INFINITELY MANY NON-CONJUGATE BRAID REPRESENTATIVES OF LINKS

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ABSTRACT. We prove that under a fairly general condition (that the edge strands are not fixed by the braid permutation) an iterated exchange move gives infinitely many non-conjugate braid representatives of links. More precisely, almost all braids obtained by iterated positive exchange moves are pairwise non-conjugate. As a consequence, every link with no trivial components has infinitely many conjugacy classes of n-braid representatives if and only if it has one admitting an exchange move.

1. OVERVIEW

The *braid groups* B_n were studied in the 1930s in the work of Artin [2]. Alexander's and Markov's theorems (see §2.1) have laid a fundament for the use, and increased the importance of braids in the theory of links with the development of polynomial [13] and quantum link invariants. In that realm, extensive study focused on the *braid representatives* of a given link L, i.e., those braids whose closure is L. *Markov's theorem* relates these representatives by two moves, the *conjugacy* in the braid group, and *(de)stabilization* (3), which passes between different braid groups. Conjugacy is, starting with Garside's [9], and later many others' work, now relatively well group-theoretically understood. In contrast, the effect of (de)stabilization on conjugacy classes of braid representatives of a given link is in general difficult to grasp. Only in very special situations can these conjugacy classes be well described, e.g., [7].

Some non-conjugate braids close to isotopic links. Birman had observed [4] that stabilizations of different sign are non-conjugate, because of different exponent sum. However, it is well known now that for any link L and $n \geq b(L)$, there are only finitely many exponent sums of n-braid representatives of L for given n. It was also known from [7] that only finitely many conjugacy classes occur when $n \leq 3$.

In this paper we study the question when infinitely many conjugacy classes of n-braid representatives of a given link occur. The first such construction is likely due to Morton [15], who discovered an infinite sequence of conjugacy classes of 4-braids with closure being the unknot. Later different types examples of were obtained [16, 8].

Birman and Menasco [6] introduced a move called *exchange move* (see §2.2), and proved that it necessarily underlies the switch between many conjugacy classes of braid representatives of L. We will prove here that it is also sufficient for generating infinitely many such classes, under a very mild restriction. Our work extends the result for knots in [22], which is included in the statement below (and will be needed in the proof, so that it requires some detailed review in §3).

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Theorem 1.1. *Let L be a knot, or a link without trivial components, and $n \geq 4$. Then L has infinitely many conjugacy classes of n-braid representatives if and only if it has one admitting an exchange move.*

In §8 we will cover many links with trivial components. This result is a consequence of a stronger property of the exchange move we prove in Theorem 1.2, when combined with the work of Birman and Menasco.

Whenever an n-braid representative b of a link L admits such a move, one obtains by iteration from b a sequence of n-braid representatives (b_m) of L, indexed by integers $m \in \mathbb{Z}$ (with $b = b_0$). Our concern is when are b_m conjugate. The proof of the following theorem occupies §4, §5 and §6.

Theorem 1.2. *Let a link L have an n-braid representative b admitting an exchange move, such that the permutation $\pi(b)$ satisfies*

$$\pi(b)(k) \neq k \quad \textit{for} \quad k = 1, n\,. \tag{1}$$

Then

(1) *every (fixed) conjugacy class of n-braids contains b_m for at most two m, and*
(2) *b_m are pairwise non-conjugate for all $m \geq 0$, or for all $m \leq 0$.*

In the case of minimal braids, Birman had conjectured that there would always be a single conjugacy class of minimal braids representing a link. However, Murasugi and Thomas [20] disproved Birman's conjecture, exhibiting some counterexamples in B_4. Our result can be seen as such a construction of nearly exhaustive generality. The few remaining braids are more subtle, and we discuss them briefly in §7. In particular, we will notice that, in terms of $\pi(b)$ alone, Theorem 1.2 requires the weakest possible condition (1) under which the exchange move can yield non-conjugate representatives.

The special case for non-minimal braid representatives follows (in slightly greater generality) from work by the second author in [24] (which is discussed in §2, but uses an entirely different Lie group theoretic approach).

In §4, §5 we will first prove that infinitely many b_m are non-conjugate. The full statement of Theorem 1.2 is completed in §6 with Proposition 6.1. It transpires more precisely that indices m of conjugate exchanged braids must satisfy a certain reflectional symmetry in the real line. (See end of the proof of Proposition 6.1.) This symmetry occurs naturally in many instances and thus cannot be further restricted (Remark 6.2).

There is also a short treatise of composite links in §9. A few problems are summarized at the end of the paper in §10.

We conclude the introduction with further historical remarks and related updates.

Originally, we had only claimed in Theorem 1.2 that infinitely many b_m are non-conjugate (`arXiv:1103.2510`). Our proof was submitted as part of [22], but the editors requested the paper to be shortened to the form later published. Later, T. Ito [12] found a very similar version of this theorem. He uses a simpler geometric argument, based on the topological entropy corresponding to the dilation of a pseudo-Anosov map coming from a braid. His version replaces (1) by a more (in fact, the most) general assumption of "non-degeneracy". (We had also previously identified this condition. See (58) and the second author's further discussion in [27].) Then we noticed that our method easily gives the improvement of the conclusion stated in Theorem 1.2. (Ito's method can only show that at most finitely many b_m are mutually conjugate.)

An algebraic proof in a style close to Theorem 1.2 for $\pi(b)(1)=1$, $\pi(b)(n)\neq n$ (instead of (1)) is given by the second author in [28]. (A natural condition, similar to (59), must be excluded to avoid degenerate braids, but the assertion remains the same.) This also protrudes the relation to the Burau representation. The final case $\pi(b)(1)=1$, $\pi(b)(n)=n$ is work in progress [29]; it also shows how to apply these theorems for specific families of links.

2. Preliminaries

'W.l.o.g.' means 'without loss of generality', and 'w.r.t.' means 'with respect to'.

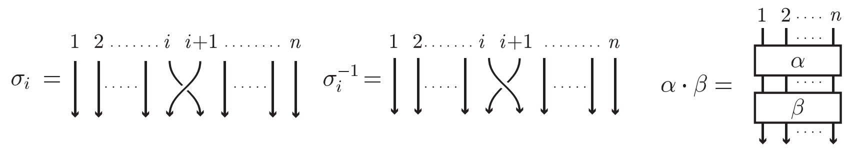
2.1. **Braids, permutations and closures.**

Definition 2.1. The *braid group* B_n on n strands (or strings) can be defined by generators and relations as

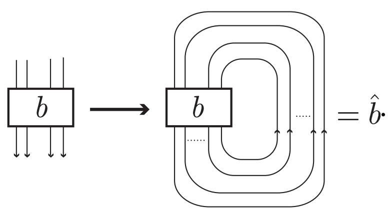
$$B_n = \left\langle \sigma_1,\dots,\sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i,\sigma_j]=1 & |i-j|>1 \\ \sigma_j\sigma_i\sigma_j=\sigma_i\sigma_j\sigma_i & |i-j|=1 \end{array} \right\rangle. \tag{2}$$

The σ_i are called *Artin standard generators*. An element $b\in B_n$ is called an *n-braid*.

There is a graphical representative of braids, where in σ_i strands i and $i+1$ cross, and multiplication is given by stacking. (We number strands from left to right and, for most of the time, orient them downward.)



The *closure* $\hat{b}$ of a braid b is a knot or link (with orientation) in S^3:



Alexander [1] proved that for every link L there is a $b\in B_n$ (when n is large enough) with $\hat{b}=L$. We call b a *braid representative* of L.

Among the different braid representatives of a link L the one with the fewest strands is called a *minimal braid*. The number of strands of a minimal braid is called the *braid index* $b(L)$ of L. Obviously it makes sense to consider braid representatives $\beta\in B_n$ of L only for $n\geq b(L)$.

Markov's theorem (see, e.g., [17]) relates these representatives by two moves, the *conjugacy* in the braid group, and *(de)stabilization*, which is the move

$$b \in B_{n-1} \longleftrightarrow b\sigma_{n-1}^{\pm 1} \in B_n\,. \tag{3}$$

As mentioned, Markov's moves and braid groups have gained importance in knot theory, among others, as a tool for defining link invariants via braids.

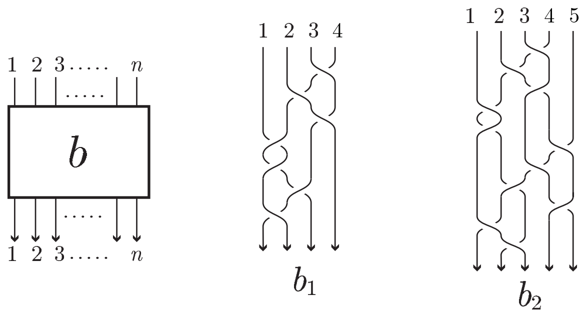


FIGURE 1. An n-braid

There is a homomorphism $B_n \to \mathbb{Z}$, given by $\sigma_i \mapsto 1$, which is the *exponent sum* mentioned in the introduction. More important for us will be the *permutation homomorphism* π of B_n,

$$\pi : B_n \to S_n\,, \quad \text{given by} \quad \pi(\sigma_i) = (i,\, i+1)\,.$$

(The permutation on the right is a transposition.) We call $\pi(b)$ the *braid permutation* of b. We call b a *pure braid* if $\pi(b) = Id$.

Let further $\overline{b}$ for $b \in B_n$ be the automorphism of B_n given by the *mirroring* $\sigma_i^{\pm 1} \mapsto \sigma_i^{\mp 1}$ and $\mathrm{rev}(b)$ be the anti-automorphism given by *word-reversal* (word written with letters $\sigma_i^{\pm 1}$ in the opposite order).

Let b be an n-braid with numbered endpoints as in Figure 1. Suppose that b has its strings connected as follows: 1 to i_1, 2 to i_2, ..., n to i_n, i.e., $\pi(b)(k) = i_k$. Then we write

$$\pi(b) = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}.$$

For example the braid b_1 in Figure 1 has the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = (\ 1 \quad 2 \quad 4 \quad 3\ )\,, \tag{4}$$

where $(1\ 2\ 4\ 3)$ is our notation for a *cyclic permutation*. (Sometimes we will comma-separate elements for readability.) Let

$$\sigma\tau = \tau \circ \sigma \tag{5}$$

be the *compositive multiplication* of permutations. Thus we can write the permutation of the braid b_2 in the figure as $\pi(b_2) = (1\ 3\ 5)(2\ 4)$. (While immaterial for cycle decomposition, this specified order of multiplication in (5) will be relevant later, and should be paid attention to.) By abuse of notation, we will often identify a cycle (subpermutation of $\pi(b)$) with its elements (subset of $\{1,\dots,n\}$).

We remark that when the closure of b is a k-component link, the braid permutation of b is a product of k disjoint cycles. The length $|C|$ of each cycle C is equal to the number of strings of b which make up a component of $\hat{b}$.

When we choose a (non-empty) subset C of $\{1,\dots,n\}$ whose elements form a subset of the cycles of $\pi(b)$, we can define a *subbraid* $b' = b_{[C]}$ of b by choosing only strings numbered in C. For subbraids b' and b'' of b one can define the *(strand) linking number* $lk(b',b'')$ by the linking number $lk(\hat{b}',\hat{b}'')$ between sublinks of $\hat{b}$. For example, in b_2 of Figure 1, we have $lk(b'_2,b''_2) = 0$, where $\hat{b}'_2$ and $\hat{b}''_2$ are the two components of $\hat{b}_2$. (The subbraid $b'_2 = (b_2)_{[\{1,3,5\}]}$ comprises the strings starting at the top as number $1,3,5$, and $b''_2 = (b_2)_{[\{2,4\}]}$ strings $2,4$.)

2.2. Exchange moves, conjugacy of braids, and axis link.

Let

$$\Delta_n^2 = (\sigma_1\cdot\ldots\cdot\sigma_{n-1})^n$$

be the (right-handed) full twist on n strands. (The square in the notation refers to the existence of a well-known square root of this element, the 'half-twist', which will not be needed here, though.) The *center* of B_n (elements that commute with all B_n) is infinite cyclic and generated by Δ_n^2. Let similarly

$$\Delta_{[i,j]}^2 = (\sigma_i\cdot\ldots\cdot\sigma_{j-1})^{j-i+1}$$

be the restricted full twist on strands i to j. Let also for $1\le i<j\le n$,

$$B_{i,j} := \langle\,\sigma_i,\dots,\sigma_{j-1}\,\rangle \tag{6}$$

be the subgroup of B_n of braids operating on strands $i\dots,j$. Where ambiguity is avoided (as indicated by diagrams we will draw), we can identify $B_{i,j}\simeq B_{j-i+1}$.

We say that $b\in B_n$ *admits an exchange move*, if b is as illustrated in Figure 2, where $\alpha\in B_{1,n-1}$, $\beta\in B_{2,n}$, and $n\ge 4$.

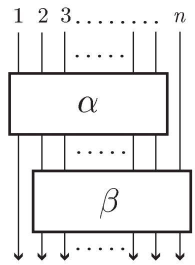


FIGURE 2. The n-braid b.

A (positive/negative) *exchange move* [6] is the transformation between the braids $b=b_0$ and $b_{\pm1}$ shown in Figure 3. When *iterated*, it can transform b into

$$b_m = \alpha\Delta_{[2,n-1]}^{2m}\beta\Delta_{[2,n-1]}^{-2m}\,. \tag{7}$$

Here m is some non-zero integer, and the boxes labeled $\pm m$ represent the full twists $\Delta_{[2,n-1]}^{\pm 2m}$ respectively, acting on the middle $n-2$ strands. (Thus a positive number of full twists are understood to be right full twists, and $-m$ full twists mean m full left-handed twists.)

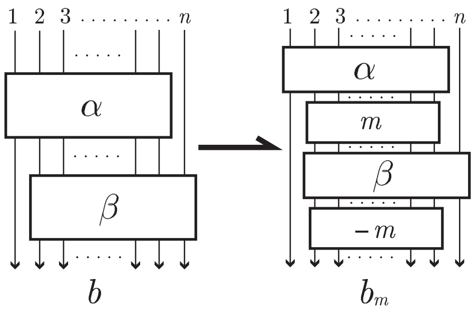


FIGURE 3. The braid b_m

Of course, no non-trivial braid on 2 strands admits an exchange move, and all exchange moves on 3 strands are trivial, so that we will naturally assume $n \geq 4$ throughout. There is another, more common, way to describe the exchange move, namely by

$$\alpha\beta \longleftrightarrow \alpha\kappa^m\beta\kappa^{-m}, \text{ where } \quad \kappa = (\sigma_1 \cdot \ldots \cdot \sigma_{n-2})(\sigma_{n-2} \cdot \ldots \cdot \sigma_1). \tag{8}$$

This description is equivalent to the previous one, because $\kappa \cdot \Delta^2_{[2,n-1]} = \Delta^2_{[1,n-1]}$, and this element commutes with α.

A further equivalent formulation of the move is

$$b_0 = \beta\sigma_{n-1}\beta'\sigma_{n-1}^{-1} \longleftrightarrow b_1 = \beta\sigma_{n-1}^{-1}\beta'\sigma_{n-1},$$

with $\beta, \beta' \in B_{1,n-1}$, which can be generalized (up to conjugacy) by

$$b_m = \Delta^{2m}_{[1,n-2]}\beta\Delta^{-2m}_{[1,n-2]}\sigma_{n-1}\beta'\sigma_{n-1}^{-1}. \tag{9}$$

This form is less convenient for our treatment, so we will not use it below.

The exchange move underlies the switch between conjugacy classes with the same closure link, in a universal way.

Theorem 2.2 (Birman-Menasco [6])**.** *The n-braid representatives of a given link decompose into a finite number of classes under the combination of exchange moves and conjugacy.*

Let us fix throughout that *failure* or *success* of the exchange move are always meant w.r.t. yielding non-conjugate representatives.

Notice that the form admitting an exchange move (for $n \geq 4$) is attained by all stabilized braids as on the right of (3) up to conjugacy, for we can set $\alpha = b$ and $\beta = \sigma_{n-1}^{\pm 1}$. Thus Theorem 1.1 can be applied for

$$n > \max(b(L), 3), \tag{10}$$

unless

$$\pi(b)(1) = 1. \tag{11}$$

This special case for non-minimal braid representatives follows (for a few more links $L = \hat{b}$, with b on the left of (3) satisfying (11)) from work in [24], which we recall now.

In [24] we addressed the case of stabilized braids, when in Figure 3 we have $\beta = \sigma_{n-1}^{\pm 1}$, and the $\pm m$ full twists can be replaced by any braid γ on strands $1, \ldots, n-1$ and its inverse. Then Theorem 1.2

was proved for all α which are not central in $B_{n-1} \simeq B_{1,n-1}$ (this obviously being the weakest possible assumption).

Theorem 2.3 ([24]). *Let $\alpha \in B_{1,n-1}$ and $\alpha \neq \Delta^{2k}_{[1,n-1]}$ for any $k \in \mathbb{Z}$. Then the set*

$$\{\gamma\alpha\gamma^{-1}\sigma_{n-1} \,:\, \gamma \in B_{1,n-1}\} \tag{12}$$

contains infinitely many non-conjugate braids.

Remark that $n \geq 4$ follows directly from the choice of α. We can conclude that for n with (10), there are infinitely many non-conjugate n-braid representatives of L, unless $n = b(L)+1$ and L is a torus link of type (m, lm) for $l \in \mathbb{Z}$ and $m = n-1 = b(L)$. (This includes, for $l = 0$, the m-component unlink L.)

Theorem 2.3 is an application of some Lie group approach showing the density of the image of braid representatives in unitary groups under the Burau and Lawrence-Krammer representation. This method has its own disadvantages; among others, it promises no satisfactory adaptation to exchange moves, i.e., cannot restrict the choice of γ in (12). Also, it cannot yield explicit estimates like (56) below. Still, we will use Theorem 2.3 to complement Theorem 1.1 by the extra case in Corollary 7.2, as well as for the treatise of trivial components in §8.

We finish this subsection with introducing one of the main objects we will work with throughout the paper.

Definition 2.4. The *axis (addition) link* of a braid b, denoted by L_b, is the oriented link consisting of the closure of b and an unknotted curve k, the axis of the closed braid, as in Figure 4. (The orientation of the axis is chosen so that its linking with the braid strands is positive.)

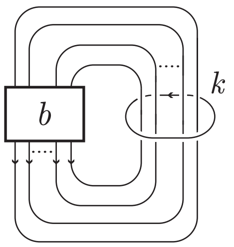


FIGURE 4. The axis-addition link of b

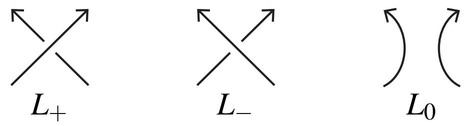
We remark that the axis-addition links of conjugate braids are ambient isotopic. Thus for a proof of non-conjugacy we will study invariants of the axis link.

2.3. Conway polynomial and graph evaluations.

As an invariant of the axis link we will employ the *Conway polynomial* ∇. The Conway polynomial is an oriented link invariant that takes values in $\mathbb{Z}[z]$. It is given by the value 1 on the unknot and the relation

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)\,. \tag{13}$$

This relation involves three links with diagrams



(14)

differing just at one crossing. They are called a *skein triple*. We write $[P]_m$ for the coefficient of z^m in $P \in \mathbb{Z}[z]$, and more shortly $a_m = [\nabla]_m$ for the coefficient of z^m in the Conway polynomial. It is well-known that for an n-component link L, all coefficients of ∇ in z-degree m vanish when $m < n-1$ or $m+n$ is even. We recall also that

$$\nabla(L) = \nabla(-L)\,, \tag{15}$$

where $-L$ is L with orientations on *all* components reversed.

We denote the *linking number* of two components of L by $lk(\cdot,\cdot)$. Now we recall a formula, given by Hoste [11], which expresses the lowest non-trivial coefficient a_{n-1} of $\nabla(L)$ in terms of component linking numbers.

Theorem 2.5 (see [11])**.** *Let $L = L_1 \cup \cdots \cup L_p$ be a p-component link of components $L_1,\dots,L_p$. Let $l_{km} = lk(L_k,L_m)$. Then the coefficient a_{p-1} of the Conway polynomial in degree $p-1$ is*

$$a_{p-1}(L) = \sum_T \prod_{(k,m)\in T} l_{km}\,. \tag{16}$$

In this formula, the sum ranges over spanning trees T of the complete graph G on the vertex set $\{1,\dots,p\}$, and (k,m) denotes the edge in G connecting the k-th and m-th vertex.

We will need several evaluations of this expression below, so let us make some preparations.

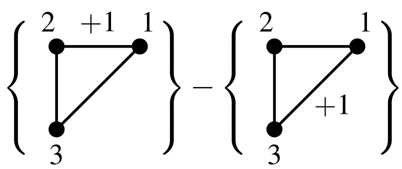
We will consider $G = G_p$ to be a complete graph on the vertex set $\{1,\dots,p\}$ and each edge (i,j) of G between vertices i and j ($1 \le i < j \le p$) will be labeled by an integer l_{ij}. Let T be a spanning tree of G. We write then

$$\{T\} := \prod_{(i,j)\in T} l_{ij}\,, \quad \{G\} := \sum_{T(G)} \{T(G)\}\,, \tag{17}$$

where $T(G)$ will indicate that we understand T as a spanning tree of G.

To save further overhead, let us now also make the following convention. When we draw G, we will assume each edge (i,j) of G has label l_{ij} if a label is not drawn near the edge (i,j), and label $l_{ij}+k$ if a label k is drawn near (i,j).

The following is a simple test for this notation.

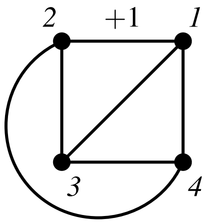


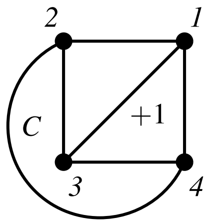
$= l_{13} - l_{12}$ (18)

This is what essentially underlines Lemma 3.2 recalled below (where indices '1' and '3' are interchanged).

We will need the extension of (18) to 4 vertices.

Lemma 2.6. *Let*[1]

$G^1 =$ 

$G^2 =$ 

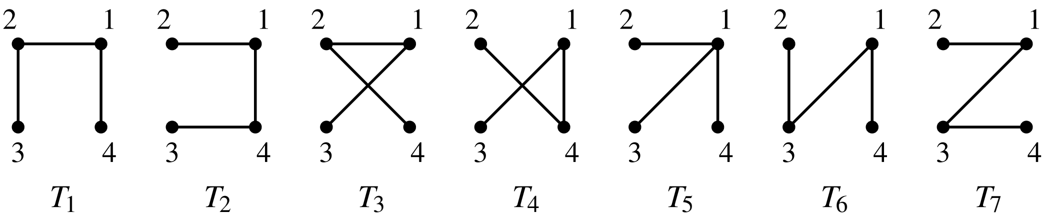
(19)

Then

$$\{G^1\} - \{G^2\} = (l_{13} - l_{12})(l_{14} + l_{24} + l_{34}) + l_{14}(l_{34} - l_{24})\,. \tag{20}$$

Proof. In applying (17), we first easily see that trees with two edges in the boundary ∂C of face C with vertices $2,3,4$ give cancelling contributions, because the labels of all edges in ∂C are equal on either side, and they complete to a tree by adding exactly one of the edges incident to vertex 1, whose labels add up to $l_{12} + l_{13} + l_{14} + 1$ on either side.

This leaves 7 of the 16 spanning trees.



By direct check, one can find that among the 7 trees T_i, the values of the difference $\{T_i(G^1)\} - \{T_i(G^2)\}$ cancel for $i = 1,6$. The terms resulting from the other 5 are given in the following table (where $[ij] = l_{ij}$).

i	$\{T_i(G^1)\} - \{T_i(G^2)\}$
2	$[14] \cdot [34]$
3	$([13] - [12]) \cdot [24]$
4	$-[14] \cdot [24]$
5	$([13] - [12]) \cdot [14]$
7	$([13] - [12]) \cdot [34]$

This shows (20). □

2.4. **Vassiliev invariants.** The concept of Vassiliev invariants, perhaps not very obviously, but very substantially, underlies our strategy of proof, and thus we make some remarks.

Our goal will be to evaluate $a_k(L_{b_m})$ for $b_m \in B_n$ with n,k fixed as a function of m. It is very well known that a_k is a Vassiliev invariant of degree (at most) k. (See [3].)

In circumstances like ours, links in the family $\{L_{b_m}\}$ (as well as similar families we derive from them) characterize themselves as exhibiting a repeated portion of a pattern, and Vassiliev invariants show a polynomial behavior. This theme has occurred in many papers; see for example [25]. It can be formulated as follows.

[1] The letter C on the right of (19) refers to a face in the planar complement of the graph, and will be needed in the proof shortly.

Lemma 2.7. *Let $D(x_1,\dots,x_n)$ for $x_i \in \mathbb{Z}$ be link diagrams which differ by insertion of tangles $\beta_i^{x_i}$ for some pure braids β_i. (Thus all $D(x_1,\dots,x_n)$ have the same number of components.) Let v be a Vassiliev invariant of degree at most d. Then*

$$(x_1,\dots,x_n) \mapsto P(x_1,\dots,x_n) = v(D(x_1,\dots,x_n))$$

is a polynomial in $x_1,\dots,x_n$ of degree at most d.

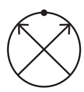
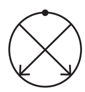
Suggestively, the second author calls this point of view on Vassiliev invariants the *braiding sequence* approach. It should be added, that the strands of the braids β_i here (unlike how we will consistently treat them elsewhere) do *not* need to have a coherent (up/downward) orientation.

Essentially our strategy will be to identify the polynomial function $m \mapsto P(m) = a_k(L_{b_m})$ as non-constant, which will automatically prove that it takes infinitely many values. (We also see at most how many times it can take a given value, which we exploit for Proposition 6.1.) It will be enough to establish that $d = \deg P(m) > 0$ by identifying the leading coefficient. Along the way we will have to argue that certain other contributions are of lower degree.

We will use the standard 'O' notation for the behavior as $m \to \infty$. Then we will be allowed to dispose of terms which we prove to be $O(m^{d-1})$. Asymptotically they give no contribution to the leading coefficient $[P]_d$ of P. Similarly to §2.3, we will write $[P(m)]_k = [P(m)]_{m^k}$ for the coefficient of degree k in a polynomial P.

In many cases, the degree of P is less than the degree of v, and in such a situation, this approach is often not helpful for estimating $\deg P$. In one specific place (proof of Lemma 4.7) we will need to appeal to a different tool, the Gauß diagram formula for the degree-2 Vassiliev knot invariant (also known as Casson's invariant). It is the easiest way to get a polynomial bound (of the desired degree) there.

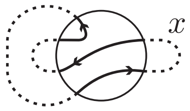
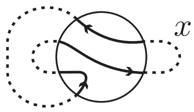
We use the Polyak-Viro formula for $v_2(K) = a_2(K)$ of a knot K, given as [26, (8)],

$v_2 = \frac{1}{2}\Big($  $+$  $\Big)$. (21)

It is obtained by symmetrization (w.r.t. taking the mirror image) from the formula [21, (4)] (see also the remarks on p. 451 therein). The details of how to read this formula have been explained in [21], [26], and many other places. Very briefly, one considers a circle S^1 parametrizing the knot and connects preimages of crossings by an arrow from the undercross to the overcross, and labels the arrow p with the writhe $w_p = \pm 1$ of the crossing. The resulting object is called a *Gauß diagram.* Then one chooses a basepoint on the circle (outside of a crossing, and as it turns out, it does not matter where), and sums the quantity $w_p w_q$ over pairs (p,q) of arrows that look like in the diagrams of (21).

In the following we will consistently stipulate that when a set of diagrams with dotted lines outside a circled spot are drawn, we assume the diagrams to be equal outside this spot, and to have strands connected as displayed by the dotted lines. With this convention, we state a lemma we will need. (Vertical bars for a set will denote number of elements.)

Lemma 2.8. *Assume we have two knot diagrams differing only at the indicated spot. Assume x denotes only the part indicated by the dotted line exiting the circles on the right. Then*

$\Bigg| -a_2 \Bigg($  $\Bigg) + a_2 \Bigg($  $\Bigg) \Bigg| \le \big|\{ \textit{ crossings on } x \,\}\big| \cdot \big|\{ \textit{ all crossings } \}\big| .$ (22)

Proof. The important implication of (21) for us is that $v_2(K)$ for a knot K can be obtained from any diagram D of K by counting pairs of crossings of D with weights 0 or $\pm 1/2$.

Looking at the left of (22), we see that the difference between the two diagrams is the relocation of a segment x of the diagram (and Gauß diagram).

It is clear from (21) that this changes v_2 at most by the number of pairs of crossings with at least one crossing (with a crossing point) on x. (The basepoint in the Polyak-Viro formula can be put at a spot not affected by the move of x.) □

3. THE CASE OF KNOTS

Proof of Theorem 1.1. The 'only if' part in Theorem 1.1 immediately follows from Theorem 2.2. The 'if' part is a consequence of Theorem 1.2, because under the assumed conditions of L, whatever braid representative b of L satisfies $\pi(b)(k) \neq k$ for $k = 1, n$. □

Proof of Theorem 1.2. We start now the proof of Theorem 1.2, which will extend over several sections until the end of the paper.

We state one more time clearly the partial case for knots, which we proved in [22]. It is also necessary to repeat some arguments and figures used in the proof, for later reference and clarification.

Theorem 3.1 ([22])**.** *Let L be a knot and $n \geq 4$. Then L has infinitely many conjugacy classes of n-braid representatives if and only if it has one admitting an exchange move.*

In order to exhibit the braids b_m in Figure 3 as non-conjugate, we evaluated the second coefficient of ∇ on the axis addition link L_{b_m} of b_m.

First, we recall a lemma from [22] needed later. A *delta move* is a local move defined in [18], and this move is equivalent to the move in Figure 5. We consider the delta move on the left-hand side in Figure 6, where the dotted arcs show how the strands connect.

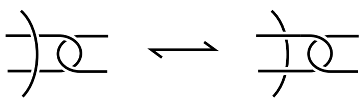


FIGURE 5. A delta move

We proved the following lemma using Theorem 2.5, as essentially mentioned in (18). (We remind that the linking number and i-th coefficient of the Conway polynomial are written $lk(\cdot,\cdot)$ and $a_i(\cdot)$, respectively.)

Lemma 3.2 ([22]). *Let L, L' and $l = k_1 \cup k_2 \cup k_3$ be oriented links related by the local moves as in Figure 6. Then $a_3(L) - a_3(L') = lk(k_2,k_3) - lk(k_3,k_1)$.*

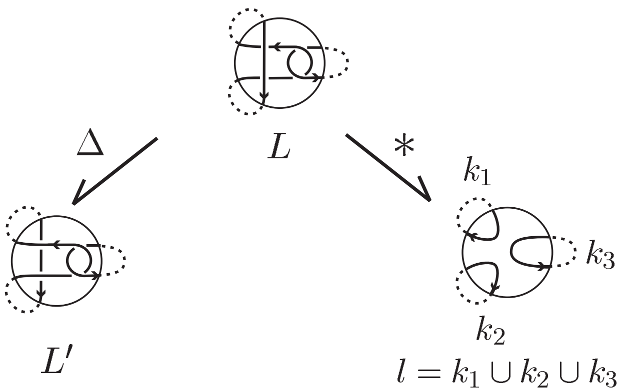


FIGURE 6. Three links related by local moves

A full twist of n strings can be deformed as in Figure 7 up to ambient isotopy. (We will henceforth write $\sim$ for ambient isotopy.) Then there is a deformation of the axis addition link L_{b_m} of b_m, which is the leftmost diagram in Figure 8, into the rightmost link in the figure, still denoted by L_{b_m}. Here k is the component corresponding to the braid axis and the boxes m and $-m$ represent m-full twists and $-m$-full twists respectively. We used this deformation for the proof of Theorem 3.1.

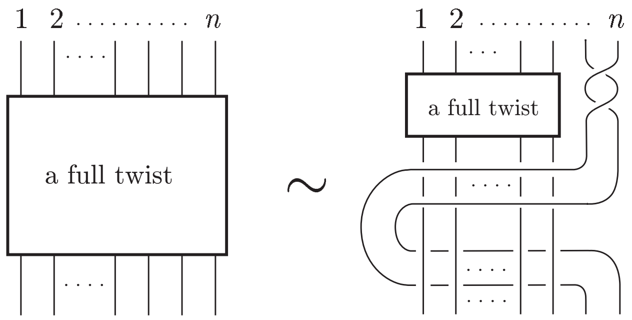


FIGURE 7. A full twist of n-strings

Then there are sequences of links $L_{b_m} = L^0, L^1, L^2, \ldots, L^{n-1} = L_{b_{m-1}}$ and $l^0, l^1, l^2, \ldots, l^{n-1}$ such that L^{i+1} resp. l^i are obtained from L^i by the delta move Δ_i resp. the move $*_i$, both illustrated in Figure 9 ($i = 0$) and 10 ($i = 1, \ldots, n-2$). In particular, there are $n-1$ delta moves transforming L_{b_m} into $L_{b_{m-1}}$: the first is chosen to undo a full twist in the box of m on the right diagram of Figure 8, and the other $n-2$ undo one full winding of the band below that box.

By Lemma 3.2, the change in a_3 resulting from Δ_0 can be obtained as follows:

$$a_3(L^1) - a_3(L^0) = lk(l_1^0, l_3^0) - lk(l_2^0, l_3^0) = n-1, \tag{23}$$

where $l^0 = l_1^0 \cup l_2^0 \cup l_3^0$ is the 3-component link illustrated in Figure 9.

Next we considered the change in a_3 resulting from Δ_i illustrated in the Figure 10 ($i = 1, 2, \ldots, n-2$). Again, as in (23), $lk(l_1^i, l_2^i)$ is turns out irrelevant. To find the two linking numbers we need, we

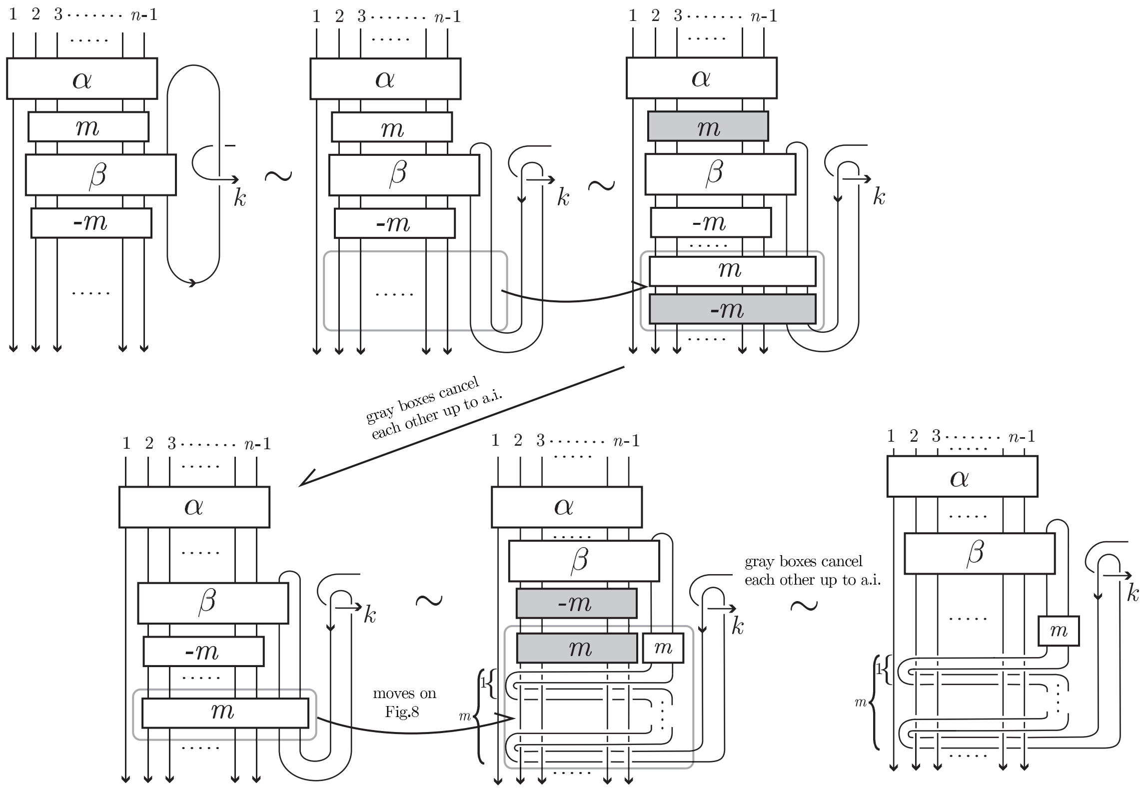


FIGURE 8. Deforming L_{b_m} by ambient isotopy ('a.i.')

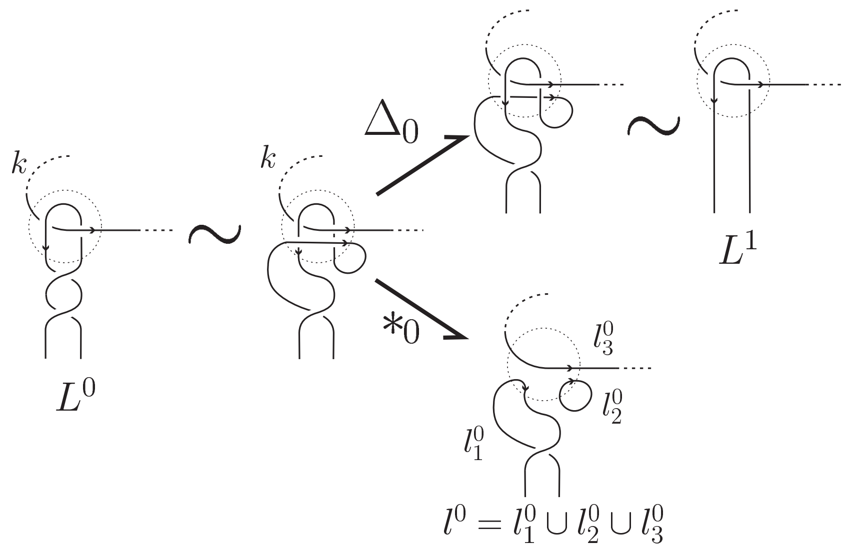


FIGURE 9. The moves Δ_0 and $*_0$

considered S_{L^i}(resp. S_{l^i}) to be a part of L^i (resp. l^i) as in the left (resp. right) diagram of Figure 11. Namely S_{L^i} and S_{l^i} are the unions of n strings and an unknotted component. Some of these $(n-1)$

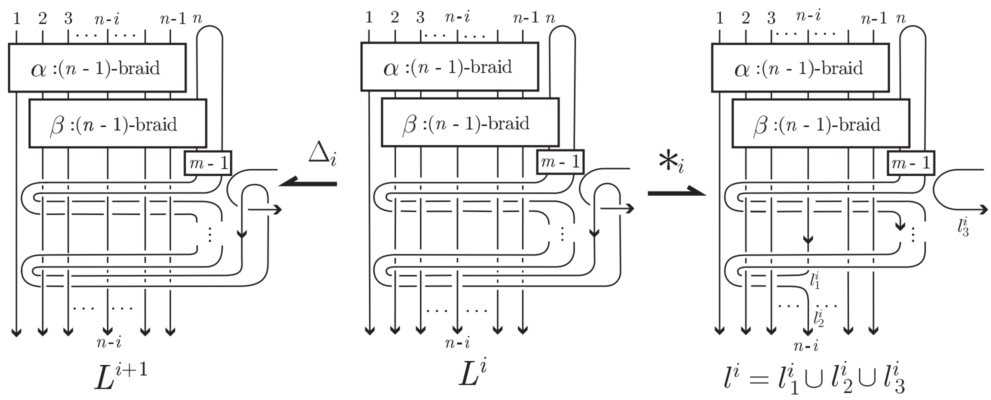


FIGURE 10. The local moves on L^i

strings of S_{l^i} belong to l_1^i and the other belong to l_2^i. The numbers of strings determine $lk(l_1^i, l_3^i)$ and $lk(l_2^i, l_3^i)$.

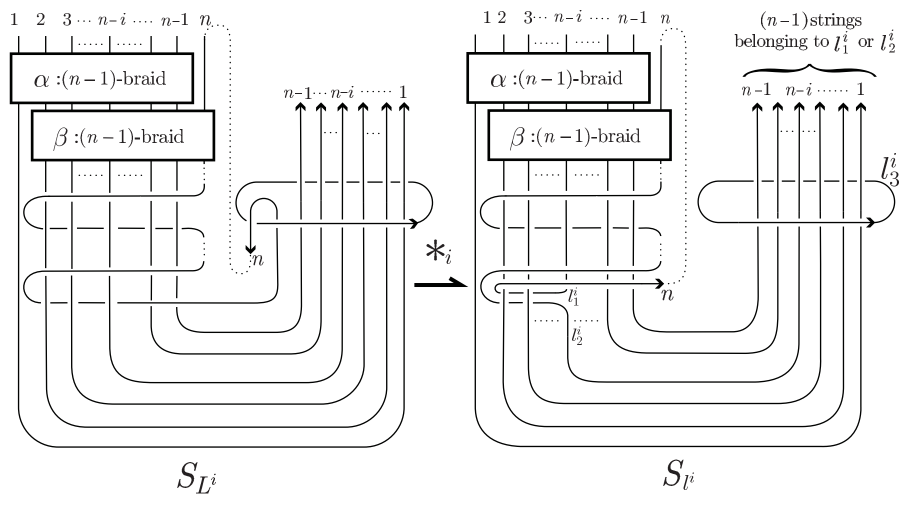


FIGURE 11

Write the braid permutation $\pi(b) = (x_1, x_2, \ldots, x_n)$, where we fix the cyclic ambiguity of x_i by letting the cycle end on $x_n = n$. Take $j = j(i)$ so that $n - i = x_j$. Then by Lemma 3.2, we found (for $i = 1, 2, \ldots, n-2$)

$$a_3(L^{i+1}) - a_3(L^i) \ = lk(l_1^i, l_3^i) - lk(l_2^i, l_3^i) \ = (j-1) - (n-j) = 2j - n - 1. \tag{24}$$

From there we succeeded in ultimately evaluating $a_3(L^{n-1}) - a_3(L^0) = a_3(L_{b_m}) - a_3(L_{b_{m-1}})$. By [22, proposition 2.3], it turns out to be non-zero, except in one restricted special case: when

$$\pi(b)^{(n+1)/2}(n) = 1\,. \tag{25}$$

In particular, in this case always n is odd.

Then we focused on this special case, and to prove that b_m are non-conjugate, we looked at b_m^2: if two braids are conjugate, so are their squares. We showed that $a_3(L_{b_m^2})$ is a quadratic polynomial in

m, with non-zero quadratic term. (Note that for n odd when $\pi(b_m)$ is a cycle, so is $\pi(b_m^2)$, and thus $L_{b_m^2}$ again has two components.) In particular, there are at most two $L_{b_m^2}$ whose a_3 is equal to some fixed value, and so at most two of (b_m^2 and) b_m are conjugate. This completed the proof of Theorem 1.1.

4. The first case of links

We now move to the case of links in Theorem 1.2. A few of the links can be easily dealt with by a sublink argument. We formulate this argument as a lemma, which will be also crucially needed for the more complicated general situation.

Let for $\beta \in B_n$ the set $B \subset \{1, \dots, n\}$ be a union of cycles of $\pi(\beta)$. Then the subbraid $\beta' = \beta_{[B]}$ (in the notation at the end of §2.1) of β obtained by taking only strands (on top or bottom) labelled by numbers in B gives a closure link $\hat{\beta}'$ which is a sublink of $\hat{\beta}$. We thus call β' a *sublink braid* of β.

Lemma 4.1. *Assume $\mathcal{B} = \{\beta_1, \beta_2, \dots\}$ is a set of conjugate braids in B_n, and β_i' are sublink braids of β_i. Then $\mathcal{B}' = \{\beta_1', \beta_2', \dots\}$ splits as a finite union*

$$\mathcal{B}' = \mathcal{B}_1' \cup \mathcal{B}_2' \cup \dots \cup \mathcal{B}_s', \tag{26}$$

such that all braids in $\mathcal{B}_i'$ are conjugate.

This means that if we like to prove that infinitely many braids in $\mathcal{B} = \{\beta_1, \beta_2, \dots\}$ are non-conjugate, we can transform the problem to any suitably chosen set of sublink braids $\{\beta_1', \beta_2', \dots\}$.

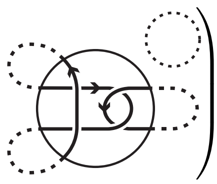
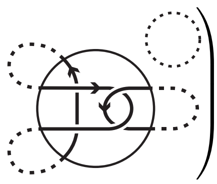
Proof. It is clear that when β and γ are conjugate, for each sublink braid β' of β there is a sublink braid γ' of γ such that β' and γ' are conjugate. Now let $\beta_{1,1}', \dots, \beta_{1,s}'$ be the sublink braids of β_1. Then each $\mathcal{B}_i$ for a fixed $i = 1, \dots, s$ can be defined as the set of β_j' conjugate to $\beta_{1,i}'$. □

We Lemma 4.1 in mind, we split the treatment of links into two major cases, depending on whether 1 and n belong to the same or to distinct cycles of $\pi(b)$.

Theorem 4.2. *Assume a braid $b \in B_n$ admits an exchange move, and 1 and n belong to the same cycle of $\pi(b)$. Then the link $\hat{b}$ has infinitely many non-conjugate n-braid representatives.*

The following is an analogue of Lemma 3.2.

Lemma 4.3. *Let L, L' and $l = l_1 \cup l_2 \cup l_3 \cup l_4$ be oriented links related by the local change in Figure 12. Then*

$$a_4(L) - a_4(L') = a_4 \left(\vphantom{\int} \right.$$  $$\left. \vphantom{\int} \right) - a_4 \left(\vphantom{\int} \right.$$  $$\left. \vphantom{\int} \right) \tag{27}$$

can be expressed in terms of linking numbers of l_i in l.

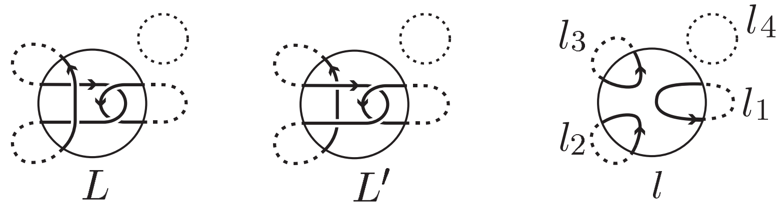
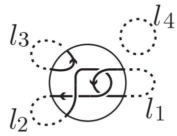
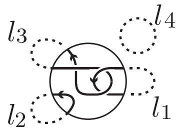


FIGURE 12. Three links related by local changes

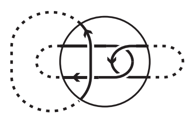
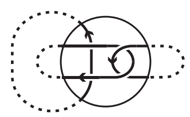
Proof.

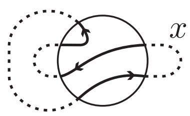
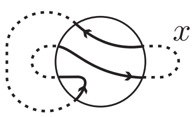
$a_4(L) - a_4(L') = -a_3 \Big($  $\Big) + a_3 \Big($  $\Big)$, (28)

and use Lemma 2.6. □

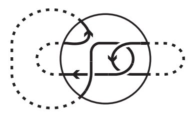
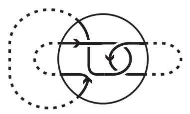
We note that later we will use Lemma 2.6 to express (27) concretely in terms of linking numbers of l_i, rather than using the lemma in its vague general formulation. (The label x in (29) will only be needed later; see above (35).)

Lemma 4.4. *We have*

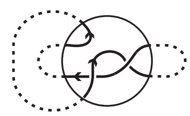
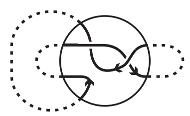
$a_4 \Big($  $\Big) - a_4 \Big($  $\Big)$

$= -a_2 \Big($  $\Big) + a_2 \Big($  $\Big)$. (29)

Proof. We transform the left hand side of (29). By switching the negative crossings on the strands in the two tangles, we see that the expression is equivalent to

$-a_3 \Big($  $\Big) + a_3 \Big($  $\Big)$.

By switching one positive crossing in the clasp on either side, we see that this is in turn equivalent to

$-a_2 \Big($  $\Big) + a_2 \Big($  $\Big)$.

The claimed equality now follows from resolving the clasps. □

Proof of Theorem 4.2. It is easy to see from the shape in Figure 2 that the cycle C of $\pi(b)$ containing 1 and n cannot be a transposition. If it is has length > 3, then looking at a sublink of L_{b_m} or $L_{b_m^2}$, and using the argument in the proof of Theorem 1.2 for knots together with Lemma 4.1, we are done. Therefore, assume that C is of length 3.

We will choose a subbraid $b_{[C\cup C']}$ of b by taking the strands corresponding to elements in C and one other cycle C' of $\pi(b)$. We can choose this cycle C' arbitrarily, and forget about the other components of $\hat{b}$. It is enough (by Lemma 4.1) to show that the so constructed b_m are non-conjugate.

Now we consider again the axis link L_{b_m} of b_m. Again, we can move a permutation of strands $3,\dots,n-1$ between α and β. Thus we assume now

$$\alpha=\sigma_1^{-1}\cdot\alpha' \quad\text{and}\quad \beta=\sigma_2\cdot\dots\cdot\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}\cdot\dots\cdot\sigma_2^{-1}\,\cdot\,\sigma_3\cdot\dots\cdot\sigma_{n-2}\,\cdot\,\beta',$$

with α',β' pure. In particular,

$$\pi(b)=C\cdot C',\quad\text{with}\quad C=(1\;2\;n),\;\; C'=(n-1\;n-2\;\cdots\;3)\,. \tag{30}$$

The link L_{b_m} for $n=7$ is shown on Figure 13. We must clarify that we made some changes in this figure as compared to Figure 8 (for visibility). First, we replaced m by $-m$. Then we also changed the strand orientation to upward. The latter has the effect of interchanging α and β in Figure 3 and replacing them by the letter-order-reversed words and $\sigma_i^{\pm1}$ by $\sigma_{n-i}^{\pm1}$. Neither of these changes affects the following arguments (only on the right of (31) the sign changes).

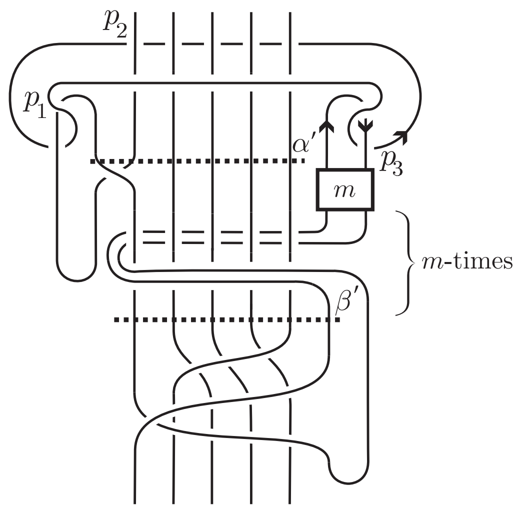


FIGURE 13. The link L_{b_m}. The dotted lines indicate spots of insertion of a pure braid. By p_i we designate three specific crossings, to be used in (37).

Now L_{b_m} can be turned into L_b by undoing m twists in the box, and then undoing the m band turns.

Each undoing of a band turn requires first one move as in (27). This move occurs when the band passes through strand 3 (counted from the downgoing leftmost strand), because it belongs to the same link component as the band. When the band passes then through one of the strands $4,\dots,n$, which belong to a different component, we have $n-3$ moves as on the left (upper) hand-side of (29).

Our attitude will be now that, to obtain L_b from L_{b_m},

1) to undo the m band turns, we first undo in L_{b_m} the m turns of the band around strand 3 (in Lemma 4.3), obtaining a link $L^*_{b_m}$,
2) then we undo the m twists in the box in $L^*_{b_m}$, obtaining a link L'_{b_m},

3) and then care about the $m(n-3)$ moves in Lemma 4.4, which turn L'_{b_m} into L_b.

See Figure 14.

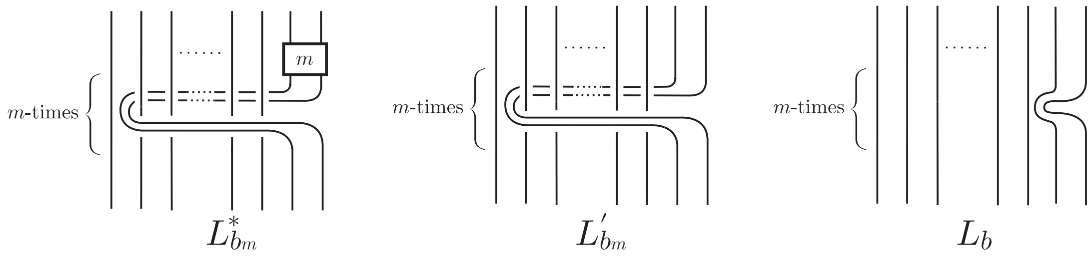


FIGURE 14. The links $L^*_{b_m}$, L'_{b_m} and L_b.

We claim the following.

Lemma 4.5. *We have that $a_4(L_{b_m}) - a_4(L^*_{b_m})$ is quadratic in m, and*

$$[a_4(L_{b_m}) - a_4(L^*_{b_m})]_{m^2} = (n-3)^2. \tag{31}$$

Lemma 4.6. *The difference $a_4(L^*_{b_m}) - a_4(L'_{b_m})$ is linear in m.*

Lemma 4.7. *Finally, $a_4(L'_{b_m}) - a_4(L_b)$ is quadratic in m, and*

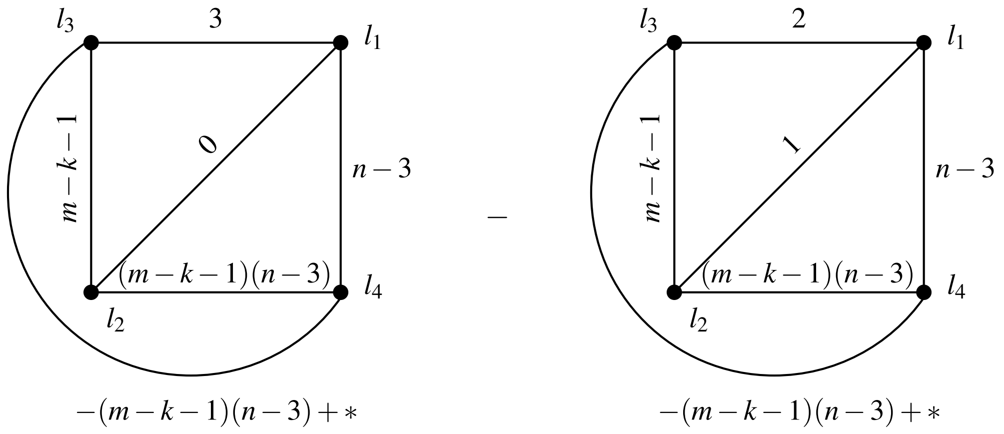
$$[a_4(L'_{b_m}) - a_4(L_b)]_{m^2} = -2(n-3)^2. \tag{32}$$

Proof of Lemma 4.5. So, we first undo in $L^0 = L_{b_m}$ the m turns of the band around strand 3 (in Lemma 4.3), obtaining a sequence of links $L^0, L^1, \dots, L^m = L^*_{b_m}$. We consider the $k+1$-st of the moves in Lemma 4.3, for $k = 0, \dots, m-1$, which transforms L^k into L^{k+1}.

One move in Lemma 4.3, as we will show, changes a_4 by a quantity $a_4(L^{k+1}) - a_4(L^k)$ we can evaluate. Thus to show (31), we use Lemma 4.3 and look at the two links on the right of (28), using Figure 15.

The labeling of components in Figure 15 is chosen to conform to (28), in particular l_1 is the braid axis, and $l_4 = P$ is the component coming from the length-$(n-3)$ cycle C' of $\pi(b)$ in (30).

The linking graphs of the two links on the right of (28) (taken with the signs there) look thus:



(33)

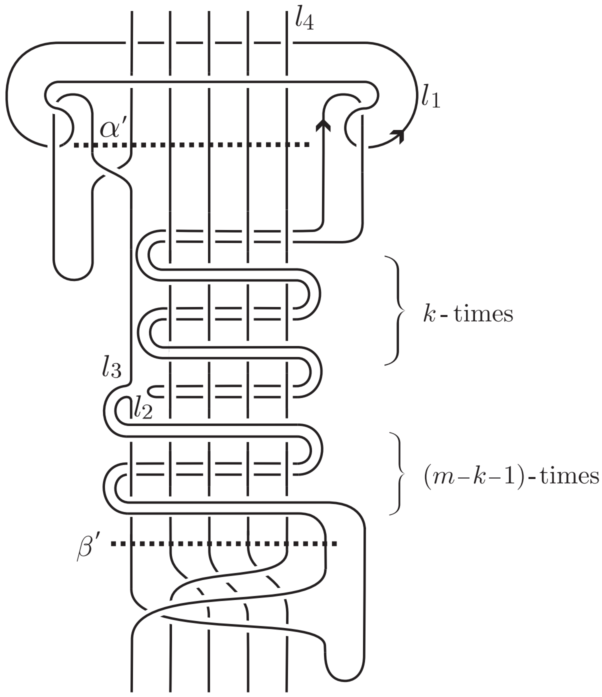


FIGURE 15. A link on the stage between $L^*_{b_m}$ and L'_{b_m} (for $m=4$), modified according to the change to the link l of Figure 12, except that the extra component (part of the band) hanging on l_1 is still depicted (l_1 is the braid axis). By k (here $k=2$) we designate the number of changes in (27) already performed.

Here $*$ is a contribution (constant in m) from the linking number of α' and β' – both (pure) braids contribute only to $lk(l_3,l_4)$. With these signs, we must compare with (27), where now $L=L^k$ and $L'=L^{k+1}$.

In applying Lemma 2.6 to (33), we have

$$a_4(L^k)-a_4(L^{k+1}) \,=\, 2((n-3)+*)+(n-3)(-2(m-k-1)(n-3)+*)\,.$$

(In the diagrams of (19) vertices 2,3 are interchanged, which cancels with the opposite signs in (33).) Then

$$a_4(L_{b_m})-a_4(L^*_{b_m}) \,=\, \sum_{k=0}^{m-1} -2((n-3)+*)+(n-3)(2(m-k-1)(n-3)-*) \,=\, m^2(n-3)^2+O(m).$$

This shows (31). □

Proof of Lemma 4.6. Now, undoing m twists in the box on Figure 13 contributes a quantity linear in m, because smoothing out any crossing in the box gives the same link which is independent of m. □

Proof of Lemma 4.7. With Lemmas 4.5 and 4.6 proved, it is enough for Theorem 4.2 to show that

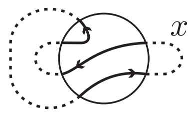
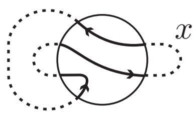
$$a_4(L'_{b_m})-a_4(L_b) \tag{34}$$

is quadratic in m, with a quadratic coefficient which does not cancel (31).

First, by Lemma 2.7, (34) must be some polynomial expression in m. We will start by showing that this polynomial is at most quadratical, by proving that it is quadratically bounded.

For this, we study the change of a_4 under one application of the move on the left of (29), and use the expression on the right of that equation. Note that the links on the right are knots, and it involves values of the Casson invariant $a_2 = v_2$.

Note that x is the remainder of the braid axis, like the component l_3^i on the right of Figure 11, and thus the number of crossings on x is $2(n-1)$, independent of m. Thus by lemma 2.8,

$-a_2\Big($  $\Big) + a_2\Big($  $\Big)$

$$\leq \Big|\{\text{ crossings on } x\ \}\Big| \cdot \Big|\{\text{ all crossings }\}\Big| = O(1)\cdot O(m) = O(m) \qquad (35)$$

(with 'O' depending on n, which is, however, fixed). Very strictly speaking, one must move the vertical strand close to the clasp, past other vertical strands, in order to look as on the left of (29). But this can be done at the cost of $O(m)$ (in fact, even $O(1)$) extra crossings (outside of x), and does not affect (35).

By iterating this estimate $(n-3)m$ times, we see that (34) is $O(m^2)$, and hence a polynomial at most quadratic[2] in m.

Next, note that, since x does not involve crossings in α' and β', the contribution of these crossings in (35) is bounded in m, and hence contributes to (34) only $O(m)$. This means that the m^2-term of (34),

$$[a_4(L'_{b_m}) - a_4(L_b)]_{m^2}\,, \qquad (36)$$

does not depend on α' and β', and hence can be evaluated when α' and β' are trivial.

Thus it is enough to determine (36) when we think away the dotted lines in Figure 13, and keep in mind that in L'_{b_m} the box with label 'm' is gone and the bands look like Figure 16. That is, they are 'unhooked' from both leftmost braid strands (those whose crossings with the axis are labeled p_1, p_2).

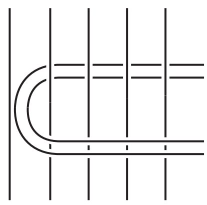
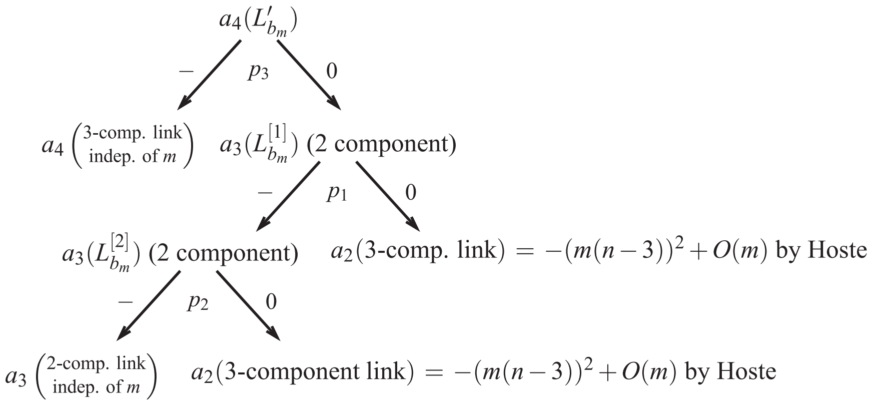


FIGURE 16

Now, in this simple L'_{b_m}, we can evaluate (36) more directly. Ignoring the (m-)constant $a_4(L_b)$, we use the following skein tree resolution for $a_4(L'_{b_m})$, at the indicated crossings p_i in Figure 13.

[2]Note that, while (21) readily gives a good degree estimate, it does not clarify easily by itself the polynomial behavior of (34). Thus the braiding sequence and Gauß diagram formula approach complement each other very usefully here.



(37)

In applying Hoste's formula, we notice that in L'_{b_m} (and therewith also in the links of the above diagram) the only linking numbers depending on m involve the component P coming from the cycle C' in (30), which are $\pm m(n-3)$. (Thus the stated m^2-terms come from the product of these two linking numbers.)

Moreover, since the crossings the skein relation is applied at are all positive, the contributions of these two links to (34) are both positive, showing (32). □

By the preceding three lemmas, the coefficient $[a_4(L_{b_m}) - a_4(L_b)]_{m^2} = [a_4(L_{b_m})]_{m^2}$ is non-zero, as desired, and thus we conclude the proof of Theorem 4.2. □

5. The second case of links

The situation when 1 and n belong to distinct cycles of $\pi(b)$ is the final case needed to complete the proof of Theorem 1.2.

Theorem 5.1. *Let $b \in B_n$ admit an exchange move, and let* 1 *and n belong to distinct non-trivial cycles of $\pi(b)$. Then infinitely many of the b_m are non-conjugate.*

Proof. Let n_1 be the length of the cycle of $\pi(b)$ containing 1, and n_2 the length of the cycle containing n.

By the sublink argument in Lemma 4.1, and by adjusting the permutations of the cycles involving 1 and n, it is enough to consider b in Figure 2, where α, β are given by

$$\alpha = \sigma_1 \cdot \ldots \cdot \sigma_{n_1-1} \cdot \alpha' \quad \text{and} \quad \beta = \sigma_{n_1+1} \cdot \ldots \cdot \sigma_{n-1} \cdot \beta', \tag{38}$$

and α' and β' are pure braids. In particular,

$$n_1 + n_2 = n, \tag{39}$$

that is, $\pi(b)$ has only the two relevant cycles.

We will evaluate $a_4(L_{b_m})$ for fixed α and β as a (polynomial) function in m. (Note that L_{b_m} is a 3-component link.) Let us from the outset take the attitude that the linear and absolute term in m are irrelevant.

Throughout the treatment of this final case, we use the description of the exchange move in (8).

Lemma 5.2. *The function $m \mapsto a_4(L_{b_m})$ is an at most cubic polynomial in m. The cubic term does not depend on α, β. The quadratic term depends on α, β only via linear combinations of linking numbers of strands in α', β'.*

Proof. It is enough to work with $m > 0$. Otherwise we can multiply α and β by a proper power of κ. The argument we give below for $m > 0$ applied on the modified α and β will give the result for the original α and β for $m > -k$, where k can be chosen arbitrarily. Thus the property holds then for all integers m.

We describe a method for doing a recursive skein calculation of $a_4(L_{b_m})$, which will be relevant also after the proof of the lemma. This calculation will be crucial throughout the treatment, and we will gradually refine it.

We consider $a_4(L_{b_m}) - a_4(L_{b_{m-1}})$, where by (8)

$$b_m = \alpha\kappa^m\beta\kappa^{-m}.$$

Now we can write

$$\begin{aligned} b_m \;&=\; \alpha\kappa^{m-1}(\sigma_1 \cdot \ldots \cdot \sigma_{n-2}\sigma_{n-1})(\underline{\sigma_{n-1}^{-1}}\sigma_{n-2} \cdot \ldots \cdot \sigma_1)\beta \times \\ &\quad \times(\sigma_1^{-1} \cdot \ldots \cdot \sigma_{n-2}^{-1}\sigma_{n-1}^{-1})(\underline{\sigma_{n-1}}\sigma_{n-2}^{-1} \cdot \ldots \cdot \sigma_1^{-1})\kappa^{1-m}. \end{aligned} \tag{40}$$

Then we have by the skein relation (14)

$$a_4(L_{b_m}) - a_4(L_{b_{m-1}}) \;=\; -a_3(L_{m-1,1}) + a_3(L_{m-1,2}), \tag{41}$$

where $L_{m-1,i}$ is the axis link of the braid obtained from the word on the right of (40) by omitting the underlined occurrences of σ_{n-1}^{-1} resp. σ_{n-1}. Let us write $[b]$ for L_b. Then

$$\begin{aligned} L_{m,1} \;&=\; [\alpha\kappa^m(\sigma_1 \cdot \ldots \cdot \sigma_{n-2}\sigma_{n-1})(\sigma_{n-2} \cdot \ldots \cdot \sigma_1)\beta \times \\ &\quad \times(\sigma_1^{-1} \cdot \ldots \cdot \sigma_{n-2}^{-1}\sigma_{n-1}^{-1})(\sigma_{n-1}\sigma_{n-2}^{-1} \cdot \ldots \cdot \sigma_1^{-1})\kappa^{-m}]. \end{aligned} \tag{42}$$

$$\begin{aligned} L_{m,2} \;&=\; [\alpha\kappa^m(\sigma_1 \cdot \ldots \cdot \sigma_{n-2}\sigma_{n-1})(\sigma_{n-1}\sigma_{n-2} \cdot \ldots \cdot \sigma_1)\beta \times \\ &\quad \times(\sigma_1^{-1} \cdot \ldots \cdot \sigma_{n-2}^{-1}\sigma_{n-1}^{-1})(\sigma_{n-2}^{-1} \cdot \ldots \cdot \sigma_1^{-1})\kappa^{-m}]. \end{aligned} \tag{43}$$

The complication now is that the links $L_{m,1}$ have two components. We need to apply the skein relation once more before we can use Hoste's formula.

We will calculate instead of $a_3(L_{m-1,i})$ the difference

$$a_3(L_{m-1,i}) - a_3(L_{0,i}). \tag{44}$$

The extra terms $a_3(L_{0,i})$ contribute only something absolute in m to $a_4(L_{b_m}) - a_4(L_{b_{m-1}})$, and hence only something linear in m to $a_4(L_{b_m})$, which we decided to ignore.

It is clear that one can determine (44) by evaluating

$$a_3(L_{m,i}) - a_3(L_{m-1,i}).$$

For this we turn around two groups of $n-2$ crossings, namely those needed to trivialize the last of the m copies of κ before β in (40) (note that we shifted $m-1$ to m) and the first of the m copies of κ^{-1} after β. We obtain

$$a_3(L_{m,i})-a_3(L_{m-1,i})=\sum_{l=2}^{n-1}a_2(L_{m,i,l})-a_2(L_{m,i,\bar l})\,. \tag{45}$$

The link $L_{m,i,l}$ is the axis link of the braid obtained from the braid in (42) (for $i=1$) or (43) (for $i=2$) by replacing the last copy of κ before β by

$$\sigma_1\dots\sigma_{l-2}\sigma_{l-1}\sigma_{l-2}\dots\sigma_1\,, \tag{46}$$

and $L_{m,i,\bar l}$ is the axis link of the braid obtained from the braid in (42) resp. (43) by replacing the first copy of κ^{-1} after β by the inverse of the braid in (46).

Now $L_{m,i,l}$ and $L_{m,i,\bar l}$ have three components, and their a_2 can be evaluated by Hoste's formula. Two of the linking numbers of the components of $L_{m,i,l}$ and $L_{m,i,\bar l}$ are independent of m, and the third one is the linear in m, with the linear term independent of α', β'. From this the claim of the lemma follows. □

Lemma 5.3. *In the function $m\mapsto a_4(L_{b_m})$ of Lemma 5.2, the cubic term vanishes.*

Proof. By Lemma 5.2, it is enough to prove this when α' and β' are trivial. Under this assumption, we claim the following:

$$L_{b_m}\sim L_{b_{-m}}\,, \tag{47}$$

up to switching orientation (of *all* components simultaneously). With (47) the lemma follows, since the function given there is even (by (15)).

To see (47), note that, assuming α' is trivial,

$$\alpha=\sigma_1\dots\sigma_{n_1-1}$$

can be conjugated to its word-reverse rev(α) (as defined in §2.1) *without using* σ_1 and σ_{n-1}, and similarly β. Then κ commutes with the subgroup generated by $\sigma_2,\dots,\sigma_{n-2}$. After α and β were reversed, flip the braid axis link by π along the horizontal axis in projection plane, conjugate by α to move it to the top, and reverse all orientations (including of the axis) to have strands pointing downward. □

We thus now are led to look at $[a_4(L_{b_m})]_{m^2}$, and our goal is to prove that is does not vanish. The skein calculation in the proof of Lemma 5.2 would be unwieldy. However, we help ourselves here by taking also the mirrored braids into account.

Let $\bar b$ be b where all σ_i and σ_i^{-1} are interchanged. Mirroring is an automorphism of B_n, thus if two braids are conjugate, so are their mirror images. Here $\overline{b_m}=\bar b_{-m}$. Consider the function

$$m\mapsto a_4(L_{b_m})+a_4(L_{\overline{b_m}})\,.$$

(Note that $L_{\overline{b_m}}$ is not the mirror image of L_{b_m}, as the braid axis retains its orientation.) Both terms on the right are polynomials of degree at most 2 in m. If we show that their sum has degree 2, then at least one of the two polynomials is non-zero, and we are done. We will thus complete the proof of Theorem 5.1, and hence also the one of Theorem 1.2, by the following lemma.

Lemma 5.4. *We have*

$$[a_4(L_{b_m})]_{m^2} + [a_4(L_{\bar{b}_m})]_{m^2} = 2(n_1-1)(n_2-1)\,. \tag{48}$$

Proof. By Lemma 5.2 we have that

$$[a_4(L_{b_m})]_{m^2}$$

depends only linearly on the linking numbers of α' and β'. Now, changing a linking number in α' for the representative (38) of α changes this linking number oppositely in the representative of $\overline{\alpha}$. We see again that the expression

$$[a_4(L_{b_m})]_{m^2} + [a_4(L_{\bar{b}_m})]_{m^2} \tag{49}$$

does not depend on α and β. We will thus evaluate it when α and β are trivial.

We have by (47) then

$$L_{\bar{b}_m} = L_{\bar{b}_{-m}} = L_{\overline{b_m}}\,. \tag{50}$$

Now we will follow the skein calculation of the proof of Lemma 5.2, simultaneously for b_m and $\overline{b_m}$. In order to distinguish the links occurring in the calculations for b_m and $\overline{b_m}$, we will write in the latter case $\overline{L}_{\dots}$, with the proper subscript, for what would have been $L_{\dots}$ in the case of b_m.

The skein calculation could be summarized by saying that we expressed $a_4(L_{b_m}) - a_4(L_{b_{m-1}})$ by a linear combination of terms

$$a_2(L_{m',i,l}) \quad \text{and} \quad a_2(L_{m',i,\bar{l}}) \tag{51}$$

for $i = 1,2$; $0 \le m' < m$; and $2 \le l \le n-1$, up to absolute terms in m. If we sum this up to express $a_4(L_{b_m})$, then we have something linear in m (and ignorable), and then for each of the four families in (51):

$$((1)) = L_{m',1,\bar{l}}\,, \quad ((2)) = L_{m',2,l}\,, \quad ((3)) = L_{m',2,\bar{l}}\,, \quad ((4)) = L_{m',1,l} \tag{52}$$

(determined by the choice $i = 1,2$ and between l and $\bar{l}$), there are

$$\frac{m^2}{2} + O(m) \tag{53}$$

terms.

Then each of the terms

$$a_2(\overline{L}_{m',i,l}) \quad \text{and} \quad a_2(\overline{L}_{m',i,\bar{l}})$$

enters into the skein calculation for $\overline{b_m}$ with the same sign as does its analogue in (51) for the calculation for b_m. This is because every time a crossing is smoothed out, the sign changes between b_m and $\overline{b_m}$, but to get (51) we smoothed out two crossings in b_m resp. $\overline{b_m}$. Combining the signs in (41) and (45), we see that the signs of families ((1)) and ((2)) in (52) are positive, for families ((3)) and ((4)) negative.

Now, the (two) component linking numbers in $\overline{L}_{m',i,l}$ involving the braid axis component are the same as for $L_{m',i,l}$, and the remaining (third) linking number is opposite. Thus in Hoste's formula for $a_2(\overline{L}_{m',i,l}) + a_2(L_{m',i,l})$, the contributions of trees containing this third linking number cancel. Likewise occurs for the index 'l' replaced by '$\bar{l}$'.

By Hoste's formula, it becomes clear that one half of (49) can be evaluated by doing again the skein calculation for b_m only (abandoning $\overline{b_m}$), and therein replacing $a_2(L_{\dots})$ in (51) by

$$\langle \pi(b_{\dots}) \rangle \,, \tag{54}$$

where $b_{\dots}$ is the braid whose axis link is $L_{\dots}$, and $\langle\sigma\rangle$ is the product of the (here always two) cycle lengths of σ. (These are the linking numbers which remain the same in $\overline{L}_{m',i,l}$ and $L_{m',i,l}$.)

Now this simplifies the calculation considerably. Note first that $\langle \pi(b_{\dots}) \rangle$ does not depend on m or m' (and that, with (53), this accords well with Lemma 5.3). Thus we can evaluate all four families in (52) just by looking at their permutations. We have then to divide by 2 following (53) to get the m^2-term. This can be compensated by the factor 2 explained in the application of Hoste's formula above (54).

From here there are two ways to get done. A "philosophical" way is to observe that by the skein calculation, the expression (49) must be some polynomial in n_1 and n_2. By using that L_{b_m} has $O(m(n_1+n_2))$ crossings, that a_4 is a Vassiliev invariant of degree 4, and the extension of the (proof of the) Lin-Wang conjecture to links in [25], we can conclude that the polynomial is of degree at most 4. (This is not implied by, but is very closely related to, Lemma 2.7.) Moreover, the triviality of the cases $n_i = 1$ explains the factor $(n_1-1)(n_2-1)$. The polynomial must also be symmetric in n_1 and n_2. From this one can obtain the formula (48) in the lemma by calculating the value of the polynomial for a few explicit (n_1,n_2). (In the realm of ascertaining the result, we did a few such computer checks which, via this argument, would establish (48) alternatively.)

Nevertheless, it is possible to make exact calculation. Now let us write $((1)),\dots,((4))$ in (52) for the contribution (54) of the link in question to $a_4(L_{b_m})$ according to (41) and (45).

Resuming the notation for permutations from (4) and (5), let $[x,y]$ be the cycle $(y, y-1, \dots, x)$. We have

$$\begin{aligned}
((1)) &= \langle (1,n)[n_1+1,n](1,l)[1,n_1] \rangle \\
&= \left\langle \begin{pmatrix} l+1 & \text{if } n \ge n_1+1 & \\ l & \text{if } l \le n_1 & ,n \end{pmatrix} (1,n)\pi(b) \right\rangle \,, \\
((2)) &= \langle (1,l)[n_1+1,n](1,n)[1,n_1] \rangle \\
&= \left\langle \begin{pmatrix} & l & \text{if } l \le n_1 \\ 1, & n & \text{if } l = n_1+1 \\ & l-1 & \text{if } l > n_1 \end{pmatrix} (1,n)\pi(b) \right\rangle \,, \\
((3)) &= \langle [n_1+1,n](1,n)(1,l)[1,n_1] \rangle \\
&= \langle (l,n)(1,n)\pi(b) \rangle \,, \\
((4)) &= \langle (1,l)(1,n)\pi(b) \rangle \,.
\end{aligned}$$

Then for $l \le n_1$ we have $((1)) = ((3))$ and $((2)) = ((4))$, and in the sum over $l > n_1$ of $((1)) - ((3))$ terms cancel with a shift of 1. Similarly for $((2)) - ((4))$.

We have then

$$\sum_l ((1)) + ((2)) - ((3)) - ((4)) = ((1))_{l=n-1} - ((3))_{l=n_1+1} + ((2))_{l=n_1+1} - ((4))_{l=n-1} \,.$$

The two permutations with positive sign are equal to $\pi(b)$, while the other two have a fixpoint (and a cycle of length $n-1$), and (with (39) in mind) the result follows. $\square$

With Lemma 5.4, Theorem 5.1 is proved, which implies that infinitely many b_m are non-conjugate. The full statement of Theorem 1.2 is completed in §6 with Proposition 6.1. $\square$

Remark 5.5. It follows from Theorem 1.2 that if $b \in B_n$ satisfies $\pi(b)(n) \neq n$, then the braids $\Delta^{2k}_{[1,n-1]} b \Delta^{-2k}_{[1,n-1]}$ are distinct (though conjugate) for any two different $k \in \mathbb{Z}$. (Compare with Theorem 2.3.)

6. THE NUMBER OF CONJUGATE EXCHANGED BRAIDS

To conclude the proof of Theorem 1.2, we show the following quantitative amplification of the infinite non-conjugacy properties of b_m. (We noted that Ito's method [12] only yields $\xi < \infty$ for (55).)

Proposition 6.1. *Under the assumption of Theorem 1.2, for every fixed conjugacy class $\mathcal{E} \subset B_n$, the number of m with in $b_m \in \mathcal{E}$,*

$$\xi \;=\; |\{ m \in \mathbb{Z} : b_m \in \mathcal{E} \}|, \tag{55}$$

satisfies the bound, for a suitable cycle C of $\pi(b)$,

$$\xi \;\leq\; \begin{cases} 1 & \text{if } 1,n \in C \text{ and } |C| \text{ is even} \\ 2 & \text{otherwise} \end{cases}. \tag{56}$$

Also, all b_m for $m \geq 0$ are pairwise non-conjugate, or all b_m for $m \leq 0$ are pairwise non-conjugate. That is, for

$$\xi_{+,\mathcal{E}} \;=\; |\{ m \geq 0 : b_m \in \mathcal{E} \}| \quad \text{and} \quad \xi_{-,\mathcal{E}} \;=\; |\{ m \leq 0 : b_m \in \mathcal{E} \}|,$$

we have that $\xi_{+,\mathcal{E}} \leq 1$ for every $\mathcal{E}$, or $\xi_{-,\mathcal{E}} \leq 1$ for every $\mathcal{E}$.

Proof. For knots $\hat{b}_m$ we have (56), because we prove in [22] that some conjugacy invariant $\eta(b_m)$ of b_m behaves as a linear or quadratic function in m (*cf.* the observation at the end of §3). Such a function will admit a value at most twice, and also be strictly monotonous for all $m \geq 0$, or for all $m \leq 0$.

For the cases we used b_m^2 for knots in [22], recall that n is odd (compare below (24)) and observe that $\eta(b_m^2)$ can of course also be regarded as a conjugacy invariant of b_m.

Now consider links $\hat{b}_m$, and we will adapt the argument for knots. We return to the proofs of theorems 4.2 and 5.1.

Consider first theorem 5.1, covering the case that $1 \in C$ and $n \in C'$ for two distinct cycles $C \neq C'$ of $\pi(b)$. We consider the conjugacy invariant η of $b \in B_n$ given by

$$\eta(b) := \sum_{C_1,C_2} a_4(L_{b_{[C_1 \cup C_2]}}), \tag{57}$$

where the sum runs over unordered pairs of distinct cycles $C_{1,2}$ of $\pi(b)$.

Now observe that when $\{1,n\} \not\subset C_1 \cup C_2$, then the exchange move trivializes to a conjugacy, and $b_{[C_1 \cup C_2]}$ and $(b_m)_{[C_1 \cup C_2]}$ are conjugate for all m, and similarly are $(\overline{b}_m)_{[C_1 \cup C_2]}$.

Thus the contribution of all (C_1,C_2) to $\eta(b_m)$ will be constant in m, except for the one pair with $1 \in C_1$ and $n \in C_2$, i.e., $(C_1,C_2) = (C,C')$. What happens for this pair was the subject of the proof of theorem 5.1. It follows that at least one of the two leading coefficients $[\eta(b_m)]_{m^2}$ and $[\eta(\overline{b}_m)]_{m^2}$ is non-zero.

Now consider theorem 4.2. Let C be the cycle of $\pi(b)$ with $1,n\in C$. We argued against $|C|=2$, so $|C|\geq 3$. If $|C|>3$, we can use the proof of theorem 3.1 (as summarized in §3). Consider

$$\eta_1(b) := \sum_{C_1} a_3(L_{b_{[C_1]}})\,,$$

which is linear in m, and unless it is constant in m, will show $\xi\leq 1$. When $|C|$ is even (and *a forteriori* $|C|\geq 4$), we can deduce from the knot case above that $\eta_1(b_m)$ is non-constant. (We could have stated in (56) and used the more restrictive, but more technical, condition (25) instead.) If $\eta_1(b_m)$ is constant in m, then evaluate on b_m the conjugacy invariant

$$\eta_1^{[2]}(b) := \eta_1(b^2) = \sum_{C_1} a_3(L_{(b^2)_{[C_1]}})\,,$$

for the sum running over cycles C_1 of $\pi(b^2)=\pi(b)^2$. Since $|C|$ is odd, $\pi(b^2)=\pi(b_m^2)$ have again $1,n\in C$ in the same cycle C. Then, because of the centrality of the full twist, for any fixed cycle $C'\neq C$ of $\pi(b_m^2)$, the braids $(b_m^2)_{[C']}$ are in fact equal for all m. It should be clear how to complete the argument.

Finally, let $|C|=3$ (with $1,n\in C$). Then we use η from (57). The contribution of (C_1,C_2) to $\eta(b_m)$ is constant in m unless w.l.o.g. $C=C_1$. Thus assume $C=C_1$. Now combine Lemmas 4.5, 4.6 and 4.7, with the attention to the meaning of 'n' therein being here $|C_1\cup C_2|=|C|+|C_2|=3+|C_2|$. This shows that the leading coefficient

$$[\eta(b_m)]_{m^2} = -\sum_{C_2\neq C} |C_2|^2 \neq 0\,,$$

as desired.

Because of the (at most) quadratic behavior of the η-s in m, all conjugate pairs $(b_m,b_{m'})$ (for $m\neq m'$, if such exist) must have the same $m+m'$. In particular, $\{\,b_m\,:\,m\geq 0\,\}$ are pairwise (distinct and) non-conjugate, and conjugate pairs in $\{\,b_m\,:\,m<0\,\}$ can only be finitely many, or the other way around. □

Remark 6.2. The condition of equal $m+m'$ (called 'subsymmetry' in [28] and considered there in detail) appears *a priori* to be somewhat artificial, transpiring from our method of proof. But in fact this turns out not to be the case at all. Several examples show that the symmetry indeed occurs, i.e., there exist b and $\mu\in\mathbb{Z}$ such that b_m and $b_{m'}$ are conjugate whenever $m+m'=\mu$. Specifically, for odd μ there is a natural construction for all $n\geq 4$. This is discussed in [27].

7. Other braids

The braids b with (11) in Theorem 1.2 are more difficult, and connected to several instances of failure of the exchange move.

Note that the exchange move in Figure 3 is trivial when the leftmost strand of α (or the rightmost strand of β) are isolated, i.e.,

$$\alpha\in B_{2,n-1}$$

(for $B_{2,n-1}$ from (6)). This immediately explains the conditions (1) Theorem 1.2 puts on $\pi(b)$ (and the remark at the end of the introduction). We observed this failure to extend to braids b with

$$\alpha \in \langle\kappa\rangle \cdot B_{2,n-1}, \tag{58}$$

for κ in (8), since this element commutes with $B_{2,n-1}$. (Angle brackets, as in (2) and (6), and unlike (54), should mean 'generated by'.)

We did not know if under exclusion of these cases (and the analogous conditions on β), the move can always yield infinitely many conjugacy classes. We understand now that this is true, but the argument is discussed in the second author's separate account [27].

However, we were aware of constructions like Stanford's [23] that allow one to 'approximate' these cases of failure by others which cannot be distinguished by any number of Vassiliev invariants (including coefficients of ∇). With this insight in advance, one must at least be careful about what conditions would allow some similar approach to distinguish the result of exchange moves applied on braids with (11). There is, though, a self-contained condition satisfied by all braids obtained from Stanford's construction applied on (58):

$$\text{strand 1 in } \alpha \text{ must have equal linking number with all strands } 2,\dots,n-1. \tag{59}$$

It is tempting to expect that under exclusion of this situation, and its analogue for β, one can always use the Conway polynomial to distinguish L_{b_m}. Our work in [28] implies that this is the case under excluding a situation only slightly more general than (59), when one allows for cables of the braid axis.

At least in one case with (11) an exact calculation is feasible.

Proposition 7.1. *Let $b \in B_n$ with* (11) *admit an exchange move and $\pi(b) = (n\ x_1\ \cdots\ x_{n-2})$ with $2 \le x_i \le n-1$. Assume further w.l.o.g. that α is a pure braid, and let lk_j be the linking number between strands* 1 *and j in α for $j = 2,\dots,n-1$. Then $a_4(L_{b_m})$ is a linear progression in m, and it is non-trivial if and only if*

$$\sum_{i=1}^{n-2} (2x_i - n - 1)\, lk_{x_i} \neq 0\,. \tag{60}$$

Proof. Again by rearrangement, we can put w.l.o.g.

$$\pi(b) = (n\ n-1\ n-2\ \cdots\ 2), \text{ i.e., } x_j = n-j. \tag{61}$$

Then we will calculate that, up to sign (which, of course, does not depend on m),

$$a_4(L_{b_m}) - a_4(L_{b_{m-1}}) = \sum_{j=2}^{n-1} lk_j \cdot (2j - n - 1)\,. \tag{62}$$

This result is a (somewhat tedious) modification of the evaluation of $a_4(L_{b_m}) - a_4(L_{b_{m-1}}) = a_3(L^{n-1}) - a_3(L^0)$ summarized in §3. We give only a few details.

The pictures in §3 remain valid, except that the connectivity of strands within α and β is different. We have an extra component, so instead of Lemma 3.2, we need to apply Lemma 2.6.

With lk_j defined above, for $j = 2, \dots, n-1$, let

$$lk_0 = \sum_{j=2}^{n-1} lk_j \tag{63}$$

be the total linking number of strand 1 in α.

The change in a_4 resulting from Δ_0 becomes instead of (23)

$$a_4(L^1) - a_4(L^0) = (n-2)(1+lk_0) + lk_0,$$

and (24) modifies for the change from Δ_i to become (with $i = j-1$ now)

$$a_4(L^{i+1}) - a_4(L^i) = (2j-n-2)(1+lk_0) + lk_2 + \dots + lk_{j-1} - lk_j - \dots - lk_{n-1}\,.$$

Then, using (63) and $\sum_{j=2}^{n-1}(2j-n-1) = 0$, we have

$$\begin{aligned} a_4(L^{n-1}) - a_4(L^0) &= (n-2)(1+lk_0) + lk_0 + \\ &\quad + \sum_{j=2}^{n-1}(2j-n-2)(1+lk_0) + \sum_{j=2}^{n-1}(n-2j)lk_j \\ &= \sum_{j=2}^{n-1}(2j-n-1)(1+lk_0) + \sum_{j=2}^{n-1}(n-2j+1)lk_j \\ &= -\sum_{j=2}^{n-1}(2j-n-1)lk_j\,, \end{aligned}$$

agreeing with (62). □

Combining with Theorem 2.3 allows us to complement Theorem 1.1 by one more self-contained case.

Corollary 7.2. *The assertion of Theorem 1.1 holds for two-component links L with even braid index and odd linking number.*

Proof. We mentioned in the introduction that [7] deals with $n \le 3$ (both sides of the equivalence are false), thus let $n \ge 4$. When $n > b(L)$, we can use Theorem 2.3, as explained. Choose $\beta = \sigma_{n-1}^{\pm 1}$, and the statement follows (with both sides of the equivalence true), unless α is central. But central braids are pure, and then the link $L = \hat{b}$ would have $n-1 \ge 3$ components.

Thus assume $n = b(L) \ge 4$, and n is even. By using Theorem 2.2 (as in the proof proof of Theorem 1.1), again we remain to show that the exchange move gives infinitely many non-conjugate b_m. If (11) fails (and similarly for strand n), we can use Theorem 1.2. Now, if (11) holds, for two component links $L = \hat{b}$, we can use Proposition 7.1. Note that the expression on the left of (60) is for even n congruent modulo 2 to lk_0 in (63), which becomes the linking number of (the two components of) L. □

One should observe that the expression on the left of (60) vanishes if all lk_j are equal, which was explained below (58) (and was used to test our calculation). The second author can address equality of all lk_j with a new (but related) method using the Burau matrix. It extends (60) in the form that, under (w.l.o.g.) (61), all b_m are pairwise non-conjugate unless the linking vector $(lk_2, \dots, lk_{n-1})$ is palindromic. This palindromicity underlies the following cautionary example illustrating the difficulties to continue with ∇. Probably some completely different (calculable) conjugacy invariant (or some method like Ito's mentioned in the introduction) may be needed to detect the success of the exchange move in some instances.

Example 7.3. Let $n=6$. We conform here to b_m shown in Figure 3 with $\beta=\sigma_2\sigma_3\sigma_4\sigma_5$ and $\alpha=\sigma_1\sigma_2\sigma_3\sigma_4^2\sigma_3^{-1}\sigma_2^{-1}\sigma_1$ (with linking vector $(1,0,0,1)$). Then as we checked (with the program [10]), $b=b_0$ is conjugate to b_{-1}, but not to b_{-2}. Nevertheless, since any link invariant, incl. ∇, coincides on L_{b_m} for $m=0,-1$, the arguments for Proposition 7.1 easily imply that $m\mapsto a_k(L_{b_m})$ will be constant in m when $k=2,4$. (We computationally checked the coincidence for $m=0,-2$.) Also one can calculate that $a_k(L_{b^2})=a_k(L_{b_{-2}^2})$ for $k=2,4$, which suffices to conclude that $m\mapsto a_k(L_{b_m^2})$ is constant as well. (For the meaning in using squares, see the remark at the end of §3 and note that here $L_{b_m^l}$ is a 3-component link whenever l is not divisible by 5.) Moreover, the lowest two terms $P_k=[P]_{z^k}$ of the skein polynomial P, in z-degrees $k=-2,0$, give $P_k(L_b)=P_k(L_{b_{-2}})$. (We have that P_k is a refinement of a_k, and for a p-component link, generally $P_k\neq 0$ already when $k\geq 1-p$.) One manifestation of the non-conjugacy is in $P_0(L_{b^2})\neq P_0(L_{b_{-2}^2})$. (The coincidence of P_{-2} will always occur for reasons similar to those for a_2.) Standard Vassiliev invariant arguments (outlined in §2.4, and applied for Proposition 6.1) then show that except for finitely many m, at most two b_m can be mutually conjugate.

8. Links with trivial components

We formulate an effectively verifiable condition for links with trivial (that is, unknotted) components to which we can apply our methods. When U is a trivial component of L, write $L_{U,k}$ for a link obtained by taking in L a k-cable of U with pattern any k-braid. (The k-braid is regarded as lying in a solid torus given by the complement of its braid axis; see, e.g., [30] for more details on this kind of construction.) In particular, any framing is allowed, and the cable of the component need not be connected (i.e., $L_{U,k}$ may have more components than L). The following can be regarded as a generalization of Theorem 1.1.

Proposition 8.1. *Let L be a link with the following property: whenever U is a trivial component, then there is a $k>1$ and a link $L'=L_{U,k}$ such that $b(L')\geq b(L)+k$. Then the assertion of Theorem 1.1 holds for L.*

Proof. If $n>b(L)$, Theorem 2.3 will apply, unless L is a torus link of the type (m,lm). But such links do not have the assumed property. This can be seen because a minimal (m-)string representative exhibits all components as closures of 1-string subbraids.

Thus take now $n=b(L)$. The assumed property of L is precisely a way to exclude that any n-braid representative has a (trivial) component as a closure of a 1-string subbraid. (Otherwise, we readily have a $b(L)+k-1$-string representative of L'.) Thus we can apply Theorem 1.2. □

Example 8.2. Take any minimal (e.g., alternating [19]) braid representative β of a link $L^*=\hat{\beta}=K\cup U$ with two components, one knotted K, and one unknoted U, such that U is the closure of a subbraid of at least 2 strings of β. For example, $\beta=\sigma_1^3(\sigma_3\sigma_2^{-1})^3\sigma_3\in B_4$. Let $L=L^*_{K,k}$ be obtained from L^* by a fully disconnected k-cable ($k>1$) along the knotted component (so that L has $k+1$ components). Then any fully disconnected k-cable $L'=L_{U,k}$ (of $2k$ components) of L along its unknotted component makes L satisfy the premise of Proposition 8.1. This is very easy to see by a sublink argument related to the multiplicativity of the braid index $b(L') = kb(L^*)$ under fully disconnected cabling (of every component of L^*).

This construction was chosen for illustration here by virtue of providing (like almost throughout this paper) for a computer-independent reasoning. (One can readily replace K by multiple knotted components in L^*, etc.) With electronic help (properly estimating $b(L')$ using the Morton-Franks-Williams inequality, for example) one could certainly add many more instances. It is presumable that this condition will be generically satisfied by sufficiently complicated links L (with trivial components), even although it fails in many simple cases.

9. Composite links

As a consequence of Theorem 4.2, we obtain the following result in [24].

Corollary 9.1. *Let L be a composite link of braid index $b(L) \geq 4$, which factors as $L_1\#L_2$ in such a way, that either components $K_{1,2}$ of $L_{1,2}$ the connected sum is performed at are not unknots that appear as 1-string subbraids in every minimal braid representative of L_i. Then L has infinitely many non-conjugate minimal braid representatives.*

E.g., knotted K_i, or K_i satisfying the condition on U in Proposition 8.1 will do. In particular, the corollary always applies for a composite *knot L.*

Proof. By the 1-subadditivity of the braid index under connected sum proved by Birman and Menasco [5], L has a composite minimal braid representative b, of the sort illustrated in Figure 17 (where $\hat{b}_i = L_i$). Such a representative admits an exchange move if it has $n = b(L) \geq 4$ strands. By assumption, the component of the common strand of b_1 and b_2 can be chosen to have at least one other strand in either of these. By conjugation of b_i it can be made to be strand 1 and n (in b), so that the cycle condition of Theorem 4.2 also holds. □

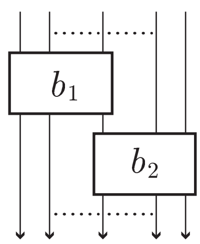


Figure 17. A composite braid

10. Problems

Apart from the difficulties discussed in §7, we conclude with two more remaining problems.

Problem 10.1. Theorem 1.2 suggests to seek braids admitting exchange moves, but the identification what links have such (minimal) braids is still difficult.

A partial study for alternating links is being worked out in [29].

Problem 10.2. We do not say anything about *Markov irreducible* $b \in B_n$ with $n > b(L)$, i.e., such not conjugate to stabilizations $b\sigma_{n-1}^{\pm 1}$ for $b \in B_{n-1}$.

Only few constructions of Markov irreducible braids are known. Morton and Fiedler [16, 8] gave examples for $n = 4$ and $K = \hat{b}$ being the unknot. In [14], some conditions using Dehornoy's ordering were given on a braid not to be a stabilization, or admit exchange moves, etc. It should be noted, though, that this ordering is not conjugacy invariant, so the conditions are not effectively testable on an entire conjugacy class. In any event, these methods require further background, and thus, also for length reasons, we may investigate their merits at a separate place.

The general problem to describe the conjugacy classes for a given link exactly, even when $n = 4$, remains a problem with no reachable (and likely no meaningful) solution. Even for $n = 3$, a simplification of (or alternative to) Birman and Menasco's substantial work has not been found for decades (except in special cases).

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References

[1] Alexander, J. W., *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. U.S.A. **9**, 93–95 (1923).
[2] Artin, E., *Theory of braids*, Ann. of Math. **48**, 101–126 (1947).
[3] Bar-Natan, D., *On the Vassiliev knot invariants*, Topology **34**, 423–472 (1995).
[4] Birman, J. S., *Non conjugate braids can define isotopic knots*, Comm. Pure Appl. Math. **22**, 239–242 (1969).
[5] Birman, J. S. and Menasco, W. W., *Studying links via closed braids IV. Composite links and split links*, Invent. Math. **102**, no. 1, 115–139 (1990).
[6] Birman, J. S. and Menasco, W. W., *Studying links via closed braids VI. A nonfiniteness theorem*, Pacific J. Math. **156**, no. 2, 265–285 (1992).
[7] Birman, J. S. and Menasco, W. W., *Studying links via closed braids III. Classifying links which are closed 3-braids*, Pacific J. Math. **161**, no. 1, 25–113 (1993).
[8] Fiedler, T., *A small state sum for knots*, Topology **32**, no. 2, 281–294 (1993).
[9] Garside, F., *The braid group and other groups*, Q. J. Math. Oxford **20**, 235–264 (1969).
[10] González-Meneses, J., *Braiding: A C++ program for computations in braid groups*, `https://personal.us.es/meneses/software.php`
[11] Hoste, J., *The first coefficient of the Conway polynomial*, Proc. Amer. Math. Soc. **95**, no. 2, 299–302 (1985).
[12] Ito, T., *A non-degenerate exchange move always produces infinitely many non-conjugate braids*, Nagoya Math. J. **243**, 205–208 (2021).
[13] Jones, V. F. R., *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126**, 335–388 (1987).
[14] Malyutin, A. and Netsvetaev, N., *Dehornoy ordering on the braid group*, St. Petersburg Math. J. **15**, no. 3, 437–448 (2004).

[15] Morton, H. R., *Infinitely many fibred knots having the same Alexander polynomial*, Topology **17**, no. 1, 101–104 (1978).
[16] Morton, H. R., *An irreducible* 4*-string braid with unknotted closure*, Math. Proc. Cambridge Philos. Soc. **93**, no. 2, 259–261 (1983).
[17] Morton, H. R., *Threading knot diagrams*, Math. Proc. Camb. Philos. Soc. **99**, 247–260 (1986).
[18] Murakami, H. and Nakanishi, Y., *On a certain move generating link-homology*, Math. Ann. **284**, 75–89 (1989).
[19] Murasugi, K., *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326**, no. 1, 237–260 (1991).
[20] Murasugi, K. and Thomas, R. S. D., *Isotopic closed nonconjugate braids*, Proc. Amer. Math. Soc. **33**, 137–139 (1972).
[21] Polyak, M. and Viro, O., *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Notes **11**, 445–454 (1994).
[22] Shinjo, R. and Stoimenow, A., *Exchange moves and non-conjugate braid representatives of knots*, Nagoya Math. J. **240**, 298–321 (2020).
[23] Stanford, T., *Braid commutators and Vassiliev invariants*, Pacific Jour. of Math. **174**, no. 1, 269–276 (1996).
[24] Stoimenow, A., *Non-conjugate braids with the same closure link from density of representations*, Journal de Mathématiques Pures et Appliquées **94**, no. 5, 470–496 (2010).
[25] Stoimenow, A., *Application of braiding sequences I: On the characterization of Vassiliev and polynomial link invariants*, Comm. Contemp. Math. **12**, no. 5, 681–726 (2010).
[26] Stoimenow, A., *Positive knots, closed braids and the Jones polynomial*, math/9805078, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **2**, no. 2, 237–285 (2003).
[27] Stoimenow, A., *Exchangeability and non-conjugacy of braid representatives*, International Journal of Computational Geometry and Applications **31**, no. 1, 39–73 (2021).
[28] Stoimenow, A., *Subsymmetric exchanged braids and the Burau matrix*, to appear in Proceedings Roy. Soc. Edinburgh, Section A, https://www.doi.org/10.1017/prm.2023.1.
[29] Stoimenow, A., *Exchanged braids for alternating links*, preprint, http://www.stoimenov.net/stoimeno/homepage/papers.html.
[30] Stong, R., *The Jones polynomial of parallels and applications to crossing number*, Pacific J. Math. **64**, no. 2, 383–395 (1994).

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