GAUSS SUM INVARIANTS, VASSILIEV INVARIANTS AND BRAIDING SEQUENCES

A. Stoimenow*

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Abstract. We introduce a new approach to Vassiliev invariants. This approach deals with Vassiliev invariants directly on knots and does not make use of diagrams. We give a series of applications of this approach, (re)proving some new and known facts on Vassiliev invariants.

Keywords: k-equivalence, braids, alternating knots, invertibility, chirality, mutation, Vassiliev invariants, closed 3-braids, Gauß sums, n-trivial knots, Jones polynomial, algebraic knots, positive knots and braids, intersection graph conjecture.

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*Humboldt University Berlin, Dept. of Mathematics, Ziegelstraße 13a, 10099 Berlin, Germany, e-mail: stoimeno@informatik.hu-berlin.de, WWW: http://www.informatik.hu-berlin.de/~stoimeno. All author's contributions are available on this web page.

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1. Introduction

Recently, T. Fiedler [28] defined some conceptually new knot invariants using the ansatz of Gauß sums. He generalized the approach of Polyak and Viro [55], whose invariants were known to be all of finite type (Vassiliev invariants, VI) [3, 4, 12, 85, 84]. As low degree Vassiliev invariants are contained in (cables of) the classical knot polynomials [12], this only gave some new formulas for known invariants. The generalized formulas of Fiedler were hoped not to be of classical skein and quantum theoretical origin and therefore to be able to attack the problems not (so successfully) affected by the knot polynomials, namely orientation and mutation [67].

In the following we study a maybe fundamental property of Gauß sum (GI) and Vassiliev invariants, their polynomial behaviour on certain sequences of knots, called *braiding* sequences. Braiding sequences have been considered independently in special cases by Trapp [82] and later by Stanford [64]. We introduce the notion of a braiding polynomial invariant (BPI) based on this property. Contrarily to Stanford, we are interested in knot-theoretical applications rather than nice algebraic properties. This suggests some interesting and probably completely new points of view in studying these two classes of invariants. Unfortunately, it turns out, that BPI are Vassiliev invariants, disappointing our hopes in [67].

However, the approach of braiding sequences has many further consequences.

We prove several applications of BPI to positive and alternating knots and the Jones polynomial [34]. In particular we show that if a primitive Vassiliev invariant detects orientation, it has to do so on a prime closed positive braid.

We prove, that all finite (and also very many infinite) collections of coefficients of the Jones polynomial [34] do not suffice to construct a non-trivial Vassiliev invariant, giving a sharp contrast to the result of Birman and Lin [12].

Another more unpleasant result is, that the polynomial behaviour of BPI provides obstructions to their degree for detecting orientation and mutation for low crossing number. We discuss an approach how to find such obstructions and, once they do not apply, how to decide using a finite number of tests whether a BPI detects orientation on large classes of knots, as closed 3-braids and algebraic knots. On the other hand, *sufficient* conditions for constructing a BPI out of such obstructions are yet missing. Finding them would be a crucial progress, because then it would allow, similarly to Kontsevich's theorem [42], to describe Vassiliev invariants on some knot class in purely algebraic terms.

Counting of the tests needed to perform gives upper bounds for the dimension of the space of Vassiliev invariants of given degree restricted to the given knot class. Although some of the resulting estimates (for rational knots and 3-braids) can be easily recovered (using braid or plat closures) from classical Vassiliev theory arguments for braids [6, 32], our approach extends them in several ways which seem very unclear in the chord diagram context. See [75, 74]. Unfortunately, (yet?) it seems not possible to handle the general case, though.

In some extreme cases, the obstructions for detecting orientation can be pushed ad hoc fairly far. The best example in this direction I have is, using the work of Birman and Menasco [13] on 3-braids, an infinite series of closed 3-braids, where we prove in a few lines that no Vassiliev invariant of degree ≤ 11 detects orientation, without involving excessive computer calculations as in [5, 41]. Now such knots can be easily constructed analogously to n-trivial knots [63, 52, 72], but all these constructions are not very economical w.r.t. the crossing number (the simplest example would probably produce Ng's construction with about 36 crossings), whereas the simplest knot is our series is a 14 crossing knot (14_{18676} in Thistlethwaite's tables [31]). On the other hand, we show that (somewhat sharper than the computational result of [41] for these concrete knots) in fact no Vassiliev invariant of degree ≤ 11 for 3-braids, which extends under the closure operation (that is, coincides on Markov equivalent braids), detects orientation of these knots. While it is easily provable (and done by Trapp [82]) that any Vassiliev invariant for 2-braids (which automatically extends under closure) always extends (non-uniquely) to a Vassiliev invariant of all knots, the situation is unclear (at least to me) for higher braid groups.

We show how to recover Stanford's construction [63] from our setting and we prove, generalizing Gousarov's n-triviality idea, that no Vassiliev invariant of degree 9 distinguishes the Kinoshita-Terasaka/Conway mutant knots and use this to show how Le's counterexample to the intersection graph conjecture of [22] works by n-triviality arguments much easier without Kontsevich integral and also in any non-zero (field) characteristic except 2,3,5,7.

In all this, we work over a commutative ring with unit. Only for the examples of closed 3-braids we will need it to be zero divisor-free (so as to deduce for polynomials over the ring $P_1|P \wedge P_2|P \Rightarrow P_1 \cdot P_2|P$).

2. Braiding sequences and Gauß sum invariants

2.1. Gauß sum invariants in the sphere and solid torus

First, we recall [67, 55] the concept of Gauß sum invariants (GI).

Consider a knot $K : S^1 \hookrightarrow \mathbf{R}^3$ (S^1 and $\mathbf{R}^3$ oriented). Decompose $\mathbf{R}^3 = \mathbf{R}^2 \oplus \mathbf{R}$ so that the projection (henceforth called *knot diagram*) of K into $\mathbf{R}^2$ is generic. To this projection we can assign a *Gauß diagram* (GD), a circle with oriented chords, by connecting points in S^1 mapped to a crossing and orienting the chord from the preimage of the undercrossing to the preimage of the overcrossing. See [55].

Figure 1 shows the knot 6_2 in its standard projection and the corresponding Gauß diagram.

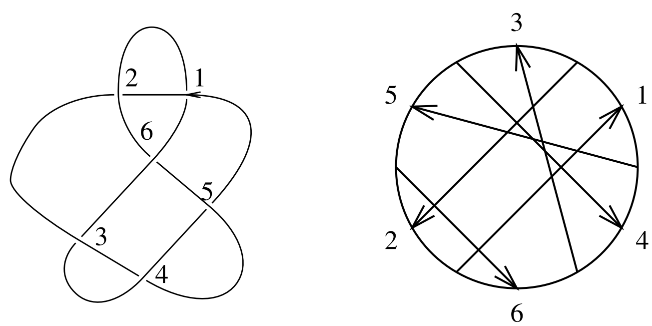


Figure 1: The knot 6_2 and its Gauß diagram.

A *Gauß sum* of degree k is a term assigned to a knot diagram, which is of the following form

$$\sum_{\substack{\text{ordered choices of } k \text{ crossings of the}\\ \text{knot diagram, whose arrows in the GD}\\ \text{form a given subdiagram}}} \text{function}\big(\text{ data, assigned to the crossings }\big)\,.$$

Each summand we will call *weight* and the function *weight function*. We will denote the summation by the subdiagram itself, which we will also call *configuration*, the choices of k crossings appearing in the sum we call *matching* the configuration.

Now we need to specify the data assigned to the crossings.

Definition 2.1 The *winding index* of a plane curve $C \subset \mathbf{R}^2 = \mathbf{C}$ around a point $p \notin C$ is

$$w(C, p) := \frac{1}{2\pi i} \oint_C \frac{1}{z-p}\, dz\,.$$

Pictorially it measures how many times the curve “walks” around p, counting reverse walk negatively.

Definition 2.2 The *Whitney index* $n(C)$ of a plane curve C is the degree of the map

$$\frac{C'}{||C'||} : S^1 \longrightarrow S^1 .$$

The Whitney index of a knot diagram is the Whitney index of its underlying plane curve.

Definition 2.3 The *writhe* $w(D)$ of a knot diagram D is the sum of the writhes of all crossings (see figure 2).

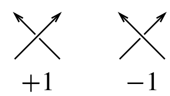
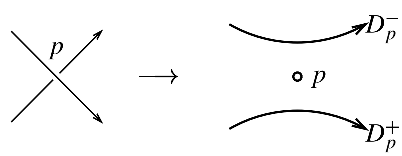


Figure 2: The writhe of a crossing.

Example 2.1 The standard projection of 6_2 on figure 1 has Whitney index 1 and writhe 2.

Definition 2.4 A *smoothing* of a crossing is the procedure



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where D_p^+ denotes the component, where the under-crossing is smoothed to the over-crossing. Note, that beside the link diagram resulted after this operation, we have the “trace” of p in its complement.

Apart from its writhe w_p, for each crossing p we have some more data:

$$i_p^{\pm} := w(D_p^{\pm}, p), \qquad n_p^{\pm} := n(D_p^{\pm}), \qquad w_p^{\pm} := w(D_p^{\pm}) .$$

Here by p we mean the trace of p in the complement of $D_p^{\pm}$, as described above. Set

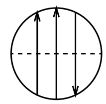
$$i_p := i_p^+ + i_p^-, \qquad \delta_p := i_p^+ - i_p^- .$$

Now consider a two component link $K \cup T$ in S^3 where T is the trivial knot (unknot). Let K, T, S^3 be oriented. Deform $K \cup T$ in $S^3 = \mathbf{R}^3 \cup \{\infty\}$ so that $\infty \in T$. This isotopy is unique up to isotopy. Such a link we can represent choosing an appropriate projection $\mathbf{R}^3 \to \mathbf{R}^2$ as knot with a point in its complement, on which T projects, assuming the orientation of T to be from the sheet of paper to the reader’s eye.

Definition 2.5 The *type* of a crossing p is $w(D_p^+, T)$ mod 2.

2.2. Braiding sequences

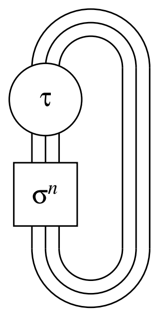
Definition 2.6 A $\uparrow\downarrow$ braid is a braid where both orientations of the strands are allowed. (More formally a $\uparrow\downarrow$ braid can be viewn as a morphism in a tangle category $\mathcal{T}$, whose objects are sequences of $\{\uparrow,\downarrow\}$). A one parameter braiding sequence is a sequence of knots $\{K_{R,\sigma,p}\}_{p\in\mathbf{Z}}$, where for some knot (diagram) K, containing a room R with k inputs and outputs and a trivial inhabitant



(i. e. an inhabitant with no crossing so that all strands intersect exactly once the horizontal axis of the room), this inhabitant is replaced by powers p of some fixed pure $\uparrow\downarrow$ braid σ with k strands (with the correspondence of inputs and outputs established by the trivial inhabitant). Parametrize this sequence by the power of the braid p. Very often R and σ will be clear from the context and hence omitted.

For rooms and inhabitants, see [44].

Alternatively spoken, we take powers of a fixed pure $\uparrow\downarrow$ braid σ, multiply them (stack them up) by a tangle τ (whose source and target object in $\mathcal{T}$ equals the one of σ), and close them up: $K_p := \widehat{\tau\sigma^p}$.



Both, Gauß and Vassiliev invariants have the following common property, because of different arguments.

Lemma 2.1 Gauß and Vassiliev invariants of deg $\leq k$ behave on braiding sequences with braid σ and tangle τ as a polynomial $P_{\tau,\sigma}$ of deg $\leq k$ in the power of the $\uparrow\downarrow$ braid.

Proof for Vassiliev invariants. As any pure $\uparrow\downarrow$ braid can be trivialized by crossing changes, $K_p^{(1)} := K_p - K_{p-1} \in \mathcal{V}^1$ is a 1-singular knot ($\mathcal{V}^m$ denotes the space [3] of m-singular knots). Then inductively over i, $K_p^{(i)} := K_p^{(i-1)} - K_{p-1}^{(i-1)}$ is an i-singular knot, and therefore each Vassiliev invariant of deg $\leq k$ vanishes on all $K_p^{(k+1)}$. This means that, defining $v_i = v_i^{(0)} := v(K_i)$ and inductively $v_i^{(p)} := v_i^{(p-1)} - v_{i-1}^{(p-1)}$, all $v_p^{(k+1)} = 0$. From this the assertion follows inductively by the fact, that if P is a polynomial of deg $= n$ in x then $\sum_{i=m}^{x} P(i) + C$ for some $m, C \in \mathbf{Z}$ is a polynomial in x of deg $= n+1$ for $x \geq m$. For more details, see [82]. $\square$

Proof for Gauß sum invariants. First consider only positive braid powers. Note that, following the definitions of [28, 67] each crossing of the pure braid σ has the same type, $n^{\pm}$ and $i^{\pm}$ in all copies of σ (we depict σ^n by stacking up n copies of σ and inserting them into the room).

Furthermore, the arrows of one and the same braid crossing s in all copies of σ in the Gauß diagram cannot be distinguished by the arrows of the tangle crossings. Therefore the fact whether a certain choice of crossings in different copies of σ and some in the tangle matches a given configuration, depends only on the order of the copies of σ. E. g., if $n = 20$ copies of σ are given, and a given configuration C of 4 arrows is matched by the choice of a crossing s in copy #3 and two crossings t, u in copy #8 of σ as well as one crossing v in the tangle, then choosing s in copy #i and t, u in copy #j

of σ as well as v in the tangle matches C for all $1 \le i < j \le n$, and for all such matchings the weights are the same.

So in the end matching configurations can be grouped into families, contributing to the value of the Gauß invariant a sum of a fixed weight over certain (ordered) choices of $k' \le k$ out of these n copies of the braid. The number of such choices is $\binom{n}{k'}$, and this is a polynomial in n of deg $\le k$.

For negative braid powers the result follows by considering the braiding sequence $\{(\widehat{\tau\sigma^{-k})\sigma^s}\}_{s\in\mathbf{N}}$ and noticing that by the previous argument for the arising polynomials we must have

$$P_{\tau\sigma^{-k},\sigma}(k+s) = P_{\tau,\sigma}(s) \quad \forall s \ge -k$$

Therefore, $P_{\tau,\sigma}$ must be a polynomial in s not only for $s \ge 0$ but for $s \ge -k$ for any k, and as a polynomial is uniquely determined by its values on positive arguments, P must be on negative arguments the extension of this polynomial for positive arguments:

$$P_{\tau,\sigma}(-k) = P_{\tau\sigma^{-k},\sigma}(0) = v(\widehat{\tau\sigma^{-k}})\,. \qquad \square$$

Now consider a regular isotopy Gauß sum invariant, i. e. one changing its value under Reidemeister I moves. As observed in [67], such an invariant behaves within an ambient isotopy class as a polynomial in the writhe w and Whitney index n, called in [67] (n,w)-polynomial. The coefficients of the polynomial carry all the ambient isotopy information of the invariant.

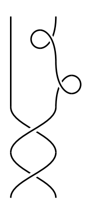
Definition 2.7 For two integers n and w with odd sum define the (n,w)-stabilization of a regular isotopy invariant v by

$$v_{n,w}(K) := v(K,n,w)\,,$$

where (K,n,w) is a representative of the ambient isotopy class of K with writhe w and Whitney index n.

Theorem 2.1 All (n,w)-stabilizations of a regular isotopy Gauß sum invariant v of deg $\le k$ are ambient isotopy invariant, satisfying the assertion of lemma 2.1 for deg $\le k$, and therefore so are all (n,w)-polynomial coefficients of v.

Proof. Consider a braiding sequence. Inserting a pure $\uparrow\downarrow$ braid σ into some room with trivial inhabitant does not change n, but may change w of the diagram. However, σ is pure and hence has even writhe. So, fixing some strand in σ we can perform on it appropriate Reidemeister I moves, so that we do not change its n, but make the writhe of the braid to 0. Here is the simplest example with the pure braid $\sigma_1^2 \in P_2$.



Call such objects "braids with loops". The inserting of the braid σ with loops into the room of the braiding sequence fixes both n and w. Do the same with all other braids in the braiding sequence (i. e., replace them by their versions with loops). Then performing some additional Reidemeister I moves outside of the rooms of the braiding sequence you can achieve any given n,w. Therefore, diagrams of knots in a braiding sequence with fixed n,w form themselves a braiding sequence "with loops". However, now the same argument as in the proof of lemma 2.1 carries over to braids with loops instead of braids, and so each Gauß sum invariant behaves polynomially in the powers of the braids with loops. $\square$

Remark 2.1 Note that the fact that the braid is pure, is important here. Else multiplication with σ would permute the segments of the circle of the Gauß diagram the arrows of τ lie on!

Now we extend this result.

Definition 2.8 A p-fold braiding sequence is a family of knots $\{K_{R_1,\dots,R_p,\sigma_1,\dots,\sigma_p,k_1,\dots,k_p}\}_{k_i\in\mathbf{Z}}$ equal except in p rooms $R_1,\dots,R_p$, where powers of some fixed pure braids $\sigma_1,\dots,\sigma_p$ are inserted ($\sigma_1,\dots,\sigma_p$ do not need to have equal strand number). Such a sequence is parametrized by the powers of the braids $k_1,\dots,k_p$.

Whenever we talk of braiding sequences, we will always mean by 'braids' $\uparrow\downarrow$ braids.

Theorem 2.2 Each Vassiliev invariant or Gauß sum invariant of deg $\le k$ behaves on a p-fold braiding sequence as a polynomial of deg $\le k$ in the parameters $x_1,\dots,x_p$.

Proof. This is just a slight modification of the proof of lemma 2.1. For Gauß sum invariants there come about multinomial coefficients rather than binomial ones, and for Vassiliev invariants the iterated summation of the polynomials has to be done with respect to different variables. $\square$

Definition 2.9 Call any knot invariant behaving on p-fold braiding sequences as a polynomial of deg $\le k$ in the parameters $x_1,\dots,x_p$ a braiding polynomial invariant of deg $\le k$.

This definition appears justified as it generalizes two classes of invariants. However, we have the

Theorem 2.3 Any BPI of degree $\le k$ is a Vassiliev invariant of degree $\le k$.

Proof. Let us start with a definition we will decisively use also in the following.

Definition 2.10 For a (singular) knot diagram K and some fixed subset S of crossings (or singularities) in K call the to K associated braiding sequence the braiding sequence in $|S|=p$ odd variables $x_1,\dots,x_p$ arising from K such that each (possibly singular) crossing i (which is viewed as a room with an inhabitant the 1 or -1 tangle or a singular tangle) is replaced by the inhabitant $\sigma_1^{x_i}$, where σ_1 is the generator of the 2 strand braid group. If S is omitted, we assume that it contains all crossings, or in case of singular knots, exactly the singularities in K.

Remark 2.2 There is some ambiguity in this definition, as the insertion of the $\sigma_1^{x_i}$ may be with parallel or antiparallel orientation of the strands (in the terminology of [60], a t_{x_i} or $\bar t_{x_i}$ move). It will be, however, often clear from the context which one of both we take (or whether we need to specify one at all).

Now take a BPI v of degree $\le k$ and a $(k+1)$-singular knot K'. Consider the braiding polynomial P of v on the braiding sequence associated to K'. This is a polynomial with more variables than its degree. Consequently, in each monomial M of P there is no x_i for some i. But then the resolution of singularity i on the level of braiding polynomials

$$M \longrightarrow M(\dots,\underset{i}{1},\dots)-M(\dots,\underset{i}{-1},\dots)$$

kills M, and so resolving all $k+1$ singularities kills P. $\square$

We see that the notions of BPI and Vassiliev invariants coincide (we will however henceforth use them both). This leads to the (deeply disappointing) fact, that contrarily to our hopes expressed in [67], Fiedler's extension of the Gauß sum idea for *knots* cannot give a new class of invariants. In fact, all formulas obtained by Fiedler [28], as far as considered for knots in S^3, are nothing but (variations of) the degree-3-Vassiliev knot invariant. To see this, recall [67], that the Fiedler invariants for knots arise by applying them to a 2 component link $K\cup T$ of a knot K and an unknotted component T, where T is the meridian (of a companion) of K. Then just consider the braiding sequence of knots so that T is the meridian (of a companion) of each knot in the sequence. It is possible to put T to be so, as it is "far away" from any braiding.

This fact was independently and simultaneously established and observed to myself by M. Polyak [54].

This fact also "trivializes" some examples given in the next sections. However, the advantage of recalling these examples is that they can be seen naturally within our context of braiding sequences. Additionally, our special proofs with orientation non-detection are decisively more straightforward than Bar-Natan's hard (and hard to verify) calculations [5], making use of the deep results of Kontsevich [42] and Drinfel'd [26].

Although this is a deep disappointment for the Gauß sum approach, on the other hand Polyak and Viro announced [56] a proof by Gousarov of their conjecture that all Vassiliev invariants admit Gauß sum formulas. Contrarily to this, Vogel [85] proved the existence of Vassiliev invariants not coming via the Reshetikhin-Turaev [62] construction from any semi-simple Lie (super)algebra and announced [86] that his examples extend to all (not necessarily semi-simple) Lie (super)algebras.

Therefore the Gauß sum approach is more powerful than the quantum approach, and at present the only known one which may produce an orientation-sensitive combinatorial invariant.

One should realize that the braiding polynomials, although friendly to work with in theory, as we will observe, are in general *very* unpleasant in practice. For this I give a somewhat striking example of a complexity, which theoretically will be often handled in the following.

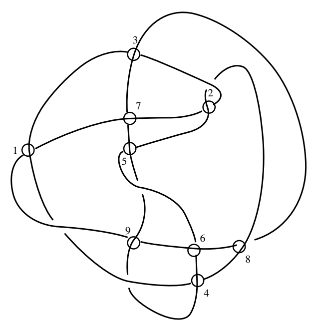


Figure 3: 13_{4233}, the knot of example 2.2.

Example 2.2 The braiding polynomial of the degree 3 Vassiliev invariant $v_3 = {}^1\!/_{36}(V'''(1)+3V''(1))$ on the braiding antiparallel sequence associated to an alternating diagram of the knot 13_{4233}, see figure 3 is, up to a reparametrization and using just one variable for crossings forming a clusp (which is explained precisely in [76, example 7.1], and is given in the figure by encircling the crossings and marking them with the index of the variable) the following rather lengthy expression:

$$-5+\frac{5x_1}{2}-\frac{x_1^2}{2}+\frac{5x_2}{2}-2x_1x_2+\frac{x_1^2x_2}{2}-\frac{x_2^2}{2}+\frac{x_1x_2^2}{2}+5x_3-$$
$$3x_1x_3+\frac{x_1^2x_3}{2}-3x_2x_3+2x_1x_2x_3+\frac{x_2^2x_3}{2}-x_3^2+\frac{x_1x_3^2}{2}+\frac{x_2x_3^2}{2}+$$
$$\frac{5x_4}{2}-2x_2x_4+x_1x_2x_4+\frac{x_2^2x_4}{2}-x_3x_4+x_2x_3x_4-\frac{x_4^2}{2}+\frac{x_2x_4^2}{2}+$$
$$\frac{5x_5}{2}-x_3x_5-2x_4x_5+x_2x_4x_5+\frac{x_4^2x_5}{2}-\frac{x_5^2}{2}+\frac{x_4x_5^2}{2}+5x_6-x_1x_6-x_2x_6-$$
$$4x_3x_6+x_1x_3x_6+x_2x_3x_6+\frac{x_3^2x_6}{2}-3x_4x_6+x_2x_4x_6+x_3x_4x_6+\frac{x_4^2x_6}{2}-3x_5x_6+$$
$$x_3x_5x_6+2x_4x_5x_6+\frac{x_5^2x_6}{2}-x_6^2+\frac{x_3x_6^2}{2}+\frac{x_4x_6^2}{2}+\frac{x_5x_6^2}{2}+5x_7-$$

$$3x_1x_7+\frac{x_1{}^2x_7}{2}-x_2x_7+x_1x_2x_7-4x_3x_7+2x_1x_3x_7+x_2x_3x_7+\frac{x_3{}^2x_7}{2}-x_4x_7-$$
$$x_5x_7-4x_6x_7+x_1x_6x_7+2x_3x_6x_7+x_4x_6x_7+x_5x_6x_7+\frac{x_6{}^2x_7}{2}-x_7{}^2+\frac{x_1x_7{}^2}{2}+$$
$$\frac{x_3x_7{}^2}{2}+\frac{x_6x_7{}^2}{2}+5x_8-x_1x_8-3x_2x_8+x_1x_2x_8+\frac{x_2{}^2x_8}{2}-4x_3x_8+x_1x_3x_8+$$
$$2x_2x_3x_8+\frac{x_3{}^2x_8}{2}-3x_4x_8+2x_2x_4x_8+x_3x_4x_8+\frac{x_4{}^2x_8}{2}-x_5x_8+x_4x_5x_8-4x_6x_8+x_2x_6x_8+$$
$$2x_3x_6x_8+2x_4x_6x_8+x_5x_6x_8+\frac{x_6{}^2x_8}{2}-4x_7x_8+x_1x_7x_8+x_2x_7x_8+2x_3x_7x_8+x_4x_7x_8+$$
$$2x_6x_7x_8+\frac{x_7{}^2x_8}{2}-x_8{}^2+\frac{x_2x_8{}^2}{2}+\frac{x_3x_8{}^2}{2}+\frac{x_4x_8{}^2}{2}+\frac{x_6x_8{}^2}{2}+$$
$$\frac{x_7x_8{}^2}{2}+5x_9-x_1x_9-x_2x_9-4x_3x_9+x_1x_3x_9+x_2x_3x_9+\frac{x_3{}^2x_9}{2}-x_4x_9-3x_5x_9+$$
$$x_3x_5x_9+x_4x_5x_9+\frac{x_5{}^2x_9}{2}-4x_6x_9+2x_3x_6x_9+x_4x_6x_9+2x_5x_6x_9+\frac{x_6{}^2x_9}{2}-4x_7x_9+$$
$$x_1x_7x_9+2x_3x_7x_9+x_5x_7x_9+2x_6x_7x_9+\frac{x_7{}^2x_9}{2}-4x_8x_9+x_2x_8x_9+2x_3x_8x_9+x_4x_8x_9+$$
$$x_5x_8x_9+2x_6x_8x_9+2x_7x_8x_9+\frac{x_8{}^2x_9}{2}-x_9{}^2+\frac{x_3x_9{}^2}{2}+\frac{x_5x_9{}^2}{2}+$$
$$\frac{x_6x_9{}^2}{2}+\frac{x_7x_9{}^2}{2}+\frac{x_8x_9{}^2}{2}$$

It can be found by examining sufficiently many values of v_3 on knots in the series and solving a linear equation system.

3. Some classical results

Although originally developed to prove theorems 2.2 and 2.3, the approach of braiding sequences has much more far reaching applications.

As a first small one, let us reprove in an elementary way a fact, which is classic. It appears originally due to Birman [10] and is treated for special cases in [25, 82, 45].

Theorem 3.1 The crossing number, unknotting number, braid index, bridge number, Seifert genus, embedding genus, signature and (any extension of) the Thistlethwaite invariant of alternating knots (the writhe of a reduced alternating diagram, cf. [38]) are not Vassiliev invariants.

Proof. For simplicity consider braid index and bridge number minus 1. Use the braiding sequence of $(2,n)$ torus knots, $n\in\mathbf{Z}$ odd. All invariants are not constantly zero on this sequence, but zero in $n=\pm1$. So the function to be a polynomial, it must grow at least quadratically in n as $n\to\infty$. But clearly no one of these invariants does so (only the signature deserves a small argument, but it can be easily seen e. g. from the principle of Murasugi [38] to compute it on alternating knots using the checkerboard shading). □

Theorem 3.1 also says that all these invariants can never be obtained via Gauß sum formulas.

The argument applied to some of the invariants in theorem 3.1 can be generalized in the following way:

Theorem 3.2 Each invariant, which is bounded, but not constant on some braiding sequence, is not a Vassiliev invariant. □

As any knot is contained in a braiding sequence containing the unknot, this implies the

Corollary 3.1 (Liouville principle) Any bounded Vassiliev invariant is constant. $\square$

Here is another known fact, which appears very natural within our context.

Theorem 3.3 The product of two Vassiliev invariants $v_{1,2}$ of degree $m_{1,2}$ is a Vassiliev invariant of degree m_1+m_2.

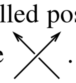
Proof. Let $K_{1,2}$ be $m_{1,2}$-singular knots, on which $v_{1,2}$ do not vanish. Then by the proof of theorem 2.3 the braiding polynomials $P_{1,2}$ of $v_{1,2}$ associated to the braiding sequence $B_{1,2}$ corresponding to $K_{1,2}$ have exactly degree $m_{1,2}$. Now consider the connected sum $K_1\#K_2$ and the braiding sequence B associated to all its (not necessarily singular) crossings. As any resolution of $K_{1,2}$ can be unknotted by crossing changes, B contains $B_{1,2}$ as subsequences and so the braiding polynomials $P'_{1,2}$ of $v_{1,2}$ on B do so with $P_{1,2}$. Hence $P'_{1,2}$ have degree $m_{1,2}$ and $P'_1\cdot P'_2$ has degree m_1+m_2. But $P'_1\cdot P'_2$ is the braiding polynomial of $v_1\cdot v_2$ on B, and so $v_1\cdot v_2$ has degree m_1+m_2. $\square$

All these facts seem far away from being deep. So we shall give more interesting applications to motivate our approach.

4. Braid positive and alternating knots, periodicity and k-equivalence

Using p-fold braiding sequences, we can formulate a series of restrictions to the values of Vassiliev invariants.

Recall the notion of positive knots.

Definition 4.1 ([28, 66]) A knot is called positive, if it has a positive diagram, i. e. a diagram with all crossings positive, i. e. looking like  .

Here we make a further

Definition 4.2 A knot is called k-positive (resp. k-alternating), if it is positive (resp. alternating) and each reduced positive (alternating) diagram does not contain (a room with) an inhabitant $=\sigma_1^k$ (resp. $\sigma_1^{\pm k}$).

Lemma 4.1 Any Vassiliev invariant of degree $\le k$ is uniquely determined by its values on $2k+2$-positive knots.

Proof. Take for some knot K some diagram of K and consider the associated braiding sequence. First observe that each BPI is uniquely determined by its values on positive knots, as any (braiding) polynomial is uniquely determined by its values on positive arguments (which is not hard to see by induction over the number of variables in the polynomial). For an invariant (and braiding polynomial) of degree $\le k$ the values of the polynomial of sequences of arguments, all arguments in $\{0,2,\dots,2k\}$ or $\{1,\dots,2k+1\}$, uniquely determine the polynomial. $\square$

By a minor modification of this argument, we immediately obtain also

Lemma 4.2 Any Vassiliev invariant of degree $\le k$ is uniquely determined by its values on $2k+2$-alternating knots. $\square$

The question of whether Vassiliev invariants detect orientation has meanwhile become an intriguing one. One recent crucial result is that of Kuperberg [43]. Here is our first small contribution to this discussion.

Remark 4.1 As a partial consequence of lemma 4.2 it follows, that if there is a Vassiliev invariant detecting orientation, there will be a primitive one, and therefore one detecting the orientation of an *alternating* prime knot. Note, that by a result of Menasco [46], if $K_1\#K_2$ is alternating, then it appears composite in *any* alternating diagram, and so both K_1 and K_2 are alternating as well.

Remark 4.2 By a result simultaneously established by Kauffman [39], Murasugi [49] and Thistlethwaite [78], the property of an alternating diagram to be reduced, is equivalent to being minimal.

Remark 4.3 As another direct consequence of lemma 4.2, the alternation degree (the minimal number of crossing changes in any diagram of the knot to obtain an alternating diagram) is not a Vassiliev invariant.

As each knot can be represented as a closed braid, we can even further sharpen our last 2 lemmas by taking a closed braid diagram of the knot. So we see that we can demand our knot to be alternating/positive *in a closed braid diagram*. Such knots were considered in [66] and were called there braid alternating/positive. There I also gave examples that braid alternating/positive is stronger than alternating/positive.

Lemma 4.3 Any Vassiliev invariant is uniquely determined by its values on braid alternating/positive knots. $\square$

Remark 4.4 It should be noted, that lemma 4.3 for braid alternating knots also follows from [63]: as any knot can be made into a braid alternating knot, not changing the value of a given Vassiliev invariant v, if v is zero on braid alternating knots, it must be zero at all. Our argument is, however, much more direct, and, of course, Stanford's construction does not work with positive braids, as any braid commutator has zero homology class and hence can never be conjugated to a positive braid.

Note, that by this fact in remark 4.1 we can restrict ourselves further to braid alternating knots, because if $K_1\#K_2$ is alternating in a closed braid diagram, then by [46] it will be composite, and so will be both K_1 and K_2. Therefore,

Lemma 4.4 If there is a Vassiliev invariant detecting orientation, there is a Vassiliev invariant detecting orientation of a braid alternating prime knot. $\square$

Recently, an analogous result to this of Menasco was obtained for braid positive knots by Cromwell [23], and so we have (in the same way):

Lemma 4.5 If there is a Vassiliev invariant detecting orientation, there is a Vassiliev invariant detecting orientation of a braid positive prime knot. $\square$

Another further consequence of braiding polynomials is

Theorem 4.1 For each even $k \in \mathbf{N}$ and fixed k-equivalence class of knots M, each Vassiliev invariant is uniquely determined by its values of M.

Note that the parity of k is only necessary M to consist of knots only, as each k-equivalence class contains some $2k$-equivalence class. k-moves and k-equivalence have been introduced by Nakanishi [51] and are described in [1]. See also [60].

For $k = 2$ there is only one k-equivalence class of knots. For $k = 4$ even if there are several, they are very large. Our result sounds however a little strange – it means that either k-equivalence classes are very large even for very high k or that we have not come very far in defining good knot invariants ...

Proof. It suffices to show that each knot K admits a projection such that crossing changes of that projection produce a member K' in M. Then consider the braiding sequence of this projection and argue by the fact, that each polynomial is uniquely determined by its values on sequences of arguments, where each argument runs over a rest class modulo $k/2$.

Take a composite projection of $K\#K'$. Then by crossing changes you can unknot once K' obtaining K and another time K, obtaining K'. □

Remark 4.5 In fact, it is even possible to combine the arguments of the proofs of lemma 4.2 / lemma 4.1 / lemma 4.3 with that of theorem 4.1 and to prove, that it suffices to consider *(braid) alternating/positive knots within some k-equivalence class for some (even) k*.

A similar result is possible for periodicity.

Theorem 4.2 Any Vassiliev invariant is uniquely determined by its values on n-periodic knots for any $n \in \mathbf{Z}$.

Proof. Fix a knot K and take some $\beta \in B_k$ with $\hat{\beta} = K$ and $(k,n) = 1$. Then β induces a k-cycle permutation and $(\widehat{\beta^{pk+1}})_{p\in\mathbf{Z}}$ is a braiding sequence. But as $(k,n) = 1$, n divides infinitely many numbers $pk+1$ and the result follows. □

The argument shows in fact that you need n-periodic closed braids, i. e., knots with an n-periodic closed braid diagram and the symmetry axis being the braid axis (and a further bit of argument can show that even n-periodic closed alternating or positive braids suffice). However, it is not immediately clear whether this is indeed a strengthening of the assertion, that is, whether there is a k-periodic knot (or link) which is not a closed k-th power of some braid.

There is already a great variety of periodicity criteria, see e. g., [17, 59, 80, 81, 90, 91]. But all they involve the linking number of the knot with the axis (if at all) only modulo the period, and hence for a periodic knot they will not be violated for arbitrarily high linking numbers, disappointing my hope to make them contradict with some lower bound coming from the braid index. In this regard much better does the classical Murasugi criterion [50].

Example 4.1 We use the simplest case $k = 2$. Take some non-trivial knot K and consider $K\#K$. For its Alexander polynomial we have

$$\Delta_{K\#K}(t) = (\Delta_K(t))^2 \equiv \Delta_K(t^2) \pmod 2\,.$$

In particular if $d = \min\deg \Delta_{K\#K}(t) \bmod 2$, then $[\Delta_{K\#K}(t) \bmod 2]_{t^{d+k}} = 0$ for odd k. But $b(K\#K) = 2b(K) - 1 \geq 3$, hence if $K\#K$ is a closed braid square, it would have a (2-)periodic diagram with linking number with the axis $\lambda > 2$. Then by [50] (or see [57, theorem 5.3]) we have

$$\Delta_{K\#K}(t) \equiv \Delta_0^2(t)\,(1 + t + \ldots + t^{\lambda-1}) \pmod 2 \tag{1}$$

for some $\Delta_0 \in \mathbf{Z}[t,1/t] \cup \sqrt{t}\mathbf{Z}[t,1/t]$. But for the same reasons as above, the second coefficient of $\Delta_0^2(t) \bmod 2$ (the one in degree one higher than the minimal degree) is zero, and hence for $\lambda > 1$, the second coefficient of the r.h.s. of (1) is one, contradicting the above observation for the l.h.s. Hence no non-trivial knot of the kind $K\#K$ has a 2-periodic braid diagram. (Note, that the same argument shows that a closed braid square knot cannot be slice.)

Remark 4.6 For $\alpha \in B_n$ positive and closing to a knot, k prime and $(n,k) = 1$, $\Delta(\widehat{\alpha^k})$ is monic (its edge coefficients are one) and by Murasugi's theorem,

$$\Delta(\widehat{\alpha^k}) \left(\frac{t^n - 1}{t - 1}\right) \bmod k \in \mathbf{Z}[t^{-k}, t^k]\,.$$

Hence any Vassiliev invariant is determined by its values on knots with such Alexander polynomial. This is not very elucidating, but it becomes more natural for example when letting $n \equiv p \pmod{k}$, $p \neq 0$ and considering just the minimal and maximal degree in the above property. Then we see that in particular any Vassiliev invariant is determined by its values on knots with $\max\deg\Delta \equiv p' \bmod k$ for $k \not| 2p'-1$. In fact, stronger $\max\deg\Delta \in S$ for any infinite $S \subset \mathbf{N}_+$ suffices. See [71]. Note, contrarily, that no finite S works – take some Vassiliev invariant of the Conway polynomial of sufficiently high degree. It is unclear whether some finite S works for *primitive* invariants as well. Maybe even $S = \{1\}$ does.

Using a well-known n-periodicity criterium for n prime (see [57, theorem 3.3 (i)] or [58, 79]), we obtain

Corollary 4.1 If a Vassiliev invariant v vanishes on all knots K with $V_K(t) \equiv V_K(t^{-1}) \bmod (n, t^n - 1)$, then $v \equiv 0$. □

This may not be of particular use, but I decided to quote it here because it nicely contrasts the obvious fact that there do exist non-trivial Vassiliev invariants vanishing on knots with self-conjugate Jones polynomial. The Polyak-Viro-Fiedler invariant [55, 28, 66] is the simplest example.

5. Obstructions for detecting orientation and mutation

In this section we give an explanation independent from [5] of the problems to detect orientation and mutation with low degree Vassiliev invariants and therefore also to our failed attempts in [67].

Henceforth in this section consider only the orientation antisymmetrization of each Vassiliev invariant v defined by $v'(K) := v(K) - v(-K)$. For all invertible K, v' is 0. For the sake of orientation detection, v' is as good as v.

5.1. n-triviality and n-invertibility

The notion of n-triviality of a knot has been introduced by Gousarov [29]. It has been known that connections exist between Gousarov and Vassiliev theory [52, 53].

Definition 5.1 ([29]) A knot is called n-trivial, if it admits a diagram and a scheme S (family of disjoint sets of crossings in the diagram) such that $\forall \varnothing \neq N \subset S$ changing crossings in $\cup N$ in this diagram, gives the unknot.

Here and in the sequel '$\subset$' denotes a not necessarily proper inclusion.

Here is a proof within our context of a well-known result for Vassiliev invariants.

Theorem 5.1 An n-trivial knot cannot be distinguished from the unknot by any BPI of deg $< n$. In particular, the existence of n-trivial knots for all n implies that no finite number of Vassiliev invariants can classify knots.

There is a very nice construction of n-trivial knots in [52].

Another series of examples was obtained by Stanford [63] using work of Birman and Menasco [13]. We will come back to these examples later.

Proof. Take a BPI v, an n-trivial diagram of K and fix a scheme S. Number the crossings in the scheme in an arbitrary way. Consider the braiding polynomial associated to this diagram, where fixing the arguments corresponding to crossings outside of S and putting one and the same variable into all arguments corresponding to crossings in one and the same set of S. You obtain a polynomial in $n = \#S$ variables. If v distinguishes the unknot from K, then you can parametrize and normalize this polynomial so that

$$P(i_1,\dots,i_n) = \begin{cases} 1 & i_1 = \dots = i_n = 0 \\ 0 & \text{else} \end{cases} \tag{2}$$

where all $i_j \in \{0,1\}$. Introduce a map

$$\Phi : \{ \text{ monomials in } x_1,\dots,x_n \} \longrightarrow \mathcal{P}(\{1,\dots,n\})$$

by $\Phi(x_i) := \{i\}$ and $\Phi(a \cdot b) := \Phi(a) \cup \Phi(b)$. Φ just indicates which indices are contained in the monomial, e. g. $\Phi(x_1^2 x_2 x_5 x_6) = \{1,2,5,6\}$.

Then (2) is equivalent to

$$\sum_{\substack{\text{monomial } X \\ \text{with } \Phi(X) \subset N}} [P]_X = \begin{cases} 0 & N \neq \varnothing \\ 1 & N = \varnothing \end{cases}$$

for all $N \subset S$, where $[P]_X$ is the coefficient of the monomial X in P. By induction over $\#\Phi(X)$ this is equivalent to

$$\sum_{\Phi(X)=N} [P]_X = (-1)^{\#N} \quad \forall N \subset S$$

In particular, setting $N = S$, P must contain a monomial containing all variables and has to be of degree $\geq n$, and so has to be v. □

In this way we see how braiding polynomials generalize Gousarov's idea of n-triviality.

Remark 5.1 This theorem *does not* yet mean that it is not decidable using Vassiliev invariants only, that 2 knots are distinct. As unlikely as it might be, it does not yet prove wrong a statement like: "If 2 knots with k and l crossings are distinct, then there is a Vassiliev invariant of degree, say, $\leq 2^{k+l}$, distinguishing them."

For the sake of orientation and mutation it appears reasonable to make the following generalizations of Gousarov's definition

Definition 5.2 A knot is called n-invertible (n-amphicheral), if it admits a diagram and a scheme S, such that $\forall \varnothing \neq N \subset S$ changing crossings in $\cup N$ in this diagram, gives an invertible (amphicheral) knot.

We shall discuss mutations a little later.

It should be possible (with some labour) to use Stanford's work [63] and that of Birman and Menasco [13] to construct non-invertible n-invertible and chiral n-amphicheral knots for all n, and maybe also to construct non-trivial n-trivial mutations.

Note that each n-trivial knot is also n-invertible and n-amphicheral.

With the same proof, theorem 5.1 carries over to n-invertible and n-amphicheral knots.

Theorem 5.2 The non-invertibility (chirality) of any (non-invertible / chiral) n-invertible/ n-amphicheral knot cannot be detected by a Vassiliev invariant of deg $< n$. □

Here are some simple examples already considered in [67], for which we will observe the origin of the problems encountered there.

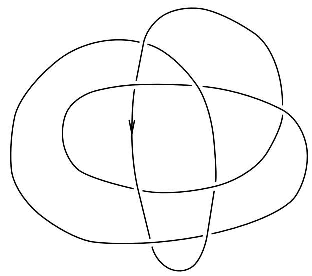


Figure 4: 8_{17}, the first non-invertible knot.

Example 5.1 The knot 8_{17} (in the usual notation of [61]) is famous with being the first non-invertible knot. Look at its standard (alternating) diagram of fig. 4. Choosing any 7 of these 8 crossings, by arbitrary crossing changes we obtain non-alternating diagrams with 8 crossings. However, 8_{17} is alternating (and prime) and therefore does not have non-alternating minimal diagrams (this follows, e. g., from the results of [40]). So all these $2^7-1=127$ knots are not 8_{17} and are therefore invertible. Therefore 8_{17} is 7-invertible its non-invertibility cannot be detected by a Vassiliev invariant of deg $<$ 7. Once established this, the argument immediately extends to 8_{17}'s both "successors" 9_{32} and 9_{33} [33], by the remark that there are no non-alternating non-invertible 9 crossing prime knots * (see [33], the invertibility of 9_{47} and 9_{49} can be directly observed from their diagrams in Rolfsen's tables [61]) and surely nor any composite ones.

Note that by the argument of [67, example 4.2] any statement deduced by reasoning using braiding polynomials immediately extends to arbitrary cable. So the cabling idea is *useless* to walk around such arguments. More precisely, the statement is as follows:

Proposition 5.1 Any p-cable (with any $b\in B_p$) of a BPI of deg $\le n$ is a BPI of deg $\le n$. $\square$

Here the "cable" of an invariant is the dualization of the operation $K\mapsto K_{p,b}$ on the level of invariants, and $K_{p,b}$ means the satellite around K with pattern being the closed p-braid b with its braid axis.

This statement appears as an exercise in [3] and follows in our context directly from the definition of BPI, as the cable of a pure braid is again a pure braid.

5.2. n-trivial mutations and the intersection graph conjecture

Definition 5.3 A mutation is called n-trivial, of both mutants admit mutated diagrams and a scheme S, such that $\forall\varnothing\ne N\subset S$ changing crossings in $\cup N$ in these diagrams gives a trivial mutation (both mutants are the same).

Example 5.2 The first non-trivial mutation is that of the Kinoshita-Terasaka/Conway knots, see fig. 5. Just by a direct argument you see that crossing changes in any of the 5 encircled crossings in fig. 5, allow a reducing Reidemeister II move, so the mutation becomes trivial (there are no mutants with fewer than 11 crossings). Therefore, the Kinoshita-Terasaka/Conway mutation cannot be detected by a Vassiliev invariant of deg <5. In fact, a direct calculation of the HOMFLY polynomials with KnotScape [31] shows, that crossing change of any $1\dots10$ crossings gives a pair of knots with different polynomial (than this of the Kinoshita-Terasaka/Conway knots and their obverses), and as this is the only non-trivial mutation with both knots of ≤ 11 crossings [47], it is 10-trivial. So, in fact, we have

*I found this argument curiously just a couple of weeks after I had erronously announced to several people a non-invertibility proof of 9_{32} with one of Fiedler's degree-3-Gauß sum invariants due to a mistake in my calculation.

Example 5.3 No Vassiliev invariant of degree 9 distinguishes the Kinoshita-Terasaka/Conway knots.

For degree ≤ 8 mutation non-sensitivity was established in general by Chmutov and Duzhin [19] using their calculations with Lando [21]. However, note, that Morton/Cromwell experimentally es-

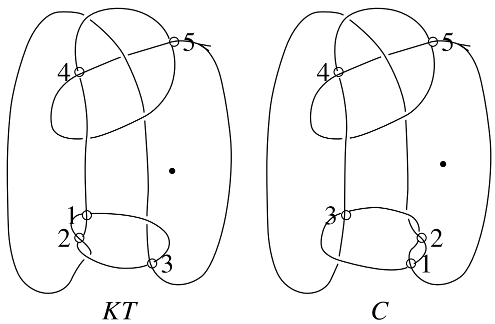


Figure 5: The Kinoshita-Terasaka/Conway knots

tablished [48], that a Vassiliev invariant of degree 11 (a 3-cable of a HOMFLY Vassiliev invariant) distinguishes both mutants. We will just say a word more on this.

Recall [12, 3, 42] that a CD (chord diagram) is a circle with $2n$ points connected in pairs by n chords. A singular knot representing a given CD is a C^1-immersion of S^1 into $\mathbf{R}^3$ with n double transverse points (singularities), such that the preimages of each singularity in S^1 are the endpoints of a given chord in the CD. Then, extending it in the classical way [12, 3, 84] to singular knots, a Vassiliev invariant of degree n has equal values on all singular knots representing one and the same CD with n chords, and so induces a function on the set of such diagrams, called weight system. In [22] it was conjectured, that any weight system of a Vassiliev invariant has equal values on CD's with the same intersection graph (where latter is a graph with vertices corresponding to chords and edges connecting intersecting chords). T. Q. T. Le pointed out (see [20]), that this would imply that no Vassiliev invariant detects mutation, contradicting the Morton/Cromwell examples. Le's argument, however, uses the Kontsevich integral and so gives a counterexample to the intersection graph conjecture only for zero characteristic.

Here we will observe, that using our (much simpler) n-triviality arguments we can extend Le's counterexample to almost any non-zero (field) characteristic.

Consider the Kinoshita-Terasaka/Conway knots in their mutated diagrams on figure 5. The procedure of mutating a diagram (that is, removing a tangle with four ends and plugging it in after some rotation around 180°) gives a bijection between the crossings of both diagrams, depicted on figure 5. We observed, that changing any $1\dots 10$ crossings (the same under this bijective correspondence) in both diagrams, the mutation becomes trivial, that is, we obtain diagrams of the same knot.

Consider now $v := b_{11}(u)$ of [48, proposition 2] on the two 11-singular knots obtained from the diagrams on figure 5 by making all crossings singular. v is a degree 11 Vassiliev invariant with values in $\mathbf{Z}[u]$, obtained by taking the coefficient of z^{11} in a variable-transformed version $\tilde{P}(u,z)$ of the HOMFLY polynomial applied on the 3-cable of a knot. Then only two resolutions of these 11 singularities (with equal choices between over- and under-crossing in bijectively corresponding crossings in both diagrams) give a non-trivial mutation, namely (KT,C) and $(!KT,!C)$. In fact, v is framing dependent, but equal crossing resolution gives equal writhe on both diagrams, and the change to some fixed writhe (say, 0) used to make a framing independent invariant out of v, multiplies its value on both knots by the same term of the form $1+O(z)$, so that the lowest non-vanishing coefficient of z in the difference of v on both resolutions is not affected. Morover, it was observed to me by Morton, that $b_{11}(u)$ negates, when we take to obverse mutant pair $(!KT,!C)$.

But as the number of singularities is odd, both pairs occur with opposite signs in the resolutions, and so the difference of v on both 11-singular knots (representing CD's with the same intersection graph) is twice the difference of v on the pair (KT, C). Now

$$2b_{11}(u) = 12u(u^2-1)(u^2-4)(u^2-9)^2\,,$$

which for $u = 4$ is not zero and has the only prime divisors 2,3,5 and 7. So that argument shows that the counterexample extends to any non-zero (field) characteristic except 2,3,5,7.

6. On the Jones polynomial

Here we give some consequences of the lemmas in §4 to the Jones polynomial V [34].

Recall an inequality of [66] I deduced using work of Fiedler [27].

Theorem 6.1 ([66]) If a knot K is braid positive, then

$$\min\deg V(K) \geq c(K)/4\,,$$

where $c(K)$ is the crossing number of K.

Remark 6.1 Combining theorem 6.1 with lemma 4.3 for braid positive knots, in the somewhat sharper version, where we exclude finitely many braid positive knots (note, that this does not spoil the argument in its proof, in particular because it follows from [27], that each braid positive knot has only finitely many diagrams as closed positive braid of fixed strand number), we immediately obtain, that the Jones polynomial as a series of Vassiliev invariants is uniquely determined by its values, where its minimal degree is higher than any given constant.

More precisely, we have

Theorem 6.2 Let $V' : \{\text{ knots }\} \longrightarrow \mathbf{Z}[t,t^{-1}]$ be a Laurent polynomial for knots. Assume that $V'^{(n)}(1)$ is a Vassiliev invariant (of any degree) for all $n \in \mathbf{N}$. If then $\exists k \in \mathbf{N}$ such that $V'(K) = V(K)$ whenever $\min\deg V(K) \geq k$, then $V' = V$. □

A further fact, which is not surprising, but at least easily provable now, is:

Theorem 6.3 The minimal degree of the Jones polynomial grows unboundedly on any k-equivalence class of knots (for any even k).

Proof. Assume there is a class without the desired property. Then this class contains only finitely many braid positive knots. But then by remark 4.5 there can only be finitely many (linearly independent) Vassiliev invariants, which is not true (see [63] for a nice argument). □

It is easy to show, e. g. using theorem 3.2, that the coefficients of the Jones polynomial are not Vassiliev invariants. However, here is another more curious consequence of remark 6.1.

Theorem 6.4 Any function f of finitely many coefficients of the Jones polynomial is either constant, or not a Vassiliev invariant of any degree. (In fact, it is even sufficient to demand the range of coefficients involved in f to be bounded in one direction, i. e. there to be a highest or lowest one).

To be more precise, what we mean by "a function of" and "depends", we say that an invariant v depends (only) on some finite or countable set of invariants $\{v_i\}_{i\in I}$, if there is a function

$$f\,:\,\mathbf{C}^{\#I} \to \mathbf{C} \tag{3}$$

with $v = f(v_i)_{i\in I}$, where $\mathbf{C}^\infty := \{(c_1,\dots,c_n,\dots)\,|\,\exists i_0 : c_i = 0 \text{ for } i > i_0\}$.

It would be reasonable to demand f to be "good" in some sense, e.g. (primitively recursively) computable, but we do not yet need to assume even something like this at this point.

Clearly, any coefficient of V can be expressed as a (formally) *in*finite linear combination of its Vassiliev invariants. But even for example a finite linear combination of infinite linear combinations can again be finite (because almost all terms cancel, or just because the resulting infinite combination forms a complicated identity), and hence, from this point of view, the result is not obvious even for f being a linear function.

Proof. Let k be the maximal coefficient involved in f. Then f is constant on all braid positive knots with $> 4k$ crossings. Then apply remark 6.1. $\square$

This should be taken as a serious warning in any attempt to understand the derivations of the Jones polynomial at 1 in terms of its coefficients (and vice versa). So, although there is an interconvention, there cannot be any simple one, and so understanding the one we will still be far away from understanding the other!

Not surprisingly, the result extends to the HOMFLY polynomial $P \in \mathbf{Z}[l^2, l^{-2}, m^2]$ [30, 44].

Theorem 6.5 Let $L \subset 2\mathbf{Z} \times 2\mathbf{N}$ be a subset such that either

$$\min_{(l,m)\in L} l > -\infty \quad \text{or} \quad \max_{(l,m)\in L} l < \infty.$$

Then any function of $\{[P]_{l^a m^b}\}_{(a,b)\in L}$ is either constant, or not a Vassiliev invariant of any degree.

Proof. Use again braid positive knots and the same idea. Instead of theorem 6.1 use [24, theorem 4(b)] and the fact (which follows also from [24] or from the Bennequin inequality [9], see [71]) that only finitely many braid positive knots have the same genus. $\square$

A similar statement is possible for evaluations at roots of unity and is a consequence of theorem 4.1. We call a k-move in an oriented diagram a t_k or $\bar{t}_k$ move according to whether the orientation of the strands is parallel or antiparallel in direction of the twist. Clearly doing the twists the one or the other way, theorem 4.1 specializes to t_k or $\bar{t}_k$ equivalence classes of knots. The invariance under $\bar{t}_{2k}$ moves of the evaluations of the Jones and HOMFLY polynomials at (complex) roots of unity was probably first noted by Przytycki [60].

Theorem 6.6 (Przytycki) Let $a^{2k} = 1$, $a \neq \pm 1$. Then $V(a) \in \mathbf{C}$ and $P(ia,m) \in \mathbf{C}[m^2]$ are $\bar{t}_{2k}$ invariant.

Corollary 6.1 Any non-constant invariant depending only on some finite set of evaluations $V(a) \in \mathbf{C}$ or $P(ia,m) \in \mathbf{C}[m^2]$ with $a \in \mathbf{C}$ being a root of unity different from ± 1, is not a Vassiliev invariant. $\square$

Note, that contrarily to this, *any in*finite set of such evaluations recovers the polynomial and all its Vassiliev invariants (as S^1 is compact and a polynomial is an analytic function).

Even worse is possible, if we pose a minimal regularity condition on the way of dependence.

Proposition 6.1 Let $C \subset D^2 \setminus \{0,-1\}$ or $C \subset \overline{Int(D^2)} \setminus \{-1\}$ be finite ($D^2 \subset \mathbf{C}$ is the closed unit disc, $\overline{C} := \mathbf{C} \setminus C$ and Int means interior). Then any non-constant invariant depending continuously only on the evaluations $\{V(a)\}_{a\in C}$, is not a Vassiliev invariant.

Here "continuously" means that f in (3) is continuous.

Proof. Let $C \subset D^2 \setminus \{0,-1\}$. The set of evaluations $\{V(a)\}$ for $a \in C$ may no longer be finite. But it was observed in [73] that this set of evaluations on the positive diagrams in the $\bar{t}_2$ braiding sequence of any diagram is bounded. A continuous image of a bounded set is bounded (if the function is defined on a neighborhood of the preimage set) and we are through as before. For $|a| \geq 1$ use the argument on the mirror images. □

It is unclear to me whether the result holds for a combination of evaluations both in and outside the unit circle, but I hope it to be sufficiently striking even this way. This gives an idea, why we need to do something different with V than just evaluating it to obtain a Vassiliev invariant out of it. The need of exclusion of the determinant (the value $V(-1)$) is yet mysterious to me (0 we exclude for obvious reasons). It is unlikely to give useful information for building a Vassiliev invariant, but ...

Remark 6.2 It is possible again to show that the above described coefficients and evaluations *combined* still do not suffice. Consider the $\bar{t}_2$ braiding sequence of braid positive diagrams with sufficiently many crossings.

7. Special diagrams and an anti-Alexander theorem

Lemmas 4.1 and 4.2, although at first glance maybe surprising, certainly do not reveal very deep. Basically they rely on the obvious fact, that any knot diagram can be made positive or alternating by crossing changes. So it is self-suggesting to ask the more substantial question, whether both cannot be combined.

Certainly, not any diagram becomes alternating when crossing-switched to a positive one. In was observed in [23, §1] that the (oriented) diagrams with this property are exactly those, which do not have separating Seifert circles (that is, Seifert circles whose both interior and exterior contain at least one other Seifert circle). Such diagrams were called there *special*. So we need to know, whether any knot has a special diagram. This is indeed the case.

Theorem 7.1 Any link has a special diagram. Moreover, this diagram can be chosen so that any two Seifert circles having more than three outer crossings each, (that is, crossings connecting them from the outside), are not connected.

Corollary 7.1 Any Vassiliev invariant in uniquely determined by its values on knots which have positive alternating diagrams. □

Note, that *a priori* per definition a knot which has a positive alternating diagram may be more special than a knot which is both positive and alternating (and therefore has a positive diagram and an alternating diagram). But this is indeed not the case, as an alternating diagram of a positive alternating knot is positive, see [73].

Alternatively, a special diagram may be described as one, in which (modulo moves in S^2) any Seifert circle has empty interior. In this sense, a special diagram is the other extreme of a (closed) braid diagram, in which (again modulo moves in S^2) for each pair of Seifert circles the one contains the other. So our theorem may be viewn as a counterpart to the famous theorem of Alexander [2], that any diagram of a knot (or link) can be transformed into a (closed) braid diagram.

Now the fundamental importance of braids [11] for knot theory has inspired numerous attempts to simplify the procedure given by Alexander, and one of these recent attempts [89] showed that this can be done without changing the number of Seifert circles of the diagram, so that one can obtain a braid diagram of minimal Seifert circle number (Vogel [87] gave a more elegant procedure with this property, which we will recall below). The procedure we will describe for our theorem by far fails to do so, and indeed there is no chance to hope to get a special diagram of minimal Seifert circle number in general. To demonstrate this, I turn to a beautiful example considered (mathematically) by Jones [35] (and rope-technically several centuries before by seamen).

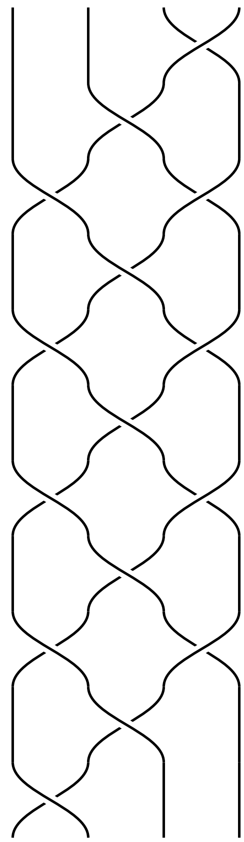
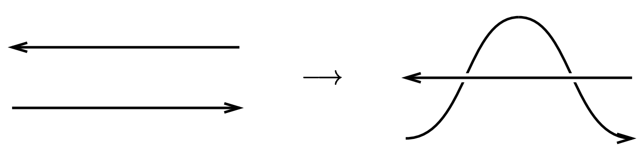


Figure 6: A fourfold platting.

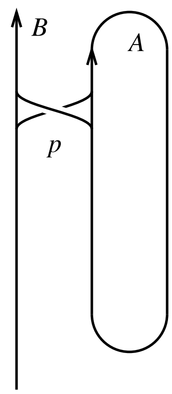
Example 7.1 The "fourfold platting" [36, p. 118] is a 4-braid, which is a power of $\sigma_1\sigma_2\sigma_3^{-1}$. Taking some diagram of its closed version we can transform it into a closed braid diagram by Vogel's algorithm [87], applying a sequence of (Vogel) moves of the kind:



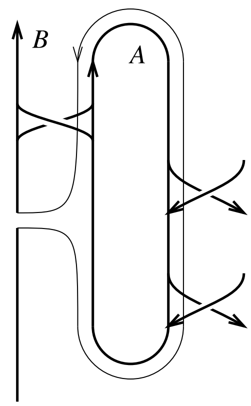
Here the fragments on the left belong to *distinct* Seifert circles, so that the move does not change the number of Seifert circles. It is easy to see that the braid axis of the resulting diagram lies in the 2-gon created by the last move of this sequence (the connected component of the complement of the diagram neighboring just two crossings). Then it is clear, that the resulting braid contains one edge (Artin) generator only twice, with powers 1 and -1 (this is what Birman and Menasco [14] call a braid admitting an exchange move). But Jones showed [35, p. 370 top] that all but finitely many powers of $\sigma_1\sigma_2\sigma_3^{-1}$ have no representation as a 4-braid admitting an exchange move. Hence, starting with a diagram of such a link with 4 Seifert circles, the sequence of Vogel moves we need to perform must be empty. Therefore, for almost all (closed) fourfold plattings any diagram with four Seifert circles must be a (closed) braid diagram and hence is not special.

Proof of theorem 7.1. We will use one move to transform a link diagram into a special one. This move is non-local (that is, it does not involve fragments of the link diagram with bounded number of crossings), but a regular isotopy move (that is, it can be achieved by a sequence of Reidemeister II and III moves). Moreover, the number of moves required is equal to the number of separating Seifert circles.

To describe the move, take some separating Seifert circle A and an outer crossing p of it, connecting A to some Seifert circle B.

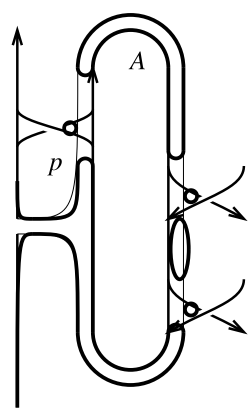


Then take the strand of the link belonging to B near p and create out of it a loop, which passes closely outside of A with reverse orientation,



so that it separates A from its outer crossings (one should realize, that, contrarily to the loop, A exists only virtually on the link diagram and is used only to specify how to place the loop).

Building the Seifert (circle) picture of the new diagram, we see that A disappears at the cost of $2n-1$ empty Seifert circles, where n is the number of outer crossings of A. Each such crossing now instead of A connects an empty Seifert circle with three outer crossings, and each segment of A between two outer crossings belongs to an empty Seifert circle with two outer crossings, except for the segment near p where the loop comes from B (this segment belongs now to B). Induction on the number of separating Seifert circles completes the proof of the first part of the theorem.



As the newly created Seifert circles have maximally three outer crossings, continuing the procedure with the (now non-separating) Seifert circles A (possibly applying a S^2 move on the outermost one) for which both A and B have more than three outer crossings, we can eliminate one of them. Hence we have inductively the second part. (B's number of outer crossings is augmented by two by a move and so it is not clear whether Seifert circles with more than three outer crossings can be completely eliminated.) □

8. The Jones Vassiliev invariants on torus knots

Torus knots are one of the most common classes of knots and hence of particular interest. Clearly, in our context, the $(m, km+n)$-torus knots $T_{m,km+n}$ for $m,n \in \mathbf{N}$ fixed form a braiding sequence in $k \in \mathbf{Z}$ and all the previous discussion applies to it. However, torus knots have more structure and the following is a remarkable property of their braiding polynomials, established by Simon Willerton [88].

Theorem 8.1 For each Vassiliev invariant v of degree k, the function $P_v(m,n) := v(T_{m,n})$ is a polynomial of $\deg_m \leq k$ and $\deg_n \leq k$.

Willerton's proof uses a deep unpublished result of Thang Le about (cabling) the Kontsevich integral. Therefore, Willerton asks for a simple proof of this fact. Here I give an alternative proof for the Vassiliev invariants of the Jones polynomial, which will later allow to determine more specifically their (torus knot braiding) polynomials. It uses Jones' formula for V of a torus knot.

Theorem 8.2 (see [35, prop. 11.19])

$$V(T_{m,n}) = \frac{t^{m+n} - t^{m+1} - t^{n+1} + 1}{t^2 - 1}\, t^{(m-1)(n-1)/2}. \tag{4}$$

Corollary 8.1 Let $v = V^{(k)}(1)$ be the Vassiliev invariant of degree k of the Jones polynomial. Then $P_v(m,n) := v(T_{m,n})$ is a polynomial in m and n with $\deg_m \leq k$ and $\deg_n \leq k$.

Definition 8.1 The change of orientation of ambient space (i. e., mirroring the knot in S^3) is an involution on the space of knots, whose dualization on the space of Vassiliev invariants decomposes it into a $+1$ (called symmetric invariants) and a -1 (called asymmetric invariants) Eigenspace.

Proof. Clearly, it is theoretically possible (at least for given k) to calculate exactly the k-th derivative of $V(T_{m,n})$ directly from (4), but practically this is easily observed to turn into a horror, so let us keep track of what happens more globally. You have

$$V(T_{m,n}) = \frac{P(t)}{Q(t)}$$

for two polynomials P and Q, with only P depending on m and n. The well-known rule of differentiating a quotient says that

$$\left(\frac{P(t)}{Q(t)}\right)' = \frac{P'(t)}{Q(t)} - \frac{P(t)Q'(t)}{Q(t)^2}\,.$$

Assume $Q(t) = (t^2-1)^n \cdot C$ for some $n \in \mathbf{N}$ and $C \in \mathbf{Z}$ and that $P(t)$ is a sum of polynomials arising by differentiating at most k times certain polynomials P_i in t, whose powers, but not coefficients, depend on m and n. In the first term the numerator involves one derivative more of the polynomials P_i, hence its coefficients have (at most) one power more of m and n (where the powers of m and n are meant here so that for each monomial we sum the powers of both variables) than in P. In the second terms the number of derivations of P does not change, and all powers of t^2-1 of the second factor $Q(t)$ in the denominator cancel those of $Q'(t)$ in the numerator except one, hence the power of t^2-1

in the denominator increased by one. Therefore, starting with $V(T_{m,n})$ and differentiating k times, you end up with a sum of polynomial fractions P_i/Q_i, where the degree of the polynomial of m and n in the coefficients of P_i plus the power of t^2-1 in Q_i is at most $k+1$. Bring all these fractions to a common denominator and call the resulting one fraction for simplicity again P_1/Q_1. Then

$$P_1 = \sum_{l=0}^{k} P_{1,l}(t)\,(t^2-1)^l$$

with $P_{1,l}(t)$ having coefficients of degree at most l in m and n, and $Q_1=(t^2-1)^{k+1}$. The denominator has a zero at one, hence to evaluate the fraction at the (potential) singularity, we need to apply l'Hospital's rule $k+1$ times (clearly P_1 must have a $k+1$-fold zero at one as well, as $v(T_{m,n})$ evidently exists). Doing so, what remains from P_1 is a sum of powers of t^2-1 multiplied with derivations of $P_{1,l}$ of order $k+1-l$ and Q_1 becomes non-zero at one. Setting $t=1$ we obtain a polynomial of degree at most $k+1$ in both m and n, so we proved the statement for $k+1$ replacing k. Now assume, there is a monomial with m^{k+1} or n^{k+1}. As P_v is symmetric in m and n (because $T_{m,n}=T_{n,m}$) assume w. l. o. g. that m^{k+1} occurs. Then the Vassiliev invariant

$$v_1(K) := \frac{v(K)-(-1)^l v(!K)}{2}$$

has a lower degree that v [77] and is a linear combination of Vassiliev invariants of the form $V^{(n)}(1)$ for $n<k$ (because $V(!K)(t)=V_K(1/t)$, so

$$\left(\frac{d}{dt}\right)^k \left(V\left(\frac{1}{t}\right)\right)\Big|_{t=1}$$

can be expressed as a linear combination of

$$\left(\frac{d}{dt}\right)^i V(t)\Big|_{t=1}$$

for $i\le k$), hence its polynomial has $\deg_m \le k$. Now, by definition $v':=v-v_1$ is symmetric for k even and asymmetric for k odd. But $v'(T_{m_0,n}) =: P_{v'}(m_0,n)$ has degree $k+1$ in n for some $m_0\in\mathbf{N}$, which is a contradiction, because $T_{m_0,n}=!T_{m_0,-n}$ and the degree $k+1$ of an (a)symmetric polynomial is even (odd), but k and $k+1$ have different parities. □

It becomes clear from the proof that the point where monomials of degree $>k+1$ in m and n come in is that the product mn appears in the powers of P in (4). This can be eliminated for example by taking $T_{m,n}\#!T_{m,n}$ and hence, rerunning the previous argument, but taking care that we have one order more of singularity at $t=1$, we obtain

Theorem 8.3 Let $v=V^{(k)}(1)$. Then $\deg_{m,n} v(T_{m,n}\#!T_{m,n}) \le k+2$. □

(Note, that the parity argument does not work here anymore.)

Corollary 8.2 Let v be a primitive Vassiliev invariant of degree k, which can be written as a linear combination of products of invariants of the form $V^{(l)}(1)$ with $l\le k$, that is

$$v = \sum_i a_i \prod_j \left[V^{(b_{i,j})}\right]^{c_{i,j}}(1)$$

with $a_i\in\mathbf{Z}$, $b_{i,j},c_{i,j}\in\mathbf{N}$, $b_{i,j}\le k$. Then if a monomial $m^{k_1}n^{k_2}$ appears in P_v with

$$k_1+k_2 > \max_i \sum_j c_{i,j}(b_{i,j}+2)$$

then k_1 and k_2 are both odd.

Proof. Use the previous theorem and that $v(K_1\#K_2)=v(K_1)+v(K_2)$, $T_{m,n}=!T_{m,-n}$ and $T_{m,n}=T_{n,m}$. $\square$

Corollary 8.3 Let v be a symmetric primitive Vassiliev invariant of degree k, which can be written as a linear combination of invariants of the form $V^{(k)}(1)$. Then $\deg_{m,n}P_v\le k+2$.

Proof. For a symmetric invariant, P_v must contain only monomials of even degree in both m and n, hence the claim follows from the previous corollary. $\square$

Example 8.1 Let v_i be the (a)symmetric prime Vassiliev invariants of degree $i=2,3,4,5$ realizable as a linear combinations of products of $V^{(k)}(1)$ with $k\le i$ and $v_i(\text{unknot})=0$. Then their polynomials are of the following form:

$$\begin{aligned}
P_{v_2} &= C\cdot(m^2-1)(n^2-1)\\
P_{v_3} &= C\cdot mn(m^2-1)(n^2-1)\\
P_{v_4} &= (C+B(m^2+n^2)+An^2m^2)\cdot(m^2-1)(n^2-1)\\
P_{v_5} &= (C+B(m^2+n^2)+An^2m^2)\cdot mn(m^2-1)(n^2-1)
\end{aligned}$$

The constants A,B and C can be found by evaluating the invariant on a few (torus) knots.

Remark 8.1 The factors $(m^2-1)(n^2-1)$ come in because $T_{\pm1,n}=T_{m,\pm1}=\text{unknot}$. It is interesting, that P_{v_2} (P_{v_3}) divide all their even (odd) degree successors, so up to a factor, $v_i/v_{2+i \bmod 2}$ is always integral on torus knots.

9. Vassiliev invariants on closed 3 braids

After work of Birman and Menasco [13], knots of braid index 3 are now well-understood. This provides the good occasion to explore the present situation on this larger class of knots. The reasoning can be applied to other large classes of knots. Here I will try to explain the approach itself and not to maximize the obstructions it produces, as this involves a bit of computer calculations.

Recall, that each conjugacy class of 3 braids has a canonical (Schreier) representative, which is (up to some special cases we will not consider) of the form

$$\beta = C^k\sigma_1^{-p_1}\sigma_2^{q_1}\dots\sigma_1^{-p_n}\sigma_2^{q_n}\,,$$

where $k,p_i,q_i\in\mathbf{Z}$ and $p_i,q_i\ge1$ and the Garside braid $C=(\sigma_1\sigma_2)^3$ is the generator of the center of B_3. We will call C^k the central part of β, the vector $(p_1,q_1,\dots,p_n,q_n)$ Schreier vector. The Schreier vector is determined up to even cyclic permutation.

It is easy to see that, in order a 3 braid (word) b to close to an alternating diagram, it must be a Schreier braid with trivial central part (note that mirroring the braid, i. e. applying $\sigma_i\mapsto\sigma_i^{-1}$, is just an odd cyclic permutation of the Schreier vector). So in view of lemma 4.2 it suffices to consider henceforth basically only braiding sequences of Schreier braids with trivial central part.

Such a braiding sequence is described by the parity of the exponents in the Schreier vector. Denote by '$+$' even and by '$-$' odd parity. Then e. g.

$$(-++-) := \left\{\widehat{\ }\left(\sigma_1^{-p_1}\sigma_2^{q_1}\sigma_1^{-p_2}\sigma_2^{q_2}\right)\ :\ p_1,q_2 \text{ odd}, p_2,q_1 \text{ even}\right\}$$

For a sequence $\vec{x}$ of $\{+,-\}$ and $k\in\mathbf{Z}$, let $P_{\vec{x},k}$ denote the braiding polynomial of a fixed Vassiliev invariant v on the braiding sequence $C^k\cdot\vec{x}$. If k is omitted, default is $k=0$.

Although the Schreier vector is canonical just for positive elements, we will henceforth allow also negative ones. In that case we have a non-canonical representative, which can be transformed into a canonical one, e. g. $(3,-5,-7,9)=(2,1,4,6,1,8)$. In the case of negative elements we will call the vector *formal* Schreier vector.

Note, that not all $\pm$vectors correspond to braids closing to a knot. In particular the number of '$-$' must be even.

Lemma 9.1 The polynomials $P_{\vec{x}}$ satisfy

1) $P_{\vec{x}^T}(x_{2p},\dots,x_1) = -P_{\vec{x}}(x_1,\dots,x_{2p})$.

2) P is cyclically supersymmetric in the sense that

$$P_{\sigma(\vec{x})}\big((-1)^{\sigma}x_{\sigma^{-1}(1)},\,(-1)^{\sigma}x_{\sigma^{-1}(2)},\dots,\,(-1)^{\sigma}x_{\sigma^{-1}(2p)}\big) \;=\; P_{\vec{x}}(x_1,\dots,x_{2p})$$

for each $\sigma\in\langle(1\dots 2p)\rangle\simeq\mathbf{Z}_{2p}\subset S_{2p}$.

In particular 2) implies that all $P_{\vec{x}}$, where $\vec{x}$ runs over a cyclic class of $\pm$vectors, are interconvertible, so we need to consider only 1 element of this class. In the case $\vec{x}$ admits cyclic symmetries, this implies inner symmetry conditions to $P_{\vec{x}}$.

Proof. 1) It follows from the antisymmetry of the invariant under change of orientation.

2) For even cyclic permutations, as noted, the braids are themselves conjugate. For odd cyclic permutations, note that negating all arguments means mirroring the braid, and the mirror image is conjugate to a Schreier braid arising from the original one by an odd cyclic permutation. $\square$

Here is another fact we will extensively use in the following.

Lemma 9.2 Let P be a polynomial in n variables $x_1,\dots,x_n$. If $\exists i_1,\dots,i_k\in\mathbf{Z}$: $P(i_1,\dots,i_k,x_{k+1},\dots,x_n)\equiv 0$ (as a polynomial in $x_{k+1},\dots,x_n$), then each top degree monomial of P contains (at least) one of the variables $x_1,\dots,x_k$.

Proof. First note, that it suffices to prove the lemma for $i_1=\dots=i_k=0$, as argument translation does not chance the top degree monomials of P.

For $i_1=\dots=i_k=0$, prove the statement by induction over k. For $k=0$ there's nothing to prove. For the induction step, note, that by induction premise

$$P(0,x_2\dots,x_n) = x_2P_2+\dots+x_kP_k\,,$$

where P_i depends on $x_i,\dots,x_n$ only, and so

$$P-x_2P_2-\dots-x_kP_k\Big|_{x_1=0} \;=\; 0\,.$$

Then

$$x_1|P-x_2P_2-\dots-x_kP_k$$

and

$$P=x_1P_1+\dots+x_kP_k \qquad \square$$

As the braids of (Schreier vector) length 2 are all invertible, look at the first non-trivial case of length 4. It will turn out, that this is simultaneously the hardest case.

Modulo cyclic permutations there are only 2 sequences closing to knots: $(----)$ and $(--++)$.

Look at the (4 variable braiding) polynomial $P_{(--++)}$. As observed in [13], setting one of x_1 or x_2 to ±1 we obtain a braid "admitting a flype", which is invertible. Therefore setting one of x_1 or x_2 to ±1 in P, the remaining 3 variable polynomial must be 0.

$$P_{(--++)}(\pm1,.,.,.) = P_{(--++)}(.,\pm1,.,.) \equiv 0\,.$$

This means that $P_{(--++)}$ must be divisible by x_2^2-1 and x_1^2-1. In the same way, setting one of x_3 or x_4 to 0, we obtain a (invertible) braid of length 2, so

$$P_{(--++)}(.,.,.,0)=P_{(--++)}(.,.,0,.)\equiv 0\,.$$

Therefore $P_{(--++)}$ must be divisible also by $x_3\cdot x_4$. Furthermore we have

$$\begin{aligned}P_{(--++)}(x_1,x_2,x_3,x_4) &= -P_{(++--)}(x_4,x_3,x_2,x_1)\\ &= -P_{(--++)}(x_2,x_1,x_4,x_3)\end{aligned}$$

Setting

$$P_{(--++)}(x_1,x_2,x_3,x_4)=:P'(x_1-x_2,x_3-x_4,x_2,x_4)\,,$$

we obtain $P'(0,0,.,.)\equiv 0$. Therefore $P'(x_1',x_2',x_3',x_4')=x_1'\cdot P_1'+x_2'\cdot P_2'$ (for some polynomials P_1' and P_2') and therefore $P_{(--++)}=(x_1-x_2)\cdot P_1+(x_3-x_4)\cdot P_2$. Summarizing, we see that the first non-trivial polynomial in question for $P_{(--++)}$ has degree 7 and therefore no Vassiliev invariant of degree <7 can detect, the orientation of a (non-invertible) 3 braid of the series $(--++)$. We have not even yet exhausted all obstructions. E. g., there are two more relations $P(.,.,-2,2)\equiv 0$ and $P(.,.,2,-2)\equiv 0$.

Look at the series $(----)$. By the same argument as above, inserting ± 1 at any position in $P_{(----)}$, we obtain an invertible braid and the remaining 3 variable polynomial must be 0. This means that $P_{(----)}$ must be divisible by x_i^2-1 for all $i=1,2,3,4$. Additionally the braid closure is invertible if $x_2=x_4$ or $x_1=x_3$. Setting

$$P_{(----)}(x_1,x_2,x_3,x_4)=:P'(x_1-x_3,x_2-x_4,x_2,x_4)\,,$$

we see that by the same argument as above P' must be divisible by $x_1'x_2'$, and so $P_{(----)}$ must be divisible also by x_2-x_4 and x_1-x_3. Furthermore,

$$\begin{aligned}P_{(----)}(x_1,x_2,x_3,x_4) &= -P_{(----)}(x_4,x_3,x_2,x_1)\\ &= -P_{(----)}(-x_2,-x_1,-x_4,-x_3)\\ &= -P_{(----)}(-x_4,-x_3,-x_2,-x_1)\end{aligned}$$

Therefore P must be 0 whenever $x_1=-x_4$ and $x_2=-x_3$ or $x_1=-x_2$ and $x_3=-x_4$. Look at the first case. We obtain $P_{(----)}=(x_1+x_4)P_1+(x_2+x_3)P_2$. In the second case, in the same way as above, $P_{(----)}=(x_1+x_2)P_1'+(x_3+x_4)P_2'$. The solution of minimal degree to all the established conditions is up to a constant

$$P=(x_1+x_2+x_3+x_4)(x_1^2-1)(x_2^2-1)(x_3^2-1)(x_4^2-1)(x_1-x_3)(x_2-x_4)\,,$$

which is the only one of degree 11. Therefore we see that no Vassiliev invariant of degree <11 can detect the orientation of a (non-invertible) 3 braid of the series $(----)$.

Remark 9.1 The failure of low degree Vassiliev invariants to detect orientation at all is known. It was first established by Bar-Natan's computer experiments [3, 5] for degree ≤ 9 and very recently also for degree ≤ 12 [41]. So this is, in a way, nothing new. However, our argument is very natural and can easily be pushed a little further to give a proof of some examples even for degree 11 Vassiliev invariants.

Proposition 9.1 A Vassiliev invariant of degree 11 detects the orientation of some closed 3 braid of the series $(----)$, exactly if it does so with the knot with formal Schreier vector $(3,-3,-5,3)=(2,1,2,4,1,2)$ (this is an alternating non-invertible 12 crossing knot).

Proof. This is basically the fact, that $P(3,-3,-5,3)\neq 0$. □

Proposition 9.2 No Vassiliev invariant of degree 11 detects the orientation of the (non-invertible amphicheral alternating 14 crossing) knot with formal Schreier vector $(3,-5,-3,5) = (2,1,4,2,1,4)$, see fig. 7.

Proof. $P(3,-5,-3,5) = 0$. □

This example can be immediately extended to an infinite series (I propose to an interested reader to do this, if he likes, as an exercise).

Using the same argument, we invite the reader to do the following

Exercise 9.1 Prove that no Vassiliev invariant of degree ≤ 8 detects orientation of a (p,q,r)-pretzel knot, with $p,q,r \in \mathbf{Z}$ odd. (Trotter [83] proved, that the knot is non-invertible for $|p|,|q|,|r|$ all distinct and greater than one.)

Hint: Use, that the knots are invertible, if two of the parameters are equal or one is equal to ± 1 (see [83]).

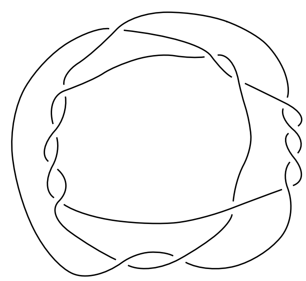


Figure 7: The $+$amphicheral non-invertible knot 14_{18676}, the closed 3 braid $\sigma_1^2\sigma_2^{-4}\sigma_1\sigma_2^{-2}\sigma_1^4\sigma_2^{-1}$.

Nevertheless, it seems that, in view of the results of [3, 5], the new approach will not give generically high ad hoc obstructions.

On the other hand, we have the more general approach of Stanford [63]. Stanford's idea was to look at iterated (pure) braid commutators. Fix some $g_1,\dots,g_k \in P_n$ and consider the braiding sequence* $[g_1^{x_1},[g_2^{x_2},\dots[g_{k-1}^{x_{k-1}},g_k^{x_k}]\dots]]$ in $x_1,\dots,x_k$. Then, whenever some $x_i = 0$, the commutator is trivial and so all x_i must divide the braiding polynomial. So starting with the unknot or an invertible or achiral knot, we obtain, inserting an n-fold interated commutator, an n-trivial (or n-invertible or n-achiral) knot. This has in turn the defect, that the knots become quickly very complicated (the examples for degree 12 will have several hundreds of crossings). However, recently Stanford proved [65] the striking result that all n-trivial knots arise by this construction. It is unclear (to me) whether the method works for n-invertible knots as well. Anyway, if existing, explicite representations of our knots in Stanford's form will be fairly unpleasant.

There is a much simpler general series of examples which is a slight generalization of the slice knots of [52], but it will take us too far aside to discuss it here.

Before looking on larger Schreier vectors, let's derive some relations between the P's, arising from braid equalities and conjugacies.

Theorem 9.1 The P's satisfy the following equations, where $a,b,c,d \in \{+,-\}$, $a \cdot b$ is the usual multiplication of signs, and $\bar{a} := - \cdot a$.

*In this sequence we insert the braid somewhere in a knot diagram with distinguished trivial inhabitant. The $g_i^{x_i}$ appear at multiple positions in the braid word, but you can consider it as a braiding sequence by letting the exponents vary in all occurences of g_i in the braid word, and then setting at each occurence the same variable x_i into the braiding polynomial.

1) $P(\ldots a \underset{l}{+} b \ldots)(x_1,\ldots,\underset{l}{0},\ldots,x_{2p}) =$
$P(\ldots(a\cdot b)\ldots)(x_1,\ldots,x_{l-1}+x_{l+1},\ldots,x_{2p})$.

2) $P(\ldots ca\underset{l}{-}b\ldots)(\ldots,x_{l-1},-1,x_{l+1},\ldots,x_{2p}) =$
$P(\ldots,\bar c - \underset{l}{a}\bar b,\ldots)(\ldots,x_{l-2}-1,1,-x_{l-1},x_{l+1}-1,\ldots,x_{2p})$

3) $P(\ldots\bar c - \underset{l}{a} - \bar b\ldots)(\ldots,x_{l-2}-1,1,x_l-2,1,x_{l+2}-1,\ldots) =$
$P_{-1}(\ldots ca\underset{l}{b}\ldots)(\ldots,x_{l-2},-x_l,x_{l+2},\ldots)$

4) $P(\ldots\underset{l}{\bar c} - \bar a\bar b - \bar d\ldots)(\ldots,x_l-1,1,x_{l+2}-1,x_{l+3}-1,1,x_{l+5}-1,\ldots) =$
$P(\ldots \underset{l}{c}abd\ldots)(\ldots,x_l,-x_{l+2},-x_{l+3},x_{l+5},\ldots)$

Proof. These are just the equalities

1) $\sigma_1^{-k}\sigma_2^0\sigma_1^{-l} = \sigma_1^{-(k+l)}$ (and analogously for σ_1^0).

2) $\sigma_1^{-k}\sigma_2^l\sigma_1\sigma_2^m = \sigma_1^{1-k}\sigma_2\sigma_1^l\sigma_2^{m-1}$, which is an iteration of the Yang-Baxter equality.

3) and 4) arise by replacing $\sigma_2^i \mapsto \sigma_2^{-i}$ and $\sigma_1^i \mapsto \sigma_1^{-i}$ in the braid word corresponding to the formal Schreier vector, and bringing the result into its Schreier normal form using the algorithm described by Birman and Menasco [13]. Note, that the conjugacy in 3) changes the central part of the braid. □

Relations 1), 3) and 4) relate special values of $P_{\vec x}$ to some $P_{\vec y}$ with $\vec y$ shorter than $\vec x$. If we take $\vec x$ of length 6 and assume that all $P_{\vec y} \equiv 0$ for $\vec y$ of length 4 (else we would be lucky to already have a non-invertibility proof of a knot), then 1), 3) and 4) provide in the same way as above divisors for $P_{\vec x}$.

Here it's useful to introduce the following notation.

Definition 9.1 A polynomial P is called to have the property "x_i" if all top degree monomials of P contain x_i.
P is called to have the property "$x_i \vee x_j$" if all top degree monomials of P contain x_i or x_j.
P is called to have the property "$x_i \leftrightarrow x_j$" if it has the property "$x_i \vee x_j$" and whenever it has a monomial M, it also has a monomial M', where M' is obtained from M either by replacing one of the copies of x_i in M to x_j or replacing one of the copies of x_j to x_i (e. g., $x_ix_j^2x_k \mapsto x_i^2x_jx_k$).

Example 9.1 Look at $P_{(----++)}$. By 1) it has the property "x_5" and "x_6", by 3) and 4) "$x_1 \vee x_3$" and "$x_2 \vee x_4$". Recall that if $P(\ldots,\underset{i}{1},\ldots,\underset{j}{1},\ldots) \equiv 0$ then $P = (x_i-1)P_1 + (x_j-1)P_2$, so P has the property "$x_i \vee x_j$".

Combining 1) and 2) of lemma 9.1, we obtain

$$\begin{aligned} P_{(----++)}(x_1,x_2,x_3,x_4,x_5,x_6) &= -P_{(++----)}(x_6,x_5,x_4,x_3,x_2,x_1) \\ &= -P_{(----++)}(x_4,x_3,x_2,x_1,x_6,x_5)\,, \end{aligned}$$

from which one deduces the property "$x_1 \leftrightarrow x_4 \vee x_2 \leftrightarrow x_3 \vee x_6 \leftrightarrow x_5$". It is easy to see that a P with these properties must have at least degree 5.

Example 9.2 Look at $P_{(--++++)}$. It has the properties "x_3,x_4,x_5,x_6" and $x_3 \leftrightarrow x_6 \vee x_5 \leftrightarrow x_4 \vee x_1 \leftrightarrow x_2$, and therefore at least degree 5 as well. $P_{(++-++-)}$ has the properties "x_1,x_2,x_4,x_5" and "$x_1 \leftrightarrow x_2 \vee x_3 \leftrightarrow x_6 \vee x_5 \leftrightarrow x_4$", and therefore also degree 5.

Note, that in all 3 examples we just explored the contents of the top degree monomials and also did not take into account the mutual relations 2) in theorem 9.1. So there are here also some open possibilities, whose detailed discussion may, however, become unattractive.

What is more important, is the following observation:

If we go over to longer and longer vectors, assuming that nothing worked for all shorter vectors, then it is easy to see, that the number of recursive relations to apply of the kind 1) and 4) in theorem 9.1 also grows beyond any extent, and so more and more divisors come into the polynomial. Therefore, if up to a certain length of the vector nothing worked for an invariant of fixed degree, then it wont work for longer vectors as well. This means, that we arrived to the following

Theorem 9.2 For any BPI it is decidable, using finitely many tests, whether it detects the orientation of a closed 3 braid. $\square$

Very probably this idea extends to (at least large subclasses of) knots of higher braid index.

10. Algebraic knots

Here is another application of the above idea.

Theorem 10.1 For any Vassiliev invariant additive under connected sum it is decidable, using finitely many tests, whether it detects the orientation of an algebraic knot.

First 2 important remarks.

Remark 10.1 Because of unfortunate circumstances, two big and completely different classes of knots received the same name “algebraic”, the first one being edges of holomorphic curves of the unit ball in **C** into $\mathbf{R}^4 = \mathbf{C}^2$ (as treated in [27, 16]), and the other being closed algebraic tangles, i. e., inter alia, rational knots (as treated in [1]; alternatively called “arborescent”). In the following we will refer by this name always to *latter* class.

Remark 10.2 Henceforth, although abusing the orientation of knots, we will always assume that each knot has a fixed orientation. So, in fact, each (unoriented) knot will stand for 2 (possibly) distinct oriented knots. This abuse, however will not spoil our arguments. The only thing to take care of is, that at the point in the following proof, when we obtain a composite diagram, we take the induced orientation of both factors.

Proof. An algebraic knot is a closed algebraic tangle, and an algebraic tangle can be obtained from the primitive tangles $0, \infty, n\,(n \in \mathbf{Z})$ by addition and multiplication. This expression of the tangle is the Conway notation, see figure 8. Call a ‘flip’ of a tangle A a new tangle A', arising from A by a

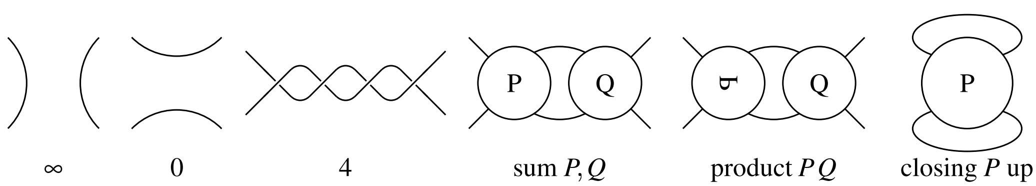
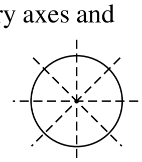


Figure 8: Operations with algebraic tangles

(possibly empty) sequence of the following operations.

1) Mirroring with respect to the 4 symmetry axes and



2) change of all crossings.

There are only finitely many ‘flip’s.

Clearly, we can remove all summations from the Conway notation using ‘flip’s:

$$a,b = \big(\mathrm{flip}\,(a)\big)\,b\,.$$

Here and throughout the whole proof, writing ‘flip’ we will never specify which flip we exactly mean (as it will turn out unimportant), but we will assume, that each symbol ‘flip’ appearing in the notation has a certain exact value assigned to it.

So, all algebraic tangles can be written as a (non-associatively parenthesized) sequence of integers with ‘flip’s applied to subexpressions. Assume, that each ‘()’ contains exactly two objects (integers or parenthesized subexpressions), so that no pair of parentheses is redundant. Call this modified notation “notation with flip”.

Consider for each notation with flip the corresponding braiding sequence. It is given by the parities of the integers. Write ‘$+$’ for even and ‘$-$’ for odd entries.

A *simplifying* or recursive relation is then an equality between special values of a braiding sequence and values of a shorter braiding sequence, where the length of a braiding sequence will be henceforth the number of integers.

The decisive point in the proof is the following

Lemma 10.1 $\#\{$ simplifying relations $\} \ge \frac{\sqrt{n}}{2}$, $n = \mathrm{len}($ notation $)$ for $n > 1$, such that each variable appears maximally in one such relation.

Proof of lemma. Start induction with $n = 2$ and $n = 3$.
For $n = 2$ there are 4 choices: $--, -+, +-, ++$, with possible ‘flip’s applied to some signs. In all cases one of the following relations applies (note, that $\infty = \mathrm{flip}\,(0)$)

$$\begin{gathered} 1\cdot 1 = 2 \\ 1\cdot 0 = 1 \\ 0\cdot 1 = \infty \\ 0\cdot 0 = \infty \end{gathered}$$

and analogously for ∞. So there is always 1 simplifying relation. In the same way argue for $n = 3$.

Now do the induction step.

Assume the inequality of the lemma is true for all notations with flip of len $< n$ and consider a notation A with len $= n$. There are 4 choices for A

$$(ab)c \quad \mathrm{flip}\,(ab)c \quad a(bc) \quad a\mathrm{flip}\,(bc) \tag{5}$$

or ‘flip’s of these 4 expressions, but it is unnecessary to consider $A = \mathrm{flip}($ something $)$ as all the simplifications we can achieve in ‘something’ carry over after flips.

Denote by $l_i, i \in \{a,b,c\}$ the lengths of the subexpressions a,b,c in (5).

Case 1. Assume, that maximally one of a,b,c has length 1. Then

$$\#\{\text{ simplifying relations }\} \ge \frac{\sqrt{l_a}}{2} + \frac{\sqrt{l_b}}{2} + \frac{\sqrt{l_c}}{2} - \frac{1}{2}\,,$$

latter term standing to equilibrate a possible uncorrect contribution from one of l_a, l_b, l_c being 1. So, as $l_a + l_b + l_c = n$,

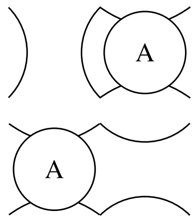
$$\frac{\sqrt{l_a}+\sqrt{l_b}+\sqrt{l_c}}{2} - \frac{1}{2} \ge \frac{\sqrt{n}}{2} \quad \text{for } n \ge 3\,.$$

Case 2. So two of a, b, c must be of length 1.

Assume, one of these two would be a '+' (or a 'flip' thereof). Then set 0 into the Conway notation. There are basically 2 possibilities (any 'flip's of the factors do not change qualitatively the picture):

2.1) $0 \cdot A$

2.2) $A \cdot 0$



In case 1 your diagram decomposes. You obtain

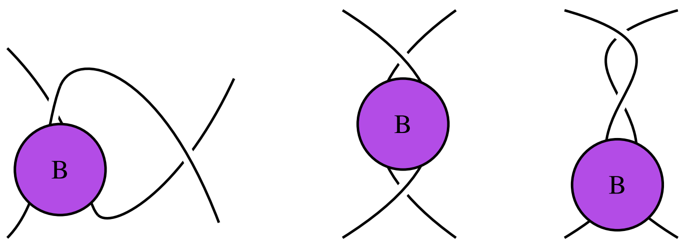
$$\dots(0 \cdot A)\dots = \dots 0 \dots \# \, \text{flip}\,(A)\,.$$

By the additivity of v under '#', it suffices to consider the (possibly trivial) factors separately. But both factors have a shorter notation and are hence dealt with by induction premise.

In case 2 we just have $A \cdot 0 = \text{flip}\,(A)$, which is also simplifying.

So in both cases there are $\geq \frac{\sqrt{n-2}}{2} + 1 \geq \frac{\sqrt{n}}{2}$ $(n \geq 3)$ recursive relations.

Case 3. Two of a, b, c are of length 1 and both are '$-$' (or flip$\,(-)$). Then by inserting appropriately 1 and -1 into these '$-$'es, you obtain modulo flips one of the following tangles



where B is some flip of the remaining tangle of len > 1 in the notation (5). But, after performing a flype, in all 3 cases you can simplify the tangles to ones, having as a notation with flip this of B and only *one* additional number (with some 'flip's performed on subexpressions). But this notation is again simpler, so it produces a relation, and you have

$$\geq \frac{\sqrt{n-2}}{2} + 1 \geq \frac{\sqrt{n}}{2}$$

simplifying relations.

Now the case distinction and the proof of the lemma are complete (note, that in our inductive procedure we never involved an entry into two recursive relations). □

The rest of the proof of the theorem follows from this lemma, applying exactly the same reasoning as for theorem 9.2: if a primitive Vassiliev invariant of deg $\leq n$ does not detect the orientation of an algebraic knot with Conway notation (with flips) of length $\leq 2n^2$, then it cannot detect the orientation of any algebraic knot. □

Remark 10.3 This lower bound for the number of recursive relations to apply is used for purely theoretical purposes and is very likely extremely unsharp. Probably (at least generically) there exists a constant $0 < C < 1$ such that $\#\{\text{ recursive relations }\} \geq C \cdot \text{len}(\text{ notation })$. So I do hope, that the algorithm can be practically realized.

Note, that the longer the Schreier vector / Conway notation become (i. e., the more complicated the knots) the more divisors of the polynomial exist and the less tolerance there is to its possible maximal degree (=the degree of the invariant) and therefore the less tests have to be performed.

It seems not hard to generalize theorem 10.1 to all knots. One way would be to extend the argument to the Conway notation of non-algebraic knots. This will be a future challenge.

11. On finite dimensionality

Now it is even possible to generalize this result. Note, that, what we did for some Vassiliev invariant v is, we looked at $v'(K) := v(K) - v(-K)$ and verified if it is $\equiv 0$. But one can see that the conditions to v' coming from its property to change the sign under orientation change in our previous discussion made up only a small fraction of the relations. So dropping this condition, we still have sufficiently many recursive relations for both algebraic knots and closed three braids.

So we see that in fact we have proved:

Theorem 11.1 Any Vassiliev invariant on algebraic knots is uniquely determined by its values on a (constructible) finite set of algebraic knots, depending only on the degree of the invariant. □

As an immediate consequence we obtain

Corollary 11.1 The space of Vassiliev invariants of given degree on algebraic knots is finite-dimensional. □

This is, of course, known [3]. What is more important here, is that our argumentation is considerably simpler (or, if you do not agree with that, at least completely independent) than the approach via the Fundamental Theorem [8], building decisively on deep work of Kontsevich [42] and Drinfel'd [26]. This classical diagrammatic approach to Vassiliev invariants can be very useful (see, e. g., [7] for a nice application). However, for this simple combinatorial structure we pay the high price, that the way back to knots is incredibly hard and complicated by lots of obstacles (see [8]). Therefore, one never found a good understanding of Vassiliev invariants on special knot classes in terms of their weight systems.

We hope, therefore, that using our approach one will be able to gain a more direct understanding of Vassiliev invariants as knot invariants.

12. A word on asymptotics of Vassiliev invariants

Here is a last small application of our approach, showing the qualitative difference of dealing directly with knots instead of diagrams.

The Universal Vassiliev invariant yield an isomorphism of the space of

$$\{\text{ Vassiliev invariants of deg} \le n\ \}\big/\{\text{ Vassiliev invariants of deg} \le n-1\}$$

to the factor space of chord diagrams modulo the $4T$ and the FI relation [3]. Although easy to define, this factor space is extremely hard to understand. As it appears in general absolutely hopeless to find out the graded dimension of this space, one aimed to give at least asymptotical bounds. This has been discussed in several papers [18, 19, 52, 69, 70]. The best lower [19] and upper [70] bounds known at present require significant effort and are surely far away from being sharp. See [68] for a summary on this topic.

Now the previous theorems give an approach to look for upper bounds by counting the necessary tests to perform. Here is a small sample consequence of this kind, showing how easy things can be in some special cases.

Theorem 12.1 The number of Vassiliev invariants of degree $\le n$ on algebraic knots with all Conway coefficients even is exponentially bounded (above) in n.

Proof. As for such Conway vectors each sign is '$+$' and gives a recursive relation, we only need to consider vectors of len $\le n$. The number of ways to parenthesize such a sequence is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is exponentially bounded in n. What happens, if we bring 'flip's into the notation? There are only finitely many 'flip's, including the trivial one. Each 'flip' stands before a pair of parentheses or before one single '$+$'. As $\#\{$ parentheses $\} = n-1$, we have $\le 2n-1$ occurences of a 'flip' in the notation and so the number of Conway notations with 'flip's is exponentially bounded in n. For each sequence we have to determine a polynomial of degree n in $k \le n$ variables. The number $\binom{n+k}{n}$ of monomials of this polynomial is also exponentially bounded in n. So we have $\le$ exponentially many polynomials with $\le$ exponentially many coefficients, and so $\le$ exponentially many tests to perform. So on algebraic knots with all Conway coefficients even you have $\le$ exponentially many primitive Vassiliev invariants of degree $\le n$, and by classical combinatorial arguments as in [68] or [19, Appendix] you obtain the same for all Vassiliev invariants. $\square$

Remark 12.1 The bound is possibly by far not sharp. It was just chosen to be better than what we know on all knots and to keep the proof minimal. Note also that such arguments may extend to all algebraic knots, if $\exists C > 0 : \#\{$ recursive relations $\} \ge C \cdot \mathrm{len}($ notation $)$, or the number of cases, where asymptotically such a C does not exist, can be controlled. This may be a future project.

Note, e. g., that on rational (2-bridge) knots (i. e., knots, where the Conway notation does not have ()'s and 'flip's) such C exists, as 2 neighbored odd entries give a recursive relation (in the same way as in the proof of theorem 10.1, by inserting 1 and -1 at the corresponding positions), and so we immediately obtain

Theorem 12.2 The number of Vassiliev invariants of degree $\le n$ on rational knots is exponentially bounded (above) in n. $\square$

In the same way one deduces

Theorem 12.3 The number of Vassiliev invariants of degree $\le n$ on closed 3 braids is exponentially bounded (above) in n. $\square$

It is hard to believe that one can so easily control the behaviour of Vassiliev invariants on the above knot classes by just looking on their weight systems and to deduce statements of this kind by chord diagrams. See Birman and Trapp [15] for such partial results for closed 3 braids and some combinatorial problems arising in the study of Vassiliev invariants on closed 3 braids via the crossing change/CD approach.

This clearly shows the limits of the graph theoretical approach to Vassiliev invariants.

13. A final question

Looking back, we see that our approach managed to give some information on what Vassiliev invariants do on special knot classes. However, one moment of thinking shows, that the question of what is a "Vassiliev invariant on a special knot class" is more subtle. Reviewing our arguments, we see that we have proven our results with the meaning of "Vassiliev invariants on some knot class" in the sense of the following

Definition 13.1 Let C be a knot class. Then a Vassiliev invariant of deg $\le m$ on C is a map $v : C \longrightarrow \mathbf{Q}$ with v vanishing on all $(m+1)$-singular knots, where *all* 2^{m+1} *possible resolutions give knots in* C.

But then there comes about a natural question.

Question 13.1 Is any Vassiliev invariant on algebraic knots the restriction of a Vassiliev invariant (on all knots) to algebraic knots? Or, in other words, does any Vassiliev invariant on algebraic knots admit an extension to a Vassiliev invariant (on all knots)?

Such a question was discussed in a very special case in [82, theorem 3.1.2]. However, in that generality, applying such straightforward arguments to confirm the question is impossible.

Surprisingly, I do not know of anyone who asked this question before. Maybe the reason is that one never got into a situation like ours, because one was never able to understand Vassiliev invariants on special knot classes. Note, that our approach naturally incorporates such (potential bizarre counter)examples to question 13.1. However, the classical graph theoretical approach will surely meet difficulties to carry over if we bring additional knot features into the game.

In that sense, it is not even (yet completely) clear, that corollary 11.1 is indeed a consequence of Kontsevich [42] and Drinfel'd [26].

I have no interpretation of what (the existence of) counterexamples to question 13.1 would mean. But maybe there is something here we do not understand ...

Remark 13.1 The careful reader will have noticed, that the generalization of our results to Vassiliev invariants on arborescent knots has a gap: we proved our theorem 10.1 for Vassiliev invariants additive under connected sum, whereas in the proof of theorem 12.1 we used primitive Vassiliev invariants (in the sense that they generate the algebra of Vassiliev invariants). But if we do not want to consider just restrictions of (global) Vassiliev invariants to arborescent knots, we don't know if primitive Vassiliev invariants on algebraic knots are really exactly the additive ones! (That's the price we pay for our attempt to generalize Vassiliev invariants and to reject the Hopf algebra structure of chord diagrams.)

But there is a way to walk around this problem. In theorem 10.1 the additivity condition can be dropped at some cost. Introduce in the proof of theorem 10.1 notation with both '$flip$' and '#', where '#' appears only outside of any parentheses, and consider with the arborescent knots all their connected sums. Then do the same proof of lemma 10.1 for the factors under connected sum separately, finally noticing, that a summand under '#' of notation of length 1 gives a recursive relation by setting 0 or 1 according to its parity (and then forgetting it, as it represents the unknot). At the point, when we obtained a composite diagram, we do not any longer need to consider the factors separately – their connected sum has already an admissible and simpler notation.

The proof of theorem 12.1 carries over also with '#', as the number of compositions of n into the lengths of the factors under connected sum in the notation is exponentially bounded in n (it is 2^{n-1}). Then for each such composition the product of the number of notations without '#' (of the factors) is exponentially bounded in n (as observed in the proof of theorem 12.1) and finally in each such composition we have $\leq n$ factors, and so $\leq 2^n$ (implicit) choices of orientation of these factors.

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