ON THE COEFFICIENTS OF THE LINK POLYNOMIALS

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. We give inequalities for the coefficients and Mahler measure of the link polynomials in terms of the crossing number of a link diagram.
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1. Introduction

Fifteen years ago, the appearance of the Jones polynomial V [J] and its immediate successors [H, PT, Ka, BLM, Ho] gave tremendous impetus to the theory of links. Similarly to the Alexander polynomial Δ, whose degree was known to estimate (from below) the genus of a link, several spectacular results have been proved which related classical link invariants as braid index [Mo, FW] or crossing number [Ka, Mu, Th, St] to the degrees of these polynomials. Some other, although less spectacular, results concerned the coefficients of the polynomials. Properties of them were proved by Thistlethwaite [Th] for the Jones polynomial of alternating links, Cromwell [Cr] gave some properties of the Conway ∇ and HOMFLY P polynomial of homogeneous links, and Kidwell [Ki] and Thistlethwaite [Th2] proved the positivity of the leading coefficient of the Brandt-Lickorish-Millett-Ho Q resp. Kauffman F polynomial of alternating links. Recently, in [St2] we showed that any coefficient of the V, P and F polynomial admits only finitely many values on positive knots.

The aim of this paper is to add to these properties some inequalities estimating exponentially the coefficients of the polynomials in terms of the number of crossings (and components) of a knot (or link) diagram, similarly to the degree estimates known for the V, Q and F polynomial.

The origin for seeking such inequalities was the consideration in previous papers (see [St2]) of generating series associated to knot polynomials, whose convergence had to be assured to justify the calculations performed with them. It was explained in [St] how estimates (from above) on the coefficients of the polynomials in certain cases yield estimates on its degrees, which in turn have applications to the braid index and Thurston-Bennequin numbers [St2]. Another motivation is that, although with comparable effort the estimates cannot be made as sharp as the ones for the degree, they are very self-contained, most of them are easy to prove, and do not seem to have been noticed, at least explicitly, before. Our approach was inspired and partially follows ideas of Kauffman [Ka2], and especially Kidwell [Ki], whose concept of the longest bridge is used.

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For special types of links (and conjecturally in full, or at least larger generality) the general estimates can be sharpened, leading in the case of the Conway and HOMFLY polynomial to another occurrence of the Fibonacci numbers in a knot theoretical context, after [St5].

We should remark that, although practical experiments (discussed in §4) suggest that the base of the exponential admits some improvement, one cannot show any subexponential bound of such type. The easiest way to see this is to consider iterated connected sums of a knot, whose polynomial is not a unit (that is, conjecturally except for the Alexander/Conway polynomial, any non-trivial knot). One can show, by a simple analytic argument (see below, or also [St]), that the powers of any non-unit in a (Laurent) polynomial ring over $\mathbb{Z}$ contain exponentially growing coefficients. We use this to derive asymptotical lower bounds on the maximal size of coefficients of polynomials of links with a given number of crossings. These estimates rely on the calculation of specific examples in Thistlethwaite's knot tables (see [HTW]).

Since the Mahler measure [Ma] of a polynomial is bounded by the size of its coefficients, a final application our estimates is that they are simultaneously estimates for the Mahler measure of the link polynomials. (See [GH] for a recent survey exposition on the Mahler measure and related topics.)

2. Definitions and preliminaries

We start by fixing some (although mostly standard) notation and terminology and recalling some well-known basic facts. A few more definitions fill follow later in the text.

2.1. Braids

The n-strand braid group B_n is generated by the elementary (Artin) braid generators σ_i for $1 \le i \le n-1$ with *braid relations* $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, called *YB relation*, and $[\sigma_i, \sigma_j] = 1$ for $|i-j| > 1$ (the brackets denoting the commutator).

We introduce an alternative notation for braid(word)s by replacing the σ_i by their subscripts and their inverses by the negated subscripts, and putting the result into brackets, e. g., $(\sigma_1\sigma_2^2)^5\sigma_1^{-1}\sigma_2 = [(12^2)^5 - 12]$.

A braid (word) is called alternating if it contains no generators σ_i and σ_j occurring with powers of opposite sign, such that $i-j$ is even. A braid word is called positive if it contains no generators occurring with negative powers.

In this paper we shall be particularly concerned with the 4-strand braid group B_4 and will consider its distinguished element $\xi_4 = [123121]$, whose square is central in B_4, and in fact generates its center. In the other B_n there are similar elements ξ_n. They are called *half twist braids*, and their squares *full twist braids*.

By $\hat{\alpha}$ we denote link, which is the braid closure of a braid α. For a braid word α' representing α, the closure operation gives a specific diagram of $\hat{\alpha}$, which we denote by $\hat{\alpha}'$.

For a word $\alpha = a_1 \dots a_n$ we say that a word $\beta = b_1 \dots b_m$ is a *subword* of α iff there is an $1 \le i \le n-m+1$ such that $b_j = a_{i-1+j}$ for any $1 \le j \le m$. We say that β is a *weak subword* of α iff there are numbers $1 \le i_1 < i_2 < \dots < i_m \le n$ with $b_j = a_{i_j}$ for any $1 \le j \le m$.

2.2. Links and diagrams

Diagrams and links will for convenience always be assumed oriented, although the orientation is not always needed. A *component* of a link (resp. link diagram) is one of the embedded circles (resp. the subdiagram representing it). *Knots* are considered links with only one component.

A *split component* of a link is an equivalence class of link components modulo the (equivalence) relation, which is the transitive expansion of the relation "inseparable", where two link components are inseparable, if there is no isotopy in $\mathbb{R}^3$ making them separable by a hyperplane (or their projections in a diagram in $\mathbb{R}^2$ separable by a line disjoint from the link). A split component may contain several link components. A split component is *trivial*, if it is the unknot.

A link is *split* if it has more than one split component. Otherwise it is *non-split*.

A split component of a link diagram is a connected component of the plane curve of the diagram. Obviously, a link diagram cannot have more split components than the link it represents.

A link is called *prime* if for any embedded sphere $S \subset \mathbb{R}^3$ intersecting L (transversely) in exactly 2 points, we have that one of the two links obtained by joining an arc on S connecting these 2 points and the intersection of the interior resp. exterior of S with L is an unknot.

A diagram D is called *prime* if there is no closed curve in the plane intersecting D transversely in exactly two points and containing in both its in- and exterior crossings or a whole (possibly trivial) split component of D. Per convention we exclude the trivial unknot diagram (no crossing, one split component) from being prime.

A crossing p of D is called *nugatory* if there is a closed curve in the plane intersecting D transversely only in p. A diagram D is called *connected* or *non-split* if it has only one split component, that is, there is no curve in its complement, such that both the in- and exterior of the curve contain parts of D. Otherwise it is called *split*.

According to our definition, the addition of a trivial split component is the connected sum with the 2 component unlink, and the addition of a non-trivial split component L is the sequence of 2 connected sums, first with the 2 component unlink, and then with L. Accordingly, the only prime split link is the 2 component unlink (as listed also in the tables of [Ro, appendix]).

A braid diagram $\hat{\alpha}'$ with $\alpha' \in B_n$ is prime iff for any $1 \le i < n-1$, α' contains a weak subword of the form $\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}$ or $\sigma_{i+1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_i^{\pm1}$, all '$\pm$' being independently choosable. A nugatory crossing in $\hat{\alpha}'$ corresponds to a letter $\sigma_i^{\pm1}$ occurring only once in α'.

Any diagram D can be written in a unique way (up to permutation) as the connected sum of prime diagrams. Latter are called the *prime components* or prime factors of D. (Note again that by our definition, the number of prime components of D is never less than the number of split components, since every split union accounts for a 2 component unlink as a prime component.) Similarly, a link can be written in a unique way as the connected sum of prime links, also called its prime factors.

A *region* of a link diagram is a connected component of the complement of the (plane curve of) the diagram. An *edge* of D is the part of the plane curve of D between two crossings (clearly each edge bounds two regions).

A link diagram is called *alternating* if each strand alternatingly passes crossings as under- and overpass. (Hence, a diagram of a closed braid word β is alternating, if and only if β is an alternating braid word.) There is always a way to switch the crossings of any link diagram so as it to become alternating, canonical up to simultaneous switch of all crossings in each split component.

A link diagram D is called alternating *along* some region R, if all of D's edges in the boundary of R join an overcrossing and an undercrossing. (Clearly, a diagram is alternating iff it is alternating along any of its regions.)

A *clasp* is a region with 2 edges. It is called *trivial* or *resolved*, if the diagram is not alternating along it.

2.3. Polynomials and sequences

For two sequences of integers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ we write

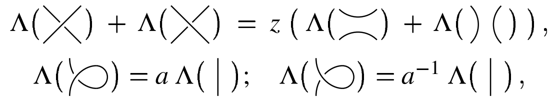
$$a_n \sim b_n, \text{ if } \lim_{n\to\infty} a_n/b_n = 1, \quad \text{and} \quad a_n = o(b_n), \text{ if } \limsup_{n\to\infty} \left|a_n/b_n\right| = 0\,.$$

For the definitions of the various link polynomials we refer to the papers [LM, Ka, BLM, J].

The skein polynomial P is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1}P\big(\text{positive crossing}\big) + l\,P\big(\text{negative crossing}\big) = -m\,P\big(\text{smoothing}\big)\,. \tag{1}$$

The Kauffman polynomial is usually defined via a regular isotopy invariant $\Lambda(a,z)$ of unoriented links with the properties



$$\Lambda(\text{unknot}) = 1\,.$$

The Kauffman polynomial $F(D)$ of a link represented by an oriented diagram D is then defined as $a^{-w(D)}\Lambda(a,z)$, where $w(D)$ is the writhe of D.

The Jones polynomial V, Brandt-Lickorish-Millett-Ho polynomial Q, (one variable) Alexander polynomial Δ and Conway polynomial ∇ are obtained from P any F by the substitutions

$$\begin{aligned} V(t) &= P(-it, i(t^{-1/2}-t^{1/2})) = F(-t^{-3/4}, t^{1/4}+t^{-1/4})\,, \\ \nabla(z) &= P(i, iz)\,, \\ \Delta(t) &= \nabla(t^{1/2}-t^{-1/2}) = P(i, i(t^{1/2}-t^{-1/2}))\,, \\ Q(z) &= F(1,z)\,. \end{aligned}$$

For P and F there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity. However, it will not be of any importance neither for the results nor for the arguments which one we use. We should only assume the invariants to be normalized so as the unknot to have polynomial 1.

We will sometimes alternatively write V_D, $\Delta_D, \dots$ for $V(D)$, $\Delta(D), \dots$ etc. Also, we will not notationally distinguish arguments to the polynomials, which are variables/numbers, diagrams, or links, since the meaning of the expression will not be ambiguous.

Let $[Y]_{t^a} = [Y]_a$ be the coefficient of t^a in a polynomial $Y \in \mathbb{Z}[t^{\pm 1}]$. Let

$$\min\deg Y = \min\{\, a \in \mathbb{Z} : [Y]_a \neq 0 \,\}, \quad \max\deg Y = \max\{\, a \in \mathbb{Z} : [Y]_a \neq 0 \,\}, \quad \operatorname{span} Y = \max\deg Y - \min\deg Y$$

be the minimal and maximal degree and *span* (or breadth) of Y, respectively. Similarly one defines for $Y \in \mathbb{Z}[x_1,\dots,x_n]$ the coefficient $[Y]_X$ for some monomial X in the x_i, and $\min\deg_{x_i} Y$ etc. $\max\mathrm{cf}\, Y$ denotes the leading coefficient of Y, i.e. $[Y]_{\max\deg Y}$.

The (1-)norm $|Y|_1 = |Y|$ of a Laurent polynomial Y is defined as

$$|Y|_1 = \sum_X \big|\, [Y]_X \,\big|\,,$$

where the (formally infinite) sum runs over all applicable monomials X.

If L is a link of n components, then for the Jones polynomial $Y = V_L$ and Alexander polynomial $Y = \Delta_L$, the applicable monomials X would be of the form $t^{(n-1)/2+m}$ for $m \in \mathbb{Z}$, for the skein polynomial $Y = P_L$, we would have $X = l^{n-1+2r}m^{n-1+2s}$ and for the Conway polynomial $X = z^{n-1+2r}$ for $r,s \in \mathbb{Z}$, for the Q polynomial $Y = Q_L$, we would have $X = z^r$ for $r \in \mathbb{Z}$, and for the Kauffman polynomial $Y = F_L$, we would have $X = a^r z^s$ with $r+s+n$ odd.

Clearly, the 1-norm is both subadditive and submultiplicative.

2.4. Knot tables

A remark on knot tables is in order. We use here the convention of Rolfsen's tables [Ro, appendix] for ≤ 10 crossing knots and that of Thistlethwaite (see [HTW]) for ≥ 11 crossing knots, which coincides with those of the first tabulators for any crossing number except 11, where the initial tables were compiled by Conway [Co]. We apologize for not using his numbering. An excuse is that all calculations have been performed by Thistlethwaite's program KnotScape, which yet does not provide a translator between its notation and that of Conway. For uniformity reasons, we will need to continue using this convention in subsequent papers, too. It is understood that alternative work on knot tabulation is being done by Aneziris [An].

3. The inequalities

3.1. Maximal bridge length

Following Kidwell [Ki], let us first recall the definition of the longest bridge.

Definition 3.1 The maximal bridge length $d(D)$ of a link diagram D is the maximal number of consecutive crossing over- or underpasses when following the orientation of some component of D.

We start with an inequality for the F and Q polynomial.

Theorem 3.1 Let D be a link diagram of $c(D)$ crossings, maximal bridge length $d(D)$ and $n(D)$ components. Then

$$\big|Q(D)\big|_1 \le \big|F(D)\big|_1 \le 5^{c(D)-d(D)}3^{n(D)-1}\,. \tag{2}$$

Proof. We use (nested) induction on $c(D)$ (outer one) and for fixed $c(D)$ (inner) induction on $c(D)-d(D)$.

For $c(D)=0$ the formula follows from the formulas of the polynomials of the unlinks.

Thus consider the induction step in $c(D)$ (outer induction). It is clear that we can work with the framed version Λ of F. We consider the resolution of D according to the Kauffman relation of Λ obtained by applying the relation to an under(over-)crossing immediately following a bridge of over(under-)crossings of length $d(D)$ in D.

If this crossing belongs to the bridge itself, then we would be able to remove a trivial loop from D (which does not augment $c(D)-d(D)$), and would be through by induction on c. This in particular always happens when for given c, $c-d$ is minimal, so that we have the induction start for the inner induction on $c-d$.

Now if $D=D_{\times}$ (the subscript referring to the crossing considered), then all of $D_{\times}$, $D_{\asymp}$ and $D_{)(}$ have smaller $c-d$ and not higher c than $D_{\times}$, so for them the inequality holds by induction.

If the crossing we consider is not mixed, that is, involves strands of the same component, the number of components of one of $D_{\asymp}$ and $D_{)(}$, say of $D_{\asymp}$, differs by ± 1 from this of D, and $n(D_{\times})=n(D_{\times})=n(D_{)(})$. Then, the contribution of $D_{\times}$ and $D_{)(}$ to $|F(D)|$ is $1/5$ of the r.h.s. of (2), while the contribution of $D_{\asymp}$ is (in the worse case that $n(D_{\asymp})=n(D_{\times})+1$) at most $3/5$ of it, so the inequality follows.

If the crossing is mixed, that is, involves strands of different components, then $n(D_{)(})=n(D_{\asymp})=n(D_{\times})-1$, with a similar conclusion. $\square$

Using the results of [Ki], we obtain

Corollary 3.1 If K is an alternating knot, then $|F_K|_1 \le 5^{\max\deg_z F_K}$. $\square$

Another consequence gives a (hypothetical, at least) obstruction against F-maximality. Analogously to [St3], where the picture for Q was considered, we call a knot (or link) K to be F*-maximal*, iff

$$\max\deg_z F_K = \min_{D \text{ diagram of } K} c(D)-d(D)\,.$$

Corollary 3.2 If $|F|_1 > 5^{\max\deg_z F_K}$ for a knot K, then it is not F-maximal. $\square$

See, however, the discussion in §4.

Let us turn back to more inequalities. The same argument as for theorem 3.1 can be applied for the P and V polynomial.

Theorem 3.2 Under the same conditions as in theorem 3.1, we have

$$\begin{aligned} |P(D)|_1 &\le 3^{c(D)-d(D)}2^{n(D)-1} \\ |V(D)|_1 &\le 5^{c(D)-d(D)}2^{n(D)-1}\,. \end{aligned} \tag{3}$$

Proof. Again this is essentially applying the previous argument, this time replacing the Kauffman relation by a skein relation. One needs to take care that for the V polynomial one of the polynomials in the skein relation is multiplied by a 2-monomial expression $(t^{1/2}-t^{-1/2})$, which forces us to increase the first base by second one. □

The same reasoning works also for the (1-variable) Conway/Alexander polynomial. Note that here the second exponential is unnecessary, as split (un)links have zero polynomial.

Theorem 3.3 Under the above conditions, we have

$$\begin{aligned} |\Delta(D)|_1 &\leq 3^{c(D)-d(D)} \qquad (4)\\ |\nabla(D)|_1 &\leq 2^{c(D)-d(D)}\,. \end{aligned}$$ □

Remark 3.1 Using the above arguments, one can give an estimate on the number of different polynomials admissible by links of given crossing number, but such an estimate would be too crude to be of any interest, and it is not clear (to me) how to enhance the method in this direction. An effective estimate, using more sophisticated tools, is given in [St4].

Remark 3.2 On the positive side, one can apply arguments similar to [KS] to give a different version of the inequalities above on diagrams which admit a tangle decomposition. If a diagram D can be obtained from another diagram $\tilde{D}$ by replacing n crossings in $\tilde{D}$ by tangles $T_1,\dots,T_n$, then in all previous formulas '$d(D)$' can be replaced by '$\sum_{i=1}^{n} d(T_i)-n+1$', with the obvious definition of $d(T_i)$. In this case some inequalities may because more relevant, because a long bridge, whose rerouting (also called 'wave move') to simplify the diagram in some obvious way, may not exist.

3.2. Canonical genus

Another type of inequalities should only be mentioned briefly here. They follow from the more elaborate inequalities involving the diagram genus, which can be found in [St].

Definition 3.2 By $\tilde{g}(K)$ we denote the weak genus of K, that is, the minimal genus of all its diagrams, and the genus $g(D)$ of a diagram D we will call the genus of the surface, obtained by applying the Seifert algorithm to this diagram:

$$\tilde{g}(K) = \min\left\{ g(D) = \frac{c(D)-s(D)+1}{2} \,:\, D \text{ is a diagram of } K \right\},$$

with $c(D)$ and $s(D)$ being the crossing and Seifert circle number of D, respectively.

We recall that by [St6], for every $g\in\mathbb{N}$ there is some number $d_g\in\mathbb{N}$ such that for any knot diagram D, the norm $\big|(t+1)^{d_{g(D)}}V_D(t)\big|_1$ can be bounded above by something depending just on $g(D)$, and *not* $c(D)$. A similar statement follows analogously for P. Using the above inequalities, and the result $d_g=O(g)$ of [STV], we obtain the following:

Proposition 3.1 There are constants C_P and C_V such that

$$\big|(t+1)^{d_{\tilde{g}(K)}}V_K(t)\big|_1 \leq C_V^{\tilde{g}(K)},$$

and

$$\big|(l^2+1)^{d_{\tilde{g}(K)}}P_K(l,m)\big|_1 \leq C_P^{\tilde{g}(K)}.$$ □

It is easy to write down explicit values for C_P and C_V, but they may not be small.

If a knot diagram D has minimal crossing number *and* minimal genus at the same time, for example if it is alternating, then we have:

Corollary 3.3 For an alternating knot K we have

$$\big|V_K\big|_1 \leq C_V^{g(K)}\,c(K)^{d_{g(K)}},$$

and

$$\big|\nabla_K\big|_1 \leq \big|P_K\big|_1 \leq C_P^{g(K)}\,c(K)^{d_{g(K)}}.$$ □

3.3. The determinant

There is another way to obtain inequalities for the Jones and Alexander polynomial, using the Kauffman bracket approach to V and the common value of V and Δ, the determinant.

To explain this, we recall a fact which is implicit in the analysis of the Kauffman bracket of alternating diagrams. The arguments are rather standard, so we will be brief. See [Ka2, Ad] for more details.

Lemma 3.1 Let D be an alternating diagram and D' be another diagram obtained from it by switching some set of crossings. Then there is some $n \in \mathbb{Z}$ such that

$$\left|[t^n V(D')]_{t^k}\right| \le \left|[V(D)]_{t^k}\right| \quad \text{for all } 2k \in \mathbb{Z}.$$

Proof. Consider the (rooted) resolution tree of D according to the bracket relation, such that only non-nugatory crossings in the node diagrams are resolved (giving a branching point in the tree), and the leafs are labeled by unlink diagrams with possible nugatory crossings. The same tree arises for D', only that the contributions of the leafs are weighted differently. However, using induction on the number or crossings, it follows that at every branching point in the tree of D both hand sides contribute to each coefficient of the bracket (and hence of the Jones polynomial) a quantity of the same sign, so no cancellations occur (as may in the tree for D'). □

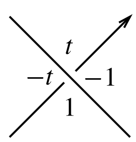
The fact that the coefficients of the Jones polynomial of an alternating connected diagram alternate in sign [Th, theorem 1] (also provable using the Kauffman bracket) implies the

Corollary 3.4 Let $\mathcal{D}$ be the set of 2^n diagrams obtained by signing in an arbitrary way the n crossings of a (closed) plane curve with n intersections. Then the two alternating diagrams in $\mathcal{D}$ are those which maximize (uniquely, if they are prime and have no nugatory crossings) the determinant $|\Delta(-1)| = |V(-1)|$. An analogous property holds for link diagrams. □

Lemma 3.2 $|\Delta(D)|_1 \le |\Delta(D')|_1$, for D and D' as above, D' alternating. Moreover, the coefficients of $\Delta(D')$ alternate in sign.

This fact follows from an observation of Kauffman [Ka3] on the way Alexander originally calculated his polynomial in [Al]. Since it is outlined only basically in [Ka4], we give a brief description.

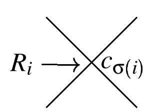
Proof. Consider an n crossing diagram D and choose n regions $R_1,\dots,R_n$ in the complement of the diagram, such that the remaining 2 regions R'_1 and R'_2 are adjacent (that is, share an edge). Number the crossings of D to be $c_1,\dots,c_n$. If c_j is not adjacent to R_i, then set $A_{i,j}=0$. Else consider the 4 regions around c_j and give each of them values of $A_{i,j}$ in ± 1, $\pm t$ depending on the side from which R_i meets c_j:



(the orientation of the overcrossing strand is irrelevant). Then $\Delta(D) \doteq \det(A_{i,j})$, '$\doteq$' denoting equality up to units in $\mathbb{Z}[t,t^{-1}]$. (This can be seen by establishing the skein relation for $\det(A_{i,j})$.) When writing

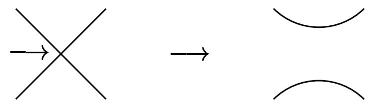
$$\det(A_{i,j}) = \sum_{\sigma\in S_n} (-1)^{\sigma} \prod_{i=1}^{n} A_{i,\sigma(i)}\,, \tag{5}$$

the contribution of a permutation σ is non-zero if and only if $R_i \mapsto c_{\sigma(i)}$ is an assignment of a crossing to a region meeting it, such that each crossing is assigned exactly once, and then this contribution is a monomial. But, when denoting this correspondence by an arrow from the region to the crossing:



,

and replacing

 , (6)

it is an easy exercise to see that these splicings of the crossings give only one plane curve. (In the context of the bracket model for the Jones polynomial this is called a 'monocyclic state' [Kr].) To see this, notice that if at some point the splicing (6) disconnects the diagram into 2 components $D_{1,2}$ of $n_{1,2}$ crossings, then the adjacency of the 2 regions $R'_{1,2}$ implies that for some $k \in \{1,2\}$, all regions of D_k except one are among the R_i's, such that σ must assign the n_k crossings of D_k to n_k+1 regions, a contradiction. Similarly, one argues that each monocyclic state can be realized only once, because a rearrangement of the arrows to give the same splicings will result in splicings which disconnect the diagram. That each monocyclic state is indeed realized by a permutation, will follow from considering the alternating diagram D'; see below.

Thus the calculation of the determinant of the matrix $(A_{i,j})$ via (5) can be interpreted as a "state sum", the non-trivial summands being units and coming from the monocyclic states in the bracket model. The number of such monocyclic states is the determinant $\Delta(-1) = V(-1)$ of the underlying alternating diagram D' (see [Kr]), and thus a cancellation of the units contributed by such monocyclic states occurs iff the diagram D is non-alternating. This can be seen directly from the construction of $(A_{i,j})$, or by using the argument for the bracket model for V and the identity $\Delta(-1) = V(-1)$. Applying this same identity to the alternating diagram D' it follows that each monocyclic state must indeed be realized by a permutation in (5), since $|\Delta|$ is always less than or equal to the number of units, adding up to Δ.

Since therefore the contribution of monomials in (5) to $\Delta(D')$ do not cancel, as they may in D, the result follows. □

Remark 3.3 It may be possible that Lemma 3.1 holds in fact also for the Alexander polynomial.

Thus one is interested in estimating the number of monocyclic states.

Lemma 3.3 Let D be a (connected) link diagram. The number of monocyclic states of D is at most $2^{c(D)-1}$ if $c(D) > 0$.

Proof. Use induction on $c(D)$, noting that if $c(D) = 1$, then (at most) one of the two splicings gives a single curve. □

Using lemmas 3.2 and 3.1 we obtain

Proposition 3.2 Let D be a link diagram of $ns(D)$ split components, each with at least one crossing. Then $|\Delta(D)| \le 2^{c(D)-1}$ for $ns(D) = 1$ (otherwise $\Delta(D) = 0$) and $|V(D)| \le 2^{c(D)+ns(D)-2}$. □

Using [Ka, Mu, Th] we obtain

Corollary 3.5 If for a knot K we have $|V_K(-1)| > 2^{\operatorname{span} V_K - 1}$, then K is not alternating. □

Remark 3.4 This further inequality for $|\Delta(D)|$ and $|V(D)|$ does not involve $d(D)$, it is more effective because of the smaller exponent base, at least when $d(D)$ is small compared to $c(D)$, which is usually the case. In [St3, appendix], Kidwell showed for *knot* diagrams that in fact if $d(D) > c(D)/3$, then D can be simplified (according to the crossing number) by rerouting the longest bridge, so that $c(D) - d(D)$ remains unchanged, unless after the rerouting another bridge becomes the longest one, in which case it decreases.

3.4. Inequalities involving Fibonacci and Lucas numbers

3.4.1. A conjecture

In the case of the Conway and HOMFLY polynomials, there exist possibly better inequalities, involving Fibonacci and Lucas numbers.

It is easy to see that if T_p is the $(2,p)$-torus knot or link (with parallel orientation), then

$$[\nabla_{T_p}]_k = \binom{\frac{p+k-1}{2}}{k},$$

and that

$$|\nabla_{T_p}| = F_p \qquad \text{and} \qquad |P_{T_p}| = L_p\,,$$

with $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ being the Fibonacci numbers, and $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ being the Lucas numbers. In this subsection we do not consider the Kauffman polynomial, so no notational confusion with the Fibonacci numbers should arise. We also set $F(n) = F_n$ and $L(n) = L_n$ to improve readability, since some indices subsequently occurring will be somewhat long.

There is some evidence for the following conjecture:

Conjecture 3.1 Let L be a link of $n(L)$ components, and $v_k = [\nabla]_k$. Then (for $2 \nmid k + n(L)$)

$$v_k(L) \le \binom{\lfloor 1/2(c(L)+k-1)\rfloor}{k}. \tag{7}$$

Also, if $k > 0$ and $2 \nmid c(L) + k$, then strict inequality holds, unless L is the $(2, c(L))$-torus knot or link (with parallel orientation).

Moreover, beside the inequality

$$|\nabla_L| \le |\nabla_{T_{c(L)}}| \tag{8}$$

following from (7), we have for any link L

$$|P_L| \le |P_{T_{c(L)}}|\,2^{b(L)-2} = L_{c(L)}\,2^{b(L)-2}, \tag{9}$$

$b(L)$ being the braid index of L, again the only links achieving equality being the $T_{c(L)}$ and their split unions with unknots.

Here we will be concerned with proving special parts of this conjecture for general k and certain classes of links. Before we start with our results, we make some remarks, in particular motivating the conjecture:

1) The inequality (7) is trivial for $k = 0$ and easy to see for $k = 1$, also including the statement of the sharpness case. By [PV, theorem 1.E], the inequality (7) is true for $k = 2$, when L is a knot (and not a 3-component link); the statement on the sharpness case could also be easily deduced from the proof.

2) The inequality (7) shows in particular that

$$\max_{c(L)\le c} v_k(L) \sim \frac{c^k}{2^k k!} + o(c)$$

as $c \to \infty$. We always have

$$\max_{c(L)\le c} v_k(L) \sim C_k c^k$$

for some explicitly computable constant C_k, following from the proof of the Lin-Wang conjecture, see [MS]. But by the methods of proof of this fact, the calculation of C_k is very complicated. We know, by [PV, theorem 1.E], that indeed $C_2 = 1/8$ on *knots*. Using [St7], we obtain that for a special alternating knot (and k even) one can choose at least

$$C_k = \frac{1}{\sqrt{8}^k}\,.$$

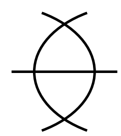
It is in fact my desire to seek an easy determination of the (optimal) constants C_k that led to the above conjecture.

3) At least for ∇, the conjectured estimate is not always better than the one of theorem 3.3, even using Kidwell's result (quoted in remark 3.4), since
$$\frac{1+\sqrt{5}}{2} > 2^{2/3}\,.$$

4) One can avoid the exponential factor in $b(L)$ in (9), but at the cost of excluding split links and augmenting the base of the exponential dependency on $c(L)$ from $\frac{1+\sqrt{5}}{2}$ at least to $\sqrt{3}$. That $\sqrt{3}$ is needed show the connected sums of Hopf links. Prime examples exist also: consider the positively oriented $(2,2,\dots,2)$-pretzel links with parallel clasps and with an even number of 2's, and let this number go to ∞.

Theorem 3.4 The inequality (8), with 'L' replaced by 'D' is true for any link diagram D, which

1) is a diagram of a (closed) 3-braid,
2) is a diagram of a (closed) 4-braid,
3) is a diagram of a (closed) positive braid,
4) is arborescent (or more generally of some Conway polyhedron [Co], if the inequalities hold for the link of this polyhedron and the polyhedra of smaller crossing number), or
5) has a clasp or two triangle regions with a common edge

 , (10)

supposed they are true for diagrams of smaller crossing number.

For any of these diagrams D, we have
$$|P_D|_1 \le L_{c(D)}\, 2^{s(D)-2}\,, \tag{11}$$
where $s(D)$ is the number of Seifert circles of D.

Also, the inequalities (7) hold except (possibly) in case 2).

Proof. We prove the special cases in a different order. The proofs of four of them are simple, but for the fifth one quite involved, so it will be stated as a separate proposition.

5) If D has a clasp, then one can choose a crossing in $D = D_+$ such that D_- can be transformed into a diagram of two crossings less. Then the claim follows.

If D has a fragment (10), then either the fragment is alternating along the two triangle regions or in one of them it is not. In latter case, one can apply a Reidemeister III move on D transforming it into a diagram with a clasp. In former case, one applies the skein relation at p, and D_- again simplifies by 2 crossings.

4) This follows immediately from the clasp part of 5) by induction on the crossing number, using the fact that each arborescent diagram has a clasp, and that smoothing out or switching a crossing in this clasp preserves arborescency (or preserves the Conway polyhedron or transforms it into one with fewer crossings).

3) This also follows from the clasp part of 5), again by induction on the crossing number, since by Boileau-Weber [BW] and Rudolph [Ru], a positive braid word can always be transformed into one with the square of a(n Artin) generator.

1) This also follows from 5), since any 3-braid contains a clasp or two triangle regions with a common edge.

2) This is the most difficult part, and follows from the proposition that will be proved in the next subsubsection.

In the case of the Conway polynomial and the (braid) diagram being composite, the estimates (14) and (15) we will prove may be worse than required (by virtue of $p(\hat\beta) > 1$). Then one needs to remark that the estimate for the polynomial of connected sums follows from the inequality for prime diagrams and the easy to prove inductively inequality $\nabla_a\nabla_b \le \nabla_{a+b}$ for any $a,b\ge 0$ (here $\nabla_a = \nabla(T_a)$ and $A\le B$ means $[A]_i \le [B]_i$ for any applicable i).

For P one argues the same way for connected sums and split unions. □

The above result settles a big part of the conjecture at least when a link in one of the above classes has a diagram of simultaneously minimal crossing and minimal Seifert circle number. In particular, by the results of [FW] and [Mu2] we have:

Corollary 3.6 The inequalities (8) and (9) are true for a link L which is the closure of an alternating braid of at most 4 strands, or of a positive braid with a full twist ($\beta = \xi^2_{s(\beta)}\alpha$ with α a positive braid). □

3.4.2. Estimates for 4-braids

Definition 3.3 For a braid word β, set $s(\beta)$ to be the number of strands of β, $p(\hat\beta)$ the number of prime factors of the diagram $\hat\beta$, and let $\tilde d(\beta) = \tilde d(\hat\beta)$ be the length of the longest bridge/tunnel in $\hat\beta$ from left to right or from right to left, that is, for $s(\beta)\le 4$ the length of the longest common subword of β or some of its cyclically letter-permuted versions and one of

$$[123]\,,\quad [-1-2-3]\,,\quad [321]\,,\quad \text{or}\quad [-3-2-1]\,. \tag{12}$$

In particular, for $s(\beta)\le 4$ always $\tilde d(\beta), p(\hat\beta)\in\{1,2,3\}$. Let

$$\hat d(\beta) = \tilde d(\beta) - 1 + sp(\hat\beta)\,,$$

where

$$sp(\hat\beta) = \#\{\text{ split components of }\hat\beta\ \} = 1+\#\{\ i\in[1,s(\beta)-1]\ :\ \beta \text{ contains no } \sigma_i^{\pm1}\ \}\,.$$

Set

$$\hat w(\beta) = c(\beta) + p(\hat\beta) - \hat d(\beta) \quad\text{and}\quad \tilde w(\beta) = c(\beta) + p(\hat\beta) - \tilde d(\beta)\,.$$

Proposition 3.3 Let β be a braid word of at least 2 and at most 4 strands. Then

$$\left|P(\hat\beta)\right|_1 \le L(\hat w(\beta))\,2^{s(\beta)-2}. \tag{13}$$

Also, for any such word β we have

$$\left|\nabla(\hat\beta)\right|_1 \le F(\tilde w(\beta))\,. \tag{14}$$

The proof is rather technical, involving many subcases, and will occupy the rest of this section.

Proof. Note first that because of the easy to see fact that $L_aL_b \le 2L_{a+b}$ whenever $a,b\ge 0$, we can recur the proof of (13) for composite diagrams to their prime factors, using that p and c are additive under connected sum, and $\tilde d$ and $\hat d$ are subadditive.

For (14) there is no problem with splitness or primeness anyway, since $F_aF_b \le F_{a+b}$ for any $a,b\ge 0$. The inequalities (7) follow in the same way inductively proving

$$[\nabla(\hat\beta)]_{z^k} \ \le\ [\nabla_{\tilde w(\beta)}]_{z^k}\,, \tag{15}$$

and using $\nabla_a\nabla_b \le \nabla_{a+b}$.

The inequalities (13) and (14) are trivial for $s(\beta)=2$ and easy to prove for $s(\beta)=3$ (by the above argument, one needs to check just a few small crossing cases).

Thus we need to consider only β with $s(\beta)=4$ and $\hat\beta$ prime (and hence non-split, so that $sp(\hat\beta)=p(\hat\beta)=1$), and to show that

$$\left|P(\hat\beta)\right|_1 \le L(w(\hat\beta))\,2^{s(\beta)-2} \quad\text{and}\quad \left|\nabla(\hat\beta)\right|_1 \le F(w(\hat\beta))\,,$$

where $w(D)$ is the weight of a braid diagram $D=\hat\beta$

$$w(D) = c(D) + 1 - \tilde d(D)\,.$$

The idea is as before to work inductively on the crossing number, and for fixed crossing number inductively on $w(D)$. Our goal is in D to find *either* a transformation into a 4-braid diagram $\tilde{D}$ with

$$c(\tilde{D}) < c(D) \quad \text{or} \quad c(\tilde{D}) = c(D) \ \text{ and } \ w(\tilde{D}) < w(D)\,,$$

or a crossing in $D = D_\pm$ such that we can transform by braid (word) moves the diagrams D_0 and $D_\mp$ into diagrams $\tilde{D}_0$ and $\tilde{D}_\mp$, for which we have

$$w(\tilde{D}_0) \le w(D_\pm) - 1 \quad \text{and} \quad w(\tilde{D}_\mp) \le w(D_\pm) - 2\,.$$

Here we call D_0 and $D_\mp$ the two diagrams obtained by nullifying (smoothing out) and switching the chosen crossing in $D = D_\pm$, the subscripts '$\pm$' and '$\mp$' being used throughout the proof without the actual specification what the sign of the crossing is.

When considering only prime diagrams D, in order the induction argument to work, we need to ensure that the diagrams $\tilde{D}_0 = \hat{\beta}'_0$ and $\tilde{D}_\mp = \hat{\beta}'_\mp$ are prime (and non-split). Since the transformations from D_0 to $\tilde{D}_0$ and from $D_\mp$ to $\tilde{D}_\mp$ will be local (that is, involving subwords of bounded length), the splitness of $\tilde{D}_0$ or $\tilde{D}_\mp$ will mean that some Artin generator occurs very few times in large portions of β. Similar is the situation with β'_0 or $\beta'_\mp$ being composite. Thus if $\tilde{D}_\mp$ or $\tilde{D}_0$ is split or composite, and if $c(\beta)$ is large, large subwords of β look like 2- and 3-braids. Then they can be simplified by the above arguments ($\tilde{d}$ can be preserved by fixing some left-right bridge of maximal length, and working outside of it), not spoiling non-splitness and primeness of D.

The few cases that remain are of small crossing number and can be checked directly. Thus, there is no need to care about splitness and primeness when describing our transformations. (Alternatively, the reader may check directly for each case when one of the resulting diagrams $\tilde{D}_\mp$ or $\tilde{D}_0$ is split or composite, and verify (13) for each one of these D separately.)

The appropriate transformations will be described by distinguishing several cases.

Write modulo cyclic permutations of β's letters

$$\beta = \gamma\,\alpha\,, \tag{16}$$

with γ being subword of one of the words of (12) of length $\tilde{d}(\beta)$.

Step I. We first try to transform α according to **5)** of theorem 3.4, thereby not affecting $\tilde{d}$. This works whenever α has a subword of the form

$$[i^{\pm 2}]\,, \quad [i^{\pm 1}(i+1)^{\pm 1}i^{\pm 1}(i+1)^{\pm 1}]\,, \quad \text{or} \quad [i^{\pm 1}(i-1)^{\pm 1}i^{\pm 1}(i-1)^{\pm 1}]\,, \tag{17}$$

with all '$\pm$' in each form independently choosable. If α has no clasp, then modulo crossing switches it is of the form

$$\alpha' = \sigma_2^{\varepsilon_0}\alpha_1\sigma_2^{-1}\alpha_2\sigma_2^{-1}\dots\alpha_n\sigma_2^{\varepsilon_1}\,, \tag{18}$$

where $\alpha_i \in \{[1],\,[13],\,[3]\}$ and $\varepsilon_i \in \{0,-1\}$.

Assume $n > 2$, the other cases have few crossings.

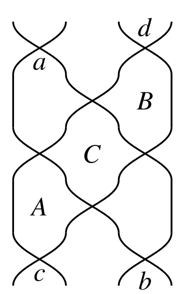
Moreover, if no one of the other subwords of (17) is in α, then either

Case 1. $\alpha_{2+\varepsilon_0} = \ldots = \alpha_{n-1-\varepsilon_1} = [13]$, or

Case 2. $\alpha_{2i} = \sigma_1,\ \alpha_{2i+1} = \sigma_3 \quad \text{or} \quad \alpha_{2i} = \sigma_3,\ \alpha_{2i+1} = \sigma_2\,.$

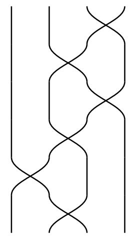
We wish to show first that we can handle all cases where α is non-alternating.

Case 1. In case 1 one has in the inner part of α fragments (parts of the diagram with crossing signs ignored) of this type:



(19)

If such a fragment is not alternating along one of the triangles A and B, then by a Reidemeister III move one obtains a fragment

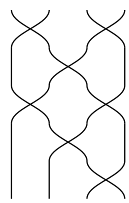


and can proceed as in **5)** of theorem 3.4. By the same argument the two unnamed triangle regions in (19) must be alternating either.

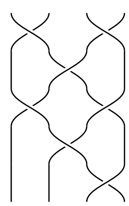
If the fragment in (19) is not alternating along the square C, then either crossings a and b (or c and d) annihelate, thus simplifying D, or one chooses one of them as the crossing at which to apply the skein relation, in which case the crossings annihelate in $D_{\mp}$, giving a diagram $\tilde{D}_{\mp}$ with $w(\tilde{D}_{\mp}) = w(D) - 2$, as desired.

A similar argument applies to the edge part of α. There are two cases, where it does not look as in (19).

Case 1.1. $\varepsilon_0 = 0$ and $\alpha_1 = \sigma_3$ (analogously one handles $\alpha_1 = \sigma_1$ or ε_1 and α_n).



By the above arguments we deduce that either it is alternating, or that (modulo switch of all crossings) looks like

 .

To deal with this case, we will involve the crossings of γ (and have to take care of what happens to $\tilde{d}$).

- If γ ends on σ_1, then there is a simplification in D, under which c goes down by 2 as with crossings a and b in (19). Note that whenever c goes down by 2, $c - \tilde{d}$ cannot increase.

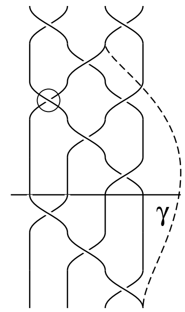
- If γ ends on $\sigma_2^{\pm 1}$ then we can apply case **5)** of theorem 3.4; γ has length 2, and as after these transformations $\tilde{D}_{\mp}$ is non-alternating, $\tilde{d}$ has not decreased.

- If γ ends on σ_3^{-1} then there is a simplification in D by resolving a trivial clasp.

- If γ ends on σ_3 and has length 2, then resolve the clasp; D_0 and $\tilde{D}_{\mp}$ remain non-alternating.

There remain the cases $\gamma = (\sigma_3^{-1})\sigma_2^{-1}\sigma_1^{-1}$ and $\gamma = [123]$.

- In case $\tilde{d} = 3$, $\gamma = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}$, choose the crossing for the skein relation as encircled, and apply the move indicated by the dotted line to $D_{\mp}$; one obtains a diagram $\tilde{D}_{\mp}$ with 4 crossings less.

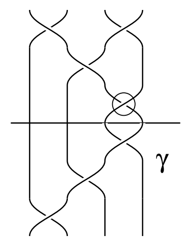


.

Note, that whenever we switch a crossing outside of γ, we always have $w(D_0) \le w(D) - 1$, so there is no need to care about D_0.

In case $\tilde{d} = 2$, $\gamma = \sigma_2^{-1}\sigma_1^{-1}$, the same transformation as for crossings a and b in (19) applies to simplify $D_{\mp}$ by 2 crossings, and $\tilde{d}$ remains (at least) two in $\tilde{D}_{\mp}$.

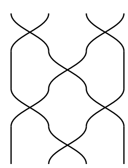
- In case $\gamma = \sigma_1\sigma_2\sigma_3$,



,

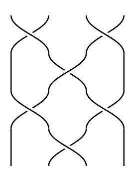
D simplifies by 6 crossings.

Case 1.2. ($\varepsilon_0 = -1$) The edge of α looks like



.

This fragment is found again to be alternating, or (modulo switch of all crossings) looking like



.

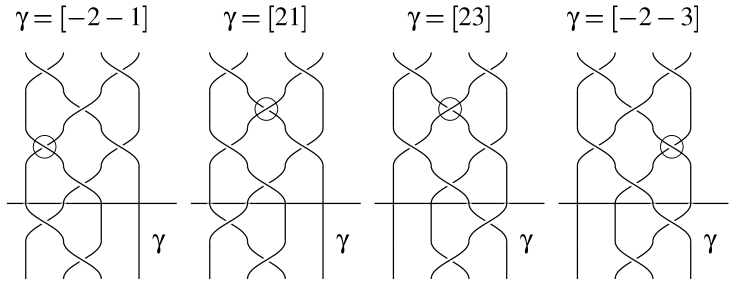
Case 1.2.1. First consider the case $\tilde{d} = \operatorname{len}\gamma = 2$.

- If γ ends on σ_2 then D simplifies by resolving a clasp.
- If γ ends on σ_2^{-1}, then the clasp corresponding to σ_2^{-2} can be resolved. We have then that $c(\tilde{D}_{\mp}) = c(D) - 2$, but since the fragment becomes alternating after resolving the clasp, we might have $\tilde{d}(\tilde{D}_{\mp}) = 1 = \tilde{d}(D) - 1$, and so only $w(\tilde{D}_{\mp}) = w(D) - 1$. However, in $D_0 = \tilde{D}_0$ we have a bridge/tunnel of length 3, so that $\tilde{d}(D_0) = \tilde{d}(D) + 1$, and hence $w(D_0) = w(D) - 2$. Then the induction argument works with the estimates for the contributions from $\tilde{D}_{\mp}$ and $\tilde{D}_0$ reversed.

There remain the four cases

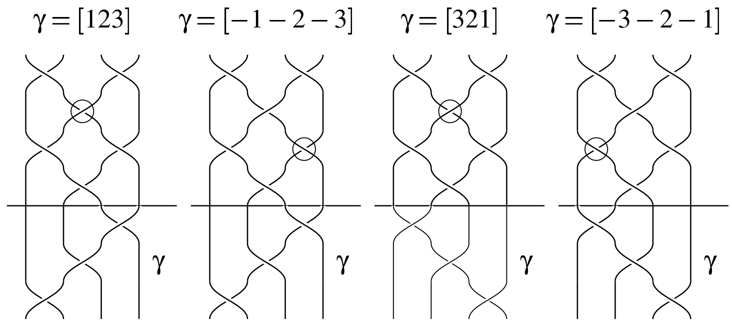
$$\gamma = [23], \quad [-2-3], \quad [21], \quad \text{or} \quad [-2-1].$$

They correspond to the diagrams



One sees that in all four cases, when switching the encircled crossings, $D_{\mp}$ simplifies by 4 crossings.

Case 1.2.2. Consider the 4 cases of γ where $\tilde{d} = \operatorname{len}\gamma = 3$.



In the first and third case, $D_{\mp}$ simplifies by 6 crossings, in the other two cases by 4.

Case 2. The case 2 is easier to handle, for if it were non-alternating, a Reidemeister III move would give a fragment (10) amenable by case **5)** of theorem 3.4, except in some cases with few crossings.

The cases where α is mirrored ($\sigma_i \leftrightarrow \sigma_i^{-1}$) are to be handled like the cases where γ is mirrored, and then applying a mirroring to the whole diagram.

Step II. Now consider β as in (16) with $\alpha = \alpha'$ in (18). Since α can no longer be handled itself, we incorporate again γ into our transformations. Again modulo a mirroring of the whole braid we can fix a mirroring convention for α and assume that α contains σ_2 with negative and $\sigma_{1,3}$ with positive exponents.

Since it will be necessary to work with some of α's letters immediately following *and* preceding γ, we cyclically permute the letters of β so that at least 4 of the last letters of α appear before γ, that is, we have the form

$$\beta \;=\; \alpha_0 \gamma \alpha, \qquad (20)$$

with $\operatorname{len}(\alpha_0) \geq 4$. We assume, modulo small crossing number cases, that also $\operatorname{len}(\alpha) \geq 4$.

We again describe some transformations by diagrams, but some also by words. In latter case we will put a '|' at the start and/or end of γ to separate it from the letters of α and/or α_0. By underlining a letter we indicate that the corresponding crossing is to be switched (that is, applied skein relation at).

We will occasionally use that fact that the form (18) of $\alpha = \alpha'$ has the following property: whenever we have σ_1 (resp. σ_3) occurring in α_k, then σ_3 (resp. σ_1) occurs in α_{k+1} for $1 \leq k < n$. Also, since σ_1 and σ_3 commute we will use the freedom to write them in our fashionable order in some α_k.

Case 1. $\operatorname{len}\gamma = 3$. By applying a *flip*, a conjugacy with the square root of the center generator of B_4, it suffices to consider only $\gamma = \sigma_1^{\pm 1}\sigma_2^{\pm 1}\sigma_3^{\pm 1}$; the mirroring convention for α is not affected by a flip.

Case 1.1. $\gamma = \sigma_1\sigma_2\sigma_3$. We consider 16 possibilities, made up of the 4 possibilities

- $\varepsilon_0 = 0$ and $\alpha_1 = \sigma_3$ (that is, $\alpha = \sigma_3\sigma_2^{-1}\ldots$),
- $\varepsilon_0 = 0$ and $\alpha_1 = \sigma_1\sigma_3$ (that is, $\alpha = \sigma_1\sigma_3\sigma_2^{-1}\ldots$),

- $\varepsilon_0 = 0$ and $\alpha_1 = \sigma_1$, and
- $\varepsilon_0 = -1$ (that is, $\alpha = \sigma_2^{-1}\dots$),

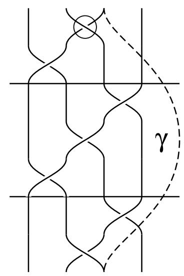
and the analogous 4 possibilities for ε_1 and α_n at the end of α_0.

Case 1.1.1. $\sigma_1\sigma_2\sigma_3|\underline{\sigma_3}\sigma_2^{-1}$. $D_\mp$ simplifies by 4 crossings. Similarly one handles $\sigma_2^{-1}\underline{\sigma_1}|\sigma_1\sigma_2\sigma_3$.

Case 1.1.2. $\sigma_1\sigma_2\sigma_3|\underline{\sigma_3}\sigma_1\sigma_2^{-1}$. $D_\mp$ simplifies by 4 crossings. Same with $\sigma_2^{-1}\sigma_3\underline{\sigma_1}|\sigma_1\sigma_2\sigma_3$.

So there are 4 cases remaining, made up of the two possibilities $\sigma_2^{-1}\sigma_3$ or σ_2^{-1} before γ (at the end of α_0 in (20)) and $\sigma_1\sigma_2^{-1}$ or σ_2^{-1} after γ (at the beginning of α).

Case 1.1.3. $\sigma_2^{-1}\sigma_3|\sigma_1\sigma_2\sigma_3|\sigma_1\underline{\sigma_2^{-1}}$.



$\tilde{D}_\pm$ has 2 crossings less than D, but still a bridge of length 3.

Case 1.1.4. $\sigma_2^{-1}\sigma_3|\sigma_1\sigma_2\sigma_3|\sigma_2^{-1}$. This simplifies to $\sigma_2^{-1}\sigma_1\sigma_2\sigma_3$. Similarly $\sigma_2^{-1}|\sigma_1\sigma_2\sigma_3|\sigma_1\sigma_2^{-1}$, which can be transformed into the other case by rotating the braid by 180° (around the axis orthogonal to the projection plane).

Case 1.1.5. $\sigma_2^{-1}|\sigma_1\sigma_2\sigma_3|\sigma_2^{-1}$. We distinguish two subcases

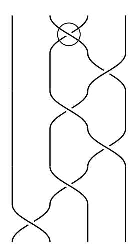
Case 1.1.5.1. $\sigma_1\sigma_2\sigma_3|\sigma_2^{-1}\underline{\sigma_3}\sigma_2^{-1}$ (that is, $\alpha_1 = \sigma_3$). $\tilde{D}_\mp$ has 4 crossings less than D. Analogously one handles $\sigma_2^{-1}\underline{\sigma_1}\sigma_2^{-1}|\sigma_1\sigma_2\sigma_3$ (rotate the braid by 180°).

Case 1.1.5.2. $\sigma_3\sigma_2^{-1}|\sigma_1\sigma_2\sigma_3|\sigma_2^{-1}\underline{\sigma_1}$. After the switch, σ_1^{-1} annihelates with the first 'σ_3'.

Case 1.2. $\gamma = [-1-2-3]$. Make the same case distinction as for case 1.1.

- In the analoga of cases 1.1.1 and 1.1.2, D simplifies (without crossing change),
- in the now modified case 1.1.3 the move applies in the same way (only the length-3 bridge being a length-3 tunnel),
- in the new cases 1.1.4 and 1.1.5.2 D also simplifies as before.

It remains to handle the new case 1.1.5.1. Then do



and $D_\mp \to \tilde{D}_\mp$ simplifies by 4 crossings.

Case 2. $\tilde{d} = \operatorname{len}\gamma = 2$. We consider γ being one of

$$[12], \quad [-1-2], \quad [23], \quad \text{or} \quad [-2-3], \tag{21}$$

and as before (by mirroring) $\alpha = \alpha'$ with σ_2^{-1}, σ_1 and σ_3. The cases for γ being [21], $[-2-1]$, [32] and $[-3-2]$ are recurred to those in (21) by flip (conjugacy with the square root of the center generator of B_4).

Case 2.1. $\gamma = \sigma_1\sigma_2$.

Case 2.1.1. $\sigma_1\sigma_2|\sigma_3$ ($\varepsilon_0 = 0$ and α_1 has a σ_3). Then $\tilde{d} = 3$, a contradiction.

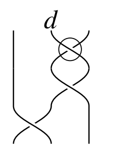
Case 2.1.2. $\sigma_1\sigma_2|\sigma_2^{-1}$ ($\varepsilon_0 = -1$). Then D simplifies.

Case 2.1.3. $\sigma_1\sigma_2|\sigma_1\sigma_2^{-1}$ ($\varepsilon_0 = 0$ and $\alpha_1 = \sigma_1$). Then D simplifies also.

Case 2.2. $\gamma = \sigma_2\sigma_3$ is analogous to case 2.1 by rotating the braid by 180°.

Case 2.3. $\gamma = \sigma_1^{-1}\sigma_2^{-1}$.

Case 2.3.1. $\sigma_1^{-1}\sigma_2^{-1}|\sigma_2^{-1}$.

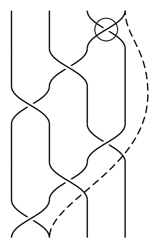


Resolve clasp. Since α is alternating, strand d continues upward by passing an undercrossing, thus $\tilde{D}_{\mp}$ after resolving the (first) clasp either has a second trivial clasp (and simplifies further), or has $\tilde{d}(\tilde{D}_{\mp}) \geq 2 = \tilde{d}(D)$, so $w(\tilde{D}_{\mp}) \leq w(D) - 2$, as desired.

Case 2.3.2. $\sigma_1|\sigma_1^{-1}\sigma_2^{-1}$. D simplifies.

Thus $\varepsilon_0 = 0$ and ($\varepsilon_1 = -1$ or $\alpha_n = \sigma_3$).

Case 2.3.3. $\sigma_1\sigma_2^{-1}\sigma_3|\sigma_1^{-1}\sigma_2^{-1}|\underline{\sigma_3}$ (α_1 has σ_3, $\varepsilon_1 = 0$ and $\alpha_n = \sigma_3$, latter in particular implying that α_{n-1} contains σ_1).

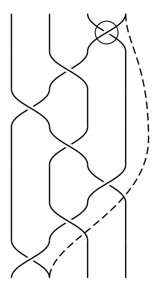


$\tilde{D}_{\mp}$ has two crossings less than D, and is non-alternating.

Case 2.3.4. $\sigma_2^{-1}|\sigma_1^{-1}\sigma_2^{-1}|\sigma_3$ (α_1 has σ_3 and $\varepsilon_1 = -1$).

Case 2.3.4.1. $\sigma_1\sigma_2^{-1}|\sigma_1^{-1}\sigma_2^{-1}|\sigma_3$ (α_n has σ_1). D simplifies.

Case 2.3.4.2. $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}|\sigma_1^{-1}\sigma_2^{-1}|\underline{\sigma_3}$ ($\alpha_n = \sigma_3$, and consequently α_{n-1} has a σ_1).



$\tilde{D}_{\mp}$ has two crossings less than D, and is non-alternating.

Case 2.3.5. $\sigma_1\sigma_2^{-1}\sigma_3|\sigma_1^{-1}\sigma_2^{-1}|\sigma_1\sigma_2^{-1}$ ($\alpha_1 = \sigma_1$, $\varepsilon_1 = 0$ and $\alpha_n = \sigma_3$, so in particular α_{n-1} has a σ_1). After a braid relation, this turns into $\sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}\sigma_2^{-1}\underline{\sigma_2^{-1}}$. Resolve the clasp; $\tilde{D}_{\mp}$ is still non-alternating, and has two crossings less than D.

Case 2.3.6. $\sigma_2^{-1}|\sigma_1^{-1}\sigma_2^{-1}|\sigma_1$. This simplifies to $\sigma_1^{-1}\sigma_2^{-1}$.

Case 2.4. $\gamma = \sigma_2^{-1}\sigma_3^{-1}$ is lead back to case 2.3 by turning around the braid by 180°.

Case 3. $\operatorname{len}\gamma = 1$. Then β is an alternating braid (or has a resolved clasp, in which case D simplifies). Then by switching some crossing corresponding to σ_2^{-1} one can create a diagram $\tilde{D}_{\mp} = D_{\mp}$ with $c(\tilde{D}_{\mp}) = c(D)$ and $\tilde{d}(\tilde{D}_{\mp}) = 3$, so that

$w(\tilde{D}_{\mp}) = w(D) - 2$, as required. As before, $\tilde{d}(D_0) = \tilde{d}(D) = 1$, but $c(D_0) = c(D) - 1$, so $w(D_0) = w(D) - 1$, also as required.

Thus the case distinction, and hence the induction argument, is now complete. □

Since even in the special case of 4-braids, the proof is already very complex, it is hard to expect the conjecture to be solvable, at least affirmatively, in general. The proof of the coefficient inequalities (7) already fails because of the need to switch the contributions from $\tilde{D}_{\mp}$ and $\tilde{D}_0$ in case 1.2.1.

On the other hand, one may hope for better. As the braids α' in (18) are basically powers of $[1-23]$ and $[1-23-2]$, one may ask whether a part of the argument could be carried out using the representation theory of the polynomials [J2]. If one could handle these cases, for example by estimating the Eigenvalues of their Burau matrices, then the proof can be considerably simplified. In fact, the involvement of γ and the (somewhat technical) quantity $\tilde{d}$, and the lengthy list of subcases arising this way, was necessary because of the few situations, where no reduction of the crossing number was possible.

Unfortunately, however, the Jones-Ocneanu parametrization X of P, arising in its Hecke algebra definition (as described in [J2]), is different from the skein parametrization, used here, and the change from one to the other is badly conditioned with respect to the 1-norm of the polynomial. The Jones-Ocneanu parametrization might be on the other hand helpful in proving similar estimates for the Jones polynomial.

4. Some experimental results

It becomes clear from the proofs of the inequalities for the V, Q and F polynomial, that the price for keeping the arguments simple is that the estimates cannot be expected to be sharp. The reason is the lack of easy control on the cancellations occurring in the resolution trees. The way I chose to get some experimental information on possible improvements was to use corollary 3.1 and proposition 3.2 and to consider for any K the smallest bases on the right of the estimate which would satisfy (in)equality, and to sort the knots of ≤ 15 crossings in Thistlethwaite's tables according to the considered numbers. The results suggest much space for improvement, but, as said, it will be likely much more elaborate.

First consider F and define

$$\delta_K := \sqrt[\max\deg F_K]{|F_K|_1}\,.$$

Note that this definition would not make sense if $\max\deg F_K = 0$ and $|F|_1 > 1$. Among the knots considered, there was no knot with such polynomial, but in general this point should be kept in mind. Note also that the obvious inequality $\delta_{K\#K'} \leq \max(\delta_K, \delta_{K'})$ with equality only if (but by far not in general if) $\delta_K = \delta_{K'}$ renders the discussion of composite knots (rather) uninteresting.

The "top ten" with respect to δ are given below.

#	knot	$\lvert F\rvert_1$	$\max\deg_z F$	$\delta_K \approx$
1	3_1	7	2	2.64575
2	4_1	11	3	2.22398
3	5_1	21	4	2.1407
4	11_{444}	395	8	2.11142
5	11_{440}	361	8	2.0878
	11_{441}	361	8	2.0878
7	6_3	37	5	2.05892
8	14_{25821}	2625	11	2.04564
9	14_{25180}	2525	11	2.03843
10	5_2	17	4	2.03054

(22)

Although the trefoil leads with sensible advance, several of its simple successors are dominated by some exotic knots. It is also striking that beyond 9 crossings non-alternating knots perform much better than alternating ones. The highest alternating knot of > 9 crossings is 12_{945} on place 272. One may expect the reason for this in the smaller value of $\max\deg_z F$, but on the other hand, the Perko knot 10_{161} with $\max\deg_z F = 6$ takes "only" place 338.

Question 4.1 Is the highest value of δ on knots the number $\sqrt{7}$ given by the trefoil? Is it the unique knot maximizing this value?

Recall that the only general fact we know about δ is that $\delta_K \le 5$ for K alternating.

I did a similar experiment for V using the the numbers

$$\gamma_K := \sqrt[\operatorname{span} V_K - 1]{|V_K|_1}\,.$$

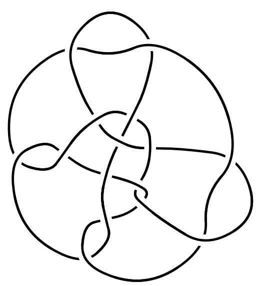
Here a similar remark on the definability of this number applies as for δ. However, we know from [FS] that if $\operatorname{span} V_K \le 1$ for a knot K, then already $V_K = 1$, so the definition of γ would be problematic only for (non-trivial) knots with unit polynomial, and such knots do not occur at ≤ 15 crossings.

As an outcome I found that

$$\max_{c(K)\le 15} \sqrt[\operatorname{span} V_K - 1]{|V|_1} = 1.89087\dots,$$

attained by some 15 crossing knot 15_{216514}.

Here the picture is entirely dominated by high crossing number knots – there is no Rolfsen knot among the first 2000. Moreover, not one single of these knots is alternating. This is explained by the fact that for non-alternating knots the crossing number bound of V is in general much worse than this of F. Thus it appears more interesting to consider just alternating knots, in which case the listing is as follows:



$V = -3\ 12\ -25\ 38\ -48\ [52]\ -48\ 40\ -26\ 13\ -4$

15_{216514}

#	knot	$\lvert V\rvert_1$	$\operatorname{span} V - 1$	$\gamma_K \approx$
1	3_1	3	2	1.73205
2	15_{82477}	2037	14	1.72328
3	8_{18}	45	7	1.72256
4	14_{17895}	1145	13	1.71907
5	13_{3478}	663	12	1.71841
6	15_{82192}	1947	14	1.71773
7	15_{81381}	1939	14	1.71722
8	14_{13618}	1115	13	1.71556
9	9_{40}	75	8	1.71547
10	15_{59607}	1903	14	1.71493

(23)

Further known knots are 4_1 on position 21, 10_{123} on position 55 and 9_{34} on position 145. From the Rolfsen knots, among the first 2000 there were only 5 further 10 crossing knots. Also, we point out that all numbers γ stay visibly below 2, which suggests possible further improvement.

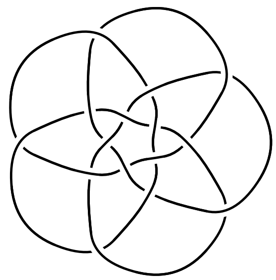
Question 4.2 1) Is there any highest value for γ_K on knots? What is it?

2) Is there any highest value for γ_K on alternating knots? Is it attained by the trefoil?

A final experiment was performed with the Q polynomial. For reasons that we will explain shortly we calculated

$$\beta_K := \left(\max_{\xi^{100}=1} |Q_K(\xi)|\right)^{1/c(K)}.$$

(In fact, instead of being a root of unity of order 100, the maximum should be taken over all ξ with $|\xi| = 1$, but this is computationally intractable.) The 10 prime knots K of ≤ 15 crossings attaining the highest β_K are shown below. It is interesting to remark that among the first 2000 knots there was not one single non-alternating one.



$Q =$ [45] -118 -94 368 170 -754 -664 1182 1680 -608 -1710 -388 522 318 52

15_{84903}

#	knot	β_K
1	15_{84903}	1.79305
2	15_{64035}	1.77796
3	12_{975}	1.77267
4	14_{14501}	1.76822
5	15_{82192}	1.76621
6	15_{82477}	1.76574
7	15_{82478}	1.76474
8	15_{73374}	1.76425
9	13_{3478}	1.76414
10	15_{61404}	1.76403

(24)

There is no *a priori* evidence the roots with maximal value modulus to be specifically distributed. Therefore, it is striking that the powers of $\xi = e^{\pi i/50}$ where the maxima are attained are strongly concentrated around 22, and decrease rapidly (and approximately equally) in both directions away from this number.

5. Lower asymptotical bounds

To motivate the preceding computations also from the theoretical point of view, we mention that any knot K will give via its iterated connected sums an *asymptotical lower* bound for a base in corollary 3.1 and proposition 3.2. For V this bound is given by

$$\sqrt[c(K)]{\max_{|\xi|=1} |V_K(\xi)|}\,.$$

This follows from

Lemma 5.1 Let $V \in \mathbb{Z}[t,t^{-1}]$ be some polynomial. Then

$$\lim_{n\to\infty} \sqrt[n]{|V^n|_1} = \max_{|\xi|=1} |V(\xi)|\,.$$

Proof. Let ξ_0 with $|\xi_0| = 1$ be such that $V(\xi_0)$ has maximal modulus

$$m(V) := \max_{|\xi|=1} |V(\xi)|\,.$$

Then by the triangle inequality,

$$\sum_{i\in\mathbb{Z}} |\,[V^n]_i\,| \geq |V^n(\xi_0)|\,,$$

whence

$$\liminf_{n\to\infty} \sqrt[n]{|V^n|_1} \geq m(V)\,.$$

On the other hand, we have the identity

$$\sum_{i=-\infty}^{\infty} [V]_{t^i}^2 = \int_0^1 \left|\,V\left(e^{2\pi i u}\right)\right|^2 du\,. \tag{25}$$

Thus

$$|V|_2 := \sqrt{\sum_{i=-\infty}^{\infty} [V]_{t^i}^2} \leq m(V)\,. \tag{26}$$

If

$$|V|_\infty := \max_{i\in\mathbb{Z}} \left|\,[V^n]_i\,\right|\,,$$

then $|V|_\infty \le |V|_2$, and $|V|_1 \le \operatorname{span} V\,|V|_\infty$. Applying this to V^n, and using $\operatorname{span} V^n = \operatorname{span} V \cdot n$, we have

$$\sqrt[n]{|V^n|_1} \le \sqrt[n]{\operatorname{span} V \cdot n\,|V^n|_2} \le \sqrt[n]{\operatorname{span} V \cdot n}\, m(V)$$

the last inequality coming from (26). This shows

$$\limsup_{n\to\infty} \sqrt[n]{|V^n|_1} \le m(V)\,,$$

and hence the assertion. □

To apply the above lemma, one needs to consider instead of γ_K

$$\gamma'_K := \sqrt[\operatorname{span} V_K]{|V_K|_1}\,.$$

Proposition 5.1

$$\liminf_{n\to\infty} \max_{c(K)=n} \sqrt[n]{|V_K|_1} \ge \sqrt[15]{2037} \approx 1.66188$$

and

$$\liminf_{n\to\infty} \max_{\operatorname{span} V_K=n} \sqrt[n]{|V_K|_1} \ge \sqrt[10]{309} \approx 1.77417\,.$$

Proof. Consider (iterated connected sums of) the knots 15_{82477} and 15_{216514} and apply the above lemma. □

For V and Δ we know that the maximum modulus on S^1 is always attained at $t=-1$, but the Q polynomial remains quite mysterious. Again the r.h.s. can be computed, at least numerically up to some decent accuracy, but this is more complicated and would pose serious difficulties in evaluating such a long knot list as above. Instead it is easier to approximate the maximum by considering just special values of ξ, as done above. Thus we have

Proposition 5.2

$$\liminf_{n\to\infty} \max_{c(K)=n} \sqrt[n]{|Q_K|_1} \ge 1.79305\dots$$

Proof. This estimate comes from taking iterated connected sums of the knot 15_{84903} and the above lemma. □

To examine F and P, one would need the 2-variable analogue of the above identity, which can be proved by an iterated integral of the type (25):

$$\lim_{n\to\infty} \sqrt[n]{|P^n|_1} = \max_{|\xi|=1,\ |\zeta|=1} |P(\xi,\zeta)|\,.$$

If prime examples are desired, they can be manufactured using the copious results and techniques available now on primeness, see for example [EKT, Me, KL], possibly augmenting the number of components and/or obtaining somewhat worse estimates.

6. The Mahler measure

Another way to measure the complexity of a polynomial was introduced by Mahler [Ma]. The Mahler measure seems to me of more number theoretic, although of less intuitive interest. It can be defined for a one-variable polynomial P by

$$M(P) = \max \operatorname{cf} P \cdot \prod_{P(\alpha)=0,\ |\alpha|>1} |\alpha|\,, \qquad (27)$$

with complex zeros $\alpha \ne 0$ counted and any zero going into the product by its multiplicity. See also [GH] and *loc. cit.*

It is useful to remark that the estimates given above also serve as estimates for the Mahler measure of the polynomials. This follows from the inequality $M(P) \le |P|_1$, which is classic, but whose proof is elementary, and so included here for completeness.

Lemma 6.1 If $P \in \mathbb{Z}[x,x^{-1}]$, then $M(P) \le |P|_2 \le |P|_1$.

Proof. The second inequality is trivial. To see the first inequality, by Jensen's integral formula [Je], we have

$$\ln M(P) = \int_0^1 \ln \left|P\big(e^{2\pi it}\big)\right| dt = \frac{1}{2}\int_0^1 \ln \left|P\big(e^{2\pi it}\big)\right|^2 dt. \tag{28}$$

Some care must be taken when P has roots on the unit circle, but since the singularities of the integrand are only logarithmical in this case, the integrals converge as improper Riemann integrals.

Since the function ln is concave, we have $\ln(a+b) \ge \ln(a)+\ln(b)$ for any $a,b>0$, and so, using (25), we obtain

$$\ln M(P) \le \frac{1}{2}\ln \int_0^1 \left|P\big(e^{2\pi it}\big)\right|^2 dt = \frac{1}{2}\ln \sum_{i=-\infty}^{\infty} [P]_{x^i}^2 = \ln |P|_2. \qquad \square$$

The Mahler measure can be defined also for polynomials in more variables (as done originally by Mahler), only that in this case only the integral formula (28), but not the product formula (27) is available. Lemma 6.1 holds in multi-variable case analogously.

One should note that the inequality in lemma 6.1 is not always very sharp. In particular, contrarily to $|\,.\,|_1$ and $|\,.\,|_2$, there are non-trivial polynomials with unit Mahler measure. By a classical result of Kronecker these are exactly the polynomials all of whose roots are roots of unity, that is, divisors of $(x^d-1)^n$ for some $d,n \in \mathbb{N}$. Lin asked in [L] (see also [Oh, p. 3 bottom]) whether the Jones polynomial V of a non-trivial knot has $M(V)>1$ (in his notation $m(V)=\ln M(V)$). As remarked, the answer to this question is negative because of the Jones polynomial of the figure 8 knot (whose roots are the primitive roots of unity of order 10). A possible relation to the achirality of the figure 8 knot was asked. There is in fact some causality, which follows from the proposition below.

Proposition 6.1 If V if the Jones polynomial of a knot and $M(V)=1$, then $V(t^{-1})=V(t)$ (that is, V is self-conjugate), and $\max\operatorname{cf} V = 1$.

Proof. If $M(V)=1$ and $\max\operatorname{cf} V=\pm 1$ and all roots of V have unit norm (this follows from Kronecker's theorem or directly considering the lowest degree coefficient of V). Since $V(1)=1$ and (hence) $V(-1)\ne 0$, all roots of V are complex, and occur in pairs $(t,\bar t = t^{-1})$ with equal multiplicities. Thus V is self-conjugate *up to a unit* in $\mathbb{Z}[t,t^{-1}]$. This unit cannot be $-t^n$ because $V(1)\ne 0$, so it is $+t^n$. Then, using again $V(1)=1$, and $V'(1)=0$ we conclude $n=0$. Finally, the lack of real zeros of V and $V(1)=1$ show that $V(t)>0$ for any $t\in\mathbb{R}_+$. Letting $t\to\infty$ shows $\max\operatorname{cf} V>0$. $\square$

One should note that much more than this may not be expected. For example, the knot 9_{42} is not achiral, but yet has a Jones polynomial with unit Mahler measure (its roots are the primitive roots of unity of order 14).

7. Questions

The inequalities proved above open several problems and questions. We conclude by proposing some of them.

The fact that the estimate for $|\Delta|_1$ gives also an estimate for the determinant $\Delta(-1)$ makes it appropriate to compile some related problems on the structure of $H_1 = H_1(D_K,\mathbb{Z})$, where D_K denotes the double branched cover of S^3 along K.

Question 7.1 Let $t(K)$ for a knot K be the number of torsion coefficients of $H_1(D_K,\mathbb{Z})$, and $t_p(K)$ for some odd natural number p the number of those divisible by p.

1. We know from (4), that $t(K)\le c(D)-d(D)$ for any diagram D of K, and from the Seifert matrix representation of H_1 that $t(K)\le 2g(K)$.
   a) Is it always true that $2g(K)\le c(D)-d(D)$ for any D, that is, is the new estimate obsolete?
   b) Is it even true that $2g(K)\le \max\deg_z F(K)$?

1.b), and hence 1.a), is true for alternating knots by [Ki] and for positive knots by [Yo]. It is also true for all examples so far decidable in the tables of [HTW], including all prime knots of ≤ 14 crossings. (For all knots up to 16 crossings we have $2\max\deg\Delta \le \max\deg_z F$.) However, such an inequality would in general imply that no non-trivial knot has $F = 1$; another odd against a general positive answer is that it is not true for Q: the knot $K = 13_{7960}$ has $2g(K) \ge \sigma(K) = 6$, but $\max\deg Q(K) = 4$.

2. a) Can one give a better estimate for $t_p(K)$ than $\log_p 3 \cdot \big(c(D) - d(D)\big)$ coming from (4)? More specifically,
   b) is $t_3(K) \le c(K)/3$ for any K? Are the only knots with equality the connected sums of trefoils? Is weaker the inequality true at least for knots K 3-equivalent (in the sense of [Pr]) to an unlink (of $t_3(K)+1$ components)?

3. Is for any non-trivial fibered positive knot $t(K) \le g(K)$? Are connected sums of (this time only positive) trefoils again the only knots satisfying equality? The answer to the first question is true from [BW] and [We] for positive braid knots, since for them $u(K) = g(K)$. It is also true for prime knots up to 16 crossings. This question is motivated (and a positive answer to it is implied) by a conjecture of [MP], where $u(K) = g(K)$ is conjectured to hold more generally for fibred positive knots.

4. Do the t_p of achiral knots have other special features except that $2 \mid t_3$? (If $p \equiv 3\,(4)$ is a prime, and the linking form does not degenerate on the subgroup generated by elements of order p in H_1, then $2 \mid t_p$. This follows by considering the action of the involution of the ambient space on H_1.)

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