GRAPHS, DETERMINANTS OF KNOTS AND HYPERBOLIC VOLUME

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. Using estimates for the number of spanning trees of planar graphs, we prove inequalities for the determinant of alternating links in terms of their hyperbolic volume.

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1. Introduction

If Δ_L denotes the (1-variable) Alexander polynomial [Al] of a link L in S^3, then $\det(L) = |\Delta_L(-1)|$ is the order of the homology group $H_1(D_L)$ (over $\mathbb{Z}$) of the double branched cover D_L of S^3 over L (or 0 if this group is infinite) and is often called "determinant" because of its expression (up to sign) as the determinant of a Seifert [Ro, page 213] or Goeritz [GL] matrix. The homology group carries much interesting information on the link (in particular conditions for sliceness [Ro] and achirality [HK, St], and unknotting number estimates [We]). The determinant can be alternatively evaluated from the Jones polynomial V [J] as $|V(-1)|$, and for alternating diagrams has a nice combinatorial interpretation (see [Kr]) in terms of the Kauffman state model [Ka2].

In [Df], Dunfield proposed (although without a very exact statement) interesting relations between the determinant and the hyperbolic volume of (complements in S^3 of) alternating knots and links, based on experimental evidence. Roughly speaking, he found that the determinant should have upper and lower bounds, which are exponential in terms of the volume. The main aim of this paper is to prove rigorous versions of Dunfield's conjectured inequalities. To formulate them, here and below we denote by $c(L)$ the crossing number of a link L, and by $\mathrm{vol}(L)$ the hyperbolic volume of (the complement in S^3 of) L, or 0 if L is not hyperbolic.

Theorem 1.1 If L is a non-trivial non-split alternating link, then

$$\det(L) \geq 2 \cdot 1.0355^{\mathrm{vol}(L)}. \tag{1}$$

Theorem 1.2 There are constants $C_1, C_2 > 0$, such that for any hyperbolic alternating link L,

$$\det(L) \leq \left[\frac{C_1 \cdot c(L)}{\mathrm{vol}(L)}\right]^{C_2 \mathrm{vol}(L)}. \tag{2}$$

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It is natural (as in [Df]) to regard such statements also as a relation between Jones polynomial and hyperbolic volume. The motivation for such a viewpoint lies in a variety of connections of the volume to "Jones-type" invariants that has become apparent recently.

One such connection worth remarking on (though beyond our scope here) is the Volume conjecture [MM], which asserts that one can determine the volume *exactly* from the Jones polynomial *and all* its cables. (This conjecture is rather of theoretical importance, since the volume evaluation it proposes is by no means practicable.) On a more related to our result (but likewise not further pursued) venue, Khovanov suggested a possible extension of Dunfield's conjectured correspondence to non-alternating links, if instead of the determinant we take the total dimension of his homology generalizing the Jones polynomial [Kh].

There is also accumulating evidence that the (ordinary) Jones polynomial might be able to provide in a different way (very practical) bounds on the volume. Such bounds, which involve degrees or coefficients of the polynomial, have been obtained for alternating links by Dasbach and Lin [DL] and later for Montesinos and 3-braid links by myself [St6]. Contrarily, the question whether one can (unboundedly) augment hyperbolic volume but preserve the Jones polynomial (or at least its Mahler measure) remains open.

Theorems 1.1 and 1.2 improve closely related, but unpublished, previous work of Chris Leininger and Ilya Kofman. They seem qualitatively close to the optimum, modulo the determination and improvement of constants. Both theorems apply the twist number of link diagrams. Theorem 1.2 is easier to deduce (with constants C_1, C_2 which are easy to make explicit; see its proof in §4). It requires a recent lower bound for the volume of alternating links, first due to Lackenby [La] and later with the constant improved by Agol, Storm and W. Thurston [AST]. Contrarily, Theorem 1.1, although similar in nature, requires considerably more effort, and it is the main content of this paper.

Our proof of Theorem 1.1 will build most decisively on the useful relationship between knot and graph theory, stating that the determinant of an alternating diagram is the number of spanning trees of a certain planar (checkerboard) graph associated to this diagram. This relationship is well-known and has been used before [C]. However, the only (general) lower estimates on the determinant known so far from it were linear in the crossing number. We need to show that we can make the linear bound exponential, if we replace crossing number by twist number (see Theorem 4.2). This will occupy us though a large part of the paper. Our work can be considered mostly independent from [La], because the upper volume bounds follow from classical facts about hyperbolic volume known before (see for example [Br]). It is only the optimal choice of constants, due to I. Agol and D. Thurston in the appendix of [La], that enters into Theorem 1.1 for a better estimate. The same comment applies on Proposition 6.2, which uses the proof of Theorem 1.1 to improve the inequality in the special case of arborescent (Conway-algebraic) links.

Some other work that relates the volume to the Alexander polynomial was done by Kalfagianni [Kf] and Silver and Whitten [SWh], and to the Jones polynomial jointly with Silver and Williams [SSW]. The growth of the order of the homology groups of the higher order cyclic branched covers of S^3 over a fixed link L has been studied by Gordon [Go], and later by González-Acuña and Short [GS] and Riley [Ri]. These numbers can still be determined from the Alexander polynomial, but lack a nice combinatorial description, and are rather of number-theoretical interest. (Riley's growth estimates in fact use some of the deepest tools from number theory.) Their growth rate is related to questions on entropy and Mahler measure.

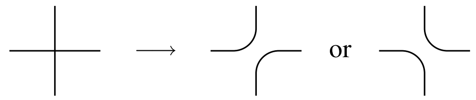
2. The determinant and twist number of alternating diagrams

Before we come to the proof of our results, we will briefly review the necessary, and previously initiated, framework to incorporate them into.

Via the relation $\Delta(-1) = V(-1)$ to the Jones polynomial (see [J2, §12]) the determinant provides a bridge between the classical Alexander polynomial and its modern successors [BLM, F&, Ka, J], whose nature is rather combinatorial. It is among the little topologically understandable information encoded in these more recent invariants. On the other hand, this relation opens combinatorial approaches for calculating the determinant.

One such approach, which is particularly nice for alternating diagrams, was given by Krebes [Kr] using the Kauffman bracket/state model for the Jones polynomial.

If D is an alternating link diagram, then consider $\hat{D} \subset \mathbb{R}^2$, the (image of) the associated immersed plane curve(s). Then $\det(D)$ is equal to the number of ways to splice the crossings (self-intersections) of $\hat{D}$



(3)

so that the resulting collection of disjoint circles has only one component (a single circle); such choices of splicings are called in [Kr] *monocyclic states*.

Definition 2.1 The diagram on the right of figure 1 is called *connected sum $A\#B$* of the diagrams A and B. If a diagram D can be represented as the connected sum of diagrams A and B, such that both A and B have at least one crossing, then D is called *composite*; otherwise it is called *prime*. Equivalently, a diagram is prime if any closed curve intersecting it in exactly two points does not contain a crossing in one of its interior or exterior.

Any diagram D can be written as $D_1\#D_2\#\cdots\#D_n$, so that all D_i are prime and have at least one crossing. Then D_i are called the *prime factors* of D. A link L is prime if for any composite diagram $D_1\#D_2$ of L, exactly one of $D_{1,2}$ is an unknot diagram. (Thus the unknot is not prime per convention, though its 0 crossing diagram is prime.)

Definition 2.2 A diagram D is *split*, or *disconnected*, if there is a closed curve which does not intersect D, but which contains parts of D in both its interior and exterior. Otherwise D is called *connected*. A link is split if it has a split diagram.

So a diagram D is connected iff its plane curve $\hat{D}$ is a connected set. Therefore, with the above explanation an alternating diagram has non-zero determinant if and only if it is connected.

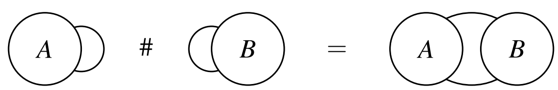


Figure 1

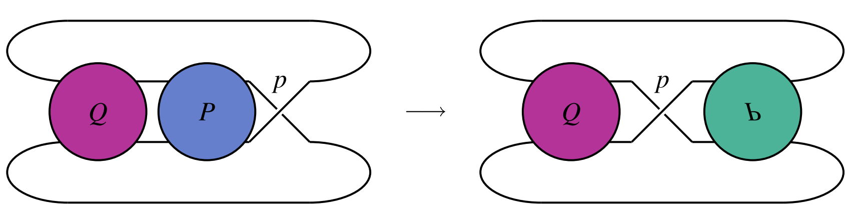
Definition 2.3 A crossing q in a link diagram D is called *nugatory* if there is a closed (smooth) plane curve intersecting D transversely in q and nowhere else. A diagram is called *reduced* if it has no nugatory crossings.

Theorem 2.1 ([Me]) An alternating reduced diagram of a link L is prime iff L is prime.

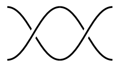
Definition 2.4 Let $c(D)$ be the *crossing number* of a link diagram D. Let $c(L)$ be the crossing number of a link L, which is the minimal crossing number of all diagrams D of L. A link L is *trivial* if $c(L) = 0$.

Theorem 2.2 ([Ka2, Mu, Th2]) Each alternating reduced diagram is of minimal crossing number (for the link it represents).

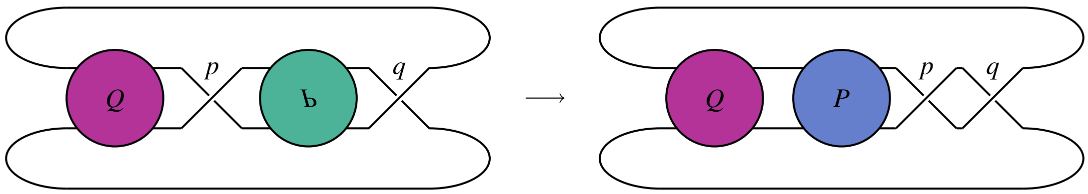
To explain the inequalities of Lackenby-Agol-Thurston, we must introduce the notion of twist equivalence of crossings. The version of this relation we present here is slightly different from that of [La]. First we introduce a diagram move called *flype*, shown on the following picture:



(4)

A *clasp*  is a tangle that consists of two crossings, that is, a pair of crossings that bound a bigonal (2-corner) region in the plane complement of a diagram.

Definition 2.5 Let D be a reduced link diagram. We say two crossings p and q of D to be *twist equivalent* if there exists a flype making p and q into a clasp.



(5)

(Note that this formulation assumes a certain correspondence between the crossings in the diagrams on both hand sides of (4), but this correspondence is quite obvious.) We call this relation *twist equivalence*. Let $t(D)$ be the *twist number* of a diagram D, which is the number of its twist equivalence classes. The twist number $t_{\min}(K)$ of a knot or link K is the minimal twist number of any diagram D of K.

For *knot* diagrams an alternative definition of twist equivalence, which follows its independently studied variants in [St4, St5], goes thus: Call two crossings p,q in a knot diagram *linked* if their crossing points are passed in cyclic order $pqpq$ (rather than $ppqq$) along the diagram plane curve. Then p and q are twist equivalent iff each crossing $r \neq p,q$ is linked with p exactly if it is so with q.

In [MT, MT2] it was proved that alternating diagrams of the same link are related by flypes. Since our definition of twist equivalence is invariant under flypes, we see that the twist number is independent of the choice of alternating diagram. So we can make

Definition 2.6 For an alternating link L, let the *twist number* of L be $t(L) = t(D)$, where D is some alternating diagram of L.

Note that for an alternating knot the (obvious) inequality $t_{\min}(K) \leq t(K)$ may be strict (that is, not an equality); an easy example is the knot 7_4. In other words, there are alternating knots with non-alternating diagrams of *smaller* twist number than the twist number of their alternating diagram(s). In particular, minimal twist number and minimal crossing number (see Theorem 2.2) may not be simultaneously attainable by a diagram.

Our notion of twist equivalence is slightly more relaxed than what was called this way in [La], the difference being that there flypes were not allowed. We call Lackenby's equivalence here *strong twist equivalence*. It was repeatedly observed that (and (5) indicates how) by flypes all twist equivalent crossings can be made strongly twist equivalent, which Lackenby formulated as the existence of *twist reduced* diagrams. Thus we can work with twist equivalence in our sense as with twist equivalence in Lackenby's sense (or strong twist equivalence in our sense), provided we can assume that the alternating diagram is twist reduced. With this remark, we can state the inequalities (into which work of Agol, Lackenby, Storm, D. Thurston and W. Thurston goes in) as follows:

Theorem 2.3 ([La, AST]) For a nontrivial prime alternating link L, we have

$$10V_0\big(t(L)-1\big) \geq \mathrm{vol}(L) \geq V_8\big(t(L)-2\big),$$

where $V_0 = \mathrm{vol}(4_1)/2 \approx 1.01494$ is the volume of the ideal tetrahedron, and $V_8 \approx 1.83193$.

Both constants are optimal; the '10' on the left is approximated asymptotically by links consisting of a certain iterative pattern, described in the appendix of [La]. The right constant V_8 makes the inequality sharp for the Borromean rings.

3. Graphs

3.1. General preliminaries

Below it will be helpful for us to translate the statements we like to prove about link diagrams into statements about planar graphs. As graphs occur throughout the rest of the paper, we first need to set up some related terminology, most of which (like *dual graph*, *deleting* or removing edges, etc.) is standard, and also explained in [MS]. For clarification, we repeat a part of these definitions, and collect several simple facts that will be used later. Some new notions seem also necessary to introduce.

Definition 3.1 A *loop edge* is an edge connecting a vertex with itself. A *multiple edge* e is the set of all n edges connecting the same pair of (distinct) vertices; we assume $n \geq 2$. The number n is called *multiplicity* of e. Edges that belong to the same multiple edge (that is, connect the same 2 vertices) are called *parallel*. A graph is called *simple* if it has no multiple edges, and *loop-free* if it has no loop edges.

Definition 3.2 For a graph G, let $E(G)$ respectively $e(G)$ be the set respectively number of edges. An edge of multiplicity n is understood to contribute n to $e(G)$. We call G *trivial* if $e(G) = 0$. An edge between vertices v and w, which is not given a separate name, will be written as vw.

We will assume that graphs are connected and have no loop edges and vertices of valence 1, unless the opposite is clear.

Definition 3.3 A cycle-free graph is a *forest*. A connected forest is a *tree*. If G is a connected graph of n vertices, a *spanning tree* of G is a tree contained in the edges of G, which is maximal, that is, has $n-1$ edges. Let for a graph G by $s(G)$ be denoted the number of its spanning trees (or 0 if G is not connected).

Definition 3.4 If G is a graph, then a set of edges $\{e_1,\dots,e_n\}$ of G is an *n-cut* (or simply *cut*) if the subgraph obtained by removing $e_1,\dots,e_n$ is disconnected. A 1-cut is called an *isthmus*. We say G is *n-connected* if it has no k-cut for $k < n$. Note that if G has more than one vertex, then the set of all edges incident to a vertex forms a cut. A *cut vertex* is a vertex for which the subgraph obtained by removing it (and all incident edges) is disconnected.

Definition 3.5 The *join* $G = G_1 * G_2$ of two graphs G_1 and G_2 is a graph G obtained by identifying a vertex in G_1 and a vertex in G_2, called *join vertices*. (The identified vertex becomes a cut vertex in G. The choice of join vertices has influence on G, but this choice is irrelevant for our purposes.) Every non-trivial graph G can be written as a join $G_1 * \dots * G_n$ for some non-trivial graphs G_i, such that no G_i has a cut vertex. We call G_i the *join factors* of the graph G. If the decomposition is non-trivial (that is, $n > 1$), we call G *decomposable*.

By definition, a graph is 2-connected iff it has no isthmus. The following is an easy observation:

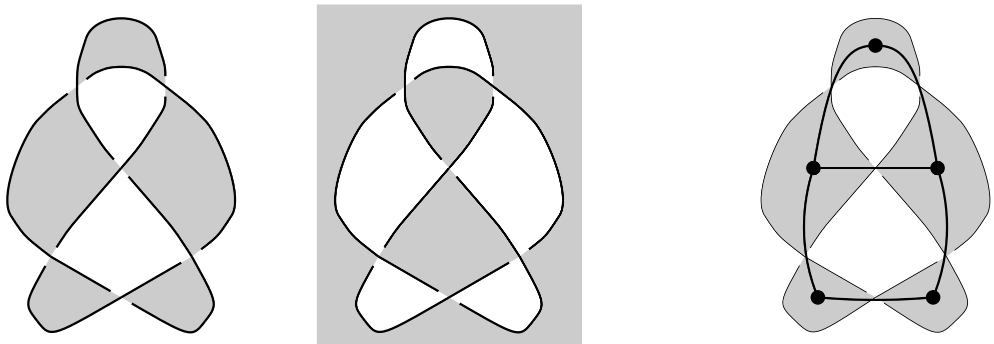
Lemma 3.1 $G_1 * G_2$ is 2-connected iff G_1 and G_2 are so. $G_1 * G_2$ has an isthmus or loop edge iff one of G_1 or G_2 has.
□

Definition 3.6 A graph G is *planar* if it is embedded in the plane. Then for each n-cut $\{e_1,\dots,e_n\}$ of G one can draw a closed curve in the plane, intersecting G transversely only in (single interior points of) the edges $e_1,\dots,e_n$. We call this curve a *cut curve* or *circuit* in G. Clearly the curve of a cut determines the cut uniquely.

Definition 3.7 Assume G is a planar graph. We call a *region* E of G a connected component of the complement of G. Its *boundary* is denoted ∂E and understood as a set of edges of G. If an edge e of G lies in the boundary of regions A and B, then we say that A is *opposite* to B at e, or that A and B are opposite.

Thus G is 2-connected if it is connected and no region is opposite to itself at some edge. Also, G is 3-connected, if it is 2-connected and for each pair of regions there is at most one edge at which they are opposite.

The join, defined so far for abstract graphs, can naturally be extended to planar graphs. (The ambiguity of choice of vertices carries over, and a new one now comes from the planar embedding; but again this will not be a problem here.)



two checkerboard colorings

the checkerboard graph
of the first coloring

Figure 2: Checkerboard coloring and checkerboard graph

Definition 3.8 The *dual* graph of a planar graph G is denoted by G^*. Observe that there is a natural bijection of edges between G and G^*; in that sense we can talk of the dual $e^* \in E(G^*)$ of an edge $e \in E(G)$. Similarly the dual A^* of a region A of G is a vertex of G^*. We consider G^* embedded in the plane in the obvious way: an edge $e \in E(G)$ and its dual e^* meet exactly once (and transversely), and other edges do not meet at all.

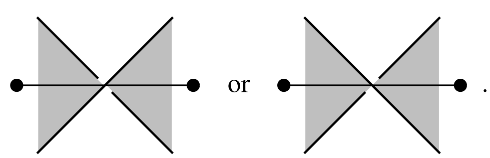
We have the following simple properties of the dual:

Lemma 3.2 For planar graphs G_1 and G_2, and proper choice of join vertices, $(G_1 * G_2)^* = G_1^* * G_2^*$. □

Lemma 3.3 Let G be a planar graph. Then
(i) an edge e of G is an isthmus if and only if the edge e^* in G^* is a loop edge,
(ii) G has a cut vertex iff G^* has one, and
(iii) if G has no isthmus and loop edge, then G is 3-connected iff G^* is simple. □

3.2. Checkerboard graphs

To any alternating link diagram D one can associate its *checkerboard graph* $G = \Gamma(D)$ in the following manner (see figure 2, or also [Ka, DH, Kr, St, Th]). Consider the plane curve $\hat{D}$ of D as a 4-valent planar graph. There is a black-white coloring of the regions of $\hat{D}$, the *checkerboard coloring*, which assigns different colors to each pair of regions opposite at some edge of $\hat{D}$. Then $\Gamma(D)$ is defined to have a vertex for each black region of $\hat{D}$ and an edge for each crossing of D as follows:



(6)

Assuming D is connected, there are only two choices of checkerboard colorings (equivalent up to the interchange of colors) and so $\Gamma(D)$ is uniquely defined *up to duality* (which corresponds to interchanging the colors).

This way the checkerboard graph is a planar graph, in general possibly with multiple edges. Conversely, any such graph is the checkerboard graph of some link diagram. A diagram D is alternating if for all crossings the same one of the two local pictures in (6) occurs. Thus the checkerboard graph determines an alternating diagram up to mirroring. The following is also easy to see:

Lemma 3.4 The checkerboard graph $G = \Gamma(D)$ of a diagram D has no cut vertex if and only if D is prime. More precisely, the connected sum of diagrams corresponds to the join of checkerboard graphs: $\Gamma(D_1\#D_2) = \Gamma(D_1) * \Gamma(D_2)$. $\square$

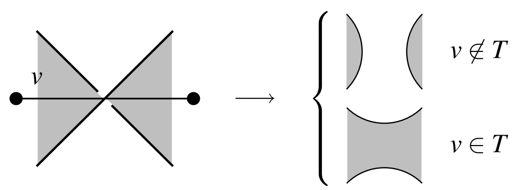
In accordance with the ambiguity of $\Gamma(D)$ up to duality, most of the graph properties we will deal with below will be symmetric with respect to taking the dual graph. We can use this symmetry to simplify our arguments at some point. Also, the operations in (3) correspond to contraction and deletion of an edge in the checkerboard graph. Note furthermore

Lemma 3.5 The property that $G = \Gamma(D)$ is 2-connected and (dually) loop-free is equivalent to D having no nugatory crossings. $\square$

The calculation of the determinant using monocyclic states can be reformulated even more naturally in terms of graphs. This principle, which we will exploit decisively later, states:

Lemma 3.6 If D is an alternating diagram of a link L, and $G = \Gamma(D)$ its checkerboard graph, then $\det(L) = s(G)$.

Proof. There is a bijection between monocyclic states of D and spanning trees T of $\Gamma(D)$, given by:



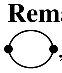
. $\square$

This lemma is well-known, and one of its first applications goes probably back to Crowell [C]. The novel(ty and) additional difficulty below will be to control the extent of the simplification under edge contraction and deletion not only in the checkerboard graph itself but also simultaneously in its planar dual. Lemma 3.6 is also a consequence of Kauffman's state models for the Jones [Ka2] and Alexander polynomial [Ka3], and was discussed extensively in [St, MS]. In this paper, we mostly use the language of [MS]. We note, however, that $s(G)$ is written in [MS] as $\Delta_G(1)$, and in [St] as $t(G)$ (a notation which will be used with a different meaning later in this paper).

3.3. Twist number of graphs

The inequalities of Lackenby-Agol-Thurston are essential in establishing a link between graph theory and the volume. Since, in applying Theorem 2.3, we want to pass from an alternating diagram D to its checkerboard graph G, it is useful to remark how twist equivalence translates from D to G. The aim of the next definition is to establish this translation. To describe it more precisely, we use some of the above introduced terminology on graphs. Later we need a few more remarks and notions for the analysis of this relation.

Definition 3.9 Let G be a planar 2-connected graph with no loop edges. We call two (simple) edges e and f of G *twist equivalent* if
(i) e and f are parallel, or
(ii) $\{e, f\}$ is a 2-cut of G.
The *twist number* $t(G)$ of G is the number of twist equivalence classes of its edges.

Remark 3.1 For $e \neq f$ only one of the two alternatives (i) and (ii) is possible, except if e and f are the two edges of , occurring as a join factor of G.

Lemma 3.7 If D is a link diagram and $G = \Gamma(D)$ its checkerboard graph, then $t(D) = t(G)$.

Proof. This is more or less by definition of $t(G)$. Look at the diagram on the left of (5). It is easy to see that p and q are in the boundary of a common pair of regions (below and above the crossings in the diagram of (5)). Depending on whether these regions are colored black or white in the checkerboard coloring, we obtain the two alternatives in Definition 3.9. $\square$

Lemma 3.8 Let G be a planar graph, and G^* its dual. Then $s(G) = s(G^*)$, and if G is loop-free and 2-connected, $t(G) = t(G^*)$.

Proof. The coincidence of s already follows from Lemma 3.6, the ambiguity of $\Gamma(D)$ up to duality, and the fact that the determinant is a link invariant. However, there is a much more appropriate way to see this equality. It is a natural bijection between spanning trees of G and G^*, which is described as follows. Let T be a spanning tree of G. To construct the corresponding spanning tree T^* of G^*, include an edge e^* of G^* into T^*, if and only if e is *not* contained in T.

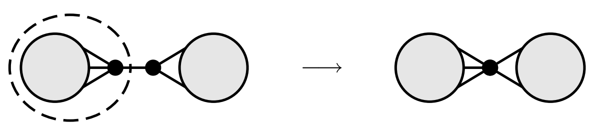
As for twist equivalence, note that duality precisely interchanges the conditions (i) and (ii) in Definition 3.9. That is, two edges e and f in G satisfy (i) (respectively (ii)) iff e^* and f^* satisfy (ii) (respectively (i)) in G^*. $\square$

Lemma 3.9 The number of spanning trees is multiplicative under join, $s(G_1 * G_2) = s(G_1) \cdot s(G_2)$. The twist number is additive under join of 2-connected loop-free graphs, $t(G_1 * G_2) = t(G_1) + t(G_2)$.

Proof. For the first claim, note that T is a spanning tree of $G_1 * G_2$, if and only if $T \cap G_i$ is a spanning tree of G_i $(i = 1, 2)$.

For the second claim, observe that if two edges are parallel in $G_1 * G_2$, they must be parallel in one of G_1 or G_2. If two edges $e_{1,2}$ in $G_1 * G_2$ form a 2-cut, and belong both to one of G_1 or G_2, they form a 2-cut in that graph too. If e_1 is in G_1 and e_2 is in G_2, then they form isthmuses therein, which we excluded. The exclusion of loop edges is needed for "dual" reasons. $\square$

Definition 3.10 For a graph G and an edge $e \in E(G)$, we write G/e for the contraction of e in G, followed by the deletion of loop edges. Similarly we write $G \setminus e$ for the graph obtained from G by deletion of e, and subsequent contraction of all resulting isthmuses:



Note again that these operations as dual to each other; $(G/e)^* = G^* \setminus e^*$.

Lemma 3.10 $s(G) = s(G/e) + s(G \setminus e)$.

Proof. Each spanning tree T of G either contains e or it does not. Exactly in former case it descends to a spanning tree G/e, and exactly in latter case to one of $G \setminus e$. Contracting isthmuses and deleting loop edges clearly does not affect the number of spanning trees. $\square$

Next we consider the behaviour of the twist number under deletion and contraction. Here particular attention must be paid to cycles and cuts of length 3.

Definition 3.11 Let G be a graph and a be an edge of G. We call length-3 cycles in G *triangles*, and denote them by $\triangle$. Abbreviate by '$a \in n\triangle$' the property 'a is contained in (at least) n triangles', and write '$a \in \triangle$' for '$a \in 1\triangle$'. The expression '$(x,a) \notin \triangle$' means that x and a are not both contained in a common triangle etc. We write $\triangle(a)$, the *triangle number* of a, for the number of $\triangle \ni a$. (So for example $a \in 2\triangle \iff \triangle(a) \geq 2$.) Similarly we write $c(a)$ for the *cut number* of a, the number of different 3-cuts containing a.

Lemma 3.11 If G is 2-connected and loop-free, so are $G\setminus e$ and G/e. Moreover, $t(G/e)\geq t(G)-1-\triangle(e)$, and $t(G\setminus e)\geq t(G)-1-c(e)$.

Proof. The first property is clear for the loop edges by definition, and from this for the isthmuses by duality. As for the second statement, the two inequalities are also dual to each other, so we just argue for the first. Contracting an edge does not create any 2-cut; if two edges $f,g\neq e$ form a 2-cut in G/e, then so they do in G. Contraction may, though, create parallel edges, which happens exactly in $\triangle\ni e$. For each such $\triangle$, the classes of the other two edges are combined into a single class in G/e. Additionally we will lose the twist equivalence class of e, except possibly when e is in a 2-cut. □

4. Properties of the volume and relation to main results

Dunfield's experimental observations [Df] result from evaluating the knot tables in [HT]. This evaluation led to evidence that for alternating links L, we have relations of the type

$$\det(L) \approx e^{a\mathrm{vol}(L)+b}, \text{ or} \tag{7}$$

$$\det(L) \approx c(L)^{a\mathrm{vol}(L)+b} \tag{8}$$

for some numbers $a>0$ and b. (Dunfield refers to the degree of the Jones polynomial, but for alternating links this degree was known to be equal to $c(L)$ from [Ka2, Mu, Th2].)

If one, however, likes to turn (7) and (8) into rigorous statements valid for *all* alternating links (and not just for some, even if "generic", subclass), one must make sure to avoid misleading conclusions from the experimental evidence. This requires to take into account some peculiar special cases.

For example, it is easy to construct alternating knots with bounded volume but $\det\to\infty$ (see [Br]), so that an inequality '$\leq$' in (7) instead of '$\approx$' is impossible. Similarly it is easy to see that an inequality '$\geq$' in (8) is impossible. Take for any a,b an alternating knot of sufficiently large volume (which exists for example from the argument of [La]). Then apply $\bar{t}_2'$ twists of [St4] successively at one and the same crossing. Under these moves, the determinant grows linearly, while the volume remains bounded. (In fact it converges to the volume of a certain link by W. Thurston's hyperbolic surgery theorem, as explained in [Br] and [La].) This in fact shows also that the inequality '$\geq$' in (7), stated in Theorem 1.1, is qualitatively close to the best possible, up to at most a linear factor in $c(L)$. That is, for any $a>1$ and any continuous function $f:\mathbb{R}_+\to\mathbb{R}_+$, the inequality

$$\det(L)>c(L)^a\cdot f(\mathrm{vol}(L))$$

cannot hold in general.

Contrarily, the right hand-side of (8) can be made valid (for proper a and b) as an upper bound on the determinant. As I was informed after my initial writing, such a bound was obtained in unpublished work of Chris Leininger and Ilya Kofman, who proved

$$\det(L)\leq c(L)^{\mathrm{vol}(L)/V_0+2}.$$

(They have apparently also obtained a special case of Theorem 1.1, with a weaker inequality; see Proposition 6.2.) In Theorem 1.2 we claimed that we can qualitatively improve such an inequality, by putting $\mathrm{vol}(L)$ into a denominator of the base. We can make the bound more explicit, thus obtaining an improvement of the inequality of Kofman and Leininger.

Theorem 4.1 For any alternating hyperbolic link L, we have (with the constants V_0 and V_8 explained in and after Theorem 2.3)

$$\det(L)\leq\left[1+\frac{c(L)}{\max\left(2,\ 1+\dfrac{\mathrm{vol}(L)}{10V_0}\right)}\right]^{\dfrac{\mathrm{vol}(L)}{V_8}+2}. \tag{9}$$

For Theorem 4.1, we will use some arguments from standard Kauffman bracket skein module theory. (Since they have been repeated many times, we will try to be short; see [St, St2] for more explanation.)

Proof of Theorem 4.1. Let D be a prime alternating diagram of L. Without loss of generality, assume all twist equivalent crossings are strongly twist equivalent (in the sense described before Theorem 2.3). Let $t = t(D)$ and $n_1,\dots,n_t$ with $\sum_{i=1}^t n_i = c(D)$ be the cardinalities of the twist equivalence classes of D. Each twist equivalence class forms a tangle T_i. For each i, it is an easy exercise to see that exactly $1+n_i$ choices of splicings of T_i result in two arcs (n_i times with one connectivity and once with the other one), and all others have additional circles. In particular, at most $1+n_i$ choices of splicings for T_i can produce a monocyclic state. So we have

$$\det(D) \le \prod_{i=1}^{t(D)} (1+n_i)\,.$$

By a computation, if we regard n_i as real variables, then for keeping fixed their sum $c(D)$, the highest possible value on the right hand-side is attained if all $n_i = c(D)/t(D)$ are equal. Then use $t(D) \ge 2$ (since L is hyperbolic) and Theorem 2.3. □

Remark 4.1 It is easy to construct diagrams D where the splicings of T_i of the connectivities occurring n_i times join to a monocyclic state, and so

$$\det(D) \ge \prod_{i=1}^{t(D)} n_i\,.$$

Therefore, an improvement of Theorem 4.1 beyond removing the additive ‘1’ in the base of the right hand-side of (9) is possible (with this type of argument), only if the inequalities of Theorem 2.3 are improved for such examples.

Proof of Theorem 1.2. We estimate base and exponent on the right of (9) separately. First consider the base. We would like to choose C_1 so that

$$\frac{C_1}{\mathrm{vol}\,(L)} \ge \frac{1}{c(L)} + \frac{1}{\max\left(2,\ 1+\dfrac{\mathrm{vol}\,(L)}{10V_0}\right)}\,. \tag{10}$$

Using the inequality $\mathrm{vol}\,(L) \le 4V_0 c(L)$, explained in [Br, La], we see that (10) would follow from

$$\frac{C_1}{\mathrm{vol}\,(L)} \ge \frac{4V_0}{\mathrm{vol}\,(L)} + \frac{1}{1+\dfrac{\mathrm{vol}\,(L)}{10V_0}}\,,$$

which is equivalent to

$$\frac{C_1 - 4V_0}{\mathrm{vol}\,(L)} \ge \frac{10V_0}{10V_0 + \mathrm{vol}\,(L)}\,.$$

So we see that we can choose $C_1 = 14V_0$.

It remains to estimate the exponent on the right of (9) (from above; keeping in mind that the base in (2) is bigger than 1, as already the one in (9) is). We use the minimality of $\mathrm{vol}\,(4_1) = 2V_0$ among all hyperbolic link volumes [CM], that is, $\mathrm{vol}\,(L) \ge 2V_0$. Then we conclude that with $C_2 = 1/V_0 + 1/V_8 \approx 1.53115$ we can achieve the stated inequality. □

Our main contribution to the proof of the first main result is

Theorem 4.2 For every non-trivial planar 2-connected graph G with no loop edges,

$$s(G) \ge 2\cdot\gamma^{t(G)-1}\,,$$

where $\gamma \approx 1.4253$ is the inverse of the (unique) real positive root of $x^5+2x^4+x^3-1=0$, given by ≈ 0.701607.

This easily implies Theorem 1.1.

Proof of Theorem 1.1. If G is the checkerboard graph of an alternating diagram D of L, then by Lemma 3.6, $\det(L) = s(G)$ and by Lemma 3.7, $t(L) = t(D) = t(G)$. By the left (Agol-Thurston) inequality in Theorem 2.3, we have

$$\mathrm{vol}\,(L) \leq 10V_0\big(t(D)-1\big) = 10V_0\big(t(G)-1\big) \leq 10V_0 \log_\gamma \frac{s(G)}{2}\,,$$

hence

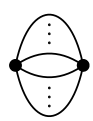
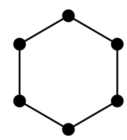
$$\det(L) = s(G) \geq 2 \cdot \gamma^{\frac{\mathrm{vol}\,(L)}{10V_0}}\,,$$

and $\gamma^{1/10V_0} \approx 1.03553$. $\square$

5. Proof of the spanning tree-twist number inequality

5.1. Induction for non-simple, 2-connected and decomposable graphs

The proof of Theorem 4.2 underlying Theorem 1.1 is more substantial, and will occupy a significant part of the paper. Let us for later use sort out a few simple cases first, in particular the graphs G with $t(G) = 1$, namely those of the types

$T_n =$  and  (11)

(two-vertex graph and *chain*).

Lemma 5.1 If G is a join of loop-free 2-connected join factors G_i, such that every G_i has at most 5 edges or $t(G_i) = 1$, then $s(G) \geq 2^{t(G)}$.

Proof. It is direct to check that $s(G_i) \geq 2^{t(G_i)}$ for each of the join factors G_i of G. The inequality for G then follows from Lemmas 3.7 and 3.9, the multiplicativity of $s(G)$ and additivity of $t(G)$ under join. $\square$

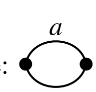
Proof of Theorem 4.2. We proceed by induction on the number $e(G)$ of edges of G. With Lemma 5.1 we can assume that $t(G) \geq 2$ (that is, G is not of the forms in (11)) and $e(G) \geq 6$.

In the following, a will be an edge in G. Excluding G from being in (11) means that both G/a and $G \setminus a$ (see Definition 3.10) are non-trivial (that is, have $t > 0$), so that induction applies on both. The aim will be now to choose a so that both $t(G/a)$ and $t(G \setminus a)$ can be controlled from below.

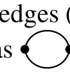
The choice of constant γ will become clear throughout the course of the proof.

We make now a(n exhaustive, but not necessarily exclusive) case distinction.

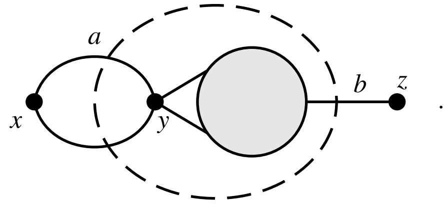
Case A. If G has a cut vertex (that is, is decomposable), then we are done inductively by the multiplicativity of $s(G)$ and additivity of $t(G)$ under join (stated in Lemma 3.9), and Lemma 3.1. So we may assume from now on, that G has no cut vertex.

Case B. Assume G has a multiple edge. Let a be an edge of this multiple edge: . Our strategy will be to use Lemma 3.10 and 3.11, and control from above $c(a)$ and $\triangle(a)$.

Note that (see the proof of Lemma 3.11) the deletion of an edge never changes the number of equivalence classes under the relation given by alternative (ii) in Definition 3.9, while the same holds for contraction of an edge and the alternative (i). Moreover, different twist equivalent edges with property (i) and (ii) are disjoint by remark 3.1.

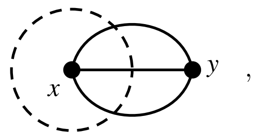
Note also that, since a has a non-trivial twist equivalent edge under property (i), deleting a creates no 1-cut edges (by remark 3.1). Also, a has a parallel edge, and so the twist equivalence class of a remains in $G \setminus a$ (unless G has  as join factor, which we excluded). Thus if deleting a creates no new 2-cut (that is, $c(a) = 0$), then $t(G \setminus a) = t(G)$, and induction applies (using Lemma 3.10 and $s(G/a) \geq 0$).

So we assume, for the rest of case B, that $c(a) > 0$. Then we have (up to the change of the ∞-region)



(12)

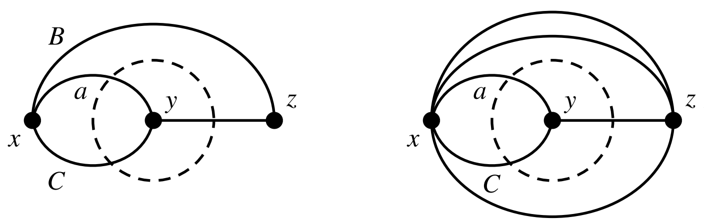
Here x and z may be connected to other vertices outside of the dotted line. Note also that a is not in a (multiple) edge of multiplicity ≥ 3, because otherwise the only option of having a in a 3-cut is



and $e(G) = 3$, or one of x or y is a cut vertex, both of which we excluded. Also note that there may be several edges b of the kind shown in (12), but any two of them form a 2-cut.

We recall the terminology in Definition 3.11 and Lemma 3.11.

Case B.1. Assume now that in a graph G as in (12), $a \in \triangle$. Then we have something of the following type:



(13)

In this case every $\triangle \ni a$ consists of a, yz and (one of) the (possibly multiple) edge between x and z.

Now consider G/a. Then $t(G/a) \geq t(G) - 2$, since contracting a we lose its twist equivalence class of edges, and at most two other twist equivalence classes unify to one class under the contraction.

Let C be the region opposite (see Definition 3.7) at the parallel edge xy to a to the bigon region enclosed by xy and a. If xz is simple, let B be the region opposite at xz to the $\triangle \ni a$. If xz is multiple, then the specification of B is easily refined, but in fact irrelevant.

Since G has more than the 4 edges drawn in (13), and x, y and z are not cut vertices, if xz is simple, regions B and C are different, that is, they are separated by edges incident to x and z but not drawn in (12) and (13). If a had now a second 3-cut, different from the dotted line in (13), the edge b in (12) would not be unique, and so yz in (13) would have a 2-cut. But such a 2-cut would pass through an edge between x and z, and thus xz must be simple and $B = C$, which we excluded.

So the dotted line in (13) is the loop of the only 3-cut in which a participates. Then $t(G \setminus a) \geq t(G) - 1$, since one identification of twist equivalence classes occurs when deleting a, but, as noted, with a being part of a double edge the twist equivalence class of a remains in $G \setminus a$. Then from Lemma 3.10 we have by induction

$$\begin{aligned} s(G) &= s(G \setminus a) + s(G/a) \geq 2\left(\gamma^{t(G\setminus a)-1} + \gamma^{t(G/a)-1}\right) \\ &\geq 2\left(\gamma^{t(G)-2} + \gamma^{t(G)-3}\right) = 2\gamma^{t(G)-1}\left(\gamma^{-1} + \gamma^{-2}\right) \geq 2\gamma^{t(G)-1}. \end{aligned} \tag{14}$$

That this last inequality holds follows from the fact that $\gamma \approx 1.4253$, and so $\gamma^{-1}+\gamma^{-2} > 1$. (Note that it would suffice here to use the inverse of the positive root of $x+x^2-1$.)

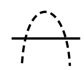
Case B.2. If $a \notin \triangle$, then we may have several possible b in (12) with which a forms a 3-cut. But we noted that they are all twist equivalent, so still $t(G \setminus a) \geq t(G)-1$, while $a \notin \triangle$ implies also that $t(G/a) \geq t(G)-1$, and thus the estimate (14) applies again.

Case C. If G has a 2-cut, then argue using case B and duality (Lemma 3.8).

5.2. Simple 3-connected graphs: properties of 3-cut circuits

Case D. Thus we can assume now that G has no 2-cut and no double edges (that is, is simple and 3-connected). Then the link diagram D (for which $G=\Gamma(D)$) is a polyhedral diagram in the sense of Conway [Co]. The simplest such diagram has 6 crossings, so we can assume that $t(G)=c(D)\geq 6$.

Now, for each 3-cut we draw a 3-cut circuit in the plane. (We consider these circuits up to homotopy, which does not change the three edges intersected, and does not change the number of intersections with each edge.) We exhibit now in several claims some properties of this collection of circuits.

Claim 1. No 3-cut circuit intersects the same edge twice:  . $\square$

Claim 2. We can homotope all 3-cut circuits so that no two of them intersect.

Proof. Assume the contrary, and consider two intersecting 3-cut circuits a and b, after homotoping all 3-cut circuits so that they have the minimal possible total number of intersections. We derive a contradiction by showing that one can further minimize this number. For this purpose, we define two simple homotopies of a pair of circuits x and y, shown in figure 3, pushing off within a region (a) and pushing off along an edge (b). (It is in fact easy to see that these types of homotopies can, and are understood to be chosen so that the total number of intersections of x and y with other circuits is not augmented.)

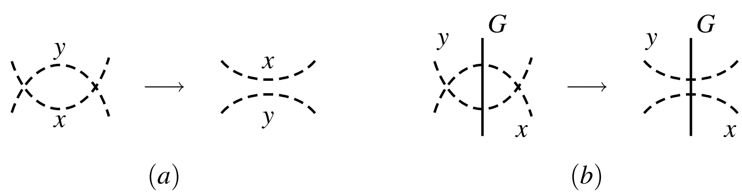


Figure 3: Cut circuit homotopies which reduce the number of intersections: pushing off within a region (a) and pushing off along an edge (b). In both cases, we assume no part of G (other than the drawn edge fragment in (b)) enters into the bigon on the left of the move.

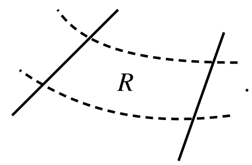
If a and b intersect in only one region of G, then they can be pushed off each other within that region. So assume a and b intersect in at least two regions of G, say A and B.

Consider G^*. Since G is simple and 3-connected, so is G^*. Any 3-cut circuit x of G corresponds in G^* (in the sense of Definition 3.8) to a cycle x^* of length 3, that is, a triangle of G^*. Thus a^* and b^* are two triangles in G^* which share a pair of common vertices A^* and B^*. Now by simplicity of G^*, any two distinct triangles in it which meet in two vertices, must share the unique edge between these vertices. Thus a and b pass between A and B through the same edge of G. Then it is clear that a and b can be pushed off each other along this edge. $\square$

We assume from now on that all 3-cut circuits are homotoped so as to be disjoint.

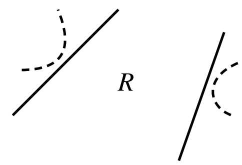
Claim 3. No two 3-cuts have a pair of common edges.

Proof. Assume there were such two 3-cuts, and pair of edges e and f, and consider the cut circuits. By 3-connectivity of G, no edge bounds the same region from both sides, and each pair of edges has at most one common region R. For e and f clearly R must exist, and then both circuits must pass (non-intersectingly by Claim 2) through R. By the Jordan curve theorem no circuit can leave R and enter it again (more than once). Thus we have



(15)

Modifying (15) to



gives a 2-cut circuit (with two different edges intersected, because the original two 3-cuts were different). This is a contradiction. □

A direct consequence of Claim 3 is

Claim 4. If E is a triangle (that is, a cycle of length 3; not necessarily the boundary of a face), then all three edges in E have cut number at most two. □

Moreover, we have

Claim 5. If E is a ≤ 5-gonal face (that is, a face bounded by at most 5 edges), then for each pair e and f of neighbored edges in ∂E there exists an edge $a \in \partial E \setminus \{e, f\}$ with $c(a) \leq 2$. (Neighbored edges means edges incident to a common vertex, which is a corner of E.)

Proof. If we assume that $c(a) \geq 3$ for all $a \notin \{e, f\}$, it is an easy exercise to check that one cannot install the pieces of the arcs of the 3-cut circuits within E so that the conditions of the first 3 claims are satisfied simultaneously. □

Claim 6. Assume l is a simple closed curve intersecting n edges of G transversely, and (among these edges) a exactly once. Then $n > \triangle(a)$.

Proof. Let B and C be the two regions bounded by a, and l be oriented so as to pass from B to C through a. This passage changes the inside/outside position of the curve l with respect to any of the triangles containing a. Thus, in order to return from C to B, the closed curve l must intersect at least one (further) edge for each triangle containing a, and these edges are all distinct by Claim 3. □

5.3. Completing the induction: finding a good edge

With the above claims prepared, we continue with case D. Let G be simple and 3-connected. It is well-known, that in every planar embedding of G there is a ≤ 5-gonal face E. Let a be an edge in its boundary ∂E.

If we choose now $a \in \partial E$ to have at most two 3-cut circuits intersecting it, by Lemma 3.11, we have $t(G \setminus a) \geq t(G) - 3$, since deleting a we lose its twist equivalence class, and at most two pairs of other twist equivalence classes identify. It remains to refine the choice of a so as to control also $t(G/a)$. To this vein, we must count again the triangles containing a. Since we assumed that G has no double edge, two such triangles have only a as a common edge.

We claim that one can find an edge a so that

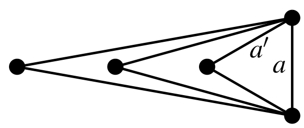
(i) $\triangle(a) \leq 2$ and $c(a) \leq 2$, but

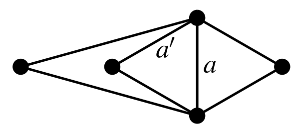
(ii) *unless* G is the graph in (18), *not* simultaneously $\triangle(a) = c(a) = 2$.

We describe an algorithm to find some edge a satisfying (i). (Here the choice of the ∞ region is kept fixed!) The condition (ii) will be dealt with subsequently.

① Start with some edge $a \in \partial E$ with $c(a) \leq 2$. Such an edge exists by Claim 5.

② If $a \in 3\triangle$, then there is always a pair among these three $\triangle$, such that one triangle encloses the other:



or


. (16)

(There may be further edges in the interior of these triangles.) Note in particular that the valence of the two vertices connected by a is higher than the number of different $\triangle \ni a$.

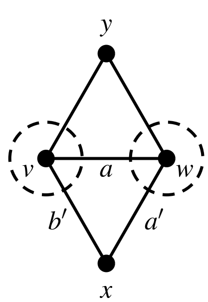
③ Take $a' \neq a$ in the boundary of an innermost triangle from the ones drawn in (16). Set $a := a'$ and go to step ②.

Since this iteration augments the number of $\triangle$ enclosing a (and for any edge of G there are only finitely many such triangles), at some point the test in ② fails, and a is in at most 2 triangles. Since $a \in \triangle$, we have $c(a) \leq 2$ by Claim 4.

This algorithm gives an edge a which is in at most two triangles, and at most two cuts. We must show now that we can avoid a being in exactly two triangles *and* exactly two cuts.

Assume that any a with $c(a) \leq 2$ and $\triangle(a) \leq 2$ has $c(a) = \triangle(a) = 2$. Let E_1, E_2 be the $2\triangle \ni a$.

Now a is in two 3-cuts, and by Claim 3 each cut must pass through one other edge of each of the $\triangle \ni a$. By Claim 6, then there are at most two triangles containing a, and both cut circuits do not pass through any further edges. Thus the vertices v and w that a connects are trivalent, and the cut circuits are going around them (that is, their interior contains this one single vertex).



(17)

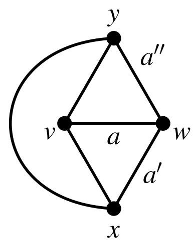
Then v, w cannot be incident to edges in the interior[1] of E_1. Now E_1 is a triangle, and its other vertex, different from v, w, is not a cut vertex (as G has no cut vertex by assumption). So E_1 must have empty interior, that is, be a triangular face. The same assertion similarly follows for E_2.

Since all the 4 edges in $\partial E_1 \cup \partial E_2 \setminus \{a\}$ are incident to a trivalent vertex, by Claim 6 they cannot belong to more than two triangles. But since they do belong to a triangle, by Claim 4 they cannot have cut number > 2. Thus assume all they belong to (exactly) two triangles and two 3-cuts – otherwise we can choose a to be one of them, and we are done.

Then consider a' in (17) instead of a. The only way in which a' can have a second 3-cut X conforming to Claims 1-3 is if X goes through a' and b'. Then the same argument as for a shows that $a' \in 2\triangle$ only if x is trivalent and connected

[1]Actually, we must possibly replace for one of E_1 or E_2 'interior' by 'exterior' in this argument; it depends on the choice of the ∞ region. We used 'interior' to keep the wording simple. Alternatively, one could avoid considering the exterior, if one continues the iteration in part ② of the above algorithm, even if $\triangle(a) = 2$, as long as one $\triangle \ni a$ encloses the other.

to a vertex, which is different from v and connected to w. The only such vertex is y. Thus we have



(18)

Now, arguing for a'' the same way as with a, we see that y is also trivalent, and that hence G has no more edges than those 6 drawn in (18). This graph G has 16 spanning trees and $t(G) = 6$, and the inequality in the theorem is directly verified. Otherwise one of the 4 edges in $\partial E_1 \cup \partial E_2 \setminus \{a\}$ has at most one cut, and we found the edge we sought.

Having found this edge a, we obtain either

$$t(G \setminus a) \geq t(G) - 2 \text{ and } t(G/a) \geq t(G) - 3, \quad \text{or} \quad t(G \setminus a) \geq t(G) - 3 \text{ and } t(G/a) \geq t(G) - 2. \tag{19}$$

This way we could already conclude the theorem with a base of the exponential being the inverse of the real positive root of $x^3 + x^2 - 1$. However, we can do a bit better in applying induction. (Recall the remarks on the polynomials we made in the beginning of the proof.)

Case D.1. If none of $c(a)$ and $\triangle(a)$ is zero, then observe that both G/a and $G \setminus a$ contain a multiple edge respectively a 2-cut. Assume that $t(G) - t(G \setminus a) \leq 3$ and $t(G) - t(G/a) \leq 2$; the reverse case is symmetric with respect to the argument that follows (only the estimates in (21) and (22) would interchange).

Case D.1.1. If G/a has a join factor G' with $e(G') \leq 5$ or $t(G') = 1$, let G'' with $G/a = G' * G''$ be the (join of the) other factor(s). Then apply Lemma 5.1 on G' and induction on G''. We obtain, using also Lemma 3.9,

$$\begin{aligned} s(G/a) &= s(G') \cdot s(G'') \geq 2^{t(G')} \cdot 2\gamma^{t(G'')-1} = 2 \cdot 2^{t(G')-1} \cdot 2\gamma^{t(G'')-1} \\ &\geq 4 \cdot \gamma^{t(G')+t(G'')-2} = 4\gamma^{t(G/a)-2} \geq 4\gamma^{t(G)-4}. \end{aligned} \tag{20}$$

We will see that this estimate subsumes the one in the next case.

Case D.1.2. If G/a has no such join factor, the multiple edge respectively 2-cut is contained in a join factor G' of G/a which is recursively subjectable to case B or C. This means that we can find an edge a' in G', and hence G/a, such that the graphs $G_{cc} = (G/a)/a'$ and $G_{cd} = (G/a) \setminus a'$, obtained by contracting respectively deleting a' in G/a, satisfy (up to interchange) $t(G_{cc}) \geq t(G/a) - 2 \geq t(G) - 4$ and $t(G_{cd}) \geq t(G/a) - 1 \geq t(G) - 3$. (Here again Lemma 3.9 is tacitly applied.) Then

$$s(G/a) \geq 2\gamma^{t(G)-5} + 2\gamma^{t(G)-4}. \tag{21}$$

Since (20) is stronger, it is enough to keep working with (21).

Similarly, using graphs $G_{dc} = (G \setminus a)/a', G_{dd} = (G \setminus a) \setminus a'$ with $t(G_{dc}) \geq t(G \setminus a) - 2 \geq t(G) - 5$ and $t(G_{dd}) \geq t(G \setminus a) - 1 \geq t(G) - 4$ (again up to interchange), we obtain for $s(G \setminus a)$ the inequality

$$s(G \setminus a) \geq 2\gamma^{t(G)-5} + 2\gamma^{t(G)-6}. \tag{22}$$

(There is no problem in writing down these quantities because we ensured in the beginning of case D that $t(G) \geq 6$. So none of $G_{cc}, G_{cd}, G_{dc}, G_{dd}$ is the trivial 1-vertex graph.) Using inequalities (21) and (22), and Lemma 3.10, we have

$$\begin{aligned} s(G) &= s(G \setminus a) + s(G/a) \\ &\geq 2\gamma^{t(G)-5} + 2\gamma^{t(G)-4} + 2\gamma^{t(G)-5} + 2\gamma^{t(G)-6} \\ &= 2\gamma^{t(G)-1}\left(\gamma^{-5} + 2\gamma^{-4} + \gamma^{-3}\right) \\ &= 2\gamma^{t(G)-1}. \end{aligned}$$

This last equality uses the fact that $1/\gamma$ is a root of $x^5 + 2x^4 + x^3 - 1$ (and so explains our choice of γ). Then we obtain the claim.

Case D.2. If some of $c(a)$ and $\triangle(a)$ is zero (and the other one is at most two), we obtain the recursive estimate

$$s(G) \ge 2\gamma^{t(G)-1}\left(\gamma^{-3}+\gamma^{-1}\right).$$

In order the right hand-side to evaluate to at least $2\gamma^{t(G)-1}$, we need $1/\gamma$ to be not smaller than the (real positive) root of the polynomial x^3+x-1. But one indeed checks that this polynomial is positive for $x=\gamma^{-1}$. In other words, the γ we have chosen so far is smaller than the one we would need for this recursive estimate.

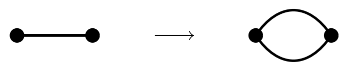
Now the proof of Theorem 4.2 is complete. $\square$

6. Possible improvements, computations and examples

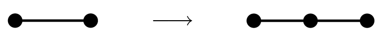
Let us conclude the topic with some observations on experimental data and partial improvements.

6.1. Alternating arborescent links

Definition 6.1 The *doubling* of an edge (installing an edge connecting the same two vertices) is the operation


.

The *bisection* of an edge (putting a valence-2 vertex on it) is the operation


.

(Other edges may be incident from the left- and rightmost vertices on both sides, and the rest of the graphs on both hand sides are assumed to be equal.)

For a graph G' let $\langle G'\rangle$ be the family of graphs obtained from G' by repeated edge bisections and doublings. A graph G is *series parallel*, if $G \in \langle \bullet\!\!-\!\!\!-\!\!\bullet \rangle$. Let us call the graph *n-series parallel*, if $G \in \langle G'\rangle$, and G' is a simple 3-connected planar graph of n edges.

Series parallel graphs correspond, via the checkerboard construction, to alternating *arborescent* link diagrams, that is, alternating diagrams with Conway polyhedron 1^* [Co]. More generally, link diagrams with an n-crossing Conway polyhedron come from n-series parallel graphs.

For a series parallel graph, one can use the proof of Theorem 4.2 to obtain, more easily, a better inequality. Here T_2 is the two-vertex graph with a double edge as in (11) (and remark 3.1), and F_i denote the *Fibonacci numbers*, defined by $F_1=1$, $F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n>2$.

Proposition 6.1 If G is series parallel, or the join of such graphs, and $G \neq T_2, T_2 * T_2$, then

$$s(G) \ge F_{t(G)+3}\,. \tag{23}$$

Proof. For a series parallel graph, case D in the induction of the proof of Theorem 4.2 never occurs. So we can use the estimate (14), which yields the recursion of the Fibonacci numbers. With that argument completed, now one needs to take care only of graphs G, for which G/a or $G\setminus a$ is one of T_2 or T_2*T_2. The proof shows that such graphs G have twist number at most 4, and they can be easily dealt with. (It suffices to check graphs with at most two edges in each twist equivalence class, and so at most 8 edges.) $\square$

Note that if $G_{1,2}$ are series parallel, then one can always perform the join $G_1 * G_2$ (by proper choice of join vertices) so that the result is series parallel, but not every possible join will be so.

The inequality (23) can be realized sharply (that is, as equality) taking (checkerboard graphs of) rational link diagrams $(21\dots12)$, so it is optimal. This inequality in particular allows us to improve the base of the exponential in Theorem 1.1 for alternating arborescent links (which seems related to the partial cases obtained also by Kofman and Leininger).

Proposition 6.2 If L is a hyperbolic alternating arborescent link, then $\det(L) \geq 1.4424 \cdot 1.0485^{\operatorname{vol}(L)} - 0.0118$.

Proof. If an alternating link L is arborescent, then an alternating diagram D of L has a Conway polyhedron 1^* or 6^* (see [Th3]). This means that its checkerboard graph $G \in \langle 1 \rangle \cup \langle G' \rangle$, where we write 1 for $\bullet\!\!-\!\!\!-\!\!\bullet$ and G' is the graph in (18). For $G \in \langle 1 \rangle$, Proposition 6.1 applies directly (the two exceptional G yield non-hyperbolic L). For $G \in \langle G' \rangle$ the same inductive argument as in the proof of Proposition 6.1 can be used, unless $G = G'$, a situation which we examined separately already. In all other cases, the proof of Theorem 4.2 shows that, with an edge a chosen properly, $t(G/a), t(G \setminus a) \geq t(G) - 3$, so if $G \in \langle G' \rangle$, then $t(G) \geq 6$, and none of G/a, $G \setminus a$ is T_2 or $T_2 * T_2$. Moreover, if $G \in \langle 1 \rangle \cup \langle G' \rangle$, then the same holds for G/a and $G \setminus a$. By adapting constants to G', where $s = 16$ and $F_9 = 34$, we have

$$\det(D) \geq \frac{8}{17} F_{t(D)+3}\,. \tag{24}$$

Now from (24), using $t(D) \geq 2$, the explicit formula

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right],$$

and the Agol-Thurston bound in Theorem 2.3, we find

$$\det(L) \geq \frac{8}{17\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{\frac{\operatorname{vol}(L)}{10V_0}+4} - \frac{8}{17\sqrt{5}} \left(\frac{\sqrt{5}-1}{2} \right)^6 \approx 1.44247 \cdot 1.04855^{\operatorname{vol}(L)} - 0.01172\,,$$

which implies the stated estimate. □

Remark 6.1 There is some annoying confusion as to the meaning of “alternating arborescent link”. As we explained, there are links which are alternating and arborescent, but whose alternating diagrams are not arborescent. (The Borromean rings, whose checkerboard graph is G', are the simplest example.) These pathologies are often tacitly ignored by many uncareful authors. If we consider alternating*ly* arborescent links, that is, links whose alternating diagrams are arborescent, then the treatment of G' and the constant $\frac{8}{17}$ in (24) are not needed, and we can slightly improve our estimate.

6.2. Experimental and computational remarks

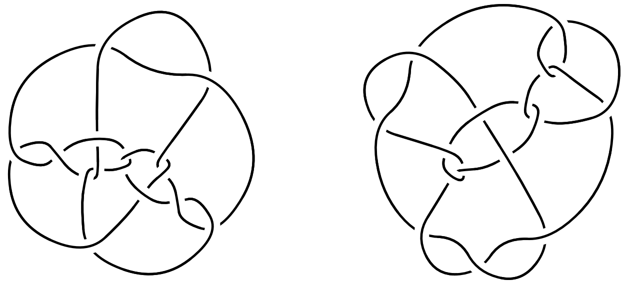
A natural question is to what extent the base in (1) can be improved in *general*. By refining the argument that led to the occurrence of the polynomial $x^5 + 2x^4 + x^3 - 1$, one could try to ‘push’ the zero below that of $x^3 + x - 1$, and then have a better (and writable in radicals) constant γ in Theorem 4.2. Such an improvement seems very burdensome and little efficient, though.

To obtain some idea about the maximum possible, we evaluated (neglecting the unnatural multiplicative factor in (1)) for alternating hyperbolic knots K the invariant $\mu(K) := \sqrt[\operatorname{vol}(K)]{\det(K)}$. The smallest value among alternating knots of at most 16 crossings was found attained by 16_{375396} with $\mu \approx 1.2507$. In general the minimal μ decreases with increasing crossing number. It is somewhat surprising that in fact the knots of largest determinant [St3] have the smallest or second smallest (for 10,11,16 crossings) μ.

Since the numeric data suggests (as expected) space for improvement, a further question is where such improvement is possible. Theorem 1.1 is a coupling of two independent parts, Theorem 4.2 and Theorem 2.3. As we observed, the base $\gamma \approx 1.42$ in Theorem 4.2 is likely not optimal, but can not be increased beyond $\frac{1+\sqrt{5}}{2} \approx 1.62$. Contrarily, while Agol-Thurston show that their inequality is (asymptotically) sharp in general, in practice it often fails considerably. We examined the inequality

$$\frac{\operatorname{vol}(K)}{10V_0\,(t(K)-1)} < \frac{(\ln\gamma) \cdot (t(K)-1)}{\ln(\det(K)/2)}\,, \tag{25}$$

whose hand-sides compare the unsharpness of the estimates in Theorems 4.2 and 2.3. For example, among the 6723 hyperbolic alternating knots K of ≤ 13 crossings, 6273 (or $\approx 93.3\%$) satisfy (25). Such evidence suggests that the



15_{79500} 16_{5435}

Figure 4: Two alternating knots with (almost?) equal volume but different determinant.

loss of quality in the constant in Theorem 1.1 is to a larger extent due to the application of Theorem 2.3, rather than Theorem 4.2. This, however, seems natural, since the volume is more complex to understand than the graphs.

We make a final remark that, apart from estimates, an exact relation between determinant and volume is unlikely to exist. Examples of (even alternating) knots with equal determinant but different volume are easy to find, and, at least including non-alternating knots, there are well-known pairs with equal volume but different determinant, for instance 9_{42} and 10_{132}. For alternating knots such pairs are not obvious, but one has situations where the volumes are very close to each other.

Example 6.1 The best pair I found, which sufficiently demonstrates the problem, is 15_{79500} and 16_{5435} (see figure 4). Their polynomials and determinants are quite different, while their volumes differ only by 10^{-10}, when calculated up to this same precision ($\mathrm{vol}(15_{79500}) \approx 27.3456887210$, and $\mathrm{vol}(16_{5435}) \approx 27.3456887209$). Since such computational deviations occur even among different diagrams of the same knot, the volumes may even well be equal. (Note that, in very contrast to discrete invariants, to decide whether two volumes are equal is not always trivial.)

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