so that now

$$Y_1 \;=(U\ 1\ 1,-2)\,m \quad \text{and} \quad Y_2 \;=\; ((2k,-2)\ 1\ 1\ ,\pm 2,\pm 2,\dots,\pm 2) \tag{20}$$

Again let T be an essential torus in $E(L)$. Since both $Y_{1,2}$ are again easily proved to be prime, we can assume w.l.o.g. that T does not intersect any tangle space X_i in disks, and so only in annuli. Still T_1 has no closed component and is subjectable to [Wu]. Then all intersections of T with the tangle sphere C are meridional disks, with respect to a proper choice of interior $S=\operatorname{int}T$. Assume again T is chosen so that $S\cap C$ has the fewest connected components.

With the same argument we have first:

Sublemma 5.1 Sublemma 4.1 holds. □

Sublemma 5.2 If T is knotted, then T is not separating, i.e. $L\subset \operatorname{int}T$.

Proof. All components of L are unknotted. Any unknot embedded in a knotted solid torus has homological degree 0. So for each pair of components $M_1\in\operatorname{int}T$, $M_2\in\operatorname{ext}T$, we must have $lk(M_1,M_2)=0$. So if T were separating, the linking graph $G(L)$ would be disconnected, which we saw is not the case. Since by incompressibility, there is always some M_1, there cannot be any M_2. □

Sublemma 5.3 If T is unknotted, then T is separating. Let P,Q be the sets of components of L in $\operatorname{int}T$ resp. $\operatorname{ext}T$. Then $G=G(L)$ has the following property. If for some $a\in P$, $b\in Q$ there is no edge between a and b in G, then there is no edge between a and b' for any $b'\in Q$, or there is no edge between a' and b for any $a'\in P$.

Proof. Clearly an unknotted torus must separate, else it would compress. Now when T is unknotted, L is a satellite of the Hopf link. Then for two components $a\in P$, $b\in Q$ of L we have $lk(a,b)=[a]\cdot[b]$, where the brackets denote the homology class in $H_1(\operatorname{int}T)=H_1(\operatorname{ext}T)=\mathbb{Z}$. So if a and b are not connected in G, one of $[a]$ or $[b]$ must be 0, and the claim is clear. □

Sublemma 5.4 Sublemma 4.2 holds still.

Proof. The proof of Sublemma 4.2 goes through with the help of now Sublemma 5.1, except for the argument why some of $T_{1,2}$ is not ∂-parallel in its exterior.

If T is knotted, then by Sublemma 5.2, its exterior is empty, so clearly none of $T_{1,2}$ can be ∂-parallel in its exterior. If T is unknotted, then all annuli of $T\cap X_i$ are unknotted too. Now since one of the Y_i, namely Y_1, still has no closed component, an outermost annulus of $T\cap X_1$ is parallel to C. Then successively all annuli of $T\cap X_1$ can be removed, so $T\cap X_1=\varnothing$. □

Back to the proof of lemma 5.2, now we can apply [Wu] to Y_1. An annulus $T\cap X_1$ must be parallel to C, provided U in (20) is not a rational tangle. Then T can be removed from X_1, so $T\subset X_2$. If U is rational and $T\cap X_1\neq\varnothing$, then we can obtain a contradiction to Oertel's result by joining Y_2 with itself properly to obtain a Montesinos link of length 4.

So we can assume $T\subset X_2$.

Now let $L'=Y_3\cup Y_2$ be a prime (non-split) link of ≥ 5 components, and Y_3 be a prime tangle with a closed component. We claim that $T\subset E(L')$ is essential. The argument is the same as in case 2 of the proof of Lemma 4.2. So again all such L' would be non-atoroidal.

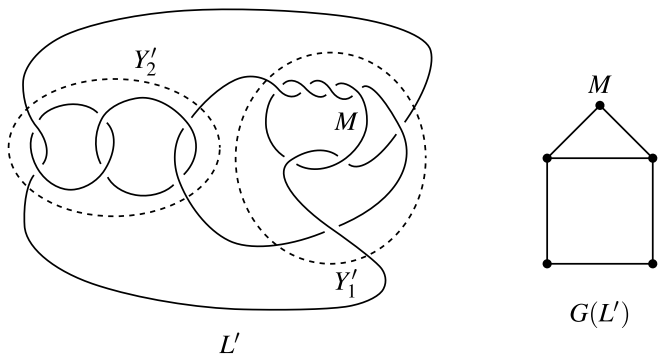
Thus we can conclude the proof of Lemma 5.2 for $g>0$ with Lemma 5.3 below. For our links of $g=0$, we can apply Oertel to the pretzel links, and the links in (18) are dealt with the same argument is those in Lemma 5.3. (See the remark at the end of its proof.) □

Lemma 5.3 The links L' with Conway notation

$$((k,-2)\ 1\ 1),\pm 2,\pm 2,\dots,\pm 2,0\,m\,,$$

of $n(L')\geq 5$ components for $k,m\in\mathbb{Z}$, $k\neq 0$ even, are atoroidal.

Here is an example L' with $m = 0$, $k = 4$ and $n = 5$ components, together with its linking graph $G(L')$ we will use shortly.



(21)

Proof. Let $Y_1' = (k,-2)\ 1\ 1\ m$ and $Y_2' = (\pm 2, \pm 2, \dots, \pm 2)$. Then $L' = Y_1' \cup Y_2'$. (Follow the diagrams in (21).)

If we remove the closed component M of Y_1', then we have a pretzel link, which is atoroidal by Oertel. (Here we may better avoid the $(2,-2,2,-2)$-pretzel link $L' \setminus M$; but we will just see that its unique essential torus still fits into the below conclusions.) Thus an essential torus T of L' must become inessential in $L' \setminus M$. Since L' is non-split, this means that one of the regions of T must contain either only M (if T compresses in $L' \setminus M$), or M and exactly one other component M' of L' (if T is ∂-parallel to M' in $L' \setminus M$). In particular, since we have $n \geq 3$ components, T is separating.

Now again all components of L' are unknotted and $G(L')$ is connected. So T separating means by Sublemma 5.2 that T is unknotted (as for the essential torus of $M(1/2, -1/2, 1/2, -1/2)$). Now we can apply Sublemma 5.3 on $G(L')$. For $n(L') \geq 5$ components, we easily see that the option T containing a component $M' \neq M$ is ruled out.

Thus T contains M alone in one region (and $n - 1 \geq 4$ components of L in the other one). Then by the argument for Sublemma 5.4, T can be isotoped (or chosen more properly) into $X_1' = X(Y_1')$ or $X_2' = X(Y_2')$. Let us explain this briefly.

First, the argument excluding $T_{1,2}$ being both ∂-parallel in their exterior applies now, because we assured that none of the regions of T contains precisely 2 components of L'. So the conclusion of Sublemma 4.2 applies. Next, the option of an annular intersection $T \cap X_i'$ is ruled out as follows.

The annuli $T \cap X_1'$ and $T \cap X_2'$ again determine an interior of T by letting the circles in $T \cap C$ collapse therein. Now T is unknotted and contains only one component in its exterior, a component which does not intersect C. Then for at least one $i = 1, 2$ the annulus $T \cap X_i'$ will be (unknotted and) with empty exterior in X_i, so ∂-parallel to C, and could be removed.

Now having T within X_1' or X_2', we can obtain the same contradiction as before by looking at $Y_1 \cup Y_3$ or $Y_2 \cup Y_3$ for proper Y_3 and applying Oertel.

Let us say a word on the links in (18). Their linking graph is the same as for our L'. Again removing M, when specifying it so as the labeling in graph on the right of (21) to be correct, gives a pretzel link. So the argument here applies unchangedly. $\square$

Let us conclude the hyperbolicity proof with a few general/historic remarks. One reason for the effort we needed to spend we see in the lack of extension of Wu's work [Wu] to tangles with closed components. This extension is a substantial program, and we were forced to go some steps along it, even though it was not our primary focus. It is clear that our method can be applied to many more examples, although the complete treatment of arborescent tangles is still far ahead.

The other main motivation for our hyperbolicity proofs was the status of Bonahon-Siebenmann's monograph [BS]. We were aware that we reprove their theorem on the classification of hyperbolic arborescent links in particular special cases. Still we were bothered by the notorious inavailability of [BS], announced decades ago, but never completed. Even for Montesinos links, written accounts needed some amendment. At least atoroidality of the link complements seemed not completely clarified. An additional complication for links is that not only torus links have Seifert fibered complements. Among the links in [BM], at least the $(2,-2,r)$ pretzel links, pointed out by Ying-Qing Wu, are Montesinos and (for $|r|\neq 1,2$) non-torus links whose complements are Seifert fibered (and atoroidal). Thus in particular the statement and proof of corollary 5 in [Oe] must be corrected accordingly (see e.g. also [St6]).

Only after we completed our work, we were informed of a recent preprint of Futer and Guéritaud [FG], which gives a written proof of Bonahon-Siebenmann's theorem characterizing the hyperbolic arborescent links. Still it seems fair to say that our effort was (almost) simultaneous, independent, shorter than the (full extent of the) work in [FG], and makes our paper more self-contained. (Some of the arguments are used also below, out of the context of [FG].) Thus we see both some right and some sense to keep the material in §4 and 5, rather than mostly avoid it by referring to [FG].

6. Tangle surgery constructions

The following constructions, which are also heavily used in [St5], show infinite families of links with given polynomial, if we focus on arborescency and χ_s, but abandon fibering and, in certain cases, minimality of the canonical surface. (Note that, in [St4] we showed that almost every monic Alexander knot polynomial of degree 2 is realized by only finitely many canonical fiber surfaces, so abandoning fibering of the canonical surface is a non-trivial relaxation. See §7 for related discussion.)

We will use some tangle surgery arguments. With the terminology of Definition 4.1, we state first

Lemma 6.1 Let S_k, for $k\in\mathbb{Z}$, $k\neq 0$, be the $(1,2k-1)$ pretzel tangle, with orientation chosen so that the twist of $2k-1$ is reverse. (S_1 is a positive parallel clasp.) Then S_k can be replaced by tangles $T_{p,q,r}$, that contain three twists of p,q,r, such that all lengths $|p|,|q|,|r|$ can be chosen arbitrarily large, and any such tangle replacement preserves the Alexander polynomial.

Proof. Consider the $(p\pm1,q,r)$-pretzel knot diagrams $D(p\pm1,q,r)$, with $p\pm1,q,r$ odd. Their Alexander polynomial is determined by $v_2={}^1\!/_2\Delta''(1)$, which is

$$v_{2,\pm}=\frac{(p\pm1)q+(p\pm1)r+qr+1}{4}.$$

Now for $p=0$, $q=1$, $r=2k-1$ we have

$$v_{2,+}=k,\qquad v_{2,-}=0. \tag{22}$$

We need to find more solutions to (22). We have

$$(p-1)q+(p-1)r+qr+1 \;=\; 0 \tag{23}$$
$$(p+1)q+(p+1)r+qr+1 \;=\; 4k \tag{24}$$

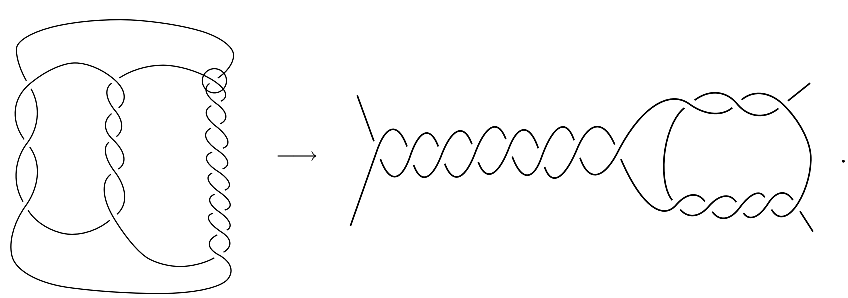
Then $(23)-(24)$ gives $q+r=2k$, and $(23)+(24)$ gives $p(2q+2r)+2qr=4k-2$, so

$$p=\frac{2k-1-qr}{2k}.$$

We would like $p\in\mathbb{Z}$ and p even. To achieve this, choose

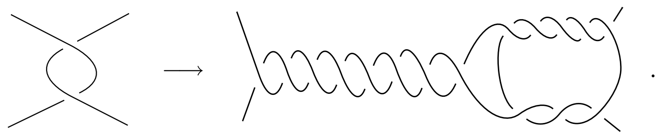
$$q=1+2nk,\qquad r=2k-1-2nk, \tag{25}$$

for $n\in\mathbb{Z}$. Let $T_{p-1,q,r}$ be the tangle obtained by cutting out from $D(p\pm1,q,r)$ the switched crossing, for example for $(p,q,r)=(8,5,-3)$:



(The shift to make the first index odd is done for future convenience.) Now we can substitute $T_{p-1,q,r}$ for S_k, so that Δ is preserved (see [Bl] or [SSW]). Also $|p|,|q|,|r| \to \infty$ when $|n| \to \infty$. □

Remark 6.1 We will use also the surgery on the mirrored tangles. The mirrored surgery for $k = 1$ and $(p,q,r) = (8,5,-3)$ is shown below:

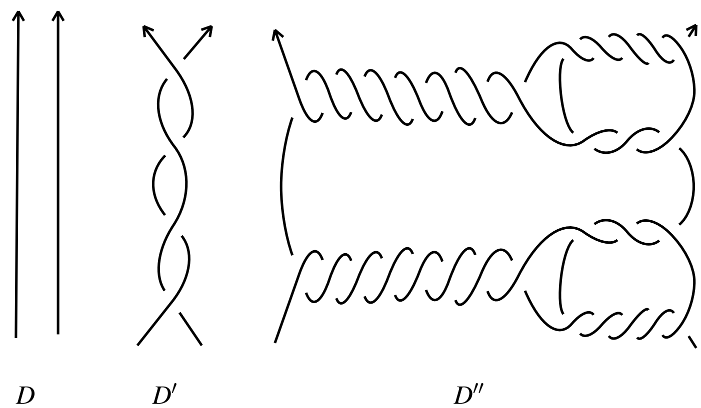


(26)

If we abandon fiberedness and relax the minimal genus condition $2\max\deg\Delta = 1-\chi$, then for example, we easily restore arborescency in corollary 5.1:

Corollary 6.1 For any admissible Alexander polynomial Δ of a link, there exists an arborescent link L with $\Delta(L) = \Delta$, $\max\deg\Delta \geq -3-\chi_c(L)$ and $\chi_s(L) \geq -1$.

Proof. Consider the link in the proof of Corollary 5.1. Let D be the diagram constructed there. We apply the modifications in (27). Create a prime diagram D' by adding a positive and negative parallel clasp. Then apply tangle surgery on these clasps in D' with two mutually mirrored tangles, so that one obtains a diagram D'' of a concordant link. For $(p,q,r) = (8,5,-3)$ the operation looks as follows:



(27)

These two tangle surgeries preserve arborescency and χ_s and augment the genus of the diagram at most by two. □

If we are interested in controlling only χ_s, there are virtually no difficulties at all in using surgeries, and we have:

Corollary 6.2 For any admissible Alexander polynomial Δ of an n-component link and $\chi \leq 0$ with $n+\chi$ even, there exists an arborescent link L with $\Delta(L) = \Delta$ and $\chi_s(L) = \chi$.

Proof. The largest χ was dealt with in corollary 6.1. Then take iterated connected sum with $(-3,5,7)$-pretzel knots and apply the (concordance) surgery (27). □

7. Infinite families of links

It is a natural question which admissible monic Alexander polynomials are realized by *infinitely many* fibered links. For knots the problem was suggested by Neuwirth and solved fully by Morton [Mo] (after previous partial results; see for example Quach [Q]). As well known, genus one fibered knots are only the trefoil and figure-8 knot. In contrast, Morton constructs for each possible monic Alexander polynomial of maximal degree greater than one an infinite sequence of distinct fibered knots with this polynomial (though without regard to any additional knot properties).

Unfortunately, extensions of Morton's construction to links seem never to have been attempted or obtained. Now we have the following analogue of Morton's result. (We use again $n = n(L)$ for the number of components, $g = g(L)$ for the genus and $\chi = \chi(L)$ for the maximal Euler characteristic of L.)

Proposition 7.1 For $n \geq 4$ components, there are infinitely many (arborescent) canonically fibered links with any given monic admissible Alexander polynomial.

Proof. We use the links of Theorem 5.1. If $g > 0$, the unknotted component created by two claspings allows to apply Stallings twists if we choose the claspings to be of opposite sign. The linking number easily distinguishes infinitely many of the resulting links, but they all have the same complements, so hyperbolicity is preserved. For $g = 0$ we can use Stallings twists for the links in (18) and for those of Gabai's type (C). (See the proof of Theorem 5.1.) His pretzel links of type (B) are already infinitely many (and all have the same polynomial). □

We know in contrast (see the discussion at the end of this section) that a generic monic Alexander *knot* polynomial of degree 2 is realized by only finitely many *canonical* fiber surfaces. So the combination of fibering and canonicalness poses non-trivial restrictions on infinite families. Assuming canonicalness and merely minimal genus property, the scope of constructible infinite families widens.

Proposition 7.2 For $n = 1$ and $g > 0$, or $n \geq 3$, any admissible Alexander link polynomial $\Delta \neq 0$ is realized by infinitely many prime arborescent n-component links with a canonical minimal genus surface and $2\max\deg\Delta = 1-\chi$.

Proof. For knots (and $\Delta \neq 1$) this can be shown by applying the surgeries of the type (26) for all admissible p, q, r at the parallel clasp * of the knots as in figure 3, constructed in the proof of Theorem 3.1. The distinction of the resulting knots is a bit subtle, but since they are arborescent, it can be done at least from [BS]. For links of ≥ 3 components and $g > 0$, as in the proof of Theorem 5.1, a parallel clasp is created by (16), and the same surgery applies. (For $n \geq 4$ the "Stallings twist" in proposition 7.1 would also apply, and the resulting links are again much less sophisticatedly distinguished by linking numbers.) The case $g = 0$ and $n \geq 3$ is again easily recovered by the pretzels. □

For 2 components, however, some new idea is needed. The parallel clasp disappears, and so far we cannot prove the claim, except for special families of polynomials (it is also false if $g = 0$).

Turning back to fiberedness, we do not know about extensions of Morton's construction, explained in the beginning of this section, to obtain infinite families of links up to 3 components. The infinite realizability is (even for general links or fiber surfaces) not fully clear. As an application of our work we can obtain at least the following additional examples.

Proposition 7.3 (1) For $n = 3$ components and a monic admissible Conway polynomial ∇ with $[\nabla]_2 = -1$, there exist infinitely many canonically fibered links realizing ∇, which are connected sums of 2 prime arborescent factors.
(2) For knots ($n = 1$), the same holds for polynomials ∇ with a multiple zero. If $\nabla = \nabla_1^2$ for some $\nabla_1 \in \mathbb{Z}[z]$, then there exist infinitely many canonically fibered prime (arborescent) knots realizing ∇.

Proof. For (1) take a prime fibered knot K with $\nabla_K = -z^{-2}\nabla(z)$, and build the connected sum with $(2,-2,2k)$-pretzel links. Part (2) is an adaptation of the observation of Quach [Q]. It suffices to consider the case $\nabla = \nabla_1^2$. If $\nabla_1 \in \mathbb{Z}[z]$ and $\nabla_1^2 \in \mathbb{Z}[z^2]$, then $\nabla_1 \in \mathbb{Z}[z^2]$ or $\nabla_1 \in z\mathbb{Z}[z^2]$. Since $[\nabla]_{z^0} = 1$, former alternative applies. Then w.l.o.g. $[\nabla_1]_{z^0} = 1$ up to taking $-\nabla_1$ for ∇_1.

So we can take a knot K as in Theorem 3.1 with $\nabla_K = \nabla_1$, and build the connected sum $K\#!K$ at the parallel clasps in figure 3. The canonical surface of the resulting diagram admits Stallings twists at the spot of the connected sum. Since smoothing out a crossing created by such Stallings twists gives a diagram of an amphicheiral 2-component link L (so that $\nabla_L = 0$), again (5) shows that the twists preserve ∇. Also it is easy to observe that the diagrams are still arborescent, so infinitely many of the knots can be distinguished using [BS]. (There is again a much less sophisticated distinction argument, which uses the leading term in the Alexander variable of the skein polynomial.) □

Since a fiber surface is connected, we must have $\chi \leq 2-n$. For $(n,\chi) = (2,0)$ we have only the Hopf links. For $(n,\chi) = (3,-1)$ and $\nabla = +z^2$ we have again only 2 (composite) links, the connected sum of two positive or two negative Hopf links (see part 1 of Remark 5.1). These observations are valid not only for canonical, but also for general fiber surfaces, as is explained in [Kn2].

For $n = 2$ and $\chi < 0$, we can observe that the knots in Morton's construction (see the proof of Theorem 4 in [Mo]) likewise have unknotting number 1, which allows to obtain analogously to our case certain fibered 2-component links.

It seems some effort needed to extend Morton's JSJ decomposition arguments and show that infinitely many of these links are different. (Fibering and control of the Alexander polynomial are again not difficult.) One would then have also (at least the obvious connected sum) examples of 3 components for any polynomial.

We also do not know how to find for general (monic or not) polynomials infinitely many (fibered or not) knots with certain specific properties (like arborescent, prime, hyperbolic etc.). For knots ($n = 1$), part (2) of proposition 7.3 implies

Corollary 7.1 In genus $g \geq 4$, then there exist infinitely many monic polynomials realized by infinitely many canonically fibered prime knots. □

To reformulate this more suitably, let for $d \geq 1$,

$$\Phi_d := \left\{ \begin{array}{c} \nabla \text{ monic of degree } 2d, \text{ realized by} \\ \text{infinitely many canonically fibered knots} \end{array} \right\}.$$

Then we can understand $\Phi_d \subset \Gamma_d := \{\pm 1\} \times \mathbb{Z}^{d-1}$. We say that Φ_d is infinite if $d \geq 4$. Contrarily, $\Phi_1 = \varnothing$, and our aforementioned result in [St4] shows that Φ_2 is finite. (We do not know about finiteness of Φ_3.) So we see that, expectedly, this result does not extend to $d \geq 4$, at least in full strength. Nevertheless, for some d still the inclusion $\Phi_d \subset \Gamma_d$ may be proper, or in fact so that $\Gamma_d \setminus \Phi_d$ is infinite. The right sort of question to ask about what polynomials are realized infinitely many times, seems to be something like:

Question 7.1 Is $\Phi_d \subset \Gamma_d$ contained in the image of finitely many $d-1$-tuples of polynomials

$$(f_1, \dots, f_{d-1}) \in \mathbb{Q}[x_1, \dots, x_k]^{\times d-1},$$

each f_i of which maps $\mathbb{Z}^k$ to $\mathbb{Z}$, with $k \leq d-2$?

There is a corresponding problem for links. The question on the maximal k needed also has some right. The bound $d-2$ may be improvable, but obviously not below 1 for $d = 4,5$, and, with the origin of corollary 7.1 in mind, expectably not below $d-4$ for $d \geq 6$.

8. Large volume knots

8.1. Arborescent knots

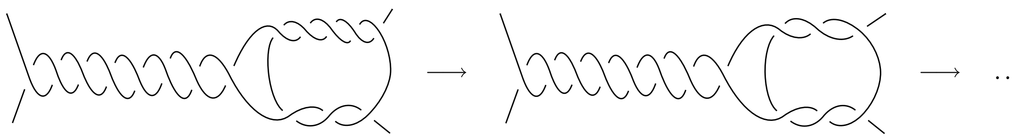
While so far we were concerned in estimating volume from above, we give, using tangle surgeries, two constructions to obtain knots of given polynomial and large volume. The case of links is left out mainly for space (rather than methodological) reasons. The first construction yields arborescent knots.

Theorem 8.1 Given an Alexander knot polynomial Δ with $d = \max\deg\Delta$ and an integer $g_s \geq \max(1, 4d-1)$, there exist hyperbolic arborescent knots of arbitrarily large volume with Alexander polynomial Δ and 4-genus g_s.

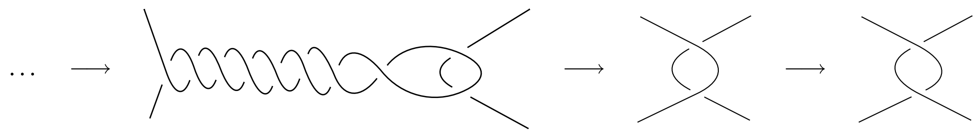
Our result is motivated by similar work of Kalfagianni [Kf], one of whose consequences (Corollary 1.1 therein) it improves. (At the end of this paper we will be able to recover Kalfagianni's full result; our tools are, however, somewhat different from hers.) A related result, that implies a certain part of the statement of Theorem 8.1, was obtained simultaneously by Silver and Whitten [SWh].

Lemma 8.1 The tangle surgeries (26) of Lemma 6.1 (for $k = 1$) alter g_s most most by ± 2.

Proof. We like to examine the change of g_s under the surgery. We change first a crossing in the twist of q.



Since $q+r=2$, applying concordance, we can cancel the remaining $q-2$ crossings with the crossings in the twist of r, and then remove the (crossings in the) twist of p. Then by switching a crossing we recreate the clasp before the tangle surgery.



Now g_s changes by at most ± 1 under a crossing change, so it changes by at most ± 2 under the tangle surgery. $\square$

Proof of Theorem 8.1. In the following we choose integer triples (p,q,r) with $p,q,-r>1$ odd, $r+q=2$ and $pq+pr+qr=-1$. We will assume that p,q,r have these properties throughout the proof.

Choose from Theorem 3.1 an arborescent knot K with $\Delta_K=\Delta$ and the arborescent diagram $\hat{D}$ constructed in the proof. Following [Ad] we call a crossing a *dealternator* if it belongs to a set of crossings whose switch makes the diagram alternating. This set is determined up to taking the complement. Since we constructed $\hat{D}$ to have at most $4d-2$ twist equivalence classes, we can choose (possibly taking the complement) the number d of twists in $\hat{D}$ consisting of dealternators to be

$$t \le 2d-1\,.$$

Now we can turn $\hat{D}$ into an arborescent diagram $\hat{D}_0$ of K, so that each of the d twist equivalence classes of dealternators in $\hat{D}$ becomes a single (dealternator) crossing in $\hat{D}_0$. Fix in $\hat{D}_0$ the set of d dealternators so obtained. Create (by a Reidemeister II move) a trivial parallel clasp near each dealternator, obtaining a diagram D_0' of K with dealternators occurring in d parallel clasps.

Now let $T_{p,q,r}$ be the tangle described in the proof of Lemma 6.1 for $k=1$, and $T_{-p,-q,-r}$ its mirror image. (So by the index shift p means now what was $p+1$ in that proof.) Let $D_0=D_0(p,q,r)$ be the result of substituting $T_{p,q,r}$ for each positive dealternator clasp tangle, and $T_{-p,-q,-r}$ for each negative dealternator clasp tangle in D_0'. Let $K_{p,q,r}$ be the knot D_0 represents. Then D_0 has all its dealternators in twists in the substituted tangles. When now the length of the twists in $T_{p,q,r}$ grows, Thurston's hyperbolic surgery theorem shows that $\mathrm{vol}\,(K_{p,q,r})$ converges (from below) to the volume of a certain link T_∞. This limit link is the same as when r has opposite sign, but then we have prime alternating diagrams. So T_∞ is an augmented alternating link (as in [Br, La]). Then in order to obtain large volume we apply Adams' result on the volume of augmented alternating links (see [Br, La]), and so it is enough to increase the number of tangles whose twist lengths we can augment unboundedly.

Simultaneously we want to carry out our construction so as to obtain large g_s. With p,q,r given, we applied the tangle surgeries of Lemma 6.1 (for $k=1$) at each clasp of dealternators in D_0' and obtained a diagram $D_0=D_0(p,q,r)$. By Lemma 8.1 we have

$$|g_s(D_0)-g_s(K)| \le 2t \le 4d-2\,. \tag{28}$$

Since $u(K)=1$, we have $g_s(K)\le 1$, so $g_s(D_0)\le 4d-1$.

We consider the pretzel knots $P(p,q,r)$, which have $\Delta=1$. By the main theorem in §1 of [Ru], these pretzel knots are quasipositive, and by Proposition 5.3 of [Ru] have slice genus 1.

Let now $D=D(l,p,q,r)$ be the diagram obtained by taking connected sum of D_0 with l copies of the (p,q,r)-pretzel diagram. (Note that now p,q,r enter into the construction of $D(l,p,q,r)$ in a second different way.) Because $P(p,q,r)$ is quasipositive of 4-genus one, we have by the Bennequin-Rudolph inequality (see [Ru2])

$$g_s(D(l,p_l,q_l,r_l)) \to \infty \tag{29}$$

when $l\to\infty$, for any sequence (p_l,q_l,r_l) of triples (p,q,r) of the above type. Moreover, the numbers (29), when taken over all $l\ge 0$, realize all integers $g_s\ge 4d-1$, again regardless of the choice of (p_l,q_l,r_l).

We apply now the moves (27). Choose the connected sum in D so that the creation of two parallel clasps in the first move in (27) gives a prime diagram D'. The second move is a tangle surgery, which preserves Δ and can be performed for any triple (p,q,r). (In (27) we show the operation for the simplest triple, which after the shift of p is now $(7,5,-3)$.) Call the resulting diagram $D''=D''(l,p,q,r)$, and $K''=K''(l,p,q,r)$ the knot it represents. Since this surgery is a concordance, we have

$$g_s(D'')=g_s(D)\,. \tag{30}$$

So from (29) and (30) we have then

$$g_s(D''(l,p_l,q_l,r_l))\to\infty\,,$$

when $l\to\infty$ and (p_l,q_l,r_l) is an arbitrary sequence of tuples (p,q,r). Moreover, all numbers above or equal to $4d-1$ are realized as 4-genera. Now D'' has all its dealternators occurring in twists whose length can be augmented arbitrarily, preserving Δ. So if for each l we choose $-r_l$ (and hence q_l,p_l) large enough, we obtain hyperbolic knots $K_l=K''(l,p_l,q_l,r_l)$ of large volume from the results of Thurston and Adams.

In order to obtain infinitely many knots of fixed 4-genus take in the construction of $D(l,p,q,r)$ connected sum with (p,q,r)-pretzel diagrams and mirror images thereof (with reverse orientation). The volume will distinguish infinitely many of the knots K_l.

To verify that K_l is arborescent, use that we chose the initial diagram $\hat{D}$ of K to be arborescent. Taking iterated connected sum with the (p_l,q_l,r_l)-pretzel knots and adding clasps can be done so as to preserve arborescency of the diagram. The same observation applies to the tangle surgeries. □

Using the upper bound in Theorem 2.1, we have a result on growing twist numbers.

Corollary 8.1 Any possible Alexander polynomial is realized by arborescent knots K_l with twist number $t(K_l)\to\infty$. □

Remark 8.1 Our construction can be easily adapted to preserve the Alexander module. Choose a prime s such that all (finitely many up to units) divisors of Δ in $\mathbb{Z}[t^{\pm1}]$ (including Δ and 1) remain distinct (up to units) when coefficients are reduced $\bmod s$. Then choose p,q,r so that $p+1,q,r\equiv 1\,(2s)$, by choosing (for $k=1$) n in (25) divisible by s. Observe that changing any of p,q,r by (multiples of) $2s$ preserves a (properly chosen) Seifert matrix $\bmod s$, and the Seifert matrix determines the Alexander module. Since our arguments incorporate concordance, we can recover most of the properties obtained by Silver and Whitten [SWh], except of course the knot group homomorphism.

8.2. Free genus

Our final result combines all the methods introduced previously to obtain an extension of a theorem of Brittenham [Br2]. He constructed knots of free genus one and arbitrary large volume. We state a similar property for free genus greater than one.

Theorem 8.2 Let Δ be an admissible Alexander knot polynomial of degree $d\geq 2$. Then there exist hyperbolic knots K_n of arbitrarily large volume with free genus $g_f(K_n)=d$ and $\Delta(K_n)=\Delta$.

Remark 8.2 As to extensions and modifications of this statement, the following can be said:

1) Our construction does not apply for free genus one. The Alexander polynomial is not of particular interest on genus one knots, so its control in Brittenham's (or some similar) construction seems only of minor use, and we will not dwell upon this here.

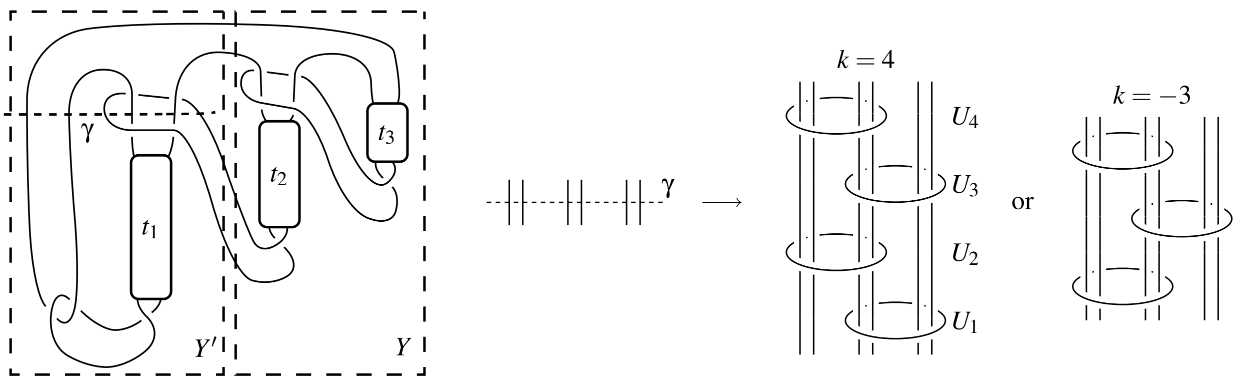
2) A justified question is whether for monic polynomial we can actually find fibered knots. We expect that it is possible, but the effort of proof would grow further, too much for the intention and length of this paper.

3) Another suggestive question, whether one can replace free by canonical genus, is to be answered negatively. Brittenham had shown [Br] that canonical genus bounds the volume (see also [St3]).

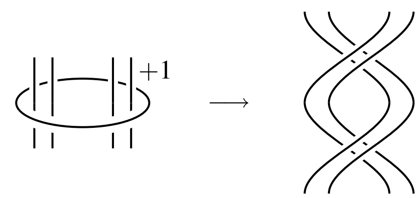
4) The knots we obtain are unlikely arborescent or of unknotting number one, but still have slice genus at most one if $g_f > 2$.

5) The case of links is, like the explanation at the beginning of this section, analogous to treat (with similar mild constraints), but also left out for space reasons.

Proof. Let first $g_f > 2$. For a given number k we consider the link $L_k = K \cup U_1 \cup \ldots \cup U_k$ given by replacing the diagram of the knot K from theorem 3.1 along the (more tightly) dashed line γ below as follows:



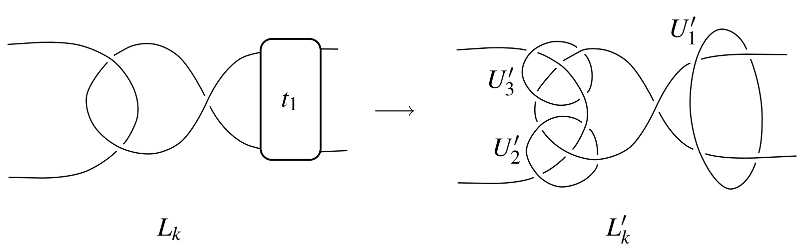
(We extend this to $k < 0$ by placing the circles U_i the other way, as shown.) Choosing a, b sufficiently large, we construct the knots $K_{k,a,b}$ from L_k for $4 \mid k$ by doing

$\left\{ \begin{array}{c} a \\ b \\ -a \\ -b \end{array} \right\}$ twists at U_k for $k \equiv \left\{ \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \end{array} \right\}$ mod 4, in the following way:  .

Here a few annotations seem proper. (i) The twists along U_k are called in the common cut-paste-language surgeries. However, we avoid this term here in order not to confuse with the tangle surgeries (which will just reenter). The "twists" may, in turn, conflict with definition 2.5, but they can be regarded here as an extension of the previous concept, and so seem the more convenient term. (ii) Twisting along U_i adds also a full twist (now in a sense directly related to definition 2.5) into the bands. However, these twists cancel each other when twisting at U_i is performed in the prescribed way, so we can ignore them.

It is easy to see now that $K_{k,a,b}$ has the same Alexander polynomial as K, since the Seifert matrix is not altered by the twisting at U_i. Similarly, the twisted Seifert surface is still free. By thickening the surface into a bicolar, we see that the twisting at U_i accounts only in braiding the various 1-handles, and this braiding can be undone by sliding the handles properly, as for the braidzel surfaces [Ru, Na3].

With this we focus on hyperbolicity. By Thurston and Adams again it suffices to show that L_k are hyperbolic for large $|k|$. (We need in fact here only $k > 0$ and $4 \mid k$, but we will soon see why it is good to have the other k around, too.) We use the tangle decomposition $Y \cup Y'$ of K, which carries over with modifications to L_k. (In order not to overwork notation, we denote Y, Y' the same way in all links, each time specifying the link.) First we use tangle surgery to remove the dependence of Y' in L_k on the number t_1 of full twists. The surgery allows us to replace the lower part of Y' as follows:



(31)

The meaning is that we can have a free surface and a desired Alexander polynomial by applying a proper, but arbitrarily augmentable, number of twists at the circles we added. Now U_1' is in fact parallel to U_1 for $k=1$. So we can, and for hyperbolicity must, omit U_1' then. This can be done with the understanding that we perform at U_1 the additional twists we would have needed to perform at U_1'.

The effect of the surgery is now that the link L_k', whose hyperbolicity it suffices to show, has a tangle Y' which does no longer depend on t_1, but only on k.

Lemma 8.2 The links $L=L_k'$ are prime.

Proof. The (only, but then indeed so because of Δ) knotted component K of L_k' is prime; e.g. it has unknotting number one. Thus if L is composite, there is a composite (possibly split) 2-component sublink $L'=K\cup O$ of L. Now, for such sublinks, the tangle Y' reduces only to finitely many cases; in fact 3 are enough to test (using that U_1 and U_1' in (31) are parallel, and U_2' and U_3' are flype-equivalent). These 4 tangles Y' can be checked to be prime by [KL], and since the same can be done for Y (despite of its dependence on $t_2,t_3,\dots$), we have that L' is prime, a contradiction. Thus L is prime. □

Lemma 8.3 The links $L=L_k'$ are atoroidal.

Proof. We first prove for $|k|\le 3$. The main point here is to remove the dependence of Y on $t_2,t_3,\dots$.

The t_2 twists can be easily removed by tangle equivalence. The argument that eliminates $t_3,t_4,\dots$ consists in a repetition of our work in applying Oertel's and Wu's results, so we just recapitulate the main points. Now $Y_1=Y$ and $Y_2=Y'$, and we have $L=Y\cup Y'$, with Y being a Montesinos tangle of length 2 for $g_f(K)=3$, or an arborescent tangle subjectable to Wu's result for $g_f(K)>3$. Assume T is an essential torus of L. Then again $T\cap X(Y)$ is empty, all of T, or an annulus A.

If A exists, then by Sublemma 4.3 and the argument after it, A is ∂-parallel to C, so can be moved out. If $T\subset X(Y)$, we have a contradiction to Wu for $g_f>3$, or by gluing Y and A to itself and Oertel's result if $g_f=3$. Thus $T\subset X(Y')$.

When $T\subset X(Y')$, then T is essential in $E(L)$ even after modifying Y, as long as Y is prime and has a closed component. Since for $|k|\le 3$, we have only finitely many Y', we can easily find a proper prime tangle Y and check the hyperbolicity of the handful of links $L=Y\cup Y'$ by SnapPea to see the contradiction to the existence of T. With this argument the atoroidality is proved for $|k|\le 3$.

Now let $|k|\ge 4$. We use induction on $|k|$ (where the cases $4\nmid k$ enter). Assume T is again an essential torus of L_k'. As for $|k|\le 3$, we can argue that $T\subset X(Y')$.

By induction, T is inessential in $L_k'\setminus U_1$ and $L_k'\setminus U_{|k|}$. (Here the use of L_k' also for $k<0$ pays off.) There are two cases.

Case 1. T is ∂-parallel in $L_k'\setminus U_1$. So T contains in one of its complementary regions either only U_1, or U_1 and exactly one other component V, to which it becomes ∂-parallel after removing U_1. In particular in latter case T must have the knot type of V. Now T separates components in $L_k'\setminus U_{|k|}$, and so it must be ∂-parallel there either. Applying the same argument to $L_k'\setminus U_{|k|}$ shows then that T must contain exactly U_1 and $U_{|k|}$ in one of its regions R, be unknotted, and have them as cores of the solid torus $R=:\operatorname{int}T$. Clearly the same conclusion follows if we assume T is ∂-parallel in $L_k'\setminus U_{|k|}$.

Now, if one removes U_2 and U_3 from L (here the assumption $|k|\ge 4$ enters), then again T must become inessential. However, $\widetilde{L}=L\setminus U_{2,3}$ is non-split by the previous lemma, and the exterior of T in $E(\widetilde{L})$ contains the knotted component K. Then T cannot compress or be ∂-parallel in its exterior, but the same applies to its interior either, a contradiction.

Case 2. T compresses in $L_k'\setminus U_1$ and $L_k'\setminus U_{|k|}$. Since either links are non-split, there is only the option that T is knotted and $\operatorname{int}T$ contains all (components) of L_k', but compresses (along a meridional disk) when removing U_1 or $U_{|k|}$ from L_k'. Let us assume, by having already ruled out the other cases, that *all* essential tori T of L_k' are of this type. The exclusion of these tori requires a bit more argument.

Lemma 8.4 Assume $L\subset S^3$ is a link and $T\subset E(L)=S^3\setminus L$ is a torus with $L\subset\operatorname{int}T$. Moreover assume $[L]=0$ in $H_1(\operatorname{int}T)=\mathbb{Z}$, and that L bounds (in S^3) a free Seifert surface S of maximal Euler characteristic (i.e. $\chi(S)=\chi(L)$). Then $T\subset E(L)$ is compressible.

Proof. The case that T is unknotted is trivial, so assume T is knotted. We consider $T \cap S$, which is a collection of disjoint curves on T. If some such curve γ is contractible on T, then either S would compress (along a disk that γ bounds on T), in contradiction to $\chi(S) = \chi(L)$, or γ could be removed by isotopy of T.

So assume all curves of $S \cap T$ are essential in T. Then for a choice of longitude l and meridian m of T, for each such curve γ we have, up to orientation, $\gamma = al + bm$, with coprime integers a, b independent on γ.

First, none of these γ bounds a disk in $\operatorname{ext} T$. Would γ bound, it would be unknotted. The only unknotted essential curve on a knotted torus is the meridian. But the meridian does not bound a disk in $\operatorname{ext} T$.

Consider two cases.

Case 2.1. Let first $a \neq 0$. Since we assume $[L] = 0$ in $H_1(\operatorname{int} T) = \mathbb{Z}$, the number of γ is even, and, with the orientation induced from $S \cap \operatorname{ext} T$ (or the one of $S \cap \operatorname{int} T$), exactly one half of them is oriented either way.

Then one can easily find a collection of annuli in $\operatorname{ext} T$ that realize these curves as boundary. Since $S \cap \operatorname{ext} T$ contains no disk component, and $\chi(S) = \chi(L)$, we see that $S \cap \operatorname{ext} T$ must be likewise a (possibly different) collection of annuli. Then one sees that there is always an innermost annulus in $S \cap \operatorname{ext} T$, which can be removed by isotopy of T (fixing the other components of $S \cap \operatorname{ext} T$).

Case 2.2. Now consider $a = 0$. Then γ are meridians m of T. These are non-trivial in $H_1(\operatorname{ext} T)$, so again the number of γ is even, and one half is oriented either way. With the same argument as in case 2.1, we move T out of S.

So now we achieved that $S \cap T = \varnothing$, that is, $S \subset \operatorname{int} T$. But S is free, so (by definition) $S^3 \setminus S$ is a handlebody, and it contains T. This means that T compresses in $S^3 \setminus S$ (see e.g. the paragraph above Proposition 3.4 in [SWh]), and hence also in $S^3 \setminus L$. $\square$

Remark 8.3 The condition $[L] = 0$ is necessary. Consider cables of positive braid knots, which are again positive braid knots or links. These links clearly bound canonical minimal genus (even fiber) surfaces.

Return to the torus $T \subset E(L'_k)$ with $L'_k \subset \operatorname{int} T$. Since T compresses in $E(L'_k \setminus U_1)$ and $E(L'_k \setminus U_{|k|})$, it has meridional disks which intersect L only in U_1 or only $U_{|k|}$. It follows that each component of L'_k is contained in a ball inside $\operatorname{int} T$, in particular $[L'_k] = 0$ in $H_1(\operatorname{int} T)$.

The assumption we comforted ourselves with, that all essential T have $L'_k \subset \operatorname{int} T$, means that in the JSJ decomposition tree of $L = L'_k$ (see e.g. figure 6 of [Mo]), all components of L lie in the same leaf v. The only vertex w adjacent to v in the tree corresponds to an essential torus $T' \subset E(L)$ such that $L \subset \operatorname{int} T'$ and $\operatorname{int} T' \setminus L$ is atoroidal. Moreover, T is knotted, and either $T = T'$ or $T' \subset \operatorname{int} T$, but T' is not contained in a ball inside $\operatorname{int} T$. Thus we see that T' is knotted, too. So we may w.l.o.g. assume that we chose an essential torus T so that $\operatorname{int} T \setminus L$ is atoroidal.

Now we can reembed T (and $\operatorname{int} T$) unknottedly to a torus $\hat{T}$, and L to a link $\hat{L} \subset \operatorname{int} \hat{T}$ (where $\operatorname{int} \hat{T}$ is chosen in the obvious way), and add the complementary (unknotted) core U to obtain an atoroidal link $\widetilde{L} = \hat{L} \cup U$ in S^3. Now since each component of L is contained in a ball inside $\operatorname{int} T$, we see that all components of $\hat{L}$ have the same knot types as the corresponding components of L. The knotted component is not a torus knot, e.g. by part 4 of remark 8.2. So by [BM], $\widetilde{L}$ is not Seifert fibered, and thus it is hyperbolic.

All unknotted components U_i of L bound mutually disjoint disks D_i in S^3, and by the argument in lemma 8.4, we may assume that $D_i \subset \operatorname{int} T$. Then the same sort of disks $\hat{D}_i$ bound $\hat{U}_i$ in $\operatorname{int} \hat{T}$. Now since $\widetilde{L}$ is hyperbolic, by Thurston's hyperbolic surgery theorem for all sufficiently large coefficients surgery at $\hat{U}_i$ gives a hyperbolic link $\tilde{K}_0 = U \cup \hat{K}_0$, where $\hat{K}_0$ is what the knotted component $\hat{K}$ of $\hat{L}$ is transformed under the surgery. This surgery does not affect $\hat{T}$, so we can clearly do the same surgery along U_i in $\operatorname{int} T$, and this commutes with the reembedding $T \to \hat{T}$.

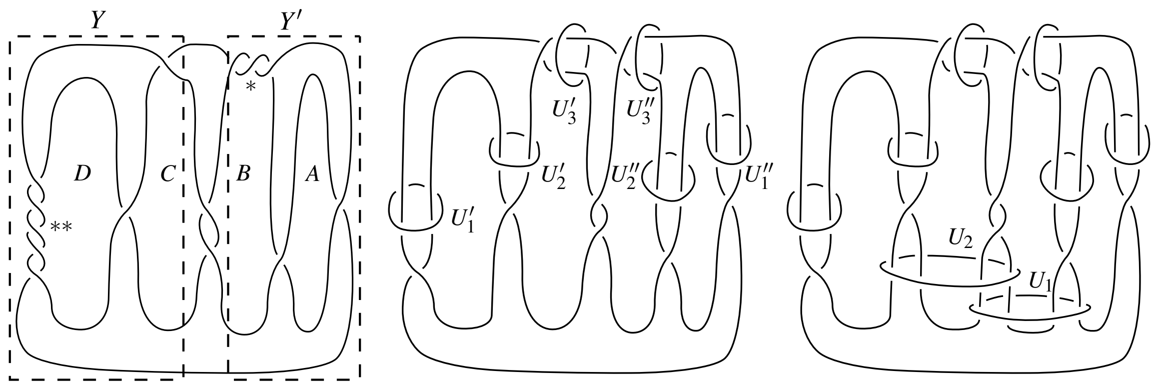
On the other hand, among any sufficiently large surgery coefficients at U_i, we can find such that the Alexander polynomial of the knotted component K of L is not altered. Thus by construction the surgered knot K_0 has a free minimal genus surface. So by lemma 8.4, we see that T compresses in $\operatorname{int} T \setminus K_0$, along a meridional disk. But then $\hat{T}$ compresses in $E(\tilde{K}_0)$, giving a splitting shpere between U and $\hat{K}_0$, which contradicts the hyperbolicity of $\tilde{K}_0$. (See also end of first paragraph of the proof of Lemma 3.3 in [SWh].)

With this argument the essential torus T in case 2 is also excluded, and lemma 8.3 is proved. $\square$

Lemma 8.5 The links $L = L'_k$ are not Seifert fibered.

Proof. Again components of Seifert fibered links are (possibly trivial) torus knots, and for our links we have a knotted component of unknotting number one. It must be then a trefoil, but then we are in the situation $g_f = 1$, which we chose not to consider. □

Now we have shown the theorem for $g_f > 2$. Our procedure does not work, though, for $g_f = 2$ (exactly the same way; for example, then Y is no longer prime). In that case, we realize V_2 of (9) as a Seifert matrix in the way shown in the diagram (a) of (32). Here we took the example with $a_1 = a_2 = 2$. The Conway polynomial is $\nabla = 1 - a_1 z^2 + a_2 z^4 = 1 - 2z^2 + 2z^4$. In general the half-twists at * are $2a_1 - 1$, and those at ** are $2a_2 + 1$. (Again -1 half-twist is a crossing of negative skein sign.)



(a) (b) (c) (32)

The rows/columns of V_2 correspond to curves that go in positive direction along the regions A, B, C, D. The curves for A and B, resp. C and D, intersect once on the lower Seifert circle; otherwise curves do not intersect.

Now observe that again we can apply a surgery in Y and Y' (where in lemma 6.1, we have $k = 1$ for Y' and $k = a_2 \neq 0$ for Y). It allows to arbitrarily augment the number of twists, keeping Δ and the surface canonical. This has the effect of eliminating the dependence on Δ (i.e. on $a_{1,2}$) of the link, whose hyperbolicity it is enough to show; see (b) in (32). Denote the triples of circles occurring for the surgery in Y by U_i', and let those for Y' be U_i''.

Finally, we must add the circles U_i around pairs of bands. This is done as shown for $k = 2$ in part (c) of (32). Since the links L_k we obtain depend only on k, we can use the same type of inductive argument to show atoroidality, checking the initial links by SnapPea. (To rule out a Seifert fibration for $\widetilde{L}$, one may need to apply tangle surgery so as to avoid the knotted component to be 5_1.)

We use then twisting at the U_i again for $4 \mid k$ in the previously specified way. It may be worth remarking that, to see the preservance of Δ, the twists along U_i' and U_i'', resulting from the tangle surgeries, must be performed before those at U_i. The U_i enter into the tangle the surgeries are performed at. Inspite of this, the resulting modifications are independent from each other, so no conflict arises.

To exclude a Seifert fibration for L_k, note that if the not obviously unknotted component K' is indeed knotted, none of the Burde-Murasugi links has such a component (even if a torus knot), and more than two unknotted ones. If K' is unknotted, the Seifert fibration for L_k is excluded using linking numbers. A look at the Burde-Murasugi list shows that there is no link with all linking numbers zero, except the trivial link (unlink). This is excluded by looking at a proper sublink of L_k. □

Remark 8.4 Observe that the twisting at the components U_i corresponds in an obvious way to a (power of the) commutator $[\sigma_1^a, \sigma_2^b] = \sigma_1^a \sigma_2^b \sigma_1^{-a} \sigma_2^{-b}$ in the 3-strand braid group B_3. Using higher order commutators (and leaving out the tangle surgeries), one can preserve, additionally to Δ, Vassiliev invariants of given degree. Then from the argument for K' being unknotted, one easily recovers the main result of Kalfagianni [Kf]: given $n > 0$, there exist hyperbolic knots K_n of arbitrary large volume with $\Delta = 1$ and trivial Vassiliev invariants of degree $\leq n$. (In our construction also $g_f(K_n) \leq 2$.)

We expoited this idea further in our subsequent work [St7], to show that K_n can be chosen to be n-similar (i.e. with Vassiliev invariants of degree $\leq n$ coinciding) and with the same Alexander polynomial as any given knot K. Still that construction demands to abandon the property $\max\deg\Delta(K) = g_f(K)$, so that the result in [St7] is not a genuine generalization of Theorem 8.2.

9. Questions and problems

We mentioned already, for example in sections 7 and 8, several problems, that may be the topic of future research. We conclude with one other group of further-going questions, concerning special knots realizing Alexander polynomials.

After we were able to incorporate arborescency into most of our constructions, it makes sense to ask in how far one can further restrict the type of knots.

Question 9.1 Are arbitrary Alexander polynomials realizable by Montesinos knots (perhaps), or even general pretzel knots (unlikely)?

The following argument shows that at least among pretzel knots restrictions on the Alexander polynomial may apply.

Proposition 9.1 There exist Alexander polynomials not realizable by any generalized pretzel knot $(a_1,\dots,a_{2n+1})$ with a_k odd, for any n.

Proof. If we use equivalently ∇, then a direct skein argument shows that all coefficients $\nabla_j = [\nabla]_{z^j}$ for even j, are polynomials in $a_1,\dots,a_{2n+1}$ of degree at most j. (One can also argue with the work in [St] and the well-known fact that ∇_j is a Vassiliev invariant of degree at most j.) Also, these polynomials are at most linear in any a_k. Furthermore, they are symmetric in all a_k, since permuting a_k accounts for mutations, that preserve ∇. So ∇_j is a linear combination of elementary symmetric polynomials σ_i in a_k for $i \leq j$. Then one also finds that σ_j indeed occurs in this linear combination, and only σ_i for even i occur. (Latter property is due to the fact that ∇ is invariant under taking the mirror image.) So, up to linear transformations, it is enough to see that some integer tuples $(\sigma_2,\sigma_4,\dots,\sigma_j)$, even for σ_i satisfying certain congruences, cannot be realized as values of elementary symmetric polynomials of any odd number of odd integers a_k. But σ_i occur as coefficients of the polynomial

$$X(x) = (x-a_1)(x-a_2)\dots(x-a_{2n+1}),$$

and it is known that the coefficients of polynomials with real roots satisfy certain inequalities; they are log-concave (see Theorem 53 in [HLP]). So for example any triple $(\sigma_2,\sigma_4,\sigma_6)$ with $0<\sigma_4<\sigma_2<\sigma_6$ will not occur. $\square$

Another question addresses an important point as to how a volume estimate can be strengthened.

Question 9.2 Is there a global constant C, such that all Alexander polynomials are realized by hyperbolic knots of volume $\leq C$?

One can pose the analogous questions also for links.

10. Result summary

Table 1 summarizes the state of knowledge about realizing (monic) Alexander polynomials by links with a canonical minimal genus (or fiber) surface, depending on the number of components, the Alexander polynomial and whether one or infinitely many such links are sought.

Acknowledgements. I would like to thank to Mikami Hirasawa, Efstratia Kalfagianni, Taizo Kanenobu, Kunio Murasugi, Takuji Nakamura, Makoto Sakuma, Dan Silver and Ying-Qing Wu for some helpful remarks, discussions, and references. This work was partly carried out during a stay in Japan under a JSPS Postdoc grant and the 21st Century COE Program.

Table 1: The realizability status of given Alexander polynomials by given number of given type of knots or links. The boldfaced entries refer to the contribution of this paper.

# comps	arbitrary $\Delta \neq 0$ $2\max\deg\Delta = 1-\chi_c$ one link	arbitrary $\Delta \neq 0$ $2\max\deg\Delta = 1-\chi_c$ ∞ many	monic Δ one canon. fibered link	monic Δ ∞ many canon. fibered links	monic Δ ∞ many fibered links
1	**yes** (**arbor.**; Theorem 3.1) **hyp.** for $g>0$ (Remark 3.2)	**yes** (**arbor.**; propos. 7.2) for $g>0$ (no for $g=0$)	**yes** (**arbor.**), **hyp.** except unknot or trefoil (Theorem 3.1)	no for $g \leq 1$ and almost all Δ in $g=2$ [St4]; **yes** for ∇ with double zero (propos. 7.3); unknown in general for $g \geq 3$	no for $g \leq 1$; yes for $g \geq 2$ [Mo]
2	**yes** (**arbor.**) **hyp.** for $g>0$ (theorem 4.1)	unknown; no for $g=0$	**yes** (**arbor.**) **hyp.** for $g>0$ (Theorem 4.1)	no for $g=0$ and almost all Δ in $g=1$ [St4]; else unknown	no for $g=0$; unknown, likely yes (modif. of Morton; see §7) if $g>0$
3	**yes** (part 3 of Remark 5.1, proposition 7.2; **hyp. arbor.**)	**yes** (propos. 7.2; **arbor.**)	**yes** (Theorem 5.1; **hyp. arbor.**) if $\nabla \neq +z^2$; only compos. exist if $\nabla = +z^2$	**yes** if $[\nabla]_2 = -1$ (propos. 7.3; compos. links); no if $\nabla = z^2$, else unknown	no if $g=0$, $\nabla = z^2$ (see rem. in [Kn2]); **yes** if $[\nabla]_2 = -1$ (compos. links); else unknown
≥ 4	**yes** (**hyp. arbor.**)	**yes** (**hyp. arbor.**)	**yes** (Theorem 5.1; **hyp. arbor.**)	**yes** (prop. 7.1; **hyp. arbor.**)	**yes** (**arbor.**)

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