GRID DIAGRAMS, LINK INDICES, AND STRONG QUASIPOSITIVITY

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Abstract. We present a viewpoint on Euler characteristic 0 braided surfaces as grid diagrams. This leads to some results regarding estimates of Thurston-Bennequin invariants of knots, strong quasipositivity of Whitehead doubles, jump numbers of slice-torus invariants, and arc and braid index. In particular we obtain rather sharp ("frying eggs in the pan"-style) information about maximal Thurston-Bennequin invariants and arc index from the HOMFLY-PT polynomial. We extend some of these results to quasipositivity. We also consider the crossing number of grid diagrams and rectilinear (planar grid) polygons, and versions of the braid index related to braided and strongly quasipositive surfaces.

Keywords: arc index, braid index, rectilinear polygon, HOMFLY-PT polynomial, Thurston-Bennequin invariant, quasipositive, strongly quasipositive, Whitehead double.

2020 AMS subject classification: 57K10 (primary), 57M50, 20F36, 05C10, 51E12, 57K33 (secondary)

1 Introduction

This investigation resulted from attempts to understand braided surfaces, in particular Bennequin and strongly quasipositive surfaces. Similar to the case of canonical surfaces [St], we were trying to develop some structural properties. As it turned out, even in the simplest case of Euler characteristic 0, the answer is revealingly complicated, in that these surfaces are essentially equivalent to grid diagrams D for knots (with an integer $\lambda(D)$ attached). It should be noted from the start that grid diagrams of links can be treated by essentially the same approach, without very major modifications, but for technical reasons we stick mostly to knots.

However, despite protruding such complexity, this connection leads to some new viewpoints, and assimilates a number of known and new results. We present some exposition on it here, in the belief that the topic could be beneficial for further study.

An outline of the paper is as follows.

After compiling preliminaries in §2, we give some simple but useful observations in §3 on crossing number and writhe of grid diagrams. In particular, we determine the maximal crossing number of a rectilinear polygon of given size exactly in the multi-component case (Lemma 3.1) and nearly exactly in the one-component case (Proposition 3.3 and Computation 3.4).

In §4 we introduce a weight model for the Thurston-Bennequin invariant from a grid diagram (Lemma 4.4), and give some simple applications, including the determination of the strong quasipositivity of twisted annuli and Whitehead doubles (Theorem 4.9 and Corollaries 4.13 and 4.15).

In §5 we discuss braid indices, and how the arc index is fundamentally connected to a braid index (see Corollary 5.1 and Conjecture 8.1). We give among others a modification of Ohyama's [Oh] inequality (Corollary 5.5), and introduce the framing diagram of a knot and its cone structure (Theorem 5.9).

Section §6 deals with the jump function of slice-torus invariants, and the Livingston-Naik [LN] estimate (Proposition 6.3), as well as its application to the Bennequin-sharpness problem (2.8) (see Corollary 6.4).

Section §7 is the longest and contains a detailed treatment of the HOMFLY-PT polynomial. The possibility exists (Conjecture 2.3) that the HOMFLY-PT polynomial determines the braid index, thus this could be true for the arc index as well. In the simplest form, we extract (in a "culinary" way) an invariant, we call $l(K)$, which gives a lower bound for the arc index of K (Proposition 7.7). It (apparently, see Question 7.12) already improves upon the Morton-Beltrami [MB] bound. For (even) better etimates, one can use cabling, and to limit complexity problems, we introduce partial cabling (Lemma 7.23). This can be complemented by some extra arguments, and shows that the HOMFLY-PT polynomial is efficient to practically determine the arc index (see Lemma 7.11 and Remark 7.26) and maximal Thurston-Bennequin number (Proposition 7.27) in most examples. We also prove the Finite-Cone-Theorem 7.3.

Section §8 mostly deals with a summary of previous considerations (including a more explicit variant of the Finite-Cone-Theorem in Proposition 8.6). We also highlight potential pathologies about non-coincidence of various types of braid indices. This comprises Rudolph's problem (5.1). We show that the l-invariant (Propositions 8.9 and 8.11) can be also used to exclude such peculiar behavior.

The final appendix §A, given by the second author, discusses what previous results on strong quasipositivity can be extended to quasipositivity. There, many considerations revolve around sliceness. For many (companion) knots, we establish that the quasipositivity and strong quasipositivity of knotted annuli are equivalent (for example, Corollary A.6), but that in general they are not (Proposition A.8). We also know that some untwisted Whitehead doubles are quasipositive but not strongly quasipositive (Remark A.10), and that Bing doubles are never strongly quasipositive (Corollary 4.15) but some are quasipositive (see Example A.11).

2 Definitions and Preliminaries

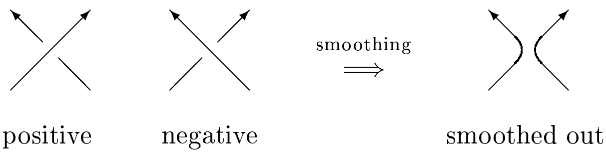
We encounter many suggestive but difficult to resolve questions, whose examination would require deepening this consideration. Developing a similar theory, of "grid-embedded graphs", for smaller Euler characteristic will thus also be a long – but nevertheless perhaps very interesting – undertaking.

2.1 Generalities

We say an inequality '$a \geq b$' is *sharp* if $a = b$ and *strict* (or *unsharp*) if $a > b$. We use $\#E$ for the cardinality of a finite set E and $\lfloor x \rfloor$ for 'greatest integer' part of $x \in \mathbb{Q}$. We also afford a few standard abbreviations like 'l.h.s.' (for 'left hand-side'), 'w.r.t.' (for 'with respect to') and 'w.l.o.g.' (for 'without loss of generality').

2.2 Links and genera

All link diagrams and links are assumed oriented. Crossings in an oriented diagram D of a knot K are called as follows.



(2.1)

The *sign* of a positive/negative crossing is assigned to be ± 1 accordingly. Let $c_\pm(D)$ be the number of positive, respectively negative crossings of a link diagram D, so that the *crossing number* of D is $c(D) = c_+(D) + c_-(D)$ and its *writhe* is $w(D) = c_+(D) - c_-(D)$. We write $s(D)$ for the *number of Seifert circles* of D, which are the circles obtained after smoothing all crossings of D. We write $c(K)$ for the crossing number of a knot K, the minimal crossing number of all diagrams of K. The mirror image of K will be written $!K$, and the mirror image of diagram D (in the form obtained by switching all crossings of D) will be $!D$. If $K = !K$ (up to orientation), we call K *amphicheiral*. We use '$\bigcirc$' to denote the *unknot* (trivial knot) in formulas. The number of components of a link L is denoted $\kappa(L)$. The *bridge number* $br(L)$ of L is the minimal number of Morse maxima of L (or equivalently, of any diagram of L). The *(Seifert) genus* $g(L)$ resp. *Euler characteristic* $\chi(L)$ of a knot or link L is said to be the minimal genus resp. maximal Euler characteristic of a *Seifert surface* of L. We have

$$2g(L) = 2 - \kappa(L) - \chi(L)\,.$$

Similarly write $\chi_4(L)$ for the *smooth 4-ball* (maximal) Euler characteristic and

$$2g_4(L) = 2 - \kappa(L) - \chi_4(L)\,.$$

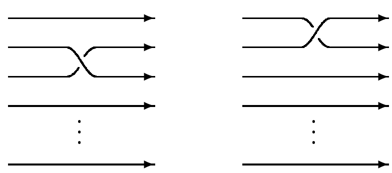
(In the following 4-ball genera and sliceness will always be understood smoothly.) A knot K is *slice* if $g_4(K) = 0$, or equivalently, $\chi_4(K) = -1$. We will refer to the following basic fact: if $\kappa(L) = 2$ and $\chi_4(L) = -2$, then both components of L must be slice (knots), and have linking number 0.

2.3 Braids and braided surfaces

We write B_n for the *braid group* on n *strands* or *strings*. The relations between the *Artin generators* σ_i, $i = 1, \dots, n-1$ are given by

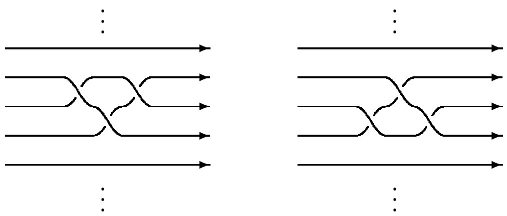
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2$ and
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \le i < j-1 \le n-2$.

In diagrams we will orient braids left to right and number strings from top to bottom, for example:

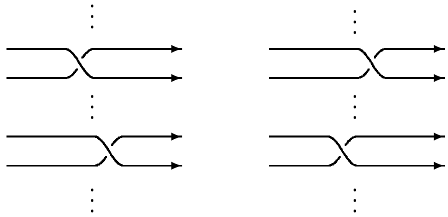


σ_2 σ_1^{-1}

The relations can then be drawn as follows:



$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$



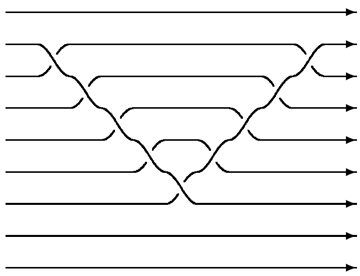
$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1$

There is a *permutation homomorphism* $\pi : B_n \to S_n$, sending each σ_i to the transposition of i and $i+1$. By a *subbraid* of $\beta \in B_n$ we mean a braid obtained by taking only a subset $C \subset \{1, \dots, n\}$ of the strands in β, which is invariant under the associated permutation $\pi(\beta)$ of β (i.e., C is a union of cycles of $\pi(\beta)$).

We define *band generators* in B_n by

$$\sigma_{i,j} \;=\; \sigma_i \dots \sigma_{j-2}\sigma_{j-1}\sigma_{j-2}^{-1} \dots \sigma_i^{-1}\,, \tag{2.2}$$

For example $\sigma_{2,7} \in B_9$ looks



Notice that $\sigma_{i,i+1} = \sigma_i$. A representation of a braid $\beta \in B_n$ in the form

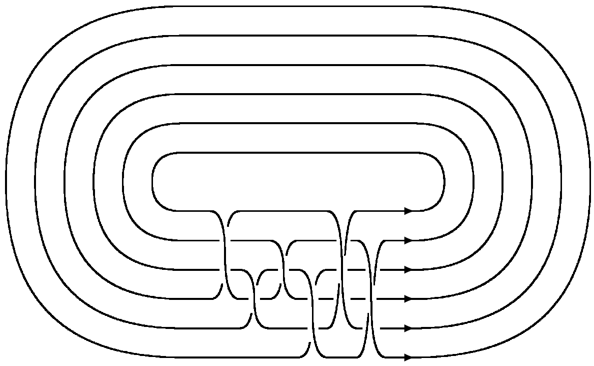
$$\beta \;=\; \prod_{k=1}^{l} \sigma_{i_k,j_k}^{\pm 1}$$

is called a *band presentation*. (See e.g. [BKL].) Usually, it will be more legible to use the symbol

$$[ij] = \sigma_{i,j}$$

when writing band generators in formulas. Similarly we use $-[ij] = \sigma_{i,j}^{-1}$. In certain cases, we even omit the brackets (see Definition 4.6 and Example 7.25). Also, when $j = i+1$, we often simply write i for σ_i and $-i$ for σ_i^{-1}, when no ambiguity arises. The image of β under the abelianization $B_n \to \mathbb{Z}$ is the *writhe* (or exponent sum) of β, and is written $w(\beta)$. A braid $\beta \in B_n$ whose *closure* $\hat{\beta}$ is the link L is a *braid representative* of L. Similarly a word for β gives a (braid closure) diagram $D = \hat{\beta}$ of L. When β is a word, then $w(\hat{\beta}) = w(\beta)$. A band presentation β naturally spans a Seifert surface of $L = \hat{\beta}$. Following

Rudolph, we call this a *braided surface* of L. For example, $n = 6$ and $l = 6$,



for the 6-braid $\beta = \sigma_{1,4}\sigma_{3,5}\sigma_{2,4}\sigma_{3,6}\sigma_{1,5}\sigma_{2,6}$ The diagram shows the closure $L = \hat{\beta}$. It is easily seen that the six 'elliptic' disks joined two by two with six twisted bands form a natural Seifert surface of L. Rudolph [Ru] proves the every Seifert surface is a braided surface. If a braided surface is of minimal genus for L, it is called a *Bennequin surface* of L [BM2].

A link is called *quasipositive* if it is the closure of a braid β of the form

$$\beta = \prod_{k=1}^{\mu} w_k \sigma_{i_k} w_k^{-1} \tag{2.3}$$

where w_k is any braid word and σ_{i_k} is a (positive) standard Artin generator of the braid group. (In [Ru4] there is some overview of this topic.) If the words $w_k \sigma_{i_k} w_k^{-1}$ are of the form σ_{i_k,j_k}, so that

$$\beta = \prod_{k=1}^{\mu} \sigma_{i_k,j_k}\,, \tag{2.4}$$

then they can be regarded as embedded bands. Links which arise this way, i.e., such with *positive band presentations*, are called *strongly quasipositive links*.

Bennequin's inequality [Be, Theorem 3] states

$$-\chi(L) \geq w - n \tag{2.5}$$

for a n-strand braid representative of L of writhe w. If there is a braid representative β of L making (2.5) an equality, we call both L and β *Bennequin-sharp*. This inequality was later extended to

$$-\chi(L) \geq -\chi_4(L) \geq w - n \tag{2.6}$$

(see e.g. [IS, St2]). In an analogous way we defined that L and β are *slice Bennequin-sharp*.

It implies that a *strongly quasipositive surface*, i.e., obtained from a positive band presentation, is minimal genus. Namely, a positive band presentation of w bands on n braid strands gives a braid of writhe w. Thus the surface S constructed from the band presentation yields, with (2.6),

$$-\chi(L) \leq -\chi(S) = w - n \leq -\chi_4(L) \leq -\chi(L)\,.$$

This also shows that a strongly quasipositive link L is always Bennequin-sharp, and

$$\chi_4(L) = \chi(L)\,. \tag{2.7}$$

The *Bennequin sharpness conjecture* (see [FLL, St2]) asserts

$$L \text{ is Bennequin-sharp} \iff L \text{ is strongly quasipositive}\,. \tag{2.8}$$

The second author's efforts to determine the quasipositivity of the (prime) 13 crossing knots [St4] also provides some evidence for a "4-ball" version of the Bennequin sharpness conjecture (2.8):

$$L \text{ is slice Bennequin-sharp} \iff L \text{ is quasipositive}. \tag{2.9}$$

In practical terms, every proof of non-quasipositivity of a knot passes via showing that it is not slice Bennequin-sharp.

Definition 2.1
- Let $b(K)$ be the braid index of K, the minimal number of strings of a braid representative of K.
- Let $b_b(K)$ be the Bennequin braid index of K, the minimal number of strings to span a Bennequin surface of K.
- When K is strongly quasipositive, let $b_{sqp}(K)$ be the minimal number of strings to span a strongly quasipositive surface of K (only positive bands).
- Further, for a Seifert surface S, let $b(S)$ be the minimal string number on which S is spanned as a braided surface.
- If S is a strongly quasipositive surface, let $b_{sqp}(S)$ be the minimal string number on which S is spanned as such (i.e., arises from a positive band presentation).

We have then (with the right inequality only valid for strongly quasipositive K)

$$b(K) \le b_b(K) \le b_{sqp}(K)\,, \tag{2.10}$$

and by definition, with S being a Seifert surface of K,

$$b_b(K) = \min\{\, b(S) \,:\, \chi(S) = \chi(K)\,\}\,, \qquad b_{sqp}(K) = \min\{\, b_{sqp}(S) \,:\, S \text{ strongly quasipositive}\,\}\,. \tag{2.11}$$

We will further discuss these relations in §5 and §8. We also feature the following result. It confirms an expectation originally formulated for $n = b(L)$ by Jones [J, end of §8] (later also referred to as the "weak" form) and subsequently extended by Kawamuro.

Theorem 2.2 (proof of the Jones-Kawamuro conjecture [DP, LaM]) For every link L, there is a number $w_{min}(L)$, so that every braid representative β of L on n strands of writhe w satisfies

$$|w - w_{min}(L)| \le n - b(L)\,. \tag{2.12}$$

Generally speaking, we will use this theorem to advance theoretical applications in our work, but for practical ones, another tool will be crucial, which we introduce next.

2.4 HOMFLY-PT polynomial

We use the *HOMFLY-PT polynomial* P [LiM], in the Morton [Mo] convention

$$P(\bigcirc) = 1\,, \qquad v^{-1}P_+ - vP_- = zP_0\,, \tag{2.13}$$

where P_+, P_- and P_0 refer to the polynomials of three links with diagrams equal except at one spot, where they contain the fragments of (2.1) from left to right. The right part of (2.13) is also called P's *skein relation*. We will use the suggestive notation $\min\deg_v P$ for minimal v-degree of (any monomial in) P, and similarly $\max\deg_v P$, and set $\operatorname{span}_v P = \max\deg_v P - \min\deg_v P$. We write $[P]_{z^k}$ for the coefficient of z^k in P, being a polynomial in v. Then, $[P]_{v^d}$ the coefficient of degree d in v (which is itself treated as a polynomial in z). Also set

$$\min\operatorname{cf}_v P = [P]_{v^{\min\deg_v P}} \tag{2.14}$$

to be the trailing (lowest degree) coefficient of P. The notation $P|_{z^{\geq k}}$ resp. $P|_{z^{\leq k}}$ resp. $P|_{z^{\neq k}}$ will mean (the polynomial consisting of) all terms in P of z-degree at least k resp. at most k resp. different from k. The z-variable is left inside. Thus $[P]_{z^k}$ is a polynomial in v, while $P|_{z^{\geq k}}$ is a polynomial in z, v. We occasionally refer to $P|_{z^{\leq k}}$ as a *(z-)truncated polynomial*. We emphasize that much of the useful information of P can be obtained from truncations thereof (like (2.20)), which are much faster (subexponentially) to compute than the full polynomial. A program that calculates such truncations was introduced in [St3], and we will extensively apply it below.

A CPU-parallelized upgrade of the truncated polynomial calculation was developed to settle the last 16 crossing prime knot standing to resolve for the below problem (5.1); it has now its own description page on [St4].

Two further standard properties of P are that for a link L of $\kappa(L)$ components, $\min\deg_z P(L) = 1-\kappa(L)$, and $P(L)$ contains only monomials $z^p v^q$ for p, q odd (resp. even) when $\kappa(L)$ is even (resp. odd). The mirroring behavior of P is (signed) v-conjugation:

$$P(!L)(v,z) = (-1)^{\kappa(L)-1} P(L)(v^{-1}, z)\,. \tag{2.15}$$

We further use the identity (see [LiM, Proposition 21])

$$P(v, v^{-1}-v) = 1\,. \tag{2.16}$$

By the MFW [Mo, FW] inequalities, the writhe w of an n-string band presentation of L satisfies

$$w+n-1 \geq \max\deg_v P(L) \geq \min\deg_v P(L) \geq w-n+1\,, \tag{2.17}$$

thus

$$\mathrm{MFW}(L) := \frac{1}{2}\,\mathrm{span}_v P(L) + 1 \leq b(L)\,, \tag{2.18}$$

where the left hand-side is the *MFW bound* for the braid index $b(L)$. If $\mathrm{MFW}(L) = b(L)$, we call L *MFW-sharp*.

When L is not MFW-sharp, there are ways to improve the braid index estimate using cables of L: when L' is a degree-c cable of L, then

$$\mathrm{MFW}(L') \leq b(L') \leq cb(L)\,,$$

thus

$$b(L) \geq \left\lceil \frac{1}{c}\,\mathrm{MFW}(L') \right\rceil\,. \tag{2.19}$$

The method is well explained in [MS] (certainly when $c=2$; some examples for $c=3,4$ can be found in [St3]). We refer to such estimates as the *cabled MFW*.

To relate this to said at the end of §2.3, we point out that MFW plus cabled versions thereof is efficient to determine the braid index of most links. In some cases alternative methods apply, but there is no link L known where (2.19) at least for sufficiently high c fails to give a sharp estimate. It it thus conjecturable that this is always the case (see [St4]):

Conjecture 2.3 For every link L there is a $c>0$ and a degree-c cable link L' of L making (2.19) sharp.

Obviously, when we can prove that a braid representative β of a link L is minimal, then we immediately also obtain $w_{min}(L) = w(\beta)$ in Theorem 2.2. However, it was also nocited in [St5] that once (2.19) (for some c) gives a sharp estimate of $b(L)$, it proves along the way that $w_{min}(L) = w(\beta)$ is unique. (And it is not too hard to derive (2.12) either from that argument.) Thus Theorem 2.2 provides a theoretical underpinning, but is neither partically helpful nor essential to determine $b(L)$ or $w_{min}(L)$ for a given L.

One main drawback of (2.19) is that in general the polynomial of a cable link L' is notoriously hard to calculate. But instead of the whole polynomial, we can use a truncation:

$$\mathrm{MFW}_d(L') = \frac{1}{2}\,\mathrm{span}_{\,v} P(L')|_{z^{\le d}} + 1 \le \mathrm{MFW}(L') \le b(L')\,. \tag{2.20}$$

We refer below to such type of estimate of the braid index as *truncated (cabled) MFW*.

Returning to surfaces, if follows from the right inequality in (2.17) that a Bennequin-sharp (in particular strongly quasipositive) link L satisfies

$$\min\deg_v P(L) \ge 1 - \chi(L)\,. \tag{2.21}$$

Morton also proves in [Mo] the *canonical genus inequality*, for any diagram D of L,

$$\max\deg_z P(L) \le c(D) - s(D) + 1\,. \tag{2.22}$$

The *Conway polynomial* ∇ is given by

$$\nabla(L)(z) = P(L)(1, z)\,. \tag{2.23}$$

The *determinant* of a knot K can be defined by

$$\det(K) = |\nabla(2\sqrt{-1})|\,. \tag{2.24}$$

This is always an odd number (when K is a knot).

The *Kauffman polynomial* $F = F(a, z)(K)$ will be needed at a few places for reference. In Remark 7.19, we use the following well-known properties: for every link L,

- $F(L)$ contains only monomials $a^p z^q$ for $p + q$ even
- $F(\sqrt{-1}, z)(L) = 1$.

2.5 Slice-torus invariants

We briefly recall Livingston's [Lv] formalism of "slice-torus invariant". An integer-valued invariant v on *knots* is a slice-torus invariant if

- $v(K) = -v(!K)$, and $v(-K) = v(K)$, where $-K$ is K with the reverse orientation
- additivity under connected sum: $v(K_1\#K_2) = v(K_1) + v(K_2)$
- crossing switch rule: $v(K_+) - v(K_-) \in \{0, 1\}$
- $v(K) \le g_4(K)$ (or equivalently $2v(K) \le 1 - \chi_4(K)$), and
- v satisfies Bennequin's inequality:
  $$2v(K) \ge w - n + 1$$
  for a braid representative of K on n strings and writhe w.

These properties are not minimal (i.e., some follow from special cases or combinations of others), but they are what we will use in §6. (We emphasize that it is not assumed v to be defined on multi-component links $\kappa(K) > 1$ in any way.)

There are two known instances of such an invariant, the Ozsváth-Szabó τ invariant, and (half of) Rasmussen's invariant s. Thus the concept was introduced mainly to give these two a uniform treatment. (The halved signature $\sigma/2$ satisfies the first four of the above five properties, but not the last.)

From the superposition of

$$2g(K) = 1 - \chi(K) \geq 1 - \chi_4(K) \geq 2v(K) \geq w - n + 1$$

with (2.5), it is straightforward that

$$\text{if } v(K) < g(K), \text{ then } K \text{ is not Bennequin-sharp}\,. \tag{2.25}$$

In relation, it follows that when K is quasipositive, then

$$v(K) = g_4(K)\,, \tag{2.26}$$

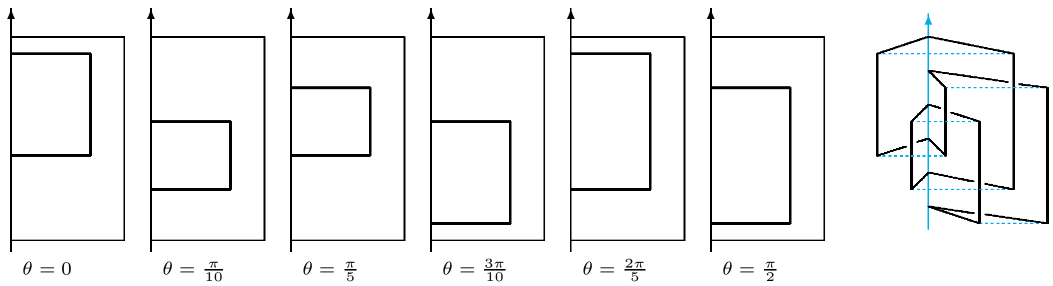
and

$$\text{when } K \text{ is strongly so, then } v(K) = g_4(K) = g(K). \tag{2.27}$$

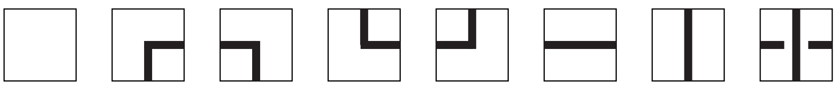
We will refer to these standard facts a few times below.

2.6 Grid diagrams and arc index

An *arc presentation* of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc.

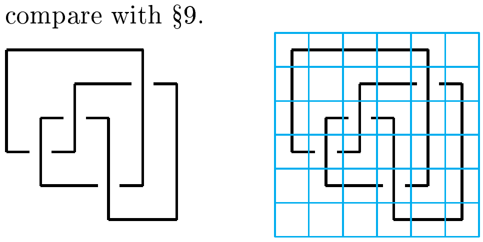


A *grid diagram* (or, for simplicity simply called *grid* often below) is a knot or link diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges. We assume that horizontal/vertical positions of vertical/horizontal edges are pairwise distinct. In particular, no edge touches another edge, no vertex lies on the interior of an edge, and vertices are pairwise distinct. Up to the adjustment of heights of horizontal and widths of vertical edges, a grid diagram is thus what can be composed in the plane by the tiles

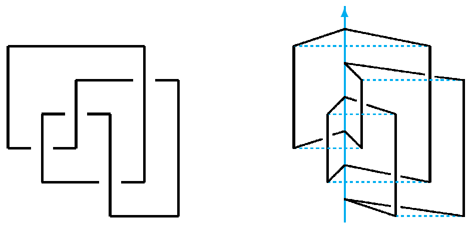


(2.28)

See, for example, [AL], and also compare with §9.



It is not hard to see that every knot admits a grid diagram (compare with the proof of Lemma 4.18). The figure below explains that every knot admits an arc presentation.



We set the *size* $\mu(D)$ of a grid diagram to be the number of vertical *or* (equivalently) horizontal segments (but *not both* together). A grid (diagram) of size μ will also be shortly called a μ*-grid*.

In general, we will afford the sloppiness of abolishing the distinction between an ordinary and a grid diagram, whenever the grid structure is unnecessary. Thus, for instance, $c(D)$ can mean the crossing number of both an ordinary and grid diagram, whereas $\mu(D)$ would imperatively assume that D is given a grid shape. Let $a(L)$ be the *arc index* of L, the minimal $\mu(D)$ over all grid diagrams D of L. It is the minimal number of pages among all arc presentations of a link L.

2.7 Knot tables

For notation from knot tables, we follow Rolfsen's [Ro, Appendix] numbering up to 10 crossings, except for the removal of the Perko duplication. For 11 and 12 crossing knots, the notation converts from [KI] by appending non-alternating knots after alternating ones of the same crossing number. Thus, for instance, $11a[k] = 11_{[k]}$ for $1 \leq k \leq 367$, and $12n[k] = 12_{1288+[k]}$ for $1 \leq k \leq 888$. For higher crossing knots, the tables of [HT] are used with the same adjustment.

If it is relevant, mirror images will be distinguished on a case-by-case basis. Specifically, for the $(2,n)$-torus knots, we will say that the knot is *positively/negatively mirrored*. The convention for 10_{132} is fixed in Example 4.10. (The knot exhibits certain phenomena that have to be treated for higher crossing knots as well, but being the only Rolfsen knot with such status, it will merit detailed attention.)

3 Upper bound of Crossing number

A *planar grid polygon* can be defined as the planar projection of a link grid diagram. Similarly we specify its size by the number of horizontal/vertical edges. It is obvious that a grid polygon determines uniquely (when crossing convention is fixed) a grid diagram, and vice versa. A grid polygon can have multiple *components* (i.e., PL-immersed circles). Such objects are of interest in discrete geometry; see for example [BOS, SG, KLA]. They are named *rectilinear* polygons, but for us it is (very) relevant that self-crossings are allowed (i.e., the polygons are *not simple*).

Lemma 3.1 Every, possibly multiple-component, planar grid polygon of size m has the following upper bounds on the number of intersections.

$$\begin{cases} (m^2-2m)/2 & m \text{ is even} \\ (m-1)^2/2-1 & m \text{ is odd} \end{cases} \tag{3.1}$$

These bounds are sharp.

Proof. What could be a conundrum becomes self-evident after introducing the right way of counting crossings. We will group crossings w.r.t. their horizontal segment l. We consider the horizontal segments from the middle high segment upward. Whenever l is such a segment and l_h is a horizontal segment above l, we define the *weight* $w_l(l_h) \in \{0,1,2\}$ to be the number of neighboring vertical edges of l_h intersecting l. Then

$$\#\{\text{ intersections of } l\,\} = \sum_{l_h \text{ above } l} w_l(l_h)\,. \tag{3.2}$$

This counting works because for each vertical edge l_v intersecting l, exactly one of its two neighboring horizontal edges l_h is above l. Thus

$$\#\{\,\text{intersections of } l\,\} \leq 2\#\{\,l_h \,:\, l_h \text{ above } l\,\}\,. \tag{3.3}$$

Now this sum will account to

$$\#\{\,\text{intersections of upper horizontal edges}\,\} \leq 2 \sum_{k=0}^{(m-1)/2} k \tag{3.4}$$

for the upper $(m+1)/2$ edges l when m odd. The lower $(m-1)/2$ edges l can be handled by choosing l_h to be below l, giving a similar sum

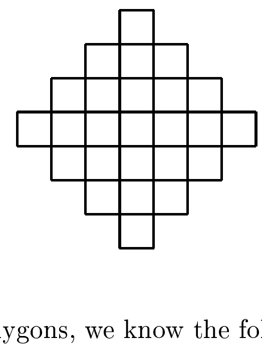
$$\#\{\,\text{intersections of lower horizontal edges}\,\} \leq 2 \sum_{k=0}^{(m-3)/2} k\,. \tag{3.5}$$

For m even one has $4 \sum_{k=0}^{m/2-1} k$ instead of (3.4)+(3.5). Direct calculation gives

$$\#\{\,\text{intersections of polygon}\,\} \leq \begin{cases} (m^2-2m)/2 & m \text{ is even} \\ (m-1)^2/2 & m \text{ is odd} \end{cases} \tag{3.6}$$

It remains to argue why for m odd, $(m-1)^2/2$ intersections are not possible. This would mean that the middle horizontal edge e satisfies $w_e(e') = 2$ for every other horizontal edge e'. But there are at least two edges e' for which this is not possible, namely those connected to the two vertical edges adjacent to e. This completes the proof of (3.1). For the sharpness of the bounds, see Example 3.2 and Proposition 3.3. □

Example 3.2 If we allow multiple components of the polygon, and consider m even, then the bound in (3.6) is sharp



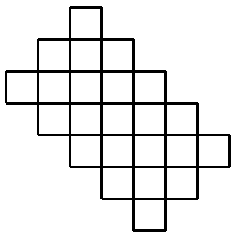
When we restrict to 1-component polygons, we know the following.

Proposition 3.3 For every $m > 2$, there exists a 1-component planar grid polygon Π_m of size m with

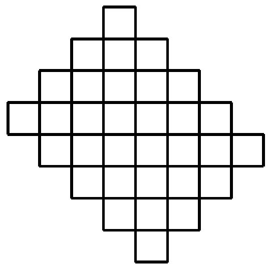
$$(m-1)^2/2 - \begin{cases} 1 & m \text{ is odd} \\ 5/2 & m \equiv 0 \bmod 4 \\ 7/2 & m \equiv 2 \bmod 4 \end{cases} \tag{3.7}$$

crossings.

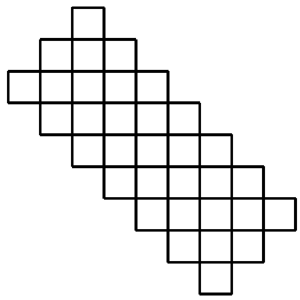
Proof. Consider the *Lissajous*[1] *polygon* $\Lambda(m_1, m_2)$.



$\Lambda(3,5)$

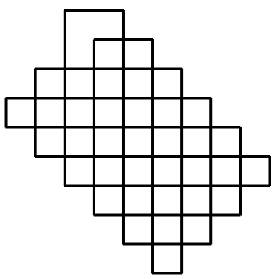


$\Lambda(4,5)$

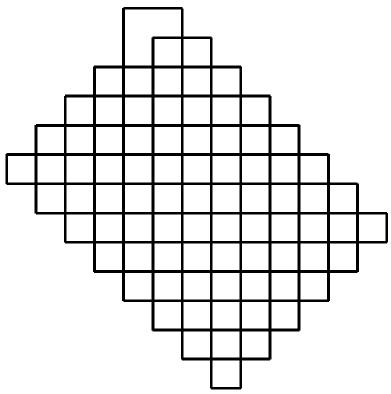


$\Lambda(3,7)$

This gives a grid polygon of size $m = m_1 + m_2$. When $m_1 - m_2 = 2$, we also need the *modified Lissajous polygon* $\Lambda'(m_1, m_2)$.



$\Lambda'(4,6)$



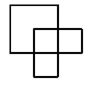
$\Lambda'(6,8)$

Then choose

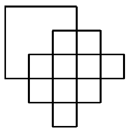
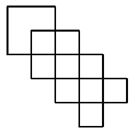
$$\Pi_m = \begin{cases} \Lambda(\frac{m-1}{2}, \frac{m+1}{2}) & m \text{ is odd} \\ \Lambda(\frac{m}{2} - 1, \frac{m}{2} + 1) & m \equiv 0 \bmod 4 \\ \Lambda'(\frac{m}{2} - 1, \frac{m}{2} + 1) & m \equiv 2 \bmod 4 \end{cases} . \tag{3.8}$$

The crossing number of these polygons (3.7) follows by explicit calculation. They are 1-component by direct inspection. (In general, $\Lambda(m_1, m_2)$ appears to be 1-component when m_1 and m_2 are relatively prime.) □

These examples leave not much room for improvement. For odd m, (3.1) settles their maximality. When m is even, by modifying the argument at the end of the proof of Lemma 3.1 to the middle two horizontal edges, it is also easy to conclude that (3.1) cannot be made sharp by a 1-component polygon. Thus the examples of Proposition 3.3 can be improved by at most 1 crossing for $m \equiv 0 \bmod 4$ and by at most 2 crossings for $m \equiv 2 \bmod 4$.

Computation 3.4 Still, verifying whether the (1-component) polygons Π_m have maximal crossings (for even m) is not entirely trivial. We wrote a computer program to test this, which in fact found the family Λ' in (3.8). For $m = 4, 6$ there are exceptional maximal crossing polygons $\Lambda'(2,2) =$ 

[1] We chose this name since they appear as rectifications of Lissajous curves, although this correspondence is not precise for every (m_1, m_2).

(as compared to $\Lambda(1,3)$), and , as compared to $\Lambda'(2,4) =$ . We know that the polygons (3.8) are maximal crossing for even m with $8 \leq m \leq 24$.

Certainly, we are interested more in grid diagrams of links, with crossing information, and in that case Lemma 3.1 easily modifies to show the following.

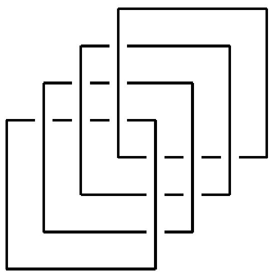
Lemma 3.5 Every grid link diagram D of size $\mu(D) = m$ has writhe $|w(D)| \leq (m-1)^2/4$.

Proof. It is essentially the same proof as for Lemma 3.1, except noting that (3.2) modifies to

$$\sum (\text{ signs of intersections of } l\,) \leq \sum_{l_h \text{ above } l,\ w_l(l_h) = 1} w_l(l_h)\,, \tag{3.9}$$

because signs of crossings on l coming from l_h with $w_l(l_h) = 2$ cancel out. Then the estimates (3.3) and (3.6) exactly halve. □

Example 3.6 If we allow multiple components of the link, and consider m even, then again the bound in Lemma 3.1 (in the form of halving (3.6)) is sharp, as shows the $(m/2, m/2)$-torus link:



At the cost of decreasing the number of crossings by $O(m)$, one may obtain a knot, like the $(m/2, m/2+1)$-torus knot, or adjust m to be odd.

Remark 3.7 Note that Lemma 3.1 immediately gives

$$c(L) < (a(L) - 1)^2/2 \tag{3.10}$$

for every link L. While some reductions may be possible under link isotopy (which is reflected in Cromwell's moves [Cr]), we have failed to significantly improve upon this estimate. This problem develops serious enough to merit a separate account if later progress is made. In contrast to the well-known bounds in [BP, JP] (see the proof of Corollary 5.5), it is striking that an *upper* estimate of the crossing number of a link in terms of its arc index has apparently never been previously considered in the literature. We note, though, that in Example 3.6, the featured torus links appear in minimal crossing number diagrams, so that the right of (3.10) cannot be reduced by more than a factor of 2 (asymptotically in $a(L)$).

4 Thurston-Bennequin invariant

4.1 Weight model for the Thurston-Bennequin invariant

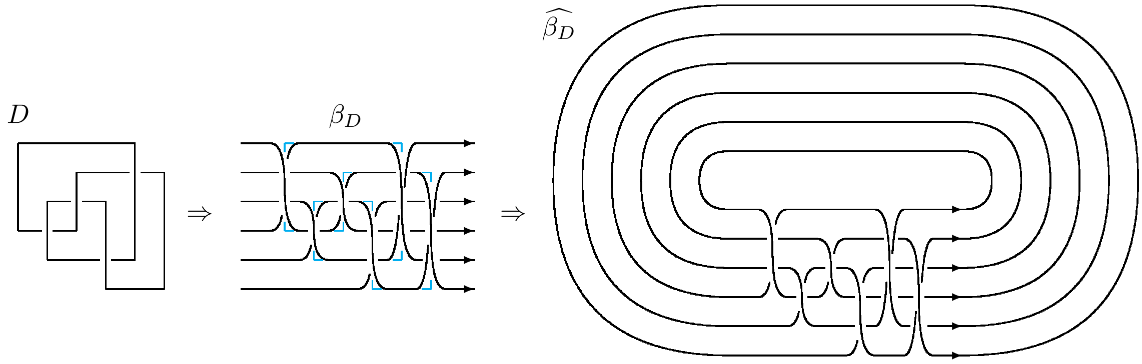
The main topic of this work starts from the observation that a braided surface of Euler characteristic 0, which is a K-knotted annulus, is essentially a grid diagram of the underlying companion knot K.

Definition 4.1 Let for a knot K and integer t,

- $A(K,t)$ be the (link of the) t-framed K-knotted annulus,
- $W_+(K,t)$ and $W_-(K,t)$ the t-framed Whitehead doubles of K with positive and negative clasp, and
- $B(K,t)$ the t-framed Bing double of K.

We will usually abuse the distinction between the annulus and the link which is its boundary. To disambiguate among different conventions for framing used elsewhere, we specify that for us, for example, $A(\bigcirc,1)$ is the positive (right-hand) Hopf link, and $A(\bigcirc,-1)$ the negative one. Also, $W_+(\bigcirc,1)$ is the positive (right-hand) trefoil, and $W_+(\bigcirc,-1)=W_-(\bigcirc,1)$ the figure-8-knot. We can understand $W_+(K,t)$ resp. $W_-(K,t)$ as the result of plumbing a positive resp. negative Hopf band into $A(K,t)$ and taking the knot which is the boundary of the resulting Seifert surface. In a similar way, we can understand $B(K,t)$ as the 2-component link which is obtained by plumbing both a positive and a negative Hopf band into $A(K,t)$ and taking the boundary. Thus for instance $B(\bigcirc,0)$ is the 2-component unlink, and $B(\bigcirc,1)$ is the Whitehead link.

Let D be a grid diagram of a knot K. Replacing each vertical segment with a half twisted band as shown below, we get a braid in band presentation, denoted by β_D. (Compare with [Nu, Theorem 3.1].) Then the closure $\widehat{\beta_D}$ bounds a twisted annulus. Therefore $\widehat{\beta_D}=A(K,t)$ for some t.



(4.1)

Consider the situation that the band presentation is positive. Then obviously $A(K,t)$ for the resulting framing t is strongly quasipositive. A question is what is the framing t, which we will write as

$$t=\lambda(D)\,, \tag{4.2}$$

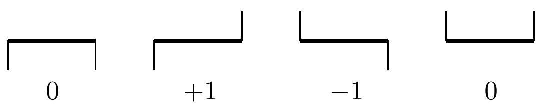
in dependence of the diagram D, and how to read $\lambda(D)$ off D. To explain the formula for $\lambda(D)$, given below as (4.5), we fix some notation.

Let the *weight* of a grid diagram D be

$$Z(D)=\frac{1}{2}\sum_{e\text{ edge of }D}\operatorname{sgn}(e)\,, \tag{4.3}$$

where the signs of the edges are determined as follows:

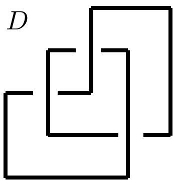
$$\operatorname{sgn}(e)=\begin{cases}1 & e\text{ is vertical}\\ \pm1,\ 0 & e\text{ is horizontal and one of the following forms}\end{cases}$$



(4.4)

Example 4.2

D



$$Z(D) = \frac{1}{2}\left(\underbrace{(1+\cdots+1)}_{\text{vertical, L to R}} + \underbrace{(0+0+1+0+0)}_{\text{horizontal, B to T}}\right) = 3$$

Remark 4.3 This weight formula (4.3) can be generalized to non-positive band presentations by letting each vertical edge have the sign of the corresponding band. But we will treat this more general case only occasionally here.

We then can identify the framing t in (4.2).

Lemma 4.4 With $w(D)$ being the writhe, we have

$$\lambda(D) = Z(D) - w(D)\,. \tag{4.5}$$

Proof. One can see than when the Seifert circles of the closed braid diagram $\hat{\beta}_D$ of $A(K,t)$ are made small, one obtains a diagram of $A(K,t)$, where the band obtained by thickening D is twisted. By a straightforward combinatorial observation, the twisting of the band is given by $Z(D)$. The untwisted band built from D carries the framing $-w(D)$ itself. This gives (4.5). $\square$

Remark 4.5 One has a certain freedom to vary the direction from which to read the bands of the braid representative β_D of $A(K,t)$ off the grid diagram D of K. We explain our convention here in an example to make clear how band presentations are used for specific knots below. While horizontal and vertical edges are easily interchangeable in grids, disks and bands are far less so in braid band presentations. The change of direction will give different t, but will change K only up to mirroring.

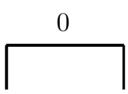
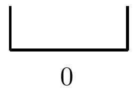
The default direction of reading will be from the left. Reading the grid diagram D in (4.1) from the left gives the word $[14][35][24][36][15][26]$, with $[ij] = \sigma_{i,j}$ of (2.2). Reading D from the right is meant to give the reverse order of (band) letters. This is the result of reading from the left a grid diagram D', which is obtained from the mirror image $!D$ after a flip (π rotation along the vertical axis). Reading D from the bottom gives the word $[46][25][13][24][36][15]$, which arises when reading from the left $!D$ after a rotation by $-\pi/2$. Reading from the top again reverses these letters and results in reading from the left D after the combination of a flip (along the horizontal axis) and $-\pi/2$ rotation. Note that thus we consider braid strands numbered right to left when reading a grid diagram from the right and from the top (while left to right otherwise).

Definition 4.6 Also, we can use the band presentation of β_D to specify the grid diagram D itself (see Example 7.25). The mirroring of D is fixed by default by saying that β_D should be obtained when reading D from the left. This means that we can write the grid diagram D in (4.1), even omitting brackets, as

`14  35  24  36  15  26`.

Since we deal with grids of size 10 or more, let us also already fix here that we use initial capital Latin letters to denote two-digit integers, so that for example, `4C` $= [4,12] = \sigma_{4,12}$.

Let $br(D)$ be the *vertical bridge number* of D, which is the number of sign-0 horizontal edges of D of one of either types in (4.4)

$br(D) := \#$ () $= \#$ ()

Lemma 4.7 We have

$$br(D) \le Z(D) \le \mu(D) - br(D)\,, \tag{4.6}$$

and thus

$$br(D) - (\mu(D)-1)^2/4 \le \lambda(D) \le (\mu(D)+1)^2/4 - br(D)\,. \tag{4.7}$$

Proof. The left inequality in (4.6) holds because each piece of the knot between two vertical extrema contributes at least 1 to the weight sum, and we have $2br(D)$ such pieces. The right inequality holds because there are $2br(D)$ edges in D with sign 0. By using Lemma 3.5,

$$-(\mu(D)-1)^2/4 \le w(D) \le (\mu(D)-1)^2/4\,. \tag{4.8}$$

Then, with (4.6), (4.5) and (4.8), we obtain (4.7). □

Let $a(K)$ be the arc index of K, the minimal $\mu(D)$ over all grid diagrams D of K. The so far best idea is, take a minimal grid diagram $\mu(D) = a(K)$. Then, with (4.6), (4.5) and (4.8), we have

$$br(D) - (a(K)-1)^2/4 \le \lambda(D) \le (a(K)+1)^2/4 - br(D)\,.$$

Thus we have:

Theorem 4.8 There exists a number $\lambda_{min}(K)$ with

$$br(K) - (a(K)-1)^2/4 \le \lambda_{min}(K) \le (a(K)+1)^2/4 - br(K)\,, \tag{4.9}$$

such that for all $t \ge \lambda_{min}(K)$, we have that $A(K,t)$ is strongly quasipositive on $b_{sqp}(A(K,t)) \le a(K) + t - \lambda_{min}(K)$ strands.

We will use $\lambda_{min}(K)$ often in the following. Two caveats are in order regarding this notation. The 'min' refers to the minimum with respect to number of strings of the surface $A(K,t)$ (or horizontal segments in the grid diagram of K), *not* the framing t itself. And, it is *not* assumed that λ_{min} is unique. At least for the unknot K,

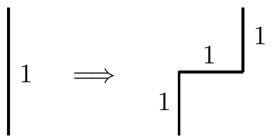
$$\text{both } b(A(\bigcirc,0)) = b(A(\bigcirc,1)) = 2, \text{ thus } \lambda_{min}(\bigcirc) = 0, 1. \tag{4.10}$$

This special behavior of unknot will require repeated attention. For a non-trivial knot K, the uniqueness and minimality of $\lambda_{min}(K)$ was settled, as will be discussed below; see Theorem 5.12. But we do not wish to exclude $K = \bigcirc$ consistently. We prefer to maintain the symbol $\lambda_{min}(K)$, stipulating that formulas involving $\lambda_{min}(K)$ are meant to hold whatever of either values (4.10) is chosen for $K = \bigcirc$. For $K \ne \bigcirc$, the reader may assume that

$$\lambda_{min}(K) = \lambda(K)\,, \tag{4.11}$$

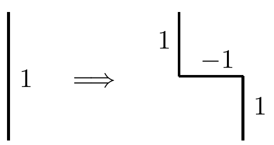
though we will not use this before stating Theorem 5.12.

Proof of Theorem 4.8. When μ is augmented by 1, we can always augment by 1,



(4.12)

resp. preserve



(4.13)

any given framing $\lambda(D)$ by the above two moves. (Apply an adjusting PL-map on the half-planes above and below the newly formed horizontal edge.) We call these moves in the following positive and negative *stabilization*, resp. Thus, $\lambda(D)$ augments by 1 under positive stabilization, and negative stabilization does not change $\lambda(D)$. (Neither stabilization changes $w(D)$. Note that the diagram D_1 of $A(K,t)$ obtained from D always has $s(D_1)=\mu(D)$ Seifert circles and writhe $w(D_1)=\mu(D)$.) The claim follows for $a(K)+t-\lambda_{min}(K)$ strands from positive stabilization, and for larger strand number by (further) negative one. $\square$

The *Thurston-Bennequin invariant* $TB(D)$ of a grid diagram D can be defined as is being identified in the following theorem.

Theorem 4.9 For any grid diagram D, the quantity $Z(D)$ counts the NE- or SW-corners of D.

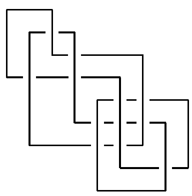
$$Z(D)=\#\left(\ulcorner\quad \text{NW-corners}\right)=\#\left(\lrcorner\quad \text{SE-corners}\right) \tag{4.14}$$

Thus

$$\lambda(D)=-TB(D)\,.$$

Proof. The first is a combinatorial observation. The second follows from the characterization of $TB(D)$ given in [LN] or [Ng]. $\square$

Example 4.10 The [J+] diagram D' of 10_{132},



10_{132}

read from the right (see Remark 4.5), gives the 9-strand band presentation

$$\beta_D=[25][14][37][26][15][48][79][38][69]\,, \tag{4.15}$$

where $D=\text{flip}\,(!D')$. We have $\mu(D)=9$, $Z(D)=3$, $w(D)=2$, $br(D)=3$ and $\lambda(D)=1$. Thus (4.15) gives a (positive) band presentation of $A(10_{132},1)$. The mirroring of 10_{132}, determined by D, is so that it *has the P polynomial of the positively mirrored* 5_1. We fix this mirroring in the sequel, since we will illustratively feature the knot a few more times. Note that it is thus *opposite* to Rolfsen's [Ro, Appendix] mirroring.

We also remark the following straightforward consequence of Theorem 4.9.

Corollary 4.11 When the grid diagram $!D$ is obtained from D by switching all crossings, and a $-\pi/2$ rotation, then $\lambda(D)+\lambda(!D)=\mu(D)$.

Proof. The writhe terms of D and $!D$ in (4.5) cancel out. Thus $\lambda(D)+\lambda(!D)=Z(D)+Z(!D)$. By taking the average of the two corner counts in (4.14) for D and its $-\pi/2$ rotation, we see that $Z(D)+Z(!D)$ is half the number of all corners of D, which is $\mu(D)$. $\square$

4.2 Application to strong quasipositivity

Let $TB(K)$ be the *maximal Thurston-Bennequin invariant* of K, an invariant often considered in contact geometry [FT, LN, Ng, Ma, Ru3]:

$$TB(K)\,:=\,\max\{\,TB(D)\,:\,D\text{ is a diagram of }K\,\}\,.$$

We also specify a region which will play an important role throughout the rest of the paper.

Definition 4.12 We define the *framing diagram* $\Phi(K)$ of K as a subset of $\mathbb{R}^2$ by

$$\Phi(K) := \{ (\mu, t) : A(K, t) \text{ has a strongly quasipositive band representation on } \mu \text{ strands} \}.$$

The following result of Rudolph [Ru3, Proposition 1] then follows directly from Theorem 4.9. (Note our different sign convention for t.)

Corollary 4.13 When K is not the unknot, then

$$\lambda(K) := \min\{ t : A(K, t) \text{ is strongly quasipositive} \} = -TB(K), \tag{4.16}$$

and more precisely,

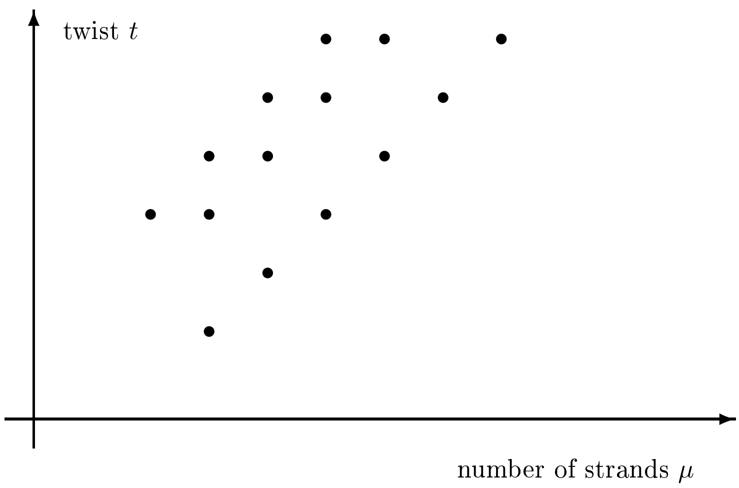
$$A(K, t) \text{ is strongly quasipositive} \iff t \geq -TB(K). \tag{4.17}$$

Proof. When $A(K, t)$ has a positive band presentation, then every disk is connected by at least two bands. Disks connected by one band can be removed, and such connected by no band do not exist unless $K = \bigcirc$ (and $t = 0$), which was excluded. Since $\chi(A(K, t)) = 0$, it follows that every disk is connected by exactly two bands, which means that the band presentation of $A(K, t)$ gives rise to a grid diagram of K. It is also well known that every integer $t \geq -TB(K)$ can be realized as Thurston-Bennequin invariant of some grid diagram of K. (We mentioned this above; see (4.12).) Note in passing that undoing the removal of disks connected by one band is, up to conjugacy, *positive braid stabilization*. This also shows that

$$(\mu, t) \in \Phi(K) \Longrightarrow (\mu + 1, t) \in \Phi(K), \tag{4.18}$$

which equals the effect of the *negative grid* stabilization (4.13). □

The following diagram illustrates the effect of the *positive* grid stabilization within $\Phi(K)$:



For the unknot K, we have

$$-TB(\bigcirc) = 1 \text{ but } \lambda(\bigcirc) = 0. \tag{4.19}$$

The problem with (4.16) there is that $A(K, 0)$ has the empty positive band presentation (on two strands), but we do not consider this band presentation corresponding to a grid diagram. For this reason, the unknot will repeatedly require special attention below. Despite the identification (4.16), $\lambda(K)$ will occur so often, that it is better to maintain the notation and avoid writing the minus sign most of the time, even when we exclude $K = \bigcirc$.

Remark 4.14 It is possible to derive similar properties for links K. Then a framing t is needed for each component, and the relationship in Corollary 4.13 becomes slightly more involved, as become the framing diagram of Definition 4.12 and its properties. We do not wish to deal extensively with links here. However, in situation where the surface structure is forgotten, the more self-contained extensions to links do emerge, as for Corollaries 5.4, 5.5, and 5.6.

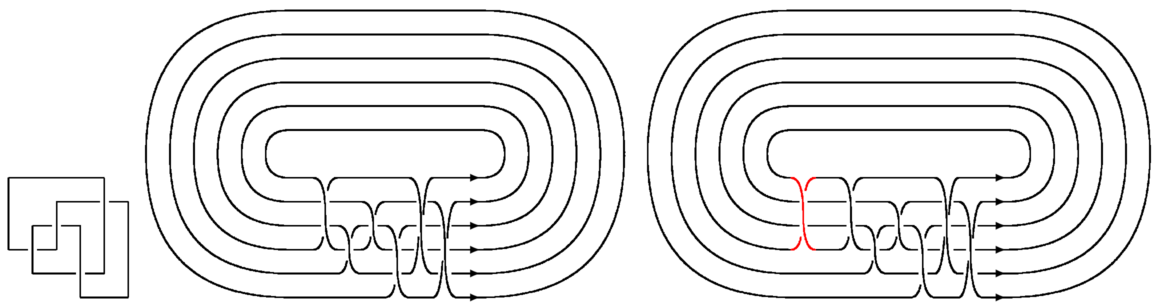
In the following application we assume that $K \neq \bigcirc$. For $K = \bigcirc$, all the links in Definition 4.1 are (alternating) 2-bridge links, and such can be handled *ad.hoc.* for strong quasipositivity (see e.g. [Ba]).

Corollary 4.15 Let K be a non-trivial knot. Then
(a) $W_+(K,t)$ is strongly quasipositive if and only if $t \geq -TB(K)$, and
(b) $W_-(K,t)$ and $B(K,t)$ are never strongly quasipositive.

Proof. The minimal genus surface of $W_\pm(K,t)$ is unique. This is proved by Whitten [Wh], but follows also from a result of Kobayashi [Ko]: the plumbing $S_1 * S_2$ is a unique minimal genus surface if and only if one of $S_{1,2}$ is a unique minimal genus surface and the other one is a fiber surface. The minimal genus surface of $W_\pm(K,t)$ is a Hopf band plumbed into the annulus $A(K,t)$ (which is obviously the unique minimal genus surface of $A(K,t)$; compare below Definition 4.1). Kobayashi's version also shows that plumbing two Hopf bands into $A(K,t)$ gives a unique minimal genus surface for $B(K,t)$. It follows from Rudolph's results on Murasugi sum [Ru2] that $W_-(K,t)$ and $B(K,t)$ are never strongly quasipositive: their unique minimal genus surfaces Muraugi desums into pieces, not all of which are strongly quasipositive. Also, $W_+(K,t)$ is strongly quasipositive if and only if $A(K,t)$ is. $\square$

Since we will need this repeatedly later, let us already here notice that the Hopf plumbing $W_+(K,t) = A(K,t) * H$ can be realized by doubling a(ny) positive band in a band presentation of $A(K,t)$.

Example 4.16



4_1 $\qquad$ $A(4_1,3)$ $\qquad$ $W_+(4_1,3)$

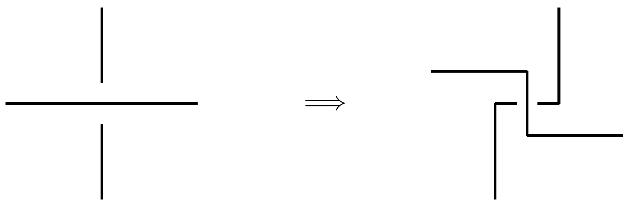
A similar remark applies to $W_-(K,t)$ whenever a band presentation of $A(K,t)$ has a negative band. However, it is important to note that this is not the only way to generate positive band presentations of Whitehead doubles. (A different example for a trefoil Whitehead double is given in [Be, fig p. 121 bottom].) We will discuss Whitehead doubles further in §6.

Here, we give a simple application of the weight model, in estimating the Thurston-Bennequin invariant. A counterpart will emerge with Lemma 7.4 from the HOMFLY-PT polynomial.

Definition 4.17 Define $pbr(D)$, the *plane-bridge number* of D as the minimal number of Morse maxima (or minima, i.e., half of the minimal number of Morse extrema) over all smooth diffeomorphic images of D in S^2. Obviously $br(K) \leq pbr(D)$.

Lemma 4.18 For any diagram D of K, we have $\lambda(K) \leq 2c_-(D) + pbr(D)$.

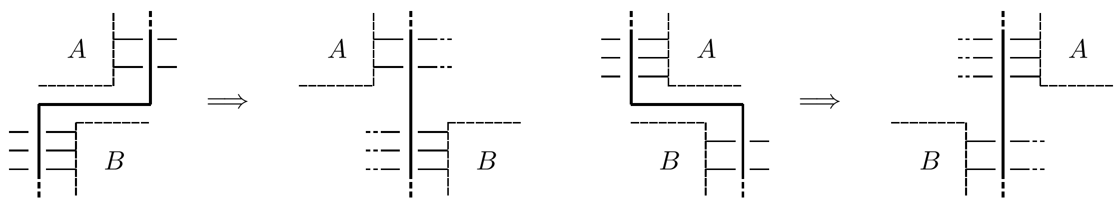
Proof. First, we can make D into a grid diagram by straightening out edges, and replacing wrong crossings, i.e., those with the horizontal strand on top, as follows.



(4.20)

(Again, as for the stabilization moves that occurred in the proof of Theorem 4.8, some small PL adjustment is needed.) This does not change the number of bridges.

Next, the horizontal adjustment technique (4.21) can be used to delete a horizontal edge e of label ± 1 without crossing on it, again by applying a suitable PL-map on the half-planes above and below e. This is the reverse of the stabilization moves, but we may need in advance to displace possible vertical edges above or below e. (If necessary, extend the box A resp. B drawn in the following figures until above resp. below the entire grid diagram, to ensure that all edges enter the box horizontally.)



(4.21)

The inequality we wish to prove about the diagram resulting after a move (4.21) is equivalent to the one about the original diagram. We may therefore assume henceforth that all ± 1 signed horizontal edges are intersected by a crossing. Thus

$$c(D) \geq \mu(D) - 2br(D)\,. \tag{4.22}$$

Also, we can see

$$Z(D) \leq \mu(D) - br(D)\,,$$

by using that in (4.3) there are $2br(D)$ edges of label 0.

Then

$$\begin{aligned} \lambda(K) &\leq Z(D) - w(D) \\ &\leq \mu(D) - br(D) - w(D) \\ &\leq c(D) + br(D) - w(D) \\ &= 2c_-(D) + br(D)\,. \end{aligned}$$

The rest follows by minimization. $\square$

5 Braid indices

We discuss some relation regarding the braid indices in Definition 2.1. (Compare[2] with [Nu, Section 3.3].) As noticed, Bennequin's inequality (2.5) implies that a strongly quasipositive surface is a Bennequin

[2] We became aware of Nutt's paper only at a very advanced stage of this work, and apologize for some overlap.

surface, thus for K strongly quasipositive, we have (2.10). We know that $b_b(K) > b(K)$ is possible [HS], but the examples K known are not strongly quasipositive. Rudolph conjectures that

$$b_{sqp}(K) = b(K) \tag{5.1}$$

when K is strongly quasipositive, and this is true, among other families, if K is a prime knot of up to 16 crossings (see [St2]). By the proof of the Jones-Kawamuro conjecture (Theorem 2.2), a Bennequin surface of a strongly quasipositive link K on $b(K)$ strands is always strongly quasipositive, so that

$$b_b(K) = b(K) \tag{5.2}$$

implies (5.1) for strongly quasipositive knots K. The problem (5.2) is extensively studied in [St2].

Since a band presentation of $A(K,t)$ always comes from a grid diagram of K, and with a confirmative notice about the unknot, we have:

Corollary 5.1

$$\min\{b_b(A(K,t))\,:\,t\in\mathbb{Z}\} = a(K)\,. \tag{5.3}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the minimum.

Proof. The case that K is the unknot can be handled directly: the minimizing t are $t=-1,0,1$.

When K is not the unknot, every maximal Euler characteristic (equal to 0) band presentation of $A(K,t)$ comes from a grid diagram of K. This shows

$$\min\{b_b(A(K,t))\,:\,t\in\mathbb{Z}\} \geq a(K)\,.$$

To see the reverse inequality, take a minimal grid diagram D of K. This gives a positive band presentation β_D of $A(K,t)$ for $t=\lambda_{min}(K)$. Now consecutively turn the $a(K)$ bands negative, which gives band presentations of Bennequin surfaces for $A(K,t)$ where $t=\lambda_{min}(K)-a(K),\dots,\lambda_{min}(K)$. □

Also, because choosing positive bands will give a band presentation of a strongly quasipositive annulus, we have with Corollary 4.13:

Corollary 5.2 $\min\{b_{sqp}(A(K,t))\,:\,t\geq\lambda(K)\} = a(K)\,.$ □

Forgetting the surface structure then yields an inequality of (ordinary) braid indices:

Corollary 5.3

$$\min\{b(A(K,t))\,:\,t\in\mathbb{Z}\} \leq \min\{b(A(K,t))\,:\,t\geq\lambda(K)\} \leq a(K) \tag{5.4}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the inequality $b(A(K,t))\leq a(K)$. □

The braid index of a link $A(K,t)$ is obviously not less than the sum of the braid indices of constituent components. Thus from Corollary 5.3, we also immediately have an inequality, which was noticed by Cromwell [Cr] (with the extension to links K as explained in Remark 4.14):

Corollary 5.4 (Cromwell) For every knot K, we have $2b(K)\leq a(K)$. □

We obtain then the following (slight) refinement of Ohyama's inequality [Oh]:

Corollary 5.5 We have $b(K)\leq c(K)/2+1$, and if K is non-alternating, then $b(K)\leq c(K)/2$.

Proof. It is known that $a(K) \leq c(K)+2$, as proved in [BP], and $a(K) \leq c(K)$ for K non-alternating [JP]. $\square$

Since $b(K) \geq br(K)$, it further follows:

Corollary 5.6 For any knot K, we have $2(br(K)-1) \leq c(K)$. If K is non-alternating, then $2br(K) \leq c(K)$. $\square$

In the obvious extension to links, connected sums of Hopf links show that the (first) inequality is sharp. But there is a more precise conjecture, apparently due to Fox [Fo], and later studied and extended by Murasugi [Mu]. For *knots* K, it states

$$3(br(K)-1) \leq c(K)\,.$$

These useful implications are worth noting, but we will see below that it is much more important to work with (5.4) rather than its simplified variant of Corollary 5.4.

We are next going discuss what (say, strongly quasipositive) framings λ are possible for given grid size μ, and in particular whether λ_{min}, the framing for a minimal (size $a(K)$) diagram (see Theorem 4.8) is unique. Since μ bounds the braid index of $A(K,t)$, and all have the same χ, Birman-Menasco [BM] imply that $\lambda(\mu)$ are finitely many. We will later prove a more precise statement (Finite-Cone-Theorem 7.3).

Question 5.7 (a) Is $b(A(K,t)) \geq a(K)$ for any t?
(b) At least, is $b(A(K,t)) \geq a(K)$ for any strongly quasipositive $A(K,t)$?

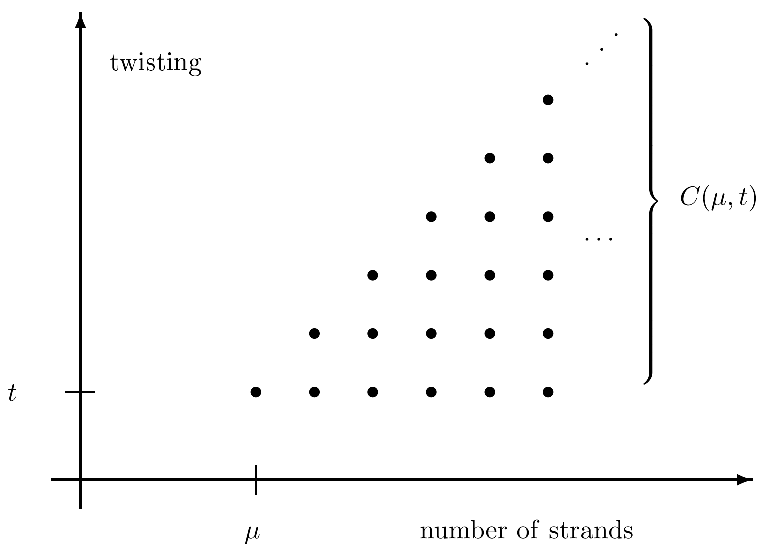
If (b) fails, then it would give an example $A(K,t)$ answering negatively Rudolph's question (5.1). This question will be further treated in Remark 7.21 and Conjecture 8.1.

To formalize this topic better, we introduce notation relating to the two grid stabilizations (4.12) and (4.13).

Definition 5.8 We define the *cone* $C(\mu,t) \subset \mathbb{Z}_+ \times \mathbb{Z}$ by

$$C(\mu,t) = \{\ (s,\lambda):\ s \geq \mu,\ t \leq \lambda \leq t+s-\mu\ \}\,.$$

We say (μ,t) is the *tip* of the cone.



We can summarize some properties we have derived regarding the region $\Phi(K)$ of Definition 4.12.

Theorem 5.9 (a) The framing diagram $\Phi(K)$ of K is a union of cones.
(b) It contains at least one cone of the form $C(a(K), \lambda_{min}(K))$ and one of the form $C(\mu, \lambda(K))$.
(c) It contains no points with $t < \lambda(K)$ and $\mu < a(K)$.
(d) Every point $(\mu, t) \in \Phi(K)$ satisfies

$$br(K) - (\mu - 1)^2/4 \leq t \leq (\mu + 1)^2/4 - br(K)\,. \tag{5.5}$$

$\square$

This estimate (5.5), that comes from (4.7), is rather crude, due to our insufficient control over the writhe. One problem with (4.8) is that, while it can be (at least asymptotically) sharp on either side, this unlikely happens (simultaneously) for diagrams D of the same link. Methods to address the writhe variation exist, based on Thistlethwaite's work on the Kauffman polynomial, but they will lead to no pleasant results here. A far more efficient technique will be introduced later, which ultimately leads to Proposition 8.6. This will yield much sharper bounds than (5.5), especially when K is fixed and μ is large. However, we emphasize that neither (5.5), nor the inequalities in Lemmas 4.18 and 7.4, follow from alternative estimates we obtain (or, to the best of our knowledge, other known results).

We announced that we will prove later (Finite-Cone-Theorem 7.3) the cones are finitely many. The following Jones-Kawamuro type of conjecture (compare with Theorem 2.2) is then suggestive.

Question 5.10 If K is non-trivial, is $\Phi(K)$ a single cone? (This cone would have to be then $C(a(K), \lambda_{min}(K))$ with $\lambda_{min}(K) = \lambda(K)$.)

Example 5.11 According to (4.10), we have

$$\Phi(\bigcirc) = C(2, 0) \cup C(2, 1)$$

being the union of two cones.

The special case for $\mu = a(K)$ in Question 5.10 (an analogue of the "weak" form of the Jones-Kawamuro conjecture) was already raised in [Ng] in the language of grid diagrams D and Thurston-Bennequin invariants $TB(D)$. It was answered in [DP, Corollary 3].

Theorem 5.12 (Dynnikov-Prasolov [DP]) The Thurston-Bennequin invariant of minimal grid diagrams of a given knot K is always equal to $TB(K)$.

We will return to this statement in §7.4 and §8.1. Note that the unknot creates no exception here, when using TB instead of λ and avoiding the discrepancy (4.19). Despite its importance, we do not use Theorem 5.12 substantially; it brings only minor simplifications, which can be worked around. We indicate this at a few places, but skip doing it consistently, because it does not seem a reasonable course of action.

6 Jump invariant

Turning to Whitehead doubles, Ozsváth-Szabó defined a number $j_\tau(K)$, the *jump invariant* of τ, with

$$\tau(W_+(K, t)) = \begin{cases} 1 & t \geq j_\tau(K) \\ 0 & t < j_\tau(K) \end{cases}\,. \tag{6.1}$$

The existence of such a number can be seen easily from Livingston's properties of slice-torus invariants §2.5. We have $g(W_+(K, t)) = 1$, so for strongly quasipositive $T = W_+(K, t)$ we have $\tau(T) = 1$ (see (2.27)). Also $W_+(K, t) \to W_+(K, t - 1)$ and $W_+(K, t) \to \bigcirc$ by a positive-to-negative crossing change, thus $\tau(T) \in \{0,1\}$. It is not immediately clear that $\tau \not\equiv 1$, i.e., $j_\tau(K) > -\infty$, but this is known, and we will also be able to derive it in Proposition 6.3.

It is important, for reasons (6.13) that will transpire below, that τ can be replaced by (half of) Rasmussen's invariant s, or any other (possible) slice-torus invariant v. In particular, for any such v we have the behavior of (6.1), leading to defining the *jump number* $j_v(K)$, as studied in [LN]. In fact, note that one can define j_σ for the signature σ as well (after some modification $(\sigma+1)/2$ to fit values $0,1$), but for obvious reasons $j_\sigma(K) = 1$ regardless of K. Corollary 4.15 shows then that there are many Whitehead doubles T which are not strongly quasipositive despite $\sigma(T) = 2g(T) = 2$. We obtain then the following.

Corollary 6.1 For any slice-torus invariant v, we have

$$j_v(K) \le \lambda(K)\,. \tag{6.2}$$

Proof. By Corollary 4.15, $W_+(K,t)$ is strongly quasipositive for $t \ge \lambda(K)$, thus from (2.27), we have $v(W_+(K,t)) = 1$ for $t \ge \lambda(K)$. □

Example 6.2 Equality does not always hold. An example for $v = \tau$ is $T = W_+(3_1, 3) = 14_{45575}$, which is a Whitehead double of the negative (left-hand) trefoil 3_1. There $\tau(T) = 1$, but $T = 14_{45575}$ is not strongly quasipositive. We have $\lambda(3_1) = 6$ (see [KI]).

Now we can also easily recover the Livingston-Naik result [LN].

Proposition 6.3 For any slice-torus invariant v, we have

$$-\lambda(!K) < j_v(K) \le \lambda(K)\,. \tag{6.3}$$

Proof. The right inequality in (6.3) was given in Corollary 6.1. To obtain the left inequality, we prove that

$$v(W_+(K,t)\#W_+(!K,-t)) \le 1\,. \tag{6.4}$$

We remind that both connected sum factors have v-invariant 0 or 1.

Assume (6.4) is proved. Then since $v(W_+(!K,-t)) = 1$ for $-t \le -\lambda(!K)$ for the same reasons as Corollary 6.1, we need from (6.4) that $v(W_+(K,t)) = 0$ for $t \le -\lambda(!K)$, so we have the left inequality in (6.3).

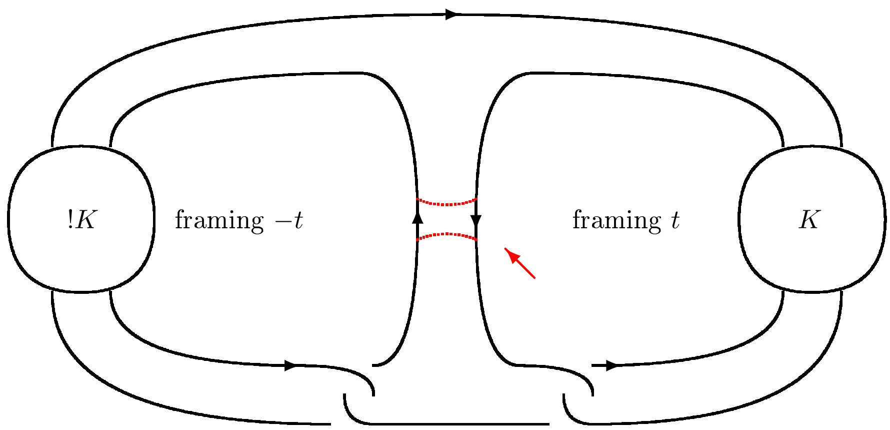
To prove (6.4), assume by contradiction that l.h.s. is 2. Thus $\chi_4(W_+(K,t)\#W_+(!K,-t)) \le -3$.

By connecting with a band as indicated in Figure 1, we obtain a 2-component link in Figure 2, with presumably

$$\chi_4[(6.7)] \le -2\,. \tag{6.5}$$

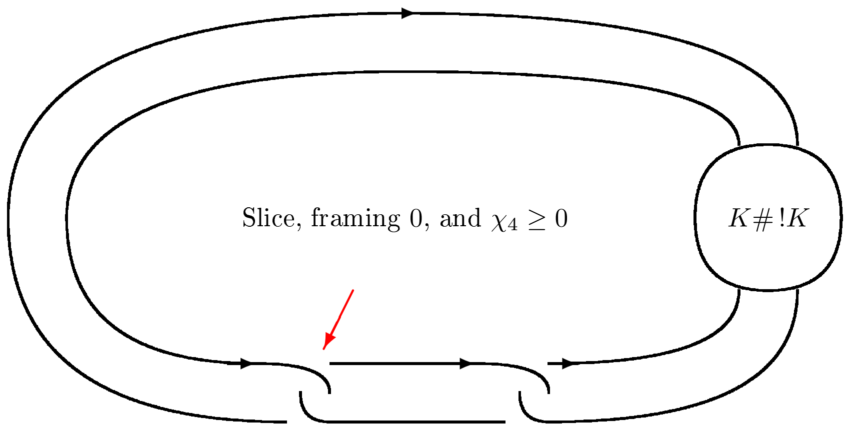
But the disk region of (6.7) represents an annulus of the slice knot $K\#!K$ with framing $t - t = 0$. However, pay attention that there is an orientation issue here. When K is non-invertible, then $K\#!K$ is slice only if $!K$ is oriented in a proper way. To resolve this issue, notice that the construction of $W_+(K,t)$ does not depend on the orientation of K, and moreover, $W_+(K,t)$ is easily seen to be invertible regardless of whether K is or not. This means one can suitably choose orientations of $W_+(K,t), W_+(!K,-t)$ when their connected sum in (6.4) is built. The v invariant obviously is not affected by this choice. Then by smoothing out any one of the four displayed crossings in (6.7), we obtain the unframed Whitehead double $(6.8) = W_+(K\#!K, 0)$ of a slice knot, in Figure 3, which must be slice itself and thus have $\chi_4 = 1$. But from (6.5), we would need $\chi_4[(6.8)] \le -1$, a contradiction. □

We have then the following contribution to the Bennequin sharpness conjecture (2.8):



(6.6)

Figure 1: Splice at the place indicated by the arrow, by adding a band



(6.7)

Figure 2: One of the four crossings should be smoothed out, and then one nugatory crossing removed

Corollary 6.4 Assume there is a slice-torus invariant v so that (6.2) is sharp for K:

$$\lambda(K) = j_v(K)\,. \tag{6.9}$$

Then for every t,

$$W_\pm(K,t) \text{ is Bennequin-sharp } \iff W_\pm(K,t) \text{ is strongly quasipositive}\,. \tag{6.10}$$

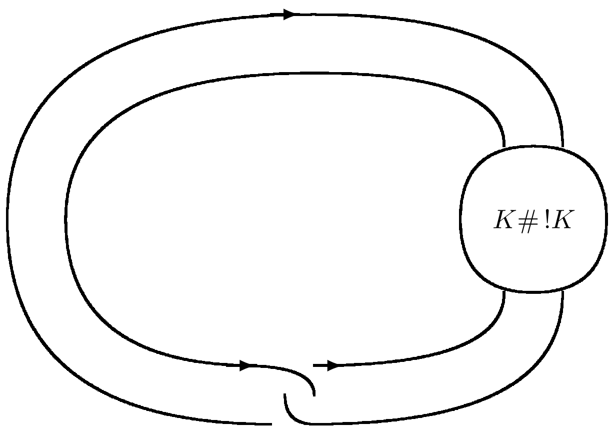
Proof. If K is the unknot, then $W_\pm(K,t)$ are twist knots, so alternating, and for them (2.8) is resolved; see [FLL, St2]. (Or one can make an explicit check.) Thus we assume below that $K \neq \bigcirc$.

We first deal with W_+. If (6.9) holds, then because of Corollary 4.15,

$$W_+(K,t) \text{ is not strongly quasipositive } \iff v(W_+(K,t)) = 0\,. \tag{6.11}$$

Furthermore, $g(W_+(K,t)) = 1$, thus by (2.27),

$$v(W_+(K,t)) = 0 \Longrightarrow W_+(K,t) \text{ is not Bennequin-sharp}\,. \tag{6.12}$$



(6.8)

Figure 3: Slice knot, $\chi_4 = 1$

Combining (6.11) and (6.12) gives the '$\Longrightarrow$' direction in (6.10). The reverse direction,

$$\text{not Bennequin-sharp} \Longrightarrow \text{not strongly quasipositive,}$$

is among the standard causalities following from Bennequin's inequality (2.5).

For W_- notice that it unknots by a negative-to-positive crossing change, so that $v(W_-) \le 0$, while $g(W_-) = 1$. Thus $W_-(K,t)$ cannot be Bennequin-sharp by (2.25). (The case that K is the unknot, and $t = 0$, can be handled extra.) $\square$

Of course, when v is effectively computable, so is $j_v(K)$. But at least for $v = \tau$, there is a more closed expression. Hedden [He] has found that

$$j_\tau(K) = 1 - 2\tau(K)\,, \tag{6.13}$$

which further elucidates Example 6.2. But the picture for Rasmussen's invariant remains less clear.

We can fit (6.13) into the general relationship

$$\lambda(K) = -TB(K) \ge 1 - 2\tau(K) = j_\tau(K) \ge \chi_4(K)\,. \tag{6.14}$$

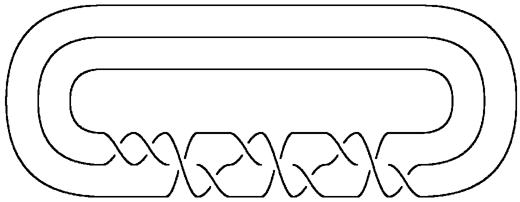
For the leftmost inequality, which is due to Plamenevskaya, see the proof of Theorem 1.5 in [He2]. One can use (6.14) to easily obtain the property (6.3) for $v = \tau$, which motivated treating there a general v rather than only focussing on this special instance. The relationship (6.13) also identifies when (6.9) holds for $v = \tau$, namely which occurs when

$$\lambda(K) = 1 - 2\tau(K)\,. \tag{6.15}$$

This raises the question what knots satisfy (6.15). The condition clearly must relate to positivity *somehow*, but the absence of the rotation term in TB (compare with [Fe, FT]) should hint to caution. Positive braid knots K will satisfy (6.15), because for them one has a positive diagram whose plane curve has Morse (say) maxima all passed in one sense of rotation. (Notice that the rotation term is zero when all cusps are left or all are right.) We do not know if for (6.15) it suffices K to be positive. But we know that strong quasipositivity is not sufficient.

Example 6.5 Consider $K = 16_{1379216}$, the closure of the 3-braid

`1 1 [13] 2 1 [13] 2 1 [13] 2,`



$16_{1379216}$

It has $\min\deg_a F(K) = 7$ (and $g_4(K) = 4$), thus by (7.27) we can conclude that (6.15) fails even for strongly quasipositive K. (This is the only strongly quasipositive example K up to 16 crossings with $\min\deg_a F(K) < 2g_4(K)$, so it underscores the value of the tabulation reported in [St2, Appendix].)

A further series of instances satisfying (6.15), which will play a special role in the appendix, are slice knots K with (A.5). They can be suspected to be quasipositive (see Remark A.4). But for (6.15) quasipositivity not necessary, as shows the below example.

Example 6.6 The knot $K = 12_{1628}$ has $\lambda = 1$ and $\tau = 0$ (see [KI]), thus were it to be quasipositive, it would have $\tau = g_4 = 0$, so it would be slice. But this is easily ruled out from the Milnor-Fox condition; the determinant $\det(12_{1628}) = 17$ (see (2.24)) is not a square.

7 HOMFLY-PT polynomial

7.1 Some degree inequalities

We now turn our attention to the HOMFLY-PT polynomial P in (2.13). Our goal is to use the polynomial to prove that when t is sufficiently small, then $A(K,t)$ is not strongly quasipositive with a good lower bound on t. The $w(D)$ term in (4.5), as we have seen, makes bounds somewhat inelegant and inefficient. We use some notation from §2.4.

Lemma 7.1 For every knot K, there exists a strongly quasipositive framing $t = \lambda_{min}(K) \ge \lambda(K)$ of $A(K,t)$, so that

$$\min\deg_v P(A(K,t)) \ge 1\,, \quad \max\deg_v P(A(K,t)) \le 2a(K) - 1\,. \tag{7.1}$$

Proof. When $K = \bigcirc$, then $t = 1$ suffices. Thus assume again below that K is non-trivial. When L is strongly quasipositive, then (2.7) and L being Bennequin-sharp mean that the right inequality in (2.6) becomes an equality. By using the right inequality in (2.17), we have

$$\min\deg_v P(L) \ge 1 - \chi(L) = 1 - \chi_4(L)\,. \tag{7.2}$$

In particular for $L = A(K,t)$, we have $\chi(L) \le 0$, so

$$\min\deg_v P(A(K,t)) > 0\,. \tag{7.3}$$

We have from the skein relation (2.13)

$$P(A(K,t)) = v^2 P(A(K,t-1)) + vz\,. \tag{7.4}$$

Notice, by further remarks from §2.4, that for the 2-component link $A(K,t))$ the only monomials in $P(A(K,t))$ that occur are $z^p v^r$ with odd p, r. Also $\min\deg_z P(A(K,t)) = -1$, and by [LiM] it is known that

$$[P(A(K,t))]_{z^{-1}} = v^{2t}(v^{-1} - v)([P(K)]_{z^0})^2 \ne 0\,. \tag{7.5}$$

We now know that there is a (at least one) framing (we denoted) $t = \lambda_{min}$, so that $b(A(K,t)) \le a(K)$. Also by MFW inequality (2.18) we have

$$\operatorname{span}_v P(A(K,t)) \le 2(a(K)-1)$$

for $t = \lambda_{\min}$. Now, the diagram D_1 of $A(K, \lambda_{\min})$ obtained from a minimal grid diagram D of K by replacing vertical segments by positive bands has $w(D_1) = \mu(D) = a(K)$ and $s(D_1) = \mu(D) = a(K)$.

Thus by MFW inequalities, we have

$$\min\deg_v P(D_1) \ge 1\,, \quad \max\deg_v P(D_1) \le 2a(K) - 1\,. \tag{7.6}$$

$\square$

Lemma 7.2 If $K \ne \bigcirc$,

$$\lambda(K) > \max\{\lambda(D) - \mu(D) \,:\, D \text{ is a grid diagram of } K\}\,, \tag{7.7}$$

with non-strict inequality if $K = \bigcirc$.

Proof. By using the right inequality (7.1) and the recursion (7.4) reversely $a(K)$ times, we see

$$\min\deg_v P(A(K, \lambda_{min} - a(K))) \le \max\deg_v P(A(K, \lambda_{min} - a(K))) \le -1\,,$$

so from (7.3), we have that

$$A(K, \lambda_{min} - a(K)) \text{ is not strongly quasipositive,}$$

if $K \ne \bigcirc$. For $K = \bigcirc$, we can conclude that

$$A(K, \lambda_{min} - a(K) - 1) \text{ is not strongly quasipositive .}$$

In a similar way, for every grid diagram D of size $\mu(D)$, the annulus $A(K, \lambda(D))$ will appear in a diagram D_1 with $w(D_1) = s(D_1) = \mu(D)$, so

$$A(K, \lambda(D) - \mu(D)) \text{ is not strongly quasipositive} \tag{7.8}$$

when $K \ne \bigcirc$, and same for $A(K, \lambda(D) - \mu(D) - 1)$ when $K = \bigcirc$. $\square$

Since this maximum is finite, we have:

Theorem 7.3 (Finite-Cone-Theorem) The framing diagram $\Phi(K)$ is a union of finitely many cones.

Proof. Note that a cone $C' = C(\mu', t')$ contains a cone $C = C(\mu, t)$ if and only if $(\mu, t) \in C'$. Thus if $C \subset \bigcup_i C_i$, then $C \subset C_{i_0}$ for some C_{i_0}.

Call a cone $C \subset \Phi(K)$ *essential* if there is no cone $C' \subset \Phi(K)$ with $C \subsetneq C'$. Now consider the essential cones $C_i = C(\mu_i, t_i)$ in $\Phi(K)$ one by one. Order them as a (first potentially infinite) sequence $C_1, C_2, \dots$ by increasing t_i, i.e., so that $t_i > t_{i-1}$. Note that there cannot be two essential cones C_i, C_j with $t_i = t_j$, since otherwise $\mu_i < \mu_j$ would lead to $C_i \supset C_j$. Also there is a smallest t_1 because $t_i \ge \lambda(K)$ for all i. Define then

$$\nu_i = \max\{t - \mu \,:\, (\mu, t) \in C_i\}\,.$$

And now argue that $\nu_i > \nu_{i-1}$. Because of (7.7), there can be only finitely many increases of ν_i. (See Proposition 8.6 for a more precise statement and argument.) $\square$

Another application of (7.7) gives an inequality we promised in stark symmetry to Lemma 4.18. (Unlike its counterpart, it thus does rely on the HOMFLY-PT polynomial in an essential way, though.)

Lemma 7.4 For any diagram D of K, we have $\lambda(K) > -2c_+(D) - pbr(D)$.

Proof. If $K = \bigcirc$, then $\lambda(K) = 0$, $pbr(D) > 0$ and $c_+(D) \geq 0$, so the inequality is trivial. Thus assume $K \neq \bigcirc$. We use the conversion (4.20) and the horizontal adjustment (4.21) of the proof of Lemma 4.18. We may then assume w.l.o.g. that D is a grid diagram and all ± 1 signed horizontal edges are intersected by a crossing. Then using (7.7), we have

$$\begin{aligned} \lambda(K) &> \lambda(D) - \mu(D) \\ &= Z(D) - w(D) - \mu(D) \\ &\geq \frac{1}{2}\big(2\mu(D) - 2c(D) - 2br(D)\big) - \mu(D) - w(D) \\ &= -c(D) - br(D) - w(D) \\ &= -2c_+(D) - br(D)\,. \end{aligned}$$

In the third line we used that each -1 edge has a crossing, and there are $2br(D)$ sign 0 edges. $\square$

Remark 7.5 The number $l(K)$, introduced later, allows for improvements of (7.8), (7.7) and Lemma 7.4. However, the present versions maintain the advantage of involving only simple geometric data of the diagram itself, without protruding algebraic constructions derived from it. Since we will find a number of (other) applications of $l(K)$, we do not like to return here to resume this specific line of argument. One place where this reasoning is incorporated is (A.13). The quantity $l(K)$ will serve as a lower estimate for the arc index, of which we put ahead a simplified variant.

Let $P = P(A(K,t))$ for some t. Keep in mind by §2.4 that $P|_{z^{\neq 1}}$ is the polynomial P with all terms of z-degree 1 removed. Because of (7.5), talking about its degrees makes sense.

Lemma 7.6 The integer

$$l'(K) := \frac{1}{2}\,\mathrm{span}_{\,v} P(A(K,t))|_{z^{\neq 1}} + 1$$

does not depend on t and satisfies

$$a(K) \geq l'(K)\,. \tag{7.9}$$

Proof. By construction, $b(A(K,\lambda_{min})) \leq a(K)$, so by MFW inequality (2.18), we see that (7.9) is true for $t = \lambda_{min}$. And for other t, note that the relation (7.4) does not add any terms of z-degree different from 1. That $l'(K)$ does not depend on t follows for this same reason. $\square$

But in fact, the z^1-term of P is also interesting, and its study relates to the "cooking" alluded to in the abstract of the paper.

7.2 Estimating $a(K)$: the pan

Like for the crossing number, there are only finitely many knots of given arc index. However, once some classical tool like (7.26) fails to give a sharp lower estimate, the method used so far, like in [J+], is to exhaustively enumerate all grids of smaller size, a feat which quicky becomes laborious and unreliable when the size increases. To change this situation here, we explain next how *not* to discard the z^1-term in (7.9), and use it to determine $a(K)$, and later $\lambda(K)$, from the P polynomial with considerable precision.

Write in the rest of this section for simplicity

$$K_t = A(K,t)$$

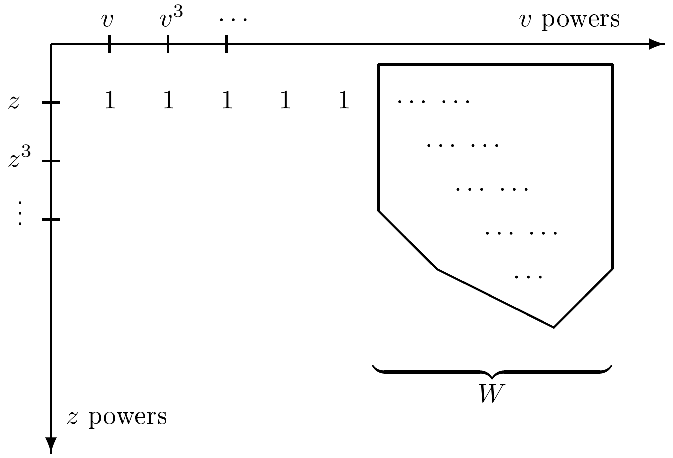
for (the boundary of) the t-framed annulus around K. We have then from (7.4),

$$P(K_t) = zv + v^2 P(K_{t-1})\,. \tag{7.10}$$

To visualize the polynomial $P(K_t)$, it will be helpful to plot its coefficients in the plane, with (odd) v degrees going from left to right and z degrees going top-down. Thus negative v-degree terms, left on the y-axis, occur, and will be considered. But negative z-degree terms, above the x-axis, occur only for z^{-1}, and we stipulate to hide them. We emphasize again that the z^{-1}-term in $P(K_t)$ is known (see (7.5))

$$[P(K_t)]_{z^{-1}} = (v^{-1} - v) \cdot v^{2t}([P(K)]_{z^0})^2 \,. \tag{7.11}$$

By iterating (7.10), we can see that for sufficiently high t, the polynomial $P(K_t)$, displayed as we just explained, starts exhibiting the pan-like shape



(7.12)

Now remove all 1's in the panhandle of (7.12). To formalize this, we consider the leftmost and rightmost column $[P]_{v^d}$ in (7.12), for the smallest $d = d_{min} > 0$ which is not of the shape

$$[P]_{v^d} = z \,, \tag{7.13}$$

and

$$d_{max} = \max \deg_v P \,. \tag{7.14}$$

We can easily treat arbitrary t, and will do below. In that case, we can modify the condition (7.13) for $d_{max} < 0$ (keep in mind that for a 2-component link, d is always odd) to

$$[P]_{v^d} = -z \tag{7.15}$$

and $d_{min} < 0$ is set as $\min \deg_v P$. But, keeping the pan shape (7.12) in mind, assume here for simplicity $t \gg 0$.

Write then

$$l(K) = \frac{d_{max} - d_{min}}{2} + 1 \tag{7.16}$$

for the *(pan) width* of W in (7.12). (For the formalization of this procedure, see the expressions given at the end of §7.4. Compare also with [Nu, Theorem 3.3].) In result, we have a way to "normalize out" the t-dependence of the degrees of $P(K_t)$ in the z^1-term, giving an improved version of the lower bound $l'(K)$ in (7.9) for $a(K)$. Due to the attention incited by the unknot, let us remark here that

$$a(\bigcirc) = l(\bigcirc) = l'(\bigcirc) = 2 \,. \tag{7.17}$$

Proposition 7.7 For every knot K, we have $l'(K) \leq l(K) \leq a(K)$.

30	1	0	14											
	-2	4							9	-33	41	-16		
	-12	4		1	1	1	1	1	25	-164	221	-80		
	-2	4							22	-342	468	-148		
	-2	4							8	-376	496	-128		
	-2	4							1	-231	286	-56		
	0	4								-79	91	-12		
	0	4								-14	15	-1		
	0	2								-1	1			
38	1	0	18											
	-2	4									16	-56	66	-25
	-16	4		1	1	1	1	1	1	1	81	-420	541	-200
	-2	4									148	-1316	1778	-610
	-2	4									128	-2248	3040	-920
	-2	4									56	-2298	3013	-771
	-2	4									12	-1457	1821	-376
	-2	4									1	-575	680	-106
	0	4										-137	153	-16
	0	4										-18	19	-1
	0	2										-1	1	

Table 1: Polynomials of the Whitehead doubles $W_+(7_1,7)$ and $W_+(9_1,9)$ of the negatively mirrored 7_1 and 9_1. The framing t can be read off, because of (7.34), from the sum of the coefficients in the second row.
It should be emphasized that what appears as a panhandle is *not* what is illustrated in (7.12). It is at the "wrong" end and will remain part of the pan when t is large.
Had the coefficients in these "false" panhandles been signed in the *opposite* way, i.e., to be -1, the polynomials of $A(!7_1,t)$ and $A(!9_1,t)$ would have instantiated the possibility (7.18). (Being signed $+1$, these coefficients will become 2 for large t.)

Proof. Obviously $l'(K) \le l(K)$, so we prove the right inequality. Because of (7.17), we also assume $K \ne \bigcirc$.

When we set (7.14), it is possible that some $P(K_t)$ for small t has MFW bound $< l(K)$. This can happen if

$$[P]_{v^d} = z \text{ for } d = d_{max} \text{ and possibly } d_{max}-2,\, d_{max}-4, \text{ etc.} \tag{7.18}$$

In particular, we would need

$$d_{max} > \max\deg_v P|_{z^{\ne 1}} \tag{7.19}$$

for such terms to occur. These terms (7.18) can be cancelled by the inverse process of (7.10) when their v degree shifts down to 1 and then goes from 1 to -1.

We pause here for some cautionary illustrations. We do not know if (7.18) can occur. But examples warn that it "almost" does. It can be seen from Table 2 that when $K = !10_{132}$, such a cancellation (when $t=1$) occurs in degree $d_{max}-2$. But it does not in degree d_{max}, which prevents a collapse in degree. Consider also the polynomials from Table 1. By smoothing out a crossing in the Whitehead double clasp and taking the mirror image, one can see that when K are positive $(2,n)$-torus knots, then terms $[P]_{v^d} = -z$ do occur in large amounts. They differ from (7.18) only by one wrong sign. In particular these terms also make a significant difference to $l'(K)=4$ in (7.9), evidencing the price tag of ignoring the z^1-term all out. This is cemented by further knots like $K = 8_{20}, 9_{43}$, with $l'(K) < l(K)$. At least, we refer here to the inequality (7.33) which, under mirroring (2.15), provides some limit on how many terms can cancel in (7.18).

Since we cannot exclude the situation (7.18), using

$$a(K) \ge \min_{t\in\mathbb{Z}} \mathrm{MFW}(K_t) \tag{7.20}$$

(for (2.18)) will not be enough, at least in theory (see, though, Remark 7.8). However, notice that the arc index, as bound for $b(K_t)$, has a certain stability: there is a number $t = \lambda_{min}$ with

$$b(K_{t'}) \le a(K) + |t' - \lambda_{min}| \tag{7.21}$$

for every t'. (We know by Theorem 5.12 that λ_{min} is unique for $K \neq \bigcirc$.) Using (7.21), we can replace (7.20) by

$$a(K) \geq \min_{t\in\mathbb{Z}} \max_{t'\in\mathbb{Z}} \mathrm{MFW}(K_{t'}) - |t' - t|\,. \tag{7.22}$$

This will prevent the sporadic collapsing of the MFW bound from deteriorating the arc index bound. It can be seen, with a bit of technical argument based on (7.10), that the right of (7.22) is precisely what was defined as $l(K)$.

For instance, there can be at most two hypothetical values of t for which $\mathrm{MFW}(K_t) < l(K)$, and for them choosing $|t' - t| = 1$ should suffice to see

$$\mathrm{MFW}(K_{t'}) - |t' - t| \geq l(K)\,.$$

An instructive example of the argument, allowing for two such t to occur, is the following sequence. We show a transformation of the $[P(K_t)]_{z^1}$ terms with increasing t, where only the coefficients are written, and vertical bar stands for the separation between v-degrees -1 and 1 (making clear the degrees of all other coefficients; even degrees are obviously omitted). The pan edge coefficients are boxed at some places (similarly to (7.40); see also (7.48)).

$$\begin{array}{c} \boxed{-1\ \ -1\ 2\ 0\ 0}\ \ -1 \,\Big|\ \ \to\ \ \boxed{-1\ \ -1\ 2\ 0\ 0}\,\Big|\ \ \to\ \ -1\ \ -1\ 2\ 0\,\Big|\,1\ \ \to\ \ \dots \\ \dots\ \ \to\ \ -1\,\Big|\,0\ 3\ 1\ 1\ \ \to\ \ \Big|\,\boxed{0\ 0\ 3\ 1\ 1}\ \ \to\ \ \Big|\,1\,\boxed{0\ 0\ 3\ 1\ 1} \end{array} \tag{7.23}$$

In that case $l(K) = 5$, while for two t, $\mathrm{MFW}(K_t) = 3$ is possible. But for either t and one of the two $|t' - t| = 1$, we have $\mathrm{MFW}(K_{t'}) = 6 = l(K) + 1$.

This argument based on (7.22) justifies that using (7.14) is appropriate to achieve $l(K)$ in (7.16) to estimate $a(K)$ as claimed. $\square$

Remark 7.8 There is a way to modify the calculation of $l(K)$ to determine the r.h.s. of (7.20). Remove all highest v-degree terms (7.13) for $d > 0$ and $[P]_{v^d} = 0$ for $d < 0$, until you reach a degree d'_{max} (with coefficient $[P]_{v^d}$) not of that form. Similarly, remove all lowest v-degree terms (7.15) for $d < 0$ and $[P]_{v^d} = 0$ for $d > 0$, finding d'_{min}. Then

$$l(K) \geq \min_{t\in\mathbb{Z}} \mathrm{MFW}(K_t) \geq \frac{d'_{max} - d'_{min}}{2} + 1\,.$$

Note that on the right there is still no equality, because when t is fixed, the just described cancellation of terms in $P(K_t)$ can only occur on one side (either for low, or for high powers of v, but not for both). Still, in the present form the estimate is good enough to allow us to confirm that in fact

$$l(K) = \min_{t\in\mathbb{Z}} \mathrm{MFW}(K_t) \tag{7.24}$$

for all prime knots K up to 10 crossings. We do not know whether this equality holds in general.

Since $P(A(K,t))$ are interconvertible for all t, one can determine $l(K)$ from $P(A(K,t))$ for any t, and then hope to determine $a(K)$ if $a(K) = l(K)$.

Definition 7.9 We say that K is *l-sharp* if $a(K) = l(K)$.

`[25] [14] [37] [26] [15] [48] [79] [38] -[69]`

`55 132 -1 17`

7	13				9	-21	16	-4	
5	15			-15	109	-186	86	31	-25
1	15	-2	0	-80	452	-724	285	169	-100
5	15			-148	870	-1493	659	272	-160
5	15			-128	895	-1771	932	202	-130
5	15			-56	520	-1256	772	76	-56
5	15			-12	170	-536	376	14	-12
5	15			-1	29	-134	106	1	-1
7	11				2	-18	16		
9	11					-1	1		

`[25] [14] [37] [26] -[15] -[48] -[79] -[38] -[69]^2`

`56 1 0 18`

-2	4				-8	21	-16	4	
-8	6	1	1	16	-108	186	-86	-31	25
-8	6	2	0	80	-452	724	-285	-169	100
-4	6			148	-870	1493	-659	-272	160
-4	6			128	-895	1771	-932	-202	130
-4	6			56	-520	1256	-772	-76	56
-4	6			12	-170	536	-376	-14	12
-4	6			1	-29	134	-106	-1	1
-2	2				-2	18	-16		
0	2					1	-1		

Table 2: Polynomial of the annulus link $A(10_{132}, 0)$ and the Whitehead double $W_-(10_{132}, -4)$ of 10_{132} and negative clasp, framing $t = -4$, together with the band presentation used, as obtained from (4.15) (where `±[ij]` stands for $\sigma_{i,j}^{\pm 1}$ in (2.2)).
The mirroring of 10_{132} can be easily confirmed from the z^{-1}-term of $P(A(10_{132}, 0))$ and (7.5) to be the one specified in Example 4.10.
For $A(10_{132}, 0)$ we see the disappearance of the (short) "false" panhandle. It comprises two monomials in z-degree 1. We call the panhandle "false" because in the same v-degrees, a term with z^3 occurs, so that this "panhandle" is not removed when reducing the framing t. Note that $A(10_{132}, 0)$ is not strongly quasipositive despite $\min \deg_v P > 0$.

Example 7.10 Among the Rolfsen [Ro, Appendix] knots, $K = 10_{132}$ is the only one which is not l-sharp. Then $l(K) = 8$ (as shown in Table 2) but [J+] (see Example 4.10) and KnotInfo [KI] report $a(K) = 9$.

There are four non-l-sharp 11 crossing knots K (up to mirror images), i.e., such with

$$a(K) > l(K)\,, \tag{7.25}$$

namely 11_{379}, 11_{424}, 11_{455}, 11_{459} (for which $l(K) = 9$ and $a(K) = 10$), and 21 further examples of 12 crossings.

In case of 10_{132} (and a series of other examples), there is a linking number argument that can help out determining the arc index. We formulate it as a lemma. (It can also be easily modified to other knots, but for simplicity we just present the prototype and leave it to the reader to adapt it.)

Lemma 7.11 We have $a(10_{132}) = 9$.

Proof. Assume $a(10_{132}) \le 8$. From the polynomial of the annulus link $A(10_{132}, 0)$ in Table 2, and (7.4), we can see that $\mathrm{MFW}(A(10_{132}, t)) \le 8$ occurs for $t = -8, \dots, 0$, and then $\mathrm{MFW}(A(10_{132}, t)) = 8$. Because of the bottom statement of Corollary 5.3, it is enough to prove that $b(A(10_{132}, 0)) \ne 8$. We claim that the polynomial of $A(10_{132}, 0)$ in Table 2 is sufficient to see that $b(A(10_{132}, 0)) > 8$, as follows.

Assume $b(A(10_{132}, 0)) = 8$, and β is an 8-braid whose closure is $A(10_{132}, 0)$. Now, the exponent sum $w(\beta)$ is made up of the exponent sums $w(\beta_i)$ of the two subbraids β_i of β, which give the individual components $\hat\beta_1 = C_1$ and $\hat\beta_2 = C_2$ of $A(10_{132}, 0)$, and the linking number $lk(C_1, C_2) = t = 0$ of these components. Since both C_1 and C_2 have the knot type of 10_{132}, and $b(10_{132}) = 4$, both components C_1 and C_2 of $A(10_{132}, 0)$ must be closures of 4-string subbraids β_i of β. Then their individual exponent sums must be $w(\beta_i) = w_{min}(10_{132})$, which is determined to be 3 (see the tables [St4] and the remarks below (2.19)). Thus

$$w(\beta) = w(\beta_1) + w(\beta_2) + lk(C_1, C_2) = 3 + 3 + 0 = 6\,.$$

But the polynomial $P = P(A(10_{132}, 0))$ in Table 2 exhibits

$$\min\deg_v P = 1 \le 15 = \max\deg_v P\,,$$

and looking at the refined inequality (2.17), we see that a braid β with $n = 8$ strands must have writhe $w = w(\beta) = 8$. This is a contradiction. □

The "classical" lower bound for $a(K)$ comes from Kauffman's polynomial F [MB]:

$$a(K) \ge \operatorname{span}_a F(K) + 2\,. \tag{7.26}$$

This can also be obtained from the bound (see [FT, Fe, Ta])

$$\lambda(K) \ge -\min\deg_a F(K) + 1 \tag{7.27}$$

and Matsuda's inequality (7.49). For all of the 26 non-l-sharp knots of Example 7.10 we have $\operatorname{span}_a F(K) + 2 = l(K)$. But $\operatorname{span}_a F(K) + 2 < l(K)$ obviously occurs for some "F-sparse" knots like $K = 9_{42}$. Likewise, $l'(K) < \operatorname{span}_a F(K) + 2$ occurs in Table 1 (due to (7.28)), thus the z-term retains its credentials.

Question 7.12 Is $\operatorname{span}_a F(K) + 2 \le l(K)$ for all non-trivial knots K?

Example 7.13 In general the approximation $l(K) \le a(K)$ is rather good. There are 2049 arc index 11 prime knots up to 16 crossings. The inequality (7.26) is sharp for 1666 of them, while 1977 (incuding all those 1666) are l-sharp.

Computation 7.14 At least up to 12 crossings no prime *alternating* non-l-sharpness example was found (i.e., $l(K) = a(K) = c(K) + 2$ holds for each such knot K). Also no anomalies occurred with the Murasugi-Przytycki 18 crossing alternating knot (and thus incl. its mutant) with unsharp (2.18).

Question 7.15 Is $a(K) = l(K)$ for all (non-trivial) alternating K?

Since Thistlethwaite [Th] proved that for K alternating

$$\operatorname{span}_a F(K) = c(K)\,, \tag{7.28}$$

this would also be a part of Question 7.12 (as can be seen from the proof of Corollary 5.5).

A further condition we temporarily consider is that

$$\max\deg_z P(A(K,t)) > 1\,. \tag{7.29}$$

Keep in mind that $\max\deg_z P(A(K,t))$ does not depend on t if for some t it is greater than 1.

The inequality (7.29) seems in practice to always hold when K is not the unknot. Indeed, it is conjectured more precisely (see [KS]) that, for $K \neq \bigcirc$,

$$\max\deg_z P(A(K,t)) = 2\max\deg_z F(K) + 1 \tag{7.30}$$

for the Kauffman polynomial F. (Often $W_\pm(K,t)$ is used instead of $A(K,t)$, but the conversions are straightforward.) In particular (7.30) subsumes the expectation that

$$\max\deg_z P(A(K,t)) = 2c(K) - 1 \tag{7.31}$$

for K prime and alternating. The conjecture (7.31) is still open. The most general results are due to Brittenham-Jensen [BJ]. We do not know about work on the extension (7.30).

Computation 7.16 We have verified using Whitehead doubles that (7.30) is true for prime knots up to 12 crossings. Thus in particular, all such knots satisfy (7.29).

By taking the reverse parallel with the blackboard framing of a minimal crossing diagram of K, counting the Seifert circles, and using Morton's inequality (2.22), one always has

$$\max\deg_z P(A(K,t)) \le 2c(K) - 1 \tag{7.32}$$

for an arbitrary non-trivial K and any t (see [BJ]). This inequality (7.32) is not always sharp. But the conjectured equality is that the canonical genus of the Whitehead double of a knot K (regardless of framing and sign of clasp) coincides with the crossing number of K. And this conjecture follows for K prime and alternating if (7.31) is confirmed. From the identity (2.16), we remark then for every non-trivial K and every t,

$$\min\deg_v P(A(K,t)) \le \max\deg_z P(A(K,t)) \le 2c(K) - 1\,. \tag{7.33}$$

This was mentioned in the proof of Proposition 7.7, and implies that the number of (potentially existing) terms in (7.18) that lead to cancellations can be at most $c(K) - 1$.

In general, $l(K)$ is not easy to calculate on infinite families of knots. Notice that it is not even evidently additive under connected sum.

Question 7.17 Is $l(K_1\#K_2) = l(K_1) + l(K_2) - 2$?

This turns out to be the case in a few examples, like $10_{132}\#(!)3_1$ and $10_{132}\#(!)10_{132}$, but as long as it is not confirmed, the possibility exists to extract further information from l as a lower arc index bound, using the relationship

$$a(K_1\#K_2) = a(K_1) + a(K_2) - 2\,,$$

which follows from (7.50) and the additivity of the Thurston–Bennequin invariant [EH, To].

We use (7.29) for at least one partial result regarding $l(K)$.

Lemma 7.18 We have $l(K)\ge 3$ for every knot K with (7.29).

Proof. Note the special form of the Conway polynomial (2.23) in our examples:

$$\nabla(K_t)=P(K_t)(1,z)=tz\,. \tag{7.34}$$

Thus in particular setting $v=1$ will collapse $P(K_t)$ in z-degrees >1.

By (7.29) and the collapsing in (7.34), if the bound $l(K)$ is at most 2, it is 2, and all P_{z^d} terms for $d>1$ are of the form

$$(c_u v^{d_{min}}-c_u v^{d_{min}+2})z^u\,. \tag{7.35}$$

By choosing t properly, let w.l.o.g. $d_{min}=1$ and $d_{max}=d_{min}+2=3$. Note that the reverse application of (7.10) will gradually annihilate all terms $[P]_{v^d}$ for $d<d_{min}$. Thus with $d_{min}=1$ and $d_{max}=3$, we have in fact

$$\min\deg_v P|_{z^{\ge 1}}=1\,,\min\deg_v P|_{z^{\ge 1}}=3\,.$$

But because of the form (7.35) and $c_u\ne 0$ for $u=\max\deg_z P\ge 3$, the substitution in (2.16) will give

$$\min\deg_v P|_{z^{\ge 1}}(v,v^{-1}-v)=1-u \tag{7.36}$$

and

$$\max\deg_v P|_{z^{\ge 1}}(v,v^{-1}-v)=3+u\,. \tag{7.37}$$

With (7.11),

$$P=P|_{z^{\ge 1}}+v^{2t}(v^{-1}-v)([P(K)]_{z^0})^2/z$$

for some t, and we have then

$$1=P(K_t)(v,v^{-1}-v)=P_1+P_2\,,$$

where

$$P_1=P|_{z^{\ge 1}}(v,v^{-1}-v)\quad\text{and}\quad P_2=v^{2t}([P(K)]_{z^0})^2\,.$$

Now we know (7.36) and (7.37) about P_1. Also $1-u<0<3+u$. Thus, adding v^{2t} times a square for any t cannot cancel all terms in v-degree $\ne 0$, unless $\operatorname{span}_v([P(K)]_{z^0})^2=2+2u$. Thus from (7.11),

$$\operatorname{span}_v[P(K_t)]_{z^{-1}}\ge 4+2u\,.$$

Since by definition, $d_{min}\le\min\deg_v[P(K_t)]_{z^{-1}}$ and $d_{max}\ge\max\deg_v[P(K_t)]_{z^{-1}}$, we have $l(K)\ge 3+u\ge 6$. This contradiction shows that $l(K)$ cannot be (at most) 2. □

Remark 7.19 By the properties of F listed in §2.4, it follows straightforwardly when $F(K)\ne 1$, then $\operatorname{span}_a F(K)+2\ge 4$. Thus, through Question 7.12, Lemma 7.18 illustrates the difficulty to control $l(K)$. This unpredictable behavior may, though, contribute to its efficiency as arc index bound.

7.3 Applications of Cabling

Conjecture 2.3 underscores the importance of cabling in settling braid, and thus also arc index issues. This is a perhaps less pleasant, but still more universal means than Lemma 7.11, to treat some l-unsharp knots K.

Computation 7.20 For $K=10_{132}$ the links L we consider with $\mathrm{MFW}(L)=8<a(10_{132})=9$ are

- $L=A(K,t)$ for $t=0,\dots,-8$,
- $L=W_+(K,t)$ for $t=0,\dots,-7$, and
- $L=W_-(K,t)$ for $t=-1,\dots,-8$.

(Of course, for the rest values of t we can conclude $\mathrm{MFW}(L) \geq 9$ using the relation (7.10), or a similar relation for Whitehead double polynomials.)

All the links listed above have $b(L) = 9$. We easily observe $b(L) \leq 9$. One can obtain a 9-string band presentation from that for $A(10_{132}, 1)$ with positive bands, given in (4.15), by making some bands negative and doubling a positive band for W_+ and a negative one for W_-. (Table 2 gives some examples.) At the opposite end, we tested $b(L) \geq 9$ with parallelized truncated 2-cable (MFW) P, as discussed in §2.4. The procedure took on a 4-CPU 10-year-old 2013 laptop between 2 and 15 h depending on individual examples: an agreeable performance, when taking into account that the diagrams resulting from 2-cabling the modifications of (4.15) have more than 200 crossings. (They depict $\uparrow\uparrow\downarrow\downarrow$ oriented degree-4 satellites of 10_{132}.)

This comparative efficiency offers the opportunity for more extensive checks (for other K). However, this option was waived on, since it still is not readily amenable to larger quantities, and it leaves unclear what insight to expect. (We will use the above compiled examples for later reference, though.)

Remark 7.21 Using Computation 7.20 for $K = 10_{132}$, and the verification of (7.24) and $l(K) = a(K)$ (see Example 7.10) for all other prime knots K up to 10 crossings, we can conclude that the answer to (both parts of) Question 5.7 is affirmative for all these 249 knots.

Remark 7.22 There are ways to "2-cable" (7.26) as well, for $a(K_{t_0}) \leq 2a(K)$ when $-t_0$ is a minimal grid writhe. (Compare with the proof of Lemma 4.4.) At least one can restrict t_0 from Lemma 3.5 (or better, from $\lambda(K)$, if it is known). And recurrence relations (analogous to (7.10)) reduce the need to calculate from diagrams $F(K_t)$ (or truncations thereof) for two values $t \in \mathbb{Z}$. Still, this is evidently strenuous but questionably effective, suggesting why such attempts are apparently lacking.

When (7.25) occurs, i.e., K is not l-sharp, the following simplification of cabling may potentially be useful. Since $\kappa(A(K,t)) = 2$, one can cable an individual component of $A(K,t)$, obtaining a $\uparrow\uparrow\downarrow$ oriented parallel $A^*(K,t,t')$ of K, where t' is the framing of the doubled component. (Here thus t' can be a half-integer when the two copies of the doubled component get connected, i.e., $\kappa(A^*(K,t,t')) = 2$ when $2t \in \mathbb{Z}$ but $t \notin \mathbb{Z}$.) Cabling an individual component only roughly doubles (and does not quadruple) the crossings in the braid word β_D for $A(K,t) = \widehat{\beta_D}$.

Lemma 7.23 For every t with $b(A(K,t)) = a(K)$ and every $2t' \in \mathbb{Z}$, we have

$$b(A^*(K,t,t')) \leq 3a(K)/2\,. \tag{7.38}$$

Proof. When $b(A(K,t)) = a(K)$, then one of the components of $A(K,t)$ in an $a(K)$-braid representative β is a subbraid on at most $a(K)/2$ strands. Thus doubling this component C, regardless of what framing t' is used, can be done by adding at most $a(K)/2$ braid strands. (The framing can be corrected by adding half-twists which do not add more strands.) This gives a braid representative of $A^*(K,t,t')$ of at most $3a(K)/2$ strands, resulting in (7.38).

Note that $A(K,t)$ is exchangeable up to simultaneous reversal of orientation of both components, which does not affect braid index arguments. Thus whether C is the component we 2-cable to obtain $A^*(K,t,t')$ from $A(K,t)$, or we cable the other component, is not relevant. (Note, though, that the framing t' of the cabled component may be different w.r.t. the blackboard framing of the diagram $\hat{\beta}$.) □

Algorithm 7.24 The following explains how one can try to use this lemma. Since the contrapositive of its statement is really used, some care is needed how to proceed, and we formulate it in several steps as an algorithm.

1. Use a band presentation β_D (as in (4.1)) for a grid diagram D of K of size μ. This gives a band presentation of $A(K,t)$ for some t.

2. Make some bands negative to ascertain that $P(A(K,t))$ has no panhandle. For example, when $K = 10_{132}$ and $\mu = 9$, then we know that there are nine values of $t \in \mathbb{Z}$ for which $\mathrm{MFW}(A(K,t)) = l(K) = 8$, namely $t = -8, \dots, 0$. The statement below (5.4) says that it is enough to treat one of these t. Thus we can consider $t = 0$ (which requires one negative band), and use the polynomial in Figure 2. In general, one can remove the panhandle (i.e., adjust t by making bands negative) only by looking at $P(A(K,t))|_{z^{\leq 1}}$.

3. Then double, with blackboard framing w.r.t. the diagram $\widehat{\beta_D}$, one of the components of the link $\widehat{\beta_D} = A(K,t)$. One obtains a link $A^*(K,t,t')$. There are in general two possibly (but not always) distinct integers t', depending on which component of $\widehat{\beta_D}$ one chooses to double. (It can be argued that these two t' will add up modulo 2 to the same parity as the "band width" sum $\sum_{k=1}^{\mu}(j_k - i_k - 1)$ in (2.4); which in turn has the same parity as μ; thus two distinct t' will in particular always occur when μ is odd.)

4. Try to prove that such a link $A^*(K,t,t')$ has braid index strictly greater than $\lfloor 3(\mu-1)/2 \rfloor$. This will prove $a(K) = \mu$.

Example 7.25 For instance, when we do this construction for $K = 10_{132}$ with (4.15) (one band needs to be made negative here), this gives $A^*(10_{132}, 0, t')$ for $t' = 3, 4$. We found (see (2.20)), though, that

$$\mathrm{MFW}_{10}(A^*(10_{132}, 0, t')) = 12$$

for both t'. Thus unfortunately, for $K = 10_{132}$, the observation (7.38) does not seem useful to show $a(10_{132}) = 9$, at least as far as (2.20) is applied (within reasonable computability).

However, there is a number of successful cases. For example, when we carry out this process for $K = 14_{27072}$, with the size-12 grid

```
13  24  58  7C  3B  1A  6C  59  8B  7A  49  26
```

(where A,B,C stand for 10,11,12; see Definition 4.6), we find $l(14_{27072}) = 11$, but making 3 bands negative, we obtain

$$\mathrm{MFW}_2(A^*(14_{27072}, 2, 1)) = 17$$

(here $t' = 1$ is the same for both choices of doubled component), which rules out $a(14_{27072}) = 11$.

Other examples, again with $\mu = 12$ (and a single t'), are

```
16 466746: 13  46  25  7A  8B  9C  3A  4B  16  7C  28  59
15 123702: 13  24  57  9C  6A  38  17  5B  49  8C  2A  6B
```

and

```
14 19935:  13  25  48  7B  3A  16  59  8B  7A  49  26
16 459158: 14  25  38  6A  7B  49  18  5A  29  6B  37
```

for $\mu = 11$ (using 5 negative bands, with two different t', both having $\mathrm{MFW}_2(A^*(K,t,t')) = 16$).

These examples do require some search, but keep in mind that even for truncated polynomials, the increase in crossing number has severe (complexity) consequences. (Here we tried only truncation degree $d = 2$, which does not cost much time and allows for testing a larger number of examples.) Thus Lemma 7.23 provides a viable option to try out.

Remark 7.26 We add the following practical hints about the determination of the arc index.

1) For more complicated knots K, it is better to approximate $l(K)$ from below by using z-truncations of the HOMFLY-PT polynomial, as explained in §2.4. This was used to assist the first and third authors' ongoing effort to tabulate the arc indices of the (non-alternating prime) 14 crossing knots. But it also emphasizes that it is useful to have a good upper estimate of $a(K)$ in advance. Once coincidence with the lower bound is reached, one can then save calculation of further truncations (and the full polynomial).

   We clarify that how an upper estimate of $a(K)$ was obtained relates to the (knot-spoke) method of [JP], finding certain proper non-alternating arcs in diagrams of K. It is not necessary (and takes extra effort) to obtain a minimal grid diagram explicitly.

2) As noticed while proving Lemma 7.11, the statement below (5.4) provides another significant shortcut to help determining $a(K)$ when $l(K)$ fails. For instance, to see (in an alternative way to Lemma 7.11) that $a(10_{132}) \neq 8$, it suffices to calculate a (truncated) 2-cable polynomial of $A(10_{132}, t)$ for only (any) one of the nine values of t that occur in the enumeration of Computation 7.20.

3) Observe that the linking number argument of Lemma 7.11 can be adapted to $A^*(K, t, t')$ as well. One has to consider instead of $lk(C_1, C_2) = t$ the total linking number of the components of $A^*(K, t, t')$, which is $2t + t'$ for $t' \in \mathbb{Z}$ (and $\kappa(A^*(K, t, t')) = 3$) and $2t$ otherwise (when $\kappa(A^*(K, t, t')) = 2$). We will give relevant examples at a separate place, where we discuss the arc indices of the 14 crossing knots.

4) Notice also Question 7.17 and the remarks below it.

To give a lookout at where we stand thus far, regarding the said at the beginning of §7.2, we have now gained a toolkit to rule out certain values of the arc index. We related it to a braid index (see Conjecture 8.1 below, although Part 2 of Remark 7.26 explains that we need a weaker statement), and then in turn to the HOMFLY-PT polynomial (compare Conjecture 2.3). These connections work out at least in a practical sense, which gives an approach to determine $a(K)$ for most K.

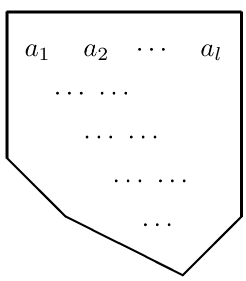
7.4 Estimating $\lambda(K)$: a cooking recipe

Returning to (7.12), we use the substitution (7.34) to extract further information from the pan.

Let $a_1, \dots, a_l$, for $l = l(K)$, be the z-degree 1 coefficients in W in (7.12):

$$[W]_{z^1} = \sum_{i=1}^{l} a_i v^{d_{min}+2i-2}\,. \tag{7.39}$$

Obviously a_i form the edge of the pan (drawn below without its handle) – whose general use is to break your eggs when frying them.



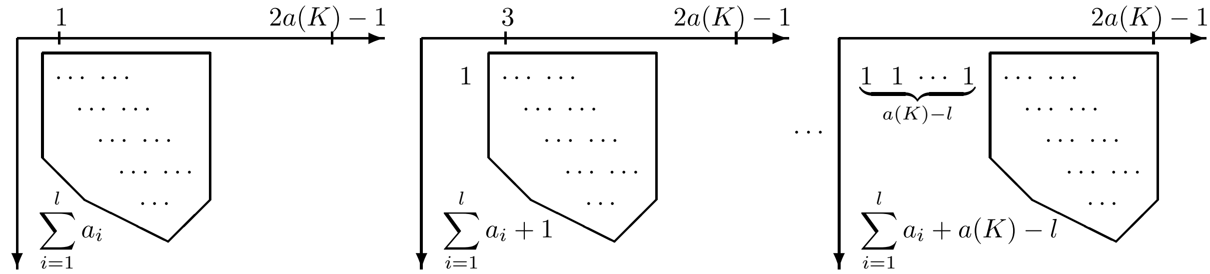
(7.40)

Note, though, that the possibility $a_1 = 0$ (or $a_l = 0$) does exist (although we did not investigate whether or how often it materializes). Furthermore, $a_0 = 1$ can occur also for $d_{min} > 0$ if $[P]_{v^{d_{min}}}$ has terms in z-degree $\neq 1$. Here is the way we put the pan edge to our own use.

Proposition 7.27

$$\sum_{i=1}^{l} a_i \leq \lambda(K) \leq \sum_{i=1}^{l} a_i + (a(K) - l(K))\,. \tag{7.41}$$

Proof. Now remember that $\min\deg_v P(K_t) > 0$ (property (7.3)) for K_t strongly quasipositive (i.e., $t \geq \lambda(K)$), as well as that there is a $t \geq \lambda(K)$, namely $t = \lambda_{min}$, so that $\max\deg_v P(K_t) \leq 2a(K) - 1$ (property (7.1)). Thus, for the polynomial $P(K_{\lambda(K)})$ we have $a(K) - l(K) + 1$ possibilities



(7.42)

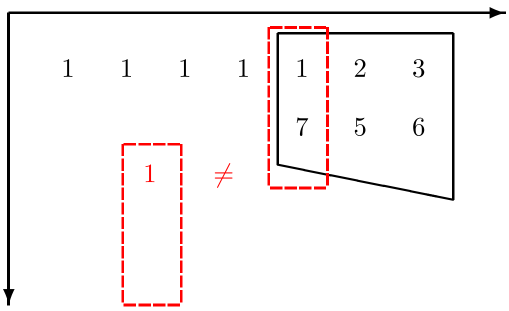
distinguished by the panhandle length $0, \ldots, a(K) - l(K)$.

The pan edge coefficients a_i are not changed for different panhandle length, and by looking at (7.34), we see (7.41). □

Thus, rather precise, information about the Thurston-Bennequin invariant manifests itself in the coefficients of the polynomial, not in its degrees[3]. It provides an additional bonus of computing $P(K_t)$ (for some t), beyond determining $l(K)$. Namely, if $l(K) = a(K)$, then one obtains $\lambda(K)$ practically for free. This "frying eggs in the pan" procedure can be useful, for instance, in comparison to Theorem 5.12, when $a(K)$ is found without constructing a minimal grid diagram explicitly (see Part 1 of Remark 7.26), or as additional information in obstructing to the existence of certain grid diagrams of a given knot. Remark 7.34 gives a hint how to proceed when $l(K) < a(K)$.

To illustrate the use of (7.41), consider the following examples.

Example 7.28 The polynomial[4]

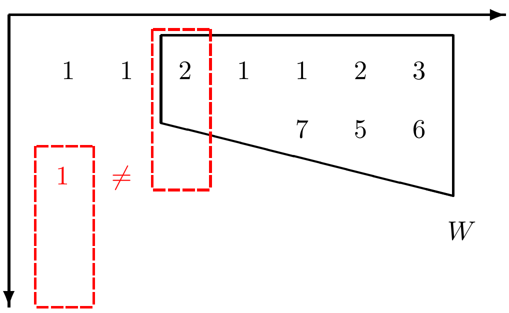


has panhandle length 4 and pan-width $l(K) = 3$. If $a(K) = 5$, then (7.41) has on the right $(5 - 3) + (1 + 2 + 3) = 8$, so (7.41) reads $6 \leq \lambda(K) \leq 8$.

[3] Of course, if one is allowed to use $[P(K)]_{z^0}$, then t can be retrieved from $[P(K_t)]_{z^{-1}}$ using (7.11) as well.

[4] We emphasize that the polynomials in this and the next example are not HOMFLY-PT polynomials of real knotted annuli, i.e., the reader should not try to guess what K they were obtained from; we just hand-invented the polynomials for illustrative purposes.

Example 7.29



has panhandle length 2 and pan-width $l(K) = 5$. If $a(K) = 6$, then (7.41) has on the right $(6-5)+(2+1+1+2+3) = 10$, so (7.41) reads $9 \le \lambda(K) \le 10$.

We have then the following "Matsuda-Dynnikov-Prasolov" (see Remark 7.32) type of relationship.

Proposition 7.30 With the notation of §2.2 for mirror image,

$$l(K) \le \lambda(K) + \lambda(!K) \le 2a(K) - l(K)\,. \tag{7.43}$$

Proof. We prove the right inequality. The argument can easily be modified to show the left one. We also assume, after inspection, that K is non-trivial. We have $(!K)_{-t} = !(K_t)$. Note that (2.15) (with $\kappa = 2$ as for $K_t = A(K,t)$) gives

$$P(!K_t)(v,z) = -P(K_t)(v^{-1},z)\,. \tag{7.44}$$

Now by mirroring property (7.1) using (7.44), we see that there is a $t = \lambda_{min}(K) \ge \lambda(K)$ with

$$\max \deg_v P((!K)_{-t}) \le -1\,, \quad \min \deg_v P((!K)_{-t}) \ge 1 - 2a(K)\,.$$

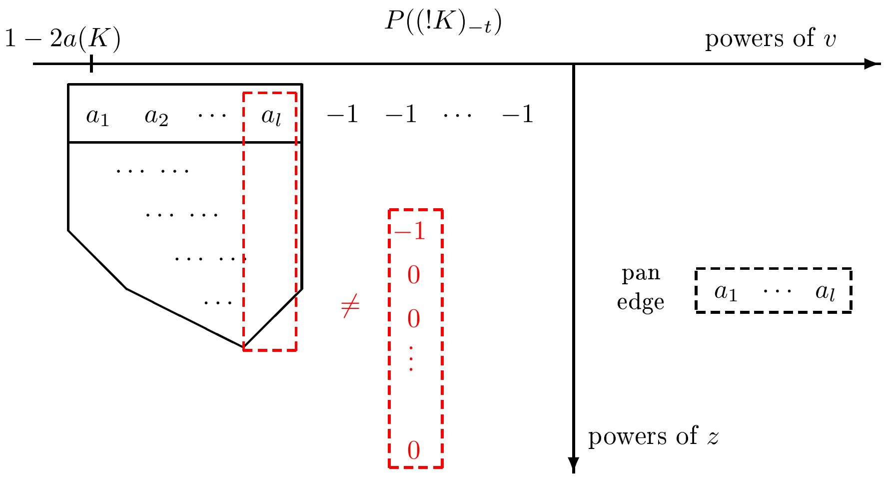
By how $l(K)$ was defined, and again using the mirroring (7.44), there is an odd

$$0 > d \ge -1 - 2a(K) + 2l(K) \tag{7.45}$$

so that

$$[P((!K)_{-t})]_{v^d} \ne -z \tag{7.46}$$

holds. (The condition (7.13) mirrors through (7.44) to (7.15).)

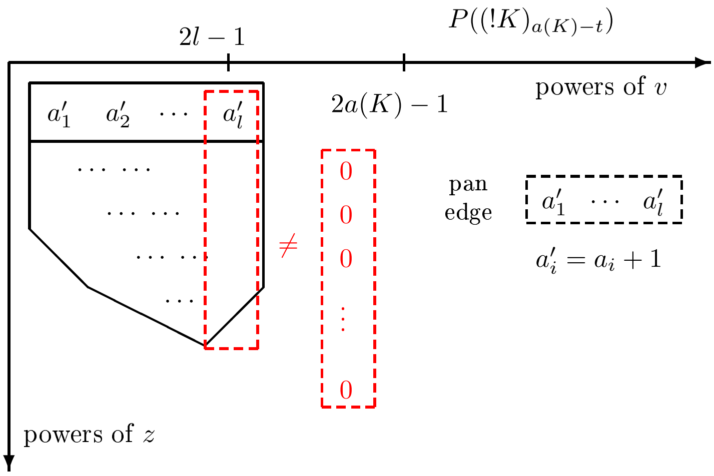


The repeated application of (7.10) then shows

$$\min \deg_v P((!K)_{a(K)-t}) \geq 1$$

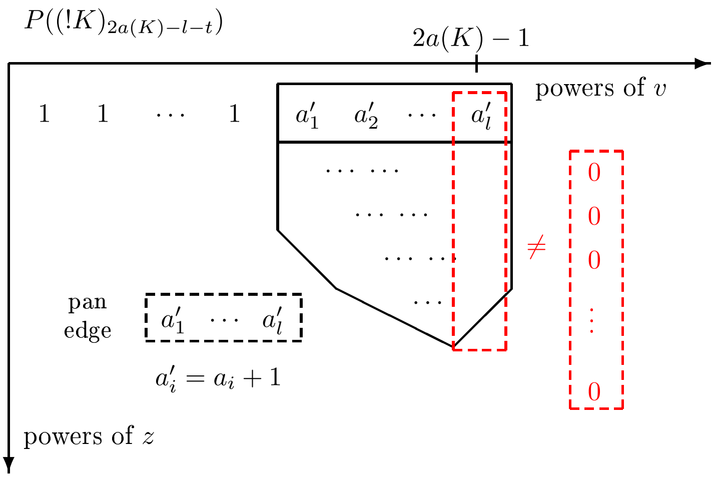
and by (7.45)

$$\max \deg_v P((!K)_{a(K)-t}) \geq 2l(K) - 1\,. \tag{7.47}$$



To see this last inequality (7.47), note that the terms annihilated by (7.10) when t increases are exactly those for $d < 0$ where (7.46) does not hold. Since $a(K) = a(!K)$, the inequality (7.47) means that the largest t' with $\max \deg_v P((!K)_{t'}) \leq 2a(!K) - 1$ satisfies

$$t' \leq 2a(K) - l(K) - t\,.$$



Now we can apply Lemma 7.1 on $!K$. We have

$$\lambda(!K) \leq t' \leq 2a(K) - l(K) - t = 2a(K) - l(K) - \lambda_{min}(K) \leq 2a(K) - l(K) - \lambda(K)\,,$$

as we claimed. $\square$

Example 7.31 We show a (fictitious) exemplary transformation of the $[P(K_t)]_{z^1}$ terms with increasing t, with the symbolics used in (7.23).

$$\boxed{5\ 4\ 1}\ -1\ -1\ \Big|\ \to\ 5\ 4\ 1\ -1\ \Big|\ \to\ 5\ 4\ 1\ \Big|\ \to \tag{7.48}$$

$$\to\ 5\ 4\ \Big|\ 2\ \to\ 5\ \Big|\ 5\ 2\ \to\ \Big|\ \boxed{6\ 5\ 2}\ \to\ \Big|\ 1\ 6\ 5\ 2\ \to\ \Big|\ 1\ 1\ 6\ 5\ 2$$

It consists of 7 steps: $a(K)=5$, $l(K)=3$, thus $2a(K)-l(K)=7$.

Remark 7.32 Matsuda [Ma] (see also [Ng]) proved

$$a(K)\geq \lambda(K)+\lambda(!K), \tag{7.49}$$

which improves the right inequality in (7.43). But in fact, Theorem 5.12 with Corollary 4.11 shows that equality holds, answering [Ng, Question 1]:

$$a(K)=\lambda(K)+\lambda(!K)\,. \tag{7.50}$$

Then Proposition 7.30 can be interpreted by saying how much the HOMFLY-PT polynomial "sees" from that geometric reasoning. But we approach (7.43) from the viewpoint of strong quasipositivity, which can later be adapted to quasipositivity (see Proposition A.2). Thus even with Theorem 5.12, our argument is not redundant.

Remark 7.33 When K is an amphicheiral knot, $K=!K$, then $A(K,0)$ is an (orientedly) amphicheiral link. One can use this and (2.15) to conclude that in that case both $l'(K)$ and $l(K)$ are even. This is compatible with the fact that $a(K)$ is even through (7.50). Furthermore, the a_i in (7.39) exhibit a shifted antisymmetry: in the normalization $d_{min}>0$, they satisfy $a_i+a_{l+1-i}=1$.

For computational purposes, we repeat here the formal self-contained (but not very pleasant) expression for $l(K)$ and the estimate (7.41) that is valid for arbitrary t. Take $P=P(K_t)$ for some $t\in\mathbb{Z}$. The quantities d_{min} and d_{max} can be determined as follows. Set

$$\widetilde{\min\deg}_v P=\left\{\begin{array}{cc}\min\deg_v P & \min\deg_v P<0\\ \min\{\,d>0\,:\,[P]_{v^d}\neq z\,\} & \min\deg_v P>0\end{array}\right.$$

and

$$\widetilde{\max\deg}_v P=\left\{\begin{array}{cc}\max\deg_v P & \max\deg_v P>0\\ \max\{\,d<0\,:\,[P]_{v^d}\neq -z\,\} & \max\deg_v P<0\end{array}\right.\,.$$

Then

$$l(K)=\frac{1}{2}\left(\widetilde{\max\deg}_v P-\widetilde{\min\deg}_v P\right)+1\,,$$

and (7.41) can be stated as

$$\lambda(K)-\left([P]_{z^1}(v=1)+\left\{\begin{array}{cc}\lfloor -1/2\min\deg_v P\rfloor & \min\deg_v P<0\\ -\left\lfloor 1/2\widetilde{\min\deg}_v P\right\rfloor & \min\deg_v P>0\end{array}\right\}\right)\in[0,a(K)-l(K)]\,.$$

Remark 7.34 Again, if (7.25) occurs, then one can adapt the arguments in Remark 7.26 to disambiguate the value for $\lambda(K)$. This gives a practical way to calculate this number for any given K.

8 Braid indices revisited (and problematized)

8.1 Framing cones and the arc index

Here we summarize some remarks provided on various braid indices, and add discussion of related natural questions. They are meant to point out a series of subtleties, which may be significant or not, but which are easy to overlook while less straightforward to resolve. One having some particular importance in this context is Question 5.7. We reformulate part (a) here as a conjecture, with the insight gained from Corollary 5.3 and Remark 7.21.

Conjecture 8.1

$$a(K) = \min_{t\in\mathbb{Z}} b(A(K,t)) \tag{8.1}$$

The following reasoning will appear in several modified versions below, thus we record it as a lemma. Compare with Theorem 5.12.

Lemma 8.2 Assume (8.1) is true. Then (4.11) holds, in particular λ_{min} is unique.

Proof. Take an $a(K)$-band positive band presentation of $A(K,t)$ for $t = \lambda_{min} \geq \lambda(K)$, and make one band negative. By Remark 4.3, one has then an $a(K)$-band presentation of $A(K,t-1)$. Now since $A(K,t)$ is strongly quasipositive, it is Bennequin-sharp. But

$$\chi(A(K,t)) = \chi(A(K,t-1))\,, \tag{8.2}$$

and thus the $a(K)$-band presentation of $A(K,t-1)$ is not Bennequin-sharp, i.e., it does not make (2.5) an equality. But still $b(A(K,t-1)) = a(K)$ by (8.1). Now, if $A(K,t-1)$ is strongly quasipositive, then because of Theorem 2.2, every minimal braid representative of $b(A(K,t-1))$ would make (2.5) an equality. Thus we have that $A(K,t-1)$ is not strongly quasipositive. This means that $t-1 < \lambda(K)$, and so $t \leq \lambda(K)$, with the reverse inequality already observed. $\square$

Remark 8.3 Note that Conjecture 8.1, when K is alternating, is related to an affirmative answer to Question 7.15. But it is not entirely implied by such answer, because of the sporadic collapsing scenario elucidated in the proof of Proposition 7.7. The way $l(K)$ was defined, $\mathrm{MFW}(K_t) < l(K)$ for some t can occur. Of course, replacing $l(K)$ with the bound $l'(K)$ in (7.9) avoids the collapsing problem. But we remind from the proof of Proposition 7.7 that we verified (7.9) to be (even very) unsharp in same cases.

More generally than (4.11), we have:

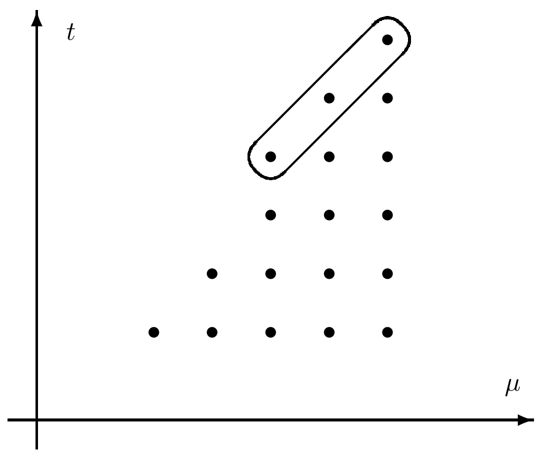
Lemma 8.4 Conjecture 8.1 implies a positive answer to Question 5.10, that $\Phi(K)$ is a single cone

$$\Phi(K) = C(a(K), \lambda(K))\,.$$

Proof. Conjecture 8.1 implies that in any band presentation on $s = a(K) + k$ strings with $> k$ negative bands will give an non-strongly quasipositive $A(K,t)$. The framing t changes with the sign of bands in an obvious way (compare with Remark 4.3). Thus if $(s,t) \in \Phi(K)$, then $t-(s-a(K)) < \lambda(K)$, in particular $(s, t-(s-a(K))) \notin \Phi(K)$. Therefore,

$$(s,t) \in \Phi(K) \Longrightarrow t \leq \lambda(K) + s - a(K)\,.$$

That is there are no points in $\Phi(K)$ like the encircled:



This shows the cone shape of $\Phi(K)$. $\square$

Lemma 8.4 pertains to the situation one may expect. But one can also use Theorem 2.2 for a version when Conjecture 8.1 is unresolved (or false).

Definition 8.5 Define the *defect* of K by

$$\delta(K) = a(K) - \min_{t\in\mathbb{Z}} b(A(K,t))$$

Then the argument for Lemma 8.4 modifies to show that an $a(K)$-band positive band presentation of $A(K,t)$ gives

$$\lambda(K) \le t \le \lambda(K) + \delta(K)\,, \tag{8.3}$$

and any positive band presentation of $A(K,t)$ on $s = a(K) + k$ strings will have

$$\lambda(K) \le t \le \lambda(K) + \delta(K) + k = \lambda(K) + \delta(K) + s - a(K)\,. \tag{8.4}$$

From this, we can conclude the following.

Proposition 8.6 For a non-trivial knot K, we have that $\Phi(K)$ is the union of at most $1 + \delta(K)$ cones.

Note that for $K = \bigcirc$, we have $\delta(K) = 0$, so that the claim is false due to the circumstance (4.10). (But, again, this case can be worked out separately: see Example 5.11.) In Remark 7.21 we have verified that $\delta(K) = 0$ for all prime knots K up to 10 crossings.

Proof. The condition (8.4) places (s,t) into a trapezoid which is the union of the cones $(a(K),t)$ for t in (8.3). Now, $\Phi(K)$ in obviously only contained in this union. Call a cone $C(\mu,t)$ in $\Phi(K)$ *essential*, if it is not properly contained in any other cone in $\Phi(K)$. Among cones $C(\mu,t)$ of fixed $t-\mu$ in $\Phi(K)$, there is always a maximal one, namely the one of the smallest μ. The same is true among cones $C(\mu,t)$ of fixed t in $\Phi(K)$. Note also that there are no values t with $\lambda(K) \le t < \lambda_{min}$, since for $K \ne \bigcirc$, we have

$$\lambda_{min} = \lambda(K)$$

by Theorem 5.12.

Also, for each value $x = \lambda(K) + 1 - a(K), \dots, \lambda(K) + \delta(K) - a(K)$ there is at most one essential cone $C(\mu,t)$ in $\Phi(K)$ with $t - \mu = x$. We call this essential cone *type X*. Obviously $C(a(K),\lambda(K))$ is also essential, and every other essential cone is of type X, by the above maximality remark. Now we have at most $\delta(K)$ type X essential cones. With $C(a(K),\lambda(K))$, this completes a set of $\delta(K)+1$ essential cones, as claimed. $\square$

Obviously, from the definition,

$$\delta(K) \le a(K) - 2b(K)\,.$$

Thus in particular from (8.4), we have

$$\lambda(K) \le t \le \lambda(K) + s - 2b(K)$$

for any positive band presentation of $A(K,t)$ on $s \ge a(K)$ strings. Note also that, for computational purposes, one may replace '$1 + \delta(K)$' in Proposition 8.6 by '$1 + a(K) - l(K)$', with an analogous proof argument. (An analogous caveat regarding $K = \bigcirc$ is needed, where $a(K) = l(K) = 2$; see (7.17).)

Proposition 8.7 When K is a non-trivial knot, then $\Phi(K)$ is the union of at most $1 + a(K) - l(K)$ cones. $\square$

8.2 Indices from braided surfaces

We return to Definition 2.1, and the inequality

$$b_{sqp}(S) \geq b(S)$$

for a strongly quasipositive surface.

Question 8.8 While it is more than suggestive, we do not know if always equality holds. I.e., is every strongly quasipositive surface always realizable on its minimal number of strings in a positive band presentation?

Because of Theorem 2.2, this is true if $b(S) = b(K)$ (where of course $K = \partial S$). This is also related to the Baker-Motegi question if all minimal genus surfaces of a strongly quasipositive knot K are strongly quasipositive (see [St2]). From [HS], we know that $b(S) > b(K)$ for some minimal genus surface S of K. But S (and K) is not strongly quasipositive in these examples. Rudolph's question (5.1) is then equivalent to asking whether

$$b_{sqp}(S) = b(K) \tag{8.5}$$

is satisfied for *some* strongly quasipositive surface S of K. It is tempting to ask if (8.5) holds in fact for *every* strongly quasipositive surface S of K.

In case of the links $L = A(K,t)$ and $W_{\pm}(K,t)$, the minimal genus surfaces S_L of L are unique (and plumbing equivalent), so there is no need to distinguish between $b_b(S_L)$ and $b_b(L)$, and between $b_{sqp}(S_L)$ and $b_{sqp}(L)$.

Proposition 8.9 We obviously have

$$\min_{t \geq \lambda(K)} b_{sqp}(A(K,t)) = a(K)\,, \tag{8.6}$$

and for $t \geq \lambda(K)$, we can incorporate Whitehead doubles into the diagram as

$$\begin{array}{ccc} b_{sqp}(A(K,t)) & \geq & b(A(K,t)) \\ (*)\ \rotatebox{-90}{$\geq$} & & \\ b_{sqp}(W_+(K,t)) & \geq & b(W_+(K,t)) \end{array} \tag{8.7}$$

Also, if K is l-sharp, then all inequalities are equalities.

Proof. The vertical inequality (*) holds because one can double any (positive) band in a strongly quasipositive band presentation of a t-twisted annulus for K (Example 4.16).

Now, consider the case that $l(K) = a(K)$. Since for $K = \bigcirc$ the equality questions in (8.7) can be settled by direct inspection, assume that $K \neq \bigcirc$, to avoid complications.

Consider $L = A(K, \lambda(K))$. We have

$$\max\deg_v P(L) = 2a(K) - 1\,, \tag{8.8}$$

and this means by (2.17) that an $a(K)$-braid (band) presentation of L cannot be of writhe less than $a(K)$. Since we did not assume $l'(K) = a(K)$, there may be a cancellation of terms in z-degree 1 (similarly to the first polynomial in Table 2). Thus $\min\deg_v P(L) > 1$ is, in principle, possible. But the writhe of an $a(K)$-braid (band) presentation of L cannot be more than $a(K)$ due to Bennequin's inequality (2.5). This means that the writhe of an $a(K)$-braid (band) presentation of L is unique, and hence $b(L) = a(K)$.

Then one can start with $t = \lambda(K)$ and propagate the bound in (2.17) through the recursion (7.4), while applying positive stabilizations (see (4.12)). $\square$

Remark 8.10 By noting that we needed in the above proof only (8.8), for which $l(K) = a(K)$ is sufficient but not necessary, one also obtains equalities in (8.7) for $K = 10_{132}$. Pictorially speaking, this extra argument succeeds because the "missing terms" in $P(A(K,t))$, accounting for the difference (7.25), are missing "at the bottom" (in low v-degrees; see the first polynomial in Table 2). Obviously, this immediately changes when v-conjugating the polynomial (by (2.15)), which explains why the trick definitely fails for the mirror image $!10_{132}$.

It follows from Computation 7.20 that all inequalities in (8.7) are equalities at least when minimum over $t \geq \lambda(K)$ is taken. This then holds for all Rolfsen knots, with mirror images (see also Example 8.12).

We can expect in (8.7) the horizontal '$\geq$' to be '$=$' in general, in accordance with Rudolph's Question (5.1). However, we do not know about (*). Obviously $S_{W_+(K,t)} = S_{A(K,t)} * H$ is a plumbing with a positive Hopf band H. But we know that

$$b_{sqp}(S * H) < b_{sqp}(S)$$

is possible, even for a strongly quasipositive fiber (in particular unique minimal genus) surface S; examples were given in [St2]. These examples, unsurprisingly, have higher genus, but they should still caution about seeing (*) as suggestive in some way.

Also, regarding (8.6), we can add

$$\min_{t \geq \lambda(K)} b_{sqp}(A(K,t)) = a(K) = \min_{t \in \mathbb{Z}} b_b(A(K,t))\,, \tag{8.9}$$

because every band presentation of Bennequin surface of $A(K,t)$ gives a grid diagram of K, and gives a strongly quasipositive surface of $A(K,t')$ for some $t' \geq \lambda(K)$ by making all bands positive.

Proposition 8.11 Then for instance for $t < \lambda(K)$, we have a similar diagram of inequalities to (8.7)

$$\begin{array}{ccc} b_b(A(K,t)) & \geq & b(A(K,t)) \\ (**)\ \ \text{\rotatebox{90}{$\geq$}} & & \\ b_b(W_-(K,t)) & \geq & b(W_-(K,t)) \end{array} \tag{8.10}$$

And if K is l-sharp, then all inequalities are equalities.

Proof. The inequality (**) results from doubling a negative band in a minimal band presentation (a negative band always exists when $t < \lambda(K)$; see the remarks following Example 4.16. And if $l(K) = a(K)$, we can infer with a similar thought to Proposition 8.9 that all inequalities are in fact equalities. (Again, exclude $K = \bigcirc$ after a direct check.) The only framing t for which cancellation may collapse the bound $\mathrm{MFW}(A(K,t))$ is when all bands in an $a(K)$-strand band presentation of $A(K,t)$ are negative. Then the argument with Bennequin's inequality applies to $!A(K,t)$. $\square$

Again (while it is tempting to suspect) we do not know if equalities hold in general.

Example 8.12 From Computation 7.20, we know that for all $t \in \mathbb{Z}$,

$$b(W_{\pm}(10_{132},t)) \geq 9 = a(10_{132})\,. \tag{8.11}$$

Obviously, as in Table 2, is it possible to write down explicit band presentations of $A(10_{132},t)$ and $W_-(10_{132},t)$ for some $t < \lambda(10_{132})$ on 9 strings, so that we have

$$b_b(A(10_{132},t)), b_b(W_-(10_{132},t)) \leq 9\,.$$

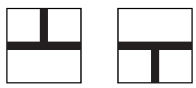
With Computation 7.20 we again know that thus for $K = 10_{132}$, the inequalities (8.10) are equalities at least when their hand sides are minimized over $t < \lambda(K)$. Under mirroring (using the computations and band presentations for $W_+(10_{132},t)$), we can conclude the same for $K =!10_{132}$, and thus for all Rolfsen knots.

When (7.25) occurs, though, this reasoning always relies on an explicit check for specific t using a 2-cable polynomial. And while we expect non-l-sharp knots to be relatively rare, such instances K clearly increase with crossing number (see Example 7.10). The method in Computation 7.20 soon becomes problematic complexity-wise, despite algorithmic optimizations. This puts a limit to the capacity of our algebraic approach to tackle a geometric issue like the sharpness of the inequalities (8.10). (But of course it is the only information we have available so far.)

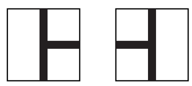
9 Conclusion

The work described here started with the simple question: how does a braided surface of Euler characteristic 0 look like? While there seems little hope to give a classification result, the attempt unfolded a connection into a variety of issues. We encountered many suggestive but difficult to resolve questions, whose examination would require deepening this consideration.

For smaller Euler characteristic, one obtains instead of a grid diagram a "grid-embedded (trivalent) graph". It can be described as a PL spatial embedding of a trivalent graph whose diagram can be built up with the tiles in (2.28), and the two extra tiles



but *not*



Developing a similar theory of grid-embedded graphs will thus also be a long – but nevertheless perhaps very interesting – undertaking.

A Remarks on quasipositivity

We have decided to strictly focus on strong quasipositivity of $W_{\pm}(K,t)$ and $A(K,t)$, essentially because of the direct relationship in Corollary 4.13 and its extensive consequences.

The problem of the *quasipositivity* of these links is far more obscure, but perhaps also very interesting.

A.1 Knotted annuli

Let us define

$$\lambda_q(K) := \min\{\ t:\ A(K,t) \text{ is quasipositive }\}\,. \tag{A.1}$$

Then we know that

$$\lambda_q(K) \le \lambda(K) = -TB(K)\,,$$

so the obvious question is: is $\lambda_q(K) = \lambda(K)$ for all K? Or, given (4.17), in other words: for every t, if is $A(K,t)$ quasipositive, is it strongly quasipositive? The answer is "no". We identify one major reason, which we call λ-sliceness (Definition A.3), but it is possible what further, more peculiar, phenomena occur.

Question A.1 For what knots is $\lambda_q(K) = \lambda(K)$?

Going further, we do not know if always

$$A(K,t) \text{ is quasipositive} \implies A(K,t+1) \text{ is quasipositive}. \tag{A.2}$$

That is, between the pans in (7.42), the quasipositivity of the links can switch on and off a few times before, for long enough panhandle length, eventually strong quasipositivity settles in. This looks like an adventurous scenario, but see Corollary A.9.

Since the (exact) connection of λ_q to the Thurston-Bennequin invariant remains elusive, so become certain methods focussing on the latter (like Theorem 5.12). But there are properties of λ_q that follow from our work above.

Many of the arguments based on the HOMFLY-PT polynomial do go through. In general, though, the preceding problems with the unknot extend here to K being slice. This changes (weakens) the inequalities a little, but to be precise, we either have to exclude sliceness, or work in cases.

Proposition 7.30 can be changed into the following form.

Proposition A.2 We have

$$\left\{ \begin{array}{ll} l(K) \le \lambda_q(K) + \lambda_q(!K) \le 2a(K) - l(K) & \text{if } K \text{ is not slice} \\ l(K) - 1 \le \lambda_q(K) + \lambda_q(!K) \le 2a(K) - l(K) + 1 & \text{if } K \text{ is slice} \end{array} \right. \tag{A.3}$$

Proof. Lemma 7.1 holds regardless of K being slice or not. But (7.3) still holds for quasipositive $A(K,t)$ only if K is not slice, or $t \neq 0$. Thus, when K is slice, and $t = 0$, then we must allow, instead of (7.3), for

$$\min \deg_v P(A(K,0)) = -1\,, \tag{A.4}$$

since $\chi_4(A(K,0)) = -2$ can occur (see end of §2.2). Then the argument goes through, but the numerics changes slightly. □

The right inequalities in (A.3) are, as explained, not very interesting now (they follow from Matsuda's result (7.49)), but the left ones have some useful implications. To better formulate them, here we define a condition which will repeatedly play some role.

Definition A.3 Call a knot K to be λ-*slice* if K is slice and

$$\lambda(K) = 1\,. \tag{A.5}$$

We say K to be λ-slice up to mirroring if one of K and $!K$ is λ-slice.

Remark A.4 The condition (A.5) and Rudolph's version of (2.6) (see the proof of Theorem 1.5 in [He2]) then also imply that a λ-slice knot K is slice Bennequin-sharp. The presumption (2.9) then possibly suggests that a λ-slice knot is quasipositive. (Example 6.6 shows that the τ invariant is insufficient to see this, though.)

Example A.5 Among the Rolfsen knots, λ-slice (up to mirroring) are $K = 9_{46}$ and 10_{140}. As Remark A.4 suggests, they are indeed quasipositive, and we fix here their quasipositive mirroring, which also satisfies

$$\min \deg_v P(K) = 0\,.$$

Corollary A.6 Assume K is l-sharp. Then either $\lambda_q(K) = \lambda(K)$, or K is λ-slice and

$$0 = \lambda_q(K) < \lambda(K) = 1\,. \tag{A.6}$$

Proof. Assume K is not slice. Then, (A.3) gives

$$l(K) \le \lambda_q(K) + \lambda_q(!K) \le \lambda(K) + \lambda(!K) \le 2a(K) - l(K)\,,$$

and $l(K) = a(K)$ implies that all inequalities are exact.

If K is slice, notice that K being l-sharp makes (A.4) relevant only if $t = 0 = \lambda(K) - 1$, as can be seen thus.

Deal with $K = \bigcirc$ by a direct check. Then take a minimal grid diagram D of K. Because of l-sharpness (or Theorem 5.12), we know that $\lambda(D) = \lambda_{min}(K) = \lambda(K)$ is unique. Obviously $t \ge \lambda_{min}(K)$ give strongly quasipositive $A(K,t)$, and are not interesting. So assume $t < \lambda_{min}(K)$. Then, $l(K) = a(K)$ also shows iteratively that

$$\min \deg_v P(A(K, \lambda(K) - k)) = 1 - 2k$$

for $k > 0$. Then $A(K, \lambda(K) - k)$ can be quasipositive (and (A.4) is relevant) only if $k = 1$. And then we need $\chi_4(A(K, \lambda(K) - 1)) = -2$, which requires $t = 0 = \lambda(K) - 1$, and means the condition (A.5). $\square$

Corollary A.6 lends further impetus to the study of $l(K)$, and a question like Question 7.15. Among others, it leaves some prospect that at least for some classes of alternating links, Question A.1 can be resolved using the present approach.

Remark A.7 Notice that a λ-slice amphicheiral knot is trivial by (7.50). Thus in particular Corollary A.6 holds for l-sharp amphicheiral knots K. But for them (7.50) is not needed: Assume K is amphicheiral. Then we have from (A.3)

$$l(K) - 1 \le 2\lambda_q(K) \le 2\lambda(K) \le 2a(K) - l(K) + 1 \;=\; l(K) + 1\,,$$

but $l(K)$ is even by Remark 7.33. This is enough to see that the middle inequality is an equality.

The slice case continues to require special attention, as witnessed by the following explanation, which shows in more detail how to handle individual examples.

Proposition A.8 If K is a prime knot of up to 10 crossings, then $\lambda_q(K) = \lambda(K)$, except (A.6) for $K = 9_{46}$ and $K = 10_{140}$.

The condition (A.5) also determines the mirroring of the knots, as fixed in Example A.5.

Proof. First consider the case $l(K) < a(K)$. This applies only to $K = 10_{132}$ and $K = !10_{132}$.

Let $K = 10_{132}$. Then $\chi_4(A(K,t)) = 0$ for all t, and we computed that $\min\deg_v P(A(K,t)) > 0$ for $t \ge 0$. When $t \ge \lambda(K) = 1$, then $A(K,t)$ is already strongly quasipositive, so consider only $t = 0$. But we computed that $b(A(K,0)) = 8$, and the minimal writhe of a braid representative of $A(K,0)$ is 6. If $A(K,0)$ were to be quasipositive, then $\chi_4(A(K,0)) = -2$, which is not the case (as K is not slice). This finishes off $K = 10_{132}$.

Next let $K = !10_{132}$. Then again $\chi_4(A(K,t)) = 0$ for all t, and we computed that $\min\deg_v P(A(K,t)) > 0$ for $t \ge 8 = \lambda(K)$. But then again $A(K,t)$ is strongly quasipositive, so this case is done either.

Now let $a(K) = l(K)$. We need to consider only K being λ-slice (and K is non-trivial), so that only $K = 9_{46}$ and $K = 10_{140}$ remain (with the mirroring in Example A.5). Their treatment continues in the argument below. $\square$

This leads then to the following cautionary tale.

Corollary A.9 For $K = 9_{46}$ and $K = 10_{140}$, we have (A.6).

Proof. We checked that, if $K = 9_{46}$ and 10_{140}, then

$$\min \deg_v P(A(K,0)) = -1\,,$$

which indeed leaves the opportunity that $L = A(K,0)$ is quasipositive. And it is easy to write down (minimal) braid representatives of L: use the grid diagrams given [J+], and make exactly one band negative (compare with Example 4.10 and Table 2).

Testing the quasipositivity of these braids is very difficult. But they satisfy, in (2.6), $w - n = -2 = \chi_4(L)$. Thus L are slice Bennequin-sharp.

In attempting a solution, I consulted Stepan Orevkov. He indeed found the following (everything but self-evident) quasipositive form (2.3) for one of the braids for 9_{46} given below[5], where x^y stands for $y^{-1}xy$.

```
[36] [58] [27] [16] [48] -[37] [25] [14] =
[3 4 5 -4 -3   5 6 7 -6 -5   2 3 4 5 6 -5 -4 -3 -2
  1 2 3 4 5 -4 -3 -2 -1   4 5 6 7 -6 -5 -4   3 4 5 -6 -5 -4 -3
   2 3 4 -3 -2   1 2 3 -2 -1] =
4^[3 -4 3 2 -3 4 3 -4 3 2 -3 4 3 -4 3 2 1 -3 4 3 2 -3 -4
-5 4 3 -4 3 6 5 4 7 6 7 6 5 6 7]*
3^[2 -3 4 3 -4 3 2 -3 4 3 -4 3 2 -3 4 3 -4 3 2 1 -3 4 3
2 -3 -4 -5 4 3 -4 3 6 5 4 7 6 7 6 5 6 7]*
4^[3 3 5 5 4 6 5 6 7]*   2^[3]*   1^[2 3 -4]*  6
```

(He also argued that the other 7 braids obtained by making some of the other bands negative are quasipositive as well.) This proves that $A(9_{46},0)$ is quasipositive, and $\lambda_q(9_{46}) = 0$. Orevkov also found quasipositive presentations for some of the similarly constructed 9-braids for $A(10_{140},0)$. $\square$

Remark A.10 By doubling a positive band (this preserves quasipositivity), we can also obtain a knot $L = W_+(9_{46},0)$ which is quasipositive, but not strongly so. A similar argument applies for $W_+(10_{140},0)$.

Example A.11 By doubling both *one* positive band and *the* negative band, one has a band presentation of $B(K,0)$ (in Definition 4.1). Orevkov also found a quasipositive form for some of the thus obtained braids for $B(9_{46},0)$, for instance

```
[36] [58]^2 [27] (-[16])^2 [48] [37] [25] [14]=
3^[2 4 4 3 2 -4 -5 4 3 4 6 4 3 7 5 4 7 6 5 7 6]*
3^[2 4 4 3 2 1 -4 -5 4 3 1 6 5 4 7 6 5 7 6 5 7 7]*
4^[3 3 5 4 6 5 6]* 4^[3 3 4 5 6 -7]* 2^[1 3]* 2^[3 -4]
```

He also proved that the below braid for $B(10_{140},0)$ is quasipositive,

$$[47]^2[69][28](-[17])^2[59][48][36][25][13]\,.$$

(It can then be argued that all other braids obtained in a similar fashion from the brand presentations of $A(K,0)$ are quasipositive as well.) Thus $B(9_{46},0)$ and $B(10_{140},0)$ are quasipositive (while not strongly so; see Corollary 4.15).

The treatment of 10_{132} for Proposition A.8 exemplifies why non-slice K are far easier to deal with. The special role of slice K is also underscored by the following fact, which can be proved similarly to Lemma 8.2. (We do not like to repeat the proof here; one mainly has to replace χ by χ_4 in (8.2).) It is a more theoretical (and less practical) version of Corollary A.6.

[5]Here the proper mirroring of 9_{46} is needed, and we chose to read the grid diagram in [J+] from the bottom, which gives this mirroring; *cf.* Remark 4.5.

Proposition A.12 If Conjecture 8.1 is true for K, then $\lambda_q(K) = \lambda(K)$, except if K is λ-slice. □

Note also that $A(K,t)$ for $K = 9_{46}$ (and other knots for which the condition (A.5) was relevant) were considered in the paper [Tr].

A.2 Whitehead doubles

The quasipositivity problem of Whitehead doubles seems not to have been treated much in the literature. The only source I know is the following.

Example A.13 ([He, Examples, (5)]) The positive clasped Whitehead double $W_+(K,t)$ is not quasipositive when

$$t \le -\lambda(!K)\,, \tag{A.7}$$

and t is *not* of the form

$$t = -p(p-1)\,, \quad p \in \mathbb{N}\,. \tag{A.8}$$

The first condition (A.7) ascertains in Proposition 6.3 that $v(W_+(K,t)) = 0$. Then excluding the values (A.8) ensures that the determinant

$$\det(W_+(K,t)) = |1-4t|$$

is not a square, hence $W_+(K,t)$ is not slice by the Milnor-Fox property (compare with Example 6.6). Then $W_+(K,t)$ is neither quasipositive by (2.26).

The complication that $W_\pm(K,t)$ (while always having genus 1) can turn slice (i.e., 4-ball genus 0) becomes more subtle for Whitehead doubles. It is clear when $\chi_4(A(K,t)) = -2$, namely when $t = 0$ and K is slice, and then always $\chi_4(W_\pm(K,t)) = -1$, but these are not all: $W_+(\bigcirc,-2) = 6_1$, the stevedore knot, is slice as well. In fact, the sliceness problem for Whitehead doubles has an illustrious history, which we only briefly mention. For $K = \bigcirc$ see [CG], and K being slice is the ***proposed*** (and admittably optimistic) answer [Ki, Problem 1.38] when $t = 0$, but the problem remains unsettled even under that constraint.

Not all these doubles are relevant here, but this hinges on the next decision problem, when such knots are quasipositive. Unlike for sliceness, mirroring is very relevant for quasipositivity, and the sign of the clasp plays a crucial role. Most negative clasped Whitehead doubles are still provably not quasipositive (see for example Computation A.21). But the situation for positive clasped Whitehead doubles seems far less uniform. Even just considering the untwisted case $t = 0$, Remark A.10 hints to extreme caution. More of the same is warranted by the following illustration.

Example A.14 The knot 8_{20} is quasipositive and slice, and thus $W_+(8_{20},0)$ is slice. But it is not quasipositive – essentially this is the reason why 8_{20} falls out of the consideration in the proof of Proposition A.8. However, if one takes the $+1/2$ twisted (connected) *2-cable* of 8_{20}, i.e., the zero-framed cable with the pattern $\sigma_1 \in B_2$ (as lying in a solid torus), then it is both slice (since 8_{20} is so) and quasipositive (since 8_{20} is so, by a result of [St2]).

Remark A.15 The provenance of 9_{46} and 10_{140} in Proposition A.8 was from being λ-slice. Remark A.4 then possibly suggests that $W_+(K,0)$ being quasipositive ***but not strongly so*** occurs only if K is quasipositive. But Example A.14 shows that quasipositivity is not sufficient.

Since Murasugi sum makes no sense in the 4-ball, there is no analogous relation for quasipositivity between $A(K,t)$ and $W_+(K,t)$, and Corollary 4.15 cannot be proved in this way for quasipositivity. A similar problem to (A.2),

$$W_+(K,t) \text{ is quasipositive} \implies W_+(K,t+1) \text{ is quasipositive}\,,$$

remains (generally) inaccessible, and we cannot extend Proposition 6.3 and Corollary 6.4 to λ_q. Obviously, one can modify (A.1) to define a number $\lambda_{q+}(K)$ etc., but instead of reiterating here a treatment analogous to $\lambda_q(K)$, it seems better to directly restrict the values of t for which $W_\pm(K,t)$ is quasipositive.

For $W_-(K,t)$, we collect the following easy remarks, that originate from previous results.

Lemma A.16 If $W_-(K,t)$ is quasipositive, then it is slice. Also, for every slice-torus invariant v,

$$t > -j_v(!K)\,,$$

and in particular

$$t \geq -2\tau(K)\,,$$

and also

$$t = p(p-1)\,,\quad p\in\mathbb{N}\,. \tag{A.9}$$

Proof. Let v be a slice-torus invariant. Because $W_-(K,t)$ unknots by a negative crossing change, we have $v(W_-(K,t))\leq 0$. Were $W_-(K,t)$ quasipositive, (2.26) implies that $v(W_-(K,t)) = g_4(W_-(K,t))\geq 0$.

This implies $g_4(W_-(K,t)) = 0$, i.e., that $W_-(K,t)$ is slice. The property (A.9) follows from the Milnor-Fox condition (A.8) under mirroring (the sign in (A.8) changes, since we changed the sign of the clasp).

But it also implies the equivalent conditions

$$\begin{aligned} v(W_-(K,t)) &= 0 \\ v(W_+(!K,-t)) &= 0 \\ -t &< j_v(!K) \\ t &> -j_v(!K)\,, \end{aligned}$$

and in particular with (6.13) also that

$$t > -j_\tau(!K) = -(1-2\tau(!K)) = -1+2\tau(!K) = -1-2\tau(K)\,. \qquad \square$$

The HOMFLY-PT polynomial does recover (albeit by entirely different means from the Murasugi sum) some parts of the complete result of Corollary 4.15 for quasipositivity. Obviously when $W_\pm(K,t)$ is strongly quasipositive, then it is also quasipositive, so here is what we obtain on the obstruction part. (Keep track of the case that $K=\bigcirc$, which is not excluded here, and which at least gives some hints to the limitations emerging.)

Theorem A.17 1. For every knot K, there is at most one value t_0 so that $W_-(K,t_0)$ is quasipositive. If this value t_0 occurs, then the following hold.

(a) We have (with (2.14))

$$\min \mathrm{cf}\,[P(K)]_{z^0} = \pm 1\,, \tag{A.10}$$

(b)

$$\min \deg_v [P(K)]_{z^0} = -p(p-1)\,,\quad p\in\mathbb{N}\,, \tag{A.11}$$

and

(c)

$$t_0 = -\min \deg_v [P(K)]_{z^0}\,. \tag{A.12}$$

(d) Moreover, $t_0\leq\lambda_q(K)$, and equality can hold only if K is slice and $\lambda_q(K)=0$.

2. $W_+(K,t)$ is not quasipositive for
$$t \le \lambda_{min}(K) - a(K) + l(K) - 2\,. \tag{A.13}$$

Proof. Part 1. If $W_-(K,t)$ is quasipositive, then $L = W_-(K,t)$ satisfies the variant of (7.2)
$$\min\deg_v P(L) \ge 1 - \chi_4(L)\,. \tag{A.14}$$
So in particular
$$\min\deg_v P(L) \ge 0\,. \tag{A.15}$$
It follows from applying the skein relation (2.13) that
$$P(W_-(K,t)) = v^{-2} - v^{-1}zP(A(K,t))\,. \tag{A.16}$$
Looking at the z^0-term in this formula (for the others, *cf.* Computation A.21), and using (7.5), we see that
$$\min\deg_v [P(W_-(K,t))]_{z^0} \le -2\,,$$
disabling (A.15), unless
$$\min\deg_v (v^{2t}(v^{-1}-v)([P(K)]_{z^0})^2) = -1\,. \tag{A.17}$$
This occurs for exactly one $t = t_0$, namely the one in (A.12), and this t_0 must be of the form (A.9), giving (A.11). But beyond (A.17), we also need
$$\min \mathrm{cf}_{\,v} (v^{2t}(v^{-1}-v)([P(K)]_{z^0})^2) = 1$$
for a cancellation to occur, and this means that (A.10) must hold.

For Part 1d notice that since when $A(K,t)$ is quasipositive, then because of (A.14) we need
$$\min\deg_v P(A(K,t)) \ge -1\,. \tag{A.18}$$
By the skein relation (2.13), we have then
$$\min\deg_v P(A(K,t')) \ge 1$$
for $t' > t$, i.e., for each $t' \ge \lambda_q(K) + 1$. Thus in (A.16), the second summand on the right will not cancel the first, and $\min\deg_v P(W_-(K,t')) < 0$ for each such t'. This shows $t_0 < \lambda_q(K) + 1$.

Also, note that (A.18) being sharp requires $\chi_4(A(K,t)) = -2$, i.e., $A(K,t)$ bounds in the 4-ball two disjoint disks. This can occur only if K is slice and $t = 0$. This shows the equality property in Part 1d.

For Part 2, this is essentially the reasoning behind (7.8), with the improvement coming from $l(K)$, as outlined in Remark 7.5.

Namely, start with a grid diagram D of K with $\mu(D) = a(K)$ and $\lambda(D) = \lambda_{min}(K)$. Then apply the skein relation (7.4) backward, decreasing t, (at least) $a(K) - l(K) + 2$ times, and see that for t in (A.13), we have
$$\min\deg_v P(A(K,t)) \le -3\,.$$
The skein relation (2.13) (as in the mirrored variant of (A.16)) then shows that
$$\min\deg_v P(W_+(K,t)) \le -2\,,$$
so that $W_+(K,t)$ is not quasipositive. □

Remark A.18 The inequality (A.14) and the mirroring property (2.15) also easily yield [He, Examples, (2)]: if both a knot K and its mirror image $!K$ are quasipositive, then K is slice. Together with (2.16), even more follows: we can see $P(K) = 1$. This tempts to suspect that the only possible K so that K and $!K$ are quasipositive is the unknot.

Remark A.19 Even disregarding (A.8), the restriction (A.13) is generally weaker than (A.7). For instance, using the left inequality of (7.43), and $\lambda_{min}(K) \ge \lambda(K)$, we have that

$$-\lambda(!K) \le \lambda_{min}(K) - a(K) + l(K) - 2$$

definitely holds if

$$2l(K) \ge a(K) + 2\,,$$

which is practically always satisfied. (It certainly is for the unknot and all prime knots through 12 crossings.)

Corollary A.20 If K is quasipositive and not slice (in particular non-trivial and strongly quasipositive), then no $W_-(K,t)$ is quasipositive.

Proof. Because of (A.14), we have that

$$\min\deg_v [P(K)]_{z^0} \ge \min\deg_v P(K) \ge 2g_4(K) > 0\,,$$

so that (A.11) cannot hold. □

Computation A.21 We now know that for no prime knot K up to 10 crossings is any $W_-(K,t)$ quasipositive. The value t_0 in Theorem A.17 exists up to mirror images for about half of the 249 knots K. For them one can explicitly compute a Whitehead double polynomial. (Note that from (A.16) we ignored in the proof all terms of positive z-degree, which of course can be retrieved from such a calculation.) This restricts

$$\min\deg_v P(W_-(K,t_0)) \ge 0$$

only to $L = W_-(10_{140}, 0)$ – the knot 10_{140} reappears. Here, though, it can be "tamed". We have

$$\min\deg_v P(L) = 0\,, \quad \max\deg_v P(L) = 14\,.$$

If quasipositive, L would have a slice Bennequin-sharp braid representative, and by the argument based on Theorem 2.2 we repeated multiple times (see the proof of Lemma 8.2), L would have braid index $b(L) = 8$ (less than $a(10_{140}) = 9$). But this can be ruled out by a truncated 2-cable HOMFLY-PT polynomial calculation (with the tool used in Computation 7.20).

This strongly suggests that maybe no (non-trivial) negatively clasped Whitehead double is quasipositive. But in fact, there is a more general question, which at least passed initial verification (crossing changes in minimal diagrams of prime knots up to 16 crossings).

Question A.22 If a knot is quasipositive (and slice), can it unknot by switching a negative crossing to positive?

A.3 Framing diagrams

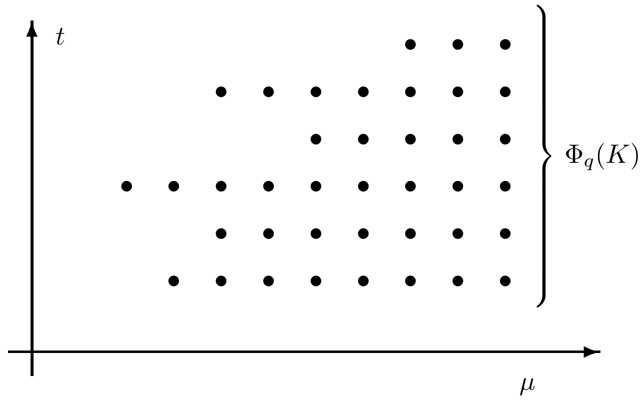
The structure of the analogue of Definition 4.12,

$$\Phi_q(K) := \{\, (\mu,t) \,:\, A(K,t) \text{ has a quasipositive braid representative on } \mu \text{ strands} \,\}\,,$$

remains less clear. Positive *braid* stabilization still implies, as in (4.18), that

$$(\mu,t) \in \Phi_q(K) \Longrightarrow (\mu+1,t) \in \Phi_q(K)\,,$$

which gives a, much poorer, *ray structure* on $\Phi_q(K)$. But the cone structure argument obviously fails so far: in a quasipositive braid representation, the grid is not evident, and grid stabilization makes no sense. That is, we cannot exclude the type of shape:



Nonetheless, some partial information on $\Phi_q(K)$ can be recovered.

Proposition A.23 When K is any knot, then $\Phi_q(K)$ is *contained in* the union of at most $2+\delta(K)$ cones, and in at most $1+\delta(K)$ cones if K is not slice.

Proof. This is obtained by the same reasoning as for Proposition 8.6. But when K is slice (and $t=0$), then we must allow for (A.4) instead of (7.3). Since $\Phi_q(K)$ has no cone structure, we can only claim that $\Phi_q(K)$ is contained in a union of cones. □

Finally, we obtain the following way to reestablish the expected shape of $\Phi_q(K)$.

Corollary A.24 If K is not λ-slice and $\delta(K)=0$, then $\Phi_q(K)=\Phi(K)$ is a single cone.

Proof. Assume K is not slice. Obviously $\Phi_q(K)\supset\Phi(K)$, and from Proposition 8.6 we have

$$\Phi_q(K)\supset\Phi(K)=C(a(K),\lambda_{min}(K))\,,$$

so that $\Phi_q(K)$ contains a cone of the form $C=C(a(K),t)$. On the opposite end, from Proposition A.23 we know that $\Phi_q(K)$ is contained in a single cone $C(\mu,\lambda_{min}(K))$, and we must have $\mu\geq a(K)$ because $\delta(K)=0$. This is only possible if $\Phi_q(K)=\Phi(K)=C$.

Now if K is slice, but not λ-slice, then the condition $\delta(K)=0$ enables us to use, instead of HOMFLY-PT as in Corollary A.6, the proved Jones-Kawamuro conjecture (Theorem 2.2), as for Lemma 8.4. It is the type of argument that allows us to state λ-slice in Proposition A.12 (and was outlined above it). □

In conclusion, we note that it was explained in [Ha] from the work in [LaM] that every quasipositive link has at least one quasipositive minimal braid representative. Thus at least a problem like (5.1) is off the table, and an analogue of much of the discussion in §8.2, for instance, is not very interesting for quasipositivity. But Orevkov's question still stands whether in fact *all* minimal braid representatives of a quasipositive link are quasipositive – an assertion which is obviously false for strong quasipositivity.

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