EXCHANGEABILITY AND NON-CONJUGACY OF BRAID REPRESENTATIVES

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ABSTRACT. We obtain some fairly general conditions on the linking numbers and geometric properties of a link, under which it has infinitely many conjugacy classes of n-braid representatives if and only if it has one admitting an exchange move. We investigate a symmetry pattern of indices of conjugate iterated exchanged braids. We then develop a test based on the Burau matrix showing examples of knots admitting no minimal exchangeable braids, admitting non-minimal non-exchangeable braids, and admitting both minimal exchangeable and minimal non-exchangeable braids. This in particular proves that conjugacy, exchange moves and destabilization do not suffice to simplify braid representatives of a general link.

1. OVERVIEW

Alexander [1] related *braids* to *links* in real 3-dimensional space (henceforth always assumed oriented), by means of a *closure* operation $\hat{\ }$. In that realm, it became important to understand the *braid representatives* of a given link L, i.e., those b with $L = \hat{b}$. *Markov's theorem* relates these representatives by two moves, the *conjugacy* in the braid group, and *(de)stabilization*, which passes between $b \in B_n$ and $b\sigma_n^{\pm 1} \in B_{n+1}$ (see, e.g., [22]). Markov's moves and braid group algebra have become fundamental in Jones' pioneering work [17] and its later continuation towards quantum invariants. Conjugacy is, starting with Garside's [11], and later many others' work, now relatively well group-theoretically understood. In contrast, the effect of (de)stabilization on conjugacy classes of braid representatives of a given link is in general difficult to understand. Only in very special situations can these conjugacy classes be well described, e.g., [7]. In order to reduce the complications of stabilization, Birman and Menasco [6] introduced and extensively studied a move called *exchange move*.

Among the different braid representatives of a link L the one with the fewest strands is called a *minimal braid*. The number of strands of a minimal braid is called the *braid index* $b(L)$ of L. Most of the paper centers around the equivalence for

$$n \geq \max(4, b(L))\,, \tag{1}$$

that

$$\begin{array}{c}L \text{ has infinitely}\\ \text{many } n\text{-braid conjugacy}\\ \text{classes}\end{array} \iff \begin{array}{c}L \text{ has an}\\ n\text{-braid admitting an}\\ \text{exchange move}\end{array} \iff \begin{array}{c}L \text{ has an}\\ n\text{-Seifert circle diagram}\\ \text{which is not a braid diagram}\end{array}. \tag{2}$$

The right equivalence bases on Example 7.1 in [37] and the Vogel move [41]. The left equivalence is true for knots and extends to links with a few exceptions for $\Longleftarrow$, coming from the condition in [29].

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This restriction is very mild; for example, it never applies when there are no trivial (i.e., unknotted) components.

The first part of the paper deals with an effort to circumvent the failure of the left equivalence for links with trivial components. We will show this equivalence unless such components satisfy some very strong divisibility properties of their linking numbers related to the braid index, and some geometric conditions on the link (Theorem 5.2). The role of the braid index in these conditions is largely due to the fact that when $n > b(L) \geq 3$, the situation is (almost) known from [31].

Next, we will investigate which iterated exchanged braids are conjugate. We will discuss several examples in §6 showing that, qualitatively, the restrictions on conjugacy we obtained in [29] are the strongest possible. In Proposition 6.6, we derive an 'almost all' version of this restriction under a very general condition.

From §7 on, we will describe efforts to decide which knots K admit minimal exchangeable braids and which not. During these calculations, it is obviously necessary to determine the braid index, which is discussed simultaneously. (We have extended the braid index computation of [33] to 13 crossings, and for $b(K) \leq 4$, also to 14 crossings.) We will develop a test outlined by Jones [17, §12], based on the Burau matrix, to prohibit exchangeability of 4-braids up to Markov equivalence or up to conjugacy. Our identification of knots admitting no minimal exchangeable braids (Example 8.3) in particular resolves some cases in [28]. On the opposite end, we benefit from the right equivalence in (2) to exhibit exchangeable braids.

We also find knots admitting non-minimal non-exchangeable braids, and admitting both minimal exchangeable and minimal non-exchangeable braids (Example 8.7). They make clear that the combination of conjugacy, exchange moves and destabilization does not suffice to transform a braid representative of a given link into a minimal one, or one in any well designated (standard) form. This, in a way, improves upon Morton's example [21], which can be interpreted saying that, for the unknot, conjugacy and destabilization do not suffice. Both contrast [8], stating that, for trivial links, conjugacy, exchange moves and destabilization do.

2. PRELIMINARIES

2.1. **Braid groups.**

The *braid groups* B_n were introduced in the 1930s in the work of Artin [2].

Definition 2.1. The *braid group* B_n on n strands can be defined by generators and relations as

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i & |i-j| = 1 \end{array} \right\rangle. \tag{3}$$

The σ_i are called *Artin standard generators*. An element $b \in B_n$ is called an *n-braid*.

We will often record a braid word as a sequence of (non-zero) integers, with i meaning σ_i for $i > 0$ and σ_{-i}^{-1} for $i < 0$.

Let

$$\delta_n = (\sigma_1 \cdot \ldots \cdot \sigma_{n-1}) \cdot (\sigma_1 \cdot \ldots \cdot \sigma_{n-2}) \cdot \ldots \cdot (\sigma_1\sigma_2) \cdot \sigma_1 \tag{4}$$

be the (right-handed) half-twist on n strands. The *center* of B_n (elements that commute with all B_n) is infinite cyclic and generated by the full twist

$$\delta_n^2 = (\sigma_1 \cdot \ldots \cdot \sigma_{n-1})^n .$$

Let similarly

$$\delta_{[i,j]}^2 = (\sigma_i \cdot \ldots \cdot \sigma_{j-1})^{j-i+1}$$

be the restricted full twist on strands i to j. Let also for $1 \le i < j \le n$,

$$B_{i,j} := \langle \sigma_i, \ldots, \sigma_{j-1} \rangle \tag{5}$$

be the subgroup of B_n of braids operating on strands $i \ldots, j$. Where ambiguity is avoided (as indicated by diagrams we will draw), we can identify $B_{i,j} \simeq B_{j-i+1}$. Specifically, B_{n-1} as a subset of B_n will by default be considered to be $B_{1,n-1}$, e.g., in (7).

There is a *permutation homomorphism* of B_n,

$$\pi : B_n \to S_n , \quad \text{given by} \quad \pi(\sigma_i) = (i,\, i+1) . \tag{6}$$

(The permutation on the right is a transposition.) We call $\pi(b)$ the *braid permutation* of b. We call b a *pure braid* if $\pi(b) = Id$. For the *combed normal form* of pure braids, see [5].

Also, there is a homomorphism $e : B_n \to \mathbb{Z}$ sending all σ_i to 1. We will write $e = e(b)$ for the image, and call it *exponent sum* or *writhe* of b.

Let further $\bar{b}$ for $b \in B_n$ be the automorphism of B_n given by the *mirroring* $\sigma_i^{\pm 1} \mapsto \sigma_i^{\mp 1}$ and $\mathrm{rev}(b) = \bar{b}^{-1} = \overline{b^{-1}}$ be the anti-automorphism given by *word-reversal* (word written with letters $\sigma_i^{\pm 1}$ in the opposite order).

Markov's theorem (see, e.g., [22]) relates braid representatives of the same link by two moves, the *conjugacy* in the braid group, and the pair of *stabilization*, which is the move to the right in

$$b \in B_{n-1} \longleftrightarrow b\sigma_{n-1}^{\pm 1} \in B_n , \tag{7}$$

together with its inverse (move to the left), called *destabilization*. As mentioned, Markov's moves have gained importance in knot theory, among others, as a tool for defining link invariants via braids.

We call a braid $b' \in B_n$ positively resp. negatively *stabilized* if $b'\sigma_{n-1}^{-1}$ resp. $b'\sigma_{n-1}$ lies in $B_{1,n-1}$. We say that $b \in B_n$ is *irreducible*, if b is not conjugate to a stabilized braid b'. Obviously for a braid minimal implies irreducible, but the converse is not true [21] (although it is for $n \le 3$ [7]). The detection of irreducible braids is one major difficulty in Markov's theorem.

The (reduced) n-strand *Burau representation* ψ_n, of dimension $n-1$, which we simply call 'Burau', can be found for example in [17, §2]. It associates to a braid $\beta \in B_n$ a matrix $\psi_n(\beta)$ of size $(n-1) \times (n-1)$ and entries in $\mathbb{Z}[t^{\pm 1}]$.

Let us for square matrices M, N write for their block sum

$$M \oplus N = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array} \right] .$$

Then ψ_n is defined by

$$\psi_n(\sigma_i)(t) = Id_{i-2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{bmatrix} \oplus Id_{n-i-2}\,, \tag{8}$$

with the first (resp. last) row and column of the 3×3 block removed for σ_1 (resp. σ_{n-1}). We will be interested only in characteristic polynomials, so different conjugacy conventions are immaterial.

2.2. **Diagrams and links.** We will only occasionally need a few terms related to (oriented) link diagrams. *Seifert circles*, *parallel* and *reverse clasps*, are standard terms and can be found defined, e.g., in [38]. We write $w(D)$ for the *writhe* of a diagram D and $s(D)$ for the number of its Seifert circles. A diagram is *positive* if all its crossings are right-hand.

A Seifert circle of a diagram D is *separating* if its interior and exterior both contain other Seifert circles. We write $s_s(D)$ for their number. A diagram D is *special* if no Seifert circle is separating, $s_s(D) = 0$, and non-special if $s_s(D) > 0$. A diagram is a *(closed) braid diagram* if all but two of its Seifert circles are separating, i.e.,

$$s_s(D) = s(D) - 2\,; \tag{9}$$

from such a diagram of a link L one can obviously read off a braid representative b of L.

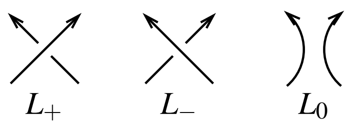
We will write $T(p,q)$ for the (p,q)*-torus link*. Also, $U_{[n]}$ is the *n-component trivial link* (unlink). (Obviously, $U_{[n]} = T(n,0)$.) $U = U_{[1]}$ is the *unknot*.

For diagrams of low-crossing links, we will refer to the table in the Rolfsen book [27, Appendix]; numbering should follow the same table, with one knot of the Perko pair discarded. Knots of more than 10 crossings are taken from the tables of KnotScape [15].

A few times we will mention the *MPC move* for reducing the number of Seifert circles found by Murasugi-Przytycki and Chalcraft; see e.g. [40].

2.3. **Link polynomials.** The various link polynomials were introduced in the papers [14, 19, 17]. We specify them here by their skein relations.

Consider links with diagrams differing just near one crossing. We call the three diagram fragments in (10) from left to right a *positive* crossing, a *negative* crossing and a *smoothed out* crossing (in the skein sense).



(10)

The *skein (HOMFLY-PT) polynomial* P is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1} P\big(L_+\big) + l\,P\big(L_-\big) \;=\; -m\,P\big(L_0\big)\,. \tag{11}$$

This convention uses the variables of [19], but differs from theirs by the interchange of l and l^{-1}.

Below Δ is the *Alexander polynomial*. It is an invariant with values in $\mathbb{Z}[t,t^{-1}]$, and can be defined by being 1 on the unknot and the relation

$$\Delta(L_+) - \Delta(L_-) \;=\; (t^{1/2} - t^{-1/2})\,\Delta(L_0)\,.$$

This is easily seen to be a special case of the skein polynomial relation. Consequently, there is the substitution formula (see [19]; i is here the complex unit),

$$\Delta(t) = P(i, i(t^{1/2} - t^{-1/2})).$$

The *Jones polynomial* V is obtained from P (in our convention) by the substitution (see [19])

$$V(t) \quad = \quad P(it, i(t^{1/2} - t^{-1/2})). \tag{12}$$

The *Conway polynomial* is an oriented link invariant that takes values in $\mathbb{Z}[z]$. It is given by the value 1 on the unknot and the skein relation

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0). \tag{13}$$

We have

$$\nabla(L)(t^{1/2} - t^{-1/2}) = \Delta(L)(t),$$

so that ∇ and Δ are interconvertible (and equivalent as invariants).

There is also the *Kauffman polynomial* which we will only briefly mention (in §6).

2.4. **Exchange move.** Birman and Menasco [6] introduced a move called *exchange move*. We say that $b \in B_n$ *admits an exchange move* or is *exchangeable*, if b is as illustrated in Figure 1, where $\alpha \in B_{1,n-1}$, $\beta \in B_{2,n}$, and $n \geq 4$.

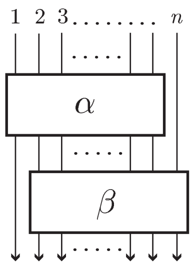


FIGURE 1. The n-braid b.

An (iterated) *exchange move* [6] is the transformation between the braids b and

$$b_m = \alpha\delta^{2m}_{[2,n-1]}\beta\delta^{-2m}_{[2,n-1]}, \tag{14}$$

shown in Figure 2. Here m is some non-zero integer, and the boxes labeled $\pm m$ represent the full twists $\delta^{\pm 2m}_{[2,n-1]}$ respectively, acting on the middle $n-2$ strands. (Thus a positive number of full twists are understood to be right full twists, and $-m$ full twists mean m full left-handed twists.) We can set $b_0 = b$.

Of course, no non-trivial braid on 2 strands admits an exchange move, and all exchange moves on 3 strands are trivial, so that we will naturally assume $n \geq 4$ throughout.

It should me kept in mind that the result b_m does depend on the decomposition

$$b = \alpha\beta \ \text{ with } \alpha \in B_{1,n-1} \text{ and } \beta \in B_{2,n}, \tag{15}$$

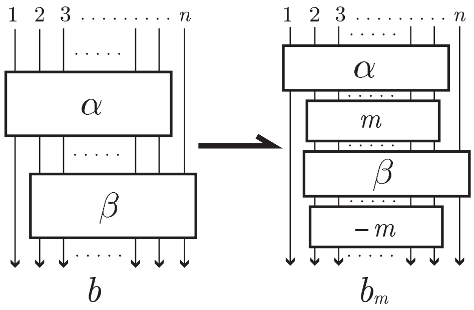


FIGURE 2. The braid b_m

although some different pairs (α,β) give equal or conjugate b_m. To formalize this, let us say that the pair

$$(\alpha,\beta)\in B_{1,n-1}\times B_{2,n}\ \text{ with (15)}$$

regarded up to the equivalences for $\gamma\in B_{2,n-1}$

$$(\alpha\gamma,\beta)\sim(\alpha,\gamma\beta)\ \text{ and }\ (\gamma\alpha,\beta)\sim(\alpha,\beta\gamma) \tag{16}$$

forms an *exchangeable structure* of b, regarded up to conjugacy in $B_{2,n-1}$. The argument in Example 3.4 shows that, if a $B_{2,n-1}$-conjugacy class admits an exchangeable structure, then it is unique.

When we consider the family $\{\, b_m : m\in\mathbb{Z}\,\}$, we will then always understand that the exchangeable structure is kept fixed. We must point out that when we later talk about braids exchangeable 'up to conjugacy', we will mean conjugacy in the full B_n, though. This leads directly to the question how to identify (all) exchangeable structures on braids in such a conjugacy class, if such exist. For instance, there is always the flip $(\alpha,\beta)\mapsto(\bar\beta,\bar\alpha)$ with $\bar\sigma_i=\sigma_{n-i}$, which in general changes the structure. How much more is possible is not immediately clear, however, the difficulty of this problem should transpire from the following simple observation.

Example 2.2. When $\alpha=1^{-k}(1\ -2\ 3)^3 1^k$ and $\beta=(3\ -4)^3$ for $n=5$, then calculation of the Jones polynomial on the axis addition link (see (20)) of b_{-1} shows that it distinguishes these links for $k=0,\dots,4$. Using the method of [37], one can then conclude that the conjugacy class of $b=\alpha\beta$ has infinitely many different exchangeable structures.

There is another, more common, way to describe the exchange move, namely by

$$\alpha\beta\longleftrightarrow\alpha\kappa^m\beta\kappa^{-m}\,,\ \text{where}\quad \kappa=(\sigma_1\cdot\ldots\cdot\sigma_{n-2})(\sigma_{n-2}\cdot\ldots\cdot\sigma_1)\,. \tag{17}$$

This description is equivalent to the previous one, because $\kappa\cdot\delta^2_{[2,n-1]}=\delta^2_{[1,n-1]}$, and this element commutes with α.

A further equivalent formulation of the move is

$$b_0=\beta\sigma_{n-1}\beta'\sigma_{n-1}^{-1}\longleftrightarrow b_1=\beta\sigma_{n-1}^{-1}\beta'\sigma_{n-1}\,,$$

with $\beta,\beta'\in B_{1,n-1}$, which can be generalized (up to conjugacy) by

$$b_m=\delta^{2m}_{[1,n-2]}\beta\delta^{-2m}_{[1,n-2]}\sigma_{n-1}\beta'\sigma_{n-1}^{-1}\,. \tag{18}$$

This form will be more convenient for our treatment of exchangeable braids from §7 on, and for the examples in §6.

In [29] we treated the question when infinitely many conjugacy classes of n-braid representatives of a given link occur. Obviously it makes sense to consider only $n \geq b(L)$. Birman and Menasco [6] proved that an exchange move necessarily underlies the switch between many conjugacy classes of braid representatives of L.

Theorem 2.3 (Birman-Menasco [6]). *The n-braid representatives of a given link decompose into a finite number of classes under the combination of exchange moves and conjugacy.*

We proved in [29] that it is also sufficient for generating infinitely many such classes, under a very mild restriction. This motivates the following definition (which was not specified in [29], but will make it very convenient to express ourselves in the following).

3. The infinite conjugacy property

3.1. **Definition and basic examples.**

Definition 3.1. We say that a pair (L,n) for a link L and $n \in \mathbb{N}$ satisfies the *infinite conjugacy property (ICP)* if the equivalence holds

$$\begin{pmatrix} L \text{ has infinitely many} \\ \text{conjugacy classes of} \\ n\text{-braid representatives} \end{pmatrix} \iff \begin{pmatrix} L \text{ has an } n\text{-braid} \\ \text{representative admitting} \\ \text{an exchange move} \end{pmatrix}.$$

We will way that $ICP(L,n)$ holds *positively*, and write $ICP(L,n)+$ if both hand sides are true. If both hand sides are false, we write $ICP(L,n)-$. We say that a link L satisfies the ICP if (L,n) satisfies the ICP for every $n \geq \max(b(L),4)$.

It transpired that almost all links satisfy this property. From the Lie group theoretic approach in [31], we had:

Theorem 3.2. *([31]) If (L,n) does not have the $ICP(L,n)+$ and $n \geq 4$, then $n = b(L)$, or $n - b(L) \in \{0,1\}$ and $L = T(p,kp)$ is a (p,kp) torus link for $k \in \mathbb{Z}$ and $p = b(L)$.*

There is then no point in signing $ICP(L)$ in Definition 3.1, since an '$ICP(L)-$', in the sense $ICP(L,n)-$ for all n with (1), never occurs by Theorem 3.2. It should be made clear, though, that deciding for $n = b(L)$ whether $ICP(L,n)+$ or $ICP(L,n)-$ holds remains a very non-trivial problem, apparently equally difficult for for links L as for knots, and will be discussed in the later sections of the paper. There are, though, a few noteworthy (and easier) examples.

Example 3.3. If $b \in B_n$ is pure and $L = \hat{b}$ has no $U_{[2]}$ sublink, then $ICP(L,n)-$ holds. All components of L must be 1-string subbraids of b for any $b \in B_n$. If b is exchangeable, then the components of strands 1 and n give a $U_{[2]}$ sublink. This argument in particular complements the case $n = b(L) = p$ and $L = T(p,kp)$ for $k \neq 0$ in Theorem 3.2. Also, these links have only finitely many minimal braid conjugacy classes.

Example 3.4. Contrarily, if $k = 0$, then $L = U_{[n]}$ is an unlink, which has the trivial representative admitting (trivially) an exchange move. But it indeed admits only this trivial exchange move, as $B_{1,n-1} \cap B_{2,n} = B_{2,n-1}$ (by combed normal form, for instance), so if $\alpha = \beta^{-1}$, then the edge strands of b can be isolated by an isotopy within α and β alone. It follows from [8] then that $U_{[n]}$ only has the trivial conjugacy class of n-braids. Thus ICP fails for $(U_{[n]},n)$.

The following table, with columns indexed by n and rows by L, summarizes the status of $ICP(L,n)$ (with 'f' meaning failure and '?' for generally undecided). We assume $k\neq 0$ and $b(L)\geq 4$. For $b(L)=3$, remove the second column and shift all its right columns one to the left; do this one more time for $b(L)\leq 2$.

	$b(L)$	$b(L)+1$	$\geq b(L)+2$
$U_{[n]}$	f	?	+
$T(p,kp)$	–	?	+
other	?	+	+

We have not clarified exactly what happens for $L=T(p,kp)$ and $n=p+1$, even if $k=0$. This situation has its difficulties and will occur in examples below (see the list in §5.3). The case is still rather special, though. It obviously becomes much more relevant to focus on $n=b(L)$ for an arbitrary link L.

Later we gained the following insight.

Theorem 3.5. *(*[29]*) Let L be a knot, or a link without trivial components. Then L satisfies the ICP, with $ICP(L,n)+$ for* (1).

This result was a consequence of a stronger property of the exchange move we proved in Theorem 3.6, when combined with the work of Birman and Menasco.

Theorem 3.6. *(*[29]*) Let a link L have an n-braid representative b admitting an exchange move, such that the permutation $\pi(b)$ satisfies*

$$\pi(b)(1)\neq 1 \quad \text{and} \quad \pi(b)(n)\neq n\,. \tag{19}$$

Then iterated exchange moves on b generate infinitely many non-conjugate braid representatives of L.

The method consisted of evaluating coefficients of the Conway polynomial ∇ of the axis addition link L_{b_m} of b_m. In fact it much more strongly restricts (possibly) conjugate b_m, as described in §6.

The *axis addition link L_b* of $b\in B_n$ can be specified by the closure of the braid

$$b\cdot(\sigma_{n-1}\cdot\ldots\cdot\sigma_1)\cdot(\sigma_1\cdot\ldots\cdot\sigma_{n-1})\in B_{n+1}\,. \tag{20}$$

3.2. **A diagrammatic condition.** It is further useful to (straightforwardly) observe that the condition (19) has the following interpretation under the right equivalence of (2).

Lemma 3.7. *Let L have a diagram D of $n=s(D)$ Seifert circles transformed into a braid diagram by Vogel moves so that the last one involves strands of two components none of which is a 0-crossing circle. Then L has an exchangeable n-braid b with* (19). ☐

Example 3.8. Consider the Rolfsen [27, Appendix] link 6^2_3 with the orientation so that the diagram is not special (3 reverse clasps).

The diagrammatic point of view may find more extensive treatment in separate work, as outlined in §10. We will discuss then the below related criterion involving the *graph index* $\mathrm{ind}\,(D)$. It follows similarly to lemma 3.7, by using besides Vogel's move the one of Murasugi-Przytycki-Chalcraft.

Proposition 3.9. *Assume a link has a diagram D and a $0 \le k \le \operatorname{ind}(D)$ with*

$$s(D) - k = n > 3 \quad \textit{and} \quad k + s_s(D) < n - 2\,. \tag{21}$$

Then L has an exchangeable braid representative on n strands.

Remark 3.10. In particular, for $k = 0$ a diagram with $s(D) = n$ Seifert circles which is not a braid diagram (*cf.* (9)) will do, giving the backward part of the right equivalence of (2). (Again, only the case $n = b(L)$ is really interesting.)

We will use the proposition to obtain criteria for proving that a link has infinitely many braid conjugacy classes. (This will further relax the constraint to knotted components in theorem 3.5; note that D in remark 3.10 is easily found for the unknot.)

4. Cases of failure of the exchange move

The exchange-move-admitting braids b with $\pi(b)(1) = 1$, or equivalently,

$$\pi(\alpha)(1) = 1 \tag{22}$$

in Theorem 3.6 are more difficult, and connected to several instances of failure of the exchange move to yield non-conjugate representatives. (In particular, Theorem 3.6 gives the weakest condition in terms of $\pi(b)$ alone under which the exchange move can work.) Note that (22) accounts for unknotted components of $\hat{b}$. In [29] we covered many links with trivial components in terms of a cabling condition allowing to exclude (22) in a minimal braid representation. However, this condition is somewhat awkward to verify.

Note that the exchange move in Figure 2 is trivial when the leftmost strand of α (or the rightmost strand of β) are isolated, i.e.,

$$\alpha \in B_{2,n-1}$$

(for $B_{2,n-1}$ from (5)). We observed in [29] this failure to extend to braids b with

$$\alpha \in \langle \kappa \rangle \cdot B_{2,n-1}, \tag{23}$$

for κ in (17), since this element commutes with $B_{2,n-1}$.

We did not know, until we found the proof of (25) given below, if under exclusion of these cases, the move can always yield infinitely many conjugacy classes. However, we were aware of constructions like Stanford's [30] that allow one to 'approximate' these cases of failure by others which cannot be distinguished by any number of Vassiliev invariants (including coefficients of ∇).

This raises the question about simple conditions that would allow some similar approach to distinguish the result of exchange moves applied on braids like (22) and all their modifications.

Then Ito [16] much more recently, about 10 years since our initial proof of Theorem 3.6, obtained using the mapping class group a very similar version of our theorem, in which (19) is replaced by the most general assumption of *non-degeneracy*, namely that in Figure 2

$$\Delta^2_{[2,n-1]}\,\alpha \neq \alpha\,\Delta^2_{[2,n-1]} \quad \text{and} \quad \Delta^2_{[2,n-1]}\,\beta \neq \beta\,\Delta^2_{[2,n-1]}\,. \tag{24}$$

Theorem 4.1. *([16]) If* (24) *holds, then infinitely many b_m are non-conjugate.*

This result could essentially do away with the difficulties surrounding ICP by restricting to *non-degenerate* exchange moves. But then it begs the question what intrinsic properties of a link allow one to decide if or not it admits degenerate exchange moves. (We may note here that (24), like (19), passes through (16), so they are defined on an exchangeable structure of b.)

While Ito's method more easily addresses, at least as far as ICP is concerned, the (few) remaining braids from Theorem 3.6, the theorem retains its merits. Our notion of 'infinitely many' in [28, 29] is stronger, and exhibits a symmetry of indices m between conjugate b_m. This is discussed in detail in §6. And while potentially (but not easily) relaxable, the assumption (19) is, unlike Ito's non-commutativity conditions (24), immediate to visually verify.

As an enhancement of this visual test, we found now that Ito's conditions (24) coincide with our previously observed instances (23) of failure:

$$\alpha \text{ is of the form (23)} \iff \alpha \text{ fails (24)}\,. \tag{25}$$

After finding a proof using Theorem 3.6, I was pointed that (25) also follows from the work of González-Meneses and Wiest on describing the centralizer in braid groups [13]. This gives an alternative (non-trivial, but at least intrinsic braid-theoretic) argument I was expecting to exist.

We will see below that the form (23) more easily protrudes some geometric implications although, for example, its counterpart in (25) may suit much better algorithmic verification from a given word for α.

5. Regularity

5.1. **Definition and property.** There is, though, a self-contained condition satisfied by all braids obtained from Stanford's construction applied on (23): strand 1 in α must have equal linking number with all strands $2,\dots,n-1$. It is tempting to expect that under exclusion of this situation, and its analogue for β, one can always use the Conway polynomial to distinguish L_{b_m}.

We try now to weaken the condition (22) on α.

Definition 5.1. We say $\alpha \in B_{1,n-1}$ is *regular* in three cases.

1. (22) does not hold.
2. Now assume (22) holds. Let lk_i be the linking number between strands 1 and i in α for $i = 2,\dots,n-1$. Then not all lk_i are equal.
3. Assume (22) holds and all (or equivalently one) $lk_i = 0$. Then the strand 1 in α is not isolated, i.e., $\alpha \notin B_{2,n-1}$. Otherwise $\alpha\kappa^{-lk_1} \notin B_{2,n-1}$.

Note that even if α is not pure, with (22) for the condition stated, in the second and third case it does not matter where strands are numbered, top or bottom.

We definite regular for $\beta \in B_{2,n}$ analogously; we can say $\beta \in B_{2,n}$ is regular if $\delta_n\beta\delta_n^{-1} \in B_{1,n-1}$ is.

Using (25), one can state Theorem 4.1 as follows.

Theorem 5.2. *Let $b \in B_n$ with (22) admit an exchange move, such that α and β are regular. Then infinitely many b_m are non-conjugate.*

This will give restrictions on non-ICP links which are much sharper compared to Theorem 3.5. The following definition tries to formalize some conditions. We will apply several simplifications later.

Definition 5.3. Let $L = U \cup K_1 \cup \cdots \cup K_m$ be a link with an unknotted component U and K_i being the components of L different from U. We write its *total linking number*

$$lk(U, L \setminus U) := \sum_{i=1}^{m} lk(U, K_i)\,.$$

For future convenience, when $U \subset L$ is an unknotted component, also write

$$\lambda_U := |lk(U, L \setminus U)|\,.$$

We define $\operatorname{sgn}(l) \in \{-1, 0, 1\}$ for l negative, zero, or positive, resp.

Definition 5.4. We say that a quadruple (L, L', U, t) is *admissible* if

1. L is a link, L' is a sublink of L, further U is an unknotted component of L', and $t \in \mathbb{N}$.
2. Let $L' \setminus U = K_1 \cup \cdots \cup K_m$ be the components of L' different from U. Then there is a $k \in \mathbb{Z}$ and $0 \le i_0 \le m$ such that for all $1 \le i \le m$,
   a. if $i_0 = 0$, then there is a component $U' \in L \setminus L'$ with $U \cup U' = U_{[2]}$,
   b. $k \mid lk(U, K_i)$,
   c. $\operatorname{sgn}(lk(U, K_i)) = \operatorname{sgn}(k)$,
   d. $|lk(U, K_i)| \ge |k| \cdot b(K_i)$ if $i \ne i_0$ and $|lk(U, K_i)| \ge |k| \cdot (b(K_i) - 1)$ for $i = i_0$, and also
   e. $lk(U, L' \setminus U) = k \cdot t$
3. L, U and k satisfy some geometric conditions.
   a. if $k = 0$, then L is the split union of U and $L \setminus U$,
   b. if $k \ne \pm 1$, then L is non-hyperbolic (with an essential unknotted torus containing U on one side); the same holds for L' and $U \cup K_i$ when K_i is knotted,
   c. if $k = \pm 1$, then twisting s times L along (and retaining) the circle U must give links L_s of $b(L_s) \le n$ with $n = t + 2$ for any $s \in \mathbb{Z}$.

In particular, if (L, L', U, t) is admissible, then $t \mid lk(U, L' \setminus U)$. Also, condition 2e uniquely determines k (to be the quotient), so it can be taken as the definition of k. The choice $i_0 = 0$ is just allowed to disable the second alternative in Condition 2d.

Theorem 5.5. *Let L not have the ICP and $L \ne T(n, pn)$ not be a torus link. Then (at least) one of the following alternatives holds:*

1. *There is an unknotted component $U_1 \subset L$ such that $(L, L, U_1, b(L) - 2)$ is admissible.*
2. *There are two (unknotted) components $U_1, U_2 \subset L$ such that $U_1 \cup U_2 = U_{[2]}$ is the trivial 2-component link, and at least one of $(L, L \setminus U_1, U_2, b(L) - 2)$ or $(L, L \setminus U_2, U_1, b(L) - 2)$ is admissible* with $i_0 = 0$.

Proof. Again, this is an easy consequence of Theorem 5.2. Assume L does not have the ICP. Then w.l.o.g. α is not regular. The two alternatives in Theorem 5.5 arise by distinguishing whether $\pi(b)(n) = n$ or not. The properties of 'admissible' were defined to be consequences of the negated second and third condition of regularity in Definition 5.1. In inequality 2d in Definition 5.4, $i_0 > 0$ is possible for the component K_{i_0} occupying strand n of b (for α, and strand 1 for β). There are a few further options we did leave out; e.g., inequality 2d can be amplified by

$$|lk(U,L^\star)| \geq |k|\cdot b(L^\star)\,, \qquad |lk(U,L^\dagger)| \geq |k|\cdot (b(L^\dagger)-1)\,,$$

for every sublink $L^\star$ of $L'\setminus U$ not containing K_{i_0} and $L^\dagger$ containing K_{i_0}, and so on. □

5.2. **Some corollaries and examples.** The implication of the theorem looks complicated at first sight, but note that complexity is owed to the number of restrictions claimed, not assumed. One can easily weaken them to obtain a series of more self-contained statements of different flavor. The proofs are rather straightforward; only a few are given thus.

Corollary 5.6. *Let L be a link without trivial (unknotted) split components so that for each unknotted component U there is a component K with $U\cup K\neq U_{[2]}$ and $lk(U,K)=0$. Then L satisfies the ICP.*

Proof. Assume L admits only degenerate exchangeable n-braids $b=\alpha\beta$. By the linking number condition $L\neq T(n,pn)$, so $n>b(L)$ is done. Let U be strand 1 of $b=\alpha\beta$. Since K cannot involve only strand n of b (because $U\cup K\neq U_{[2]}$), the condition $lk(U,K)=0$ means that in Definition 5.4 $k=0$. But then by property 3a, U is a split component. □

Corollary 5.7. *Let $L=K_1\cup K_2$ be a 2-component link with $b(L)-2\nmid lk(K_1,K_2)$. Then L satisfies the ICP.*

Proof. If $L=U_{[2]}$, then $b(L)=2$, so with (1), $n>b(L)+1$ and then Theorem 3.2 applies. Assume $L\neq U_{[2]}$, so condition 2 in Theorem 5.5 does not apply. If $n=b(L)+1$, then $b(L)\geq 3$, and since L has two components, $L\neq T(k,pk)$. Then Theorem 3.2 applies. So we can assume $n=b(L)$. If both $K_{1,2}$ are knotted, then Theorem 3.5 applies. Otherwise, for some of the unknotted components $U=K_1$ we have condition 1 in Theorem 5.5. This gives $b(L)-2\nmid lk(U,K_2)$, by property 2e in Definition 5.4, with $t=b(L)-2$. □

We have tried to capture in Definition 5.4 some geometric conditions. Here is one instance where one can use condition 3c.

Example 5.8. Consider the link $L=8^2_9$. Since it is not a torus link and $lk=\pm 2$, we assume some orientation with $n=b(L)=4$. From Theorem 5.5 we need in Definition 5.4 that $L=L'$. Then twisting along the unknotted component can be done for proper sign of s to yield the other component to become a 2-bridge knot K_s of $b(K_s)=4$, so $b(L_s)\geq 5$. With this argument one can show that 8^2_9 (with either orientation) has ICP.

As (more or less) a generalization of Corollary 5.7, we can state the following.

Corollary 5.9. *Let L be an n-component link which is not a (n,pn) torus link and such that for each unknotted component $U\subset L$ we have*

$$b(L)-2\nmid\lambda_U\,. \tag{26}$$

Then L satisfies the ICP.

Example 5.10. Consider the link $L=8^3_2$ with the non-special braid index $b(L)=4$ orientation. (Removing symmetries and simultaneous reversal of all components, the other two choices are a special braid index 4 orientation, and a braid index 3 orientation.) This link L is not a torus link, has no $U_{[2]}$ sublink, and for the 3 unknotted components U, the values of λ_U are $0,1,3$. Thus L has the ICP.

If we assume L is hyperbolic (and thus $L \neq T(n, pn)$), then (26) improves, because $k = \pm 1$:

Corollary 5.11. *Let L be a hyperbolic n-component link such that for each unknotted component $U \subset L$ we have $b(L) - 2 \neq \lambda_U$. Then L satisfies the ICP.*

Example 5.12. The link $L = 8^2_2$ has $lk = 4$ and $b(L) = 3$ or $b(L) = 4$ depending on orientation, but being a 2-bridge link (and not a $(2,k)$-torus link), it is hyperbolic. Thus it has ICP.

The below modification is useful for many low-crossing examples.

Corollary 5.13. *Let $L = K_1 \cup K_2$ be a non-split 2-component link with $|lk(K_1, K_2)| \leq 1$. Then L satisfies the ICP.*

Proof. Repeat the proof of corollary 5.7 until the conclusion $b(L) - 2 \nmid lk(U, K_2)$, with the addition that $b(L) \geq 4$, because of $n = b(L)$ and (1), so $b(L) - 2 \geq 2$. Then $|lk| \leq 1$ implies $k = 0$ in Definition 5.4, and condition 3a applies. □

Corollary 5.14. *Let L be an n-component link such that for each unknotted component $U \subset L$ there are components K_1, K_2 of $L \setminus U$ such that $lk(U, K_1) \cdot lk(U, K_2) < 0$. Then L satisfies the ICP.*

By Corollary 5.6, this can be widened providing for zero linking numbers.

Corollary 5.15. *Let L be an n-component link such that for each unknotted component $U \subset L$ there are components K_1, K_2 of $L \setminus U$ such that $U \cup K_1 \neq U_{[2]} \neq U \cup K_2$, and $\operatorname{sgn}(lk(U, K_1)) \neq \operatorname{sgn}(lk(U, K_2))$. Then L satisfies the ICP.*

The next consequence uses the divisibility condition by k.

Corollary 5.16. *Let $L \neq T(3,3)$ be an n-component link such that for each unknotted component $U \subset L$ there are components K_1, K_2 of $L \setminus U$ such that*

$$l_i = |lk(U, K_i)| \tag{27}$$

are relatively prime (and non-zero) and

$$l_1 + l_2 > b(L) - 2\,. \tag{28}$$

Then L satisfies the ICP. More generally, it is enough that for each U in $L \neq T(m+1, p(m+1))$ there is an $m \geq 1$ and components $K_i \neq U$, $i = 1, \dots, m$, so that with (27)

$$\sum_{i=1}^{m} l_i > (b(L) - 2) \cdot \gcd(l_1, l_2, \dots, l_m)\,.$$

Proof. We consider only the first assertion. Let $b = \alpha\beta$ admit a degenerate exchange move and U be (the closure of) strand 1 of α. Obviously none of $K_{1,2}$ involves only strand n of β. The condition $\gcd(l_1, l_2) = 1$ implies that $k = \pm 1$. Then $l_1 + l_2 \leq n - 2$, so for (28) we need $n > b(L)$. If $n > b(L) + 1$, theorem 3.2 always applies, so $n = b(L) + 1$, and because of (1), we need $b(L) \geq 3$ and $L = T(n-1, p(n-1))$. Next $l_1 = l_2 = p$, so by relative primeness $p = 1$. Also if $n > 4$, then because of $b(L) = n - 1$, (28) fails again, so $n = 4$, and $L = T(3,3)$. □

The final in a series of simplifications uses the inequality 2d in Definition 5.4.

Corollary 5.17. *Let L be an n-component link such that for each unknotted component $U \subset L$ there is a (knotted) component K_1 of $L \setminus U$ such that* $0 < |lk(U, K_1)| < b(K_1) - 1$. *Then L satisfies the ICP.*

Note also that it would be sufficient if among different unknotted components U of L each one satisfies some (but not necessarily the same) condition of the corollaries, etc.

By the construction of Stanford [30], inserting pure braid commutators, which do not alter linking numbers, one can make a braid into a prime alternating braid, for which the braid index is visible [26]. Subbraids of pure braid commutators are pure braid commutators, so lower degree Vassiliev invariants of component knots will not be altered. Starting with knots with non-trivial such invariants, one could use this to avoid knotted components becoming unknotted. Oppositely, to avoid unknotted components becoming knotted, make all unknotted components be closures of 1- or 2-string subbraids. One can exploit this idea then to easily construct many links to which any of these corollaries apply.

5.3. **Some more links.** We assumed more practical use of the above instrumentarium when trying it out on a number of explicit examples. It was evident that such a source is Rolfsen's link table [27, Appendix], despite that low-crossing cases are usually irrepresentative, and many corollaries become efficient only for higher crossing numbers. Since most of the infrastructure of [15] (including extensive upgrades and troubleshooting [35]) is set up for knots, we had to try many tests by hand. Issues of component orientation become relevant and more technical to settle. A few examples (from that process) were already given.

By the methods mostly explained above (noteworthy additions are flypes and the *MPC move* of §2.2), we were able to verify that most prime 2- and 3-component links up to 8 crossings have ICP. Below are the *possible* exceptions to ICP. Here L stands for the link and D for its Rolfsen diagram. Orientation is considered up to simultaneous reversal of all components, and symmetries of the link. The specifications of being positive, special or not, parallel and reverse clasps (§2.2), all refer to the Rolfsen diagram.

1. 7^2_5 with the 5 Seifert circle ($s(D) = 5$) orientation
2. 8^2_{11} with the non-special ($b(L) = 4$) orientation,
3. 8^2_{14} with both orientations
4. 8^2_{16} with both orientations
5. 6^3_1 with the special orientation
6. 6^3_3 with the positive ($L = T(3,3)$) orientation,
7. 7^3_1 with the non-special orientation (one parallel clasp, 2 reverse)
8. 8^3_2 with the special orientation
9. 8^3_4 with $b(L) = 4$ orientation (2 parallel clasps, 2 reverse)
10. 8^3_8 with $\lambda_U = 2$ orientations,
11. 8^3_9 with three $b(L) = 4$ orientations (at least 2 reverse clasps)
12. 8^3_{10} with orientation of 2 parallel clasps, 2 reverse

In all cases we do not know if $ICP(L,n)$ holds for $n = 4$. (Also $b(L) = 4$ except for $L = 6^3_3$.) Consider Remark 8.5 as well.

It should be pointed out that this is not exactly an extension of the table in [28, §5], since we did not investigate which of $ICP(L,b(L))\pm$ holds.

Note also the absence of 2-bridge links in the list. It was already apparent that for them something more can be proved. Compare with §10.

It should be stressed that, in contrast, the problem whether there is an arbitrary exchange move (as opposed to a degenerate one) appears to be little affected by multiple link components, and thus in that regard it is very natural to focus later in the paper on knots.

6. Symmetry

6.1. **Conjugacy tests.** The question which b_m are actually conjugate (when the move is non-degenerate) turns out to be an interesting one. The proof of Theorem 3.6 in [29] shows its following refinement. For later reference, we write $\mathcal{B}\subset B_n$ for a conjugacy class of n-braids, and for $S\subset\mathbb{Z}$ write

$$\nu(S,\mathcal{B}) = |\{m\in S\,:\,b_m\in\mathcal{B}\}|,\quad \nu(\mathcal{B}) = \nu(\mathbb{Z},\mathcal{B})\,.$$

Proposition 6.1 ([29]). *Under the assumption of Theorem 3.6, and for a suitable cycle C of $\pi(b)$, we have for every conjugacy class $\mathcal{B}\subset B_n$*

$$\nu(\mathcal{B})\le\left\{\begin{array}{ll}1 & \text{if } 1,n\in C \text{ and } C \text{ does not satisfy (30)}\\ 2 & \text{otherwise}\end{array}\right.. \tag{29}$$

The condition

$$\pi(b)^{(|C|+1)/2}(n)=1 \tag{30}$$

has its origins in [28] (which we do not discuss here). In particular, in this case always $|C|$ is odd.

Remark 6.2. The way (29) was obtained was by constructing a conjugacy invariant υ so that $Y(m)=\upsilon(b_m)$ is a (non-constant at most) quadratic polynomial in m. (If (30) fails, then this polynomial is linear.) Then more follows: whenever b_m is conjugate to $b_{m'}$, then

$$\mu=m+m' \tag{31}$$

is the same. In particular b_m and $b_{m'}$ are non-conjugate for all $m'>m\ge 0$ or for all $m'<m\le 0$.

The condition of equal μ in Remark 6.2 appears *a priori* to be somewhat artificial, transpiring from our method of proof. But in fact this turns out not to be the case at all. Several examples show that this symmetry indeed occurs, at various levels. Let us consider m,m' with fixed μ in (31) and assume $m<m'$. We will use the alternative version (18) of the exchange move. Note that in the form (18), the condition (19) will modify to

$$\pi(b)(k)\ne k\quad\text{ for }\quad k=n-1,n\,, \tag{32}$$

since the switch between (14) and (18) essentially accounts in exchanging the role of 1 and $n-1$.

Example 6.3. The following Table 1 summarizes some examples for

$$n=5\quad\text{and}\quad -3\le m\le 0\,, \tag{33}$$

where β, β' are as in (18), and the fourth column gives the set Ξ of values m in (33) for which conjugacy occurs between b_m and $b_{m'}$. The notation $*=\{m\,:\,m<m'\}=\{m<\mu/2\}$ stands for all (such) m. The conjugacy was tested with the program in [12]. Even with the limitation (33), the data should already make clear that there is little in improving upon the assertion of Proposition 6.1.

	β	β'	μ	Ξ	Θ
1	$\sigma_2^{-2}\sigma_1^{-1}\sigma_3^{-1}$	$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}$	1	$*$	$*$
2	$\sigma_2^{-2}\sigma_1^{-1}\sigma_3^{-1}$	$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3$	2	$*$	$*$
3	$\sigma_2^{-2}\sigma_1^{-1}\sigma_3$	$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}$	1	$-1,0$	$*$
4	$\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-2}\sigma_2^{-1}$	σ_3^{-2}	1	$*$	$*$
5	$\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-2}$	$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}$	1	0	$*$
6	$\sigma_2^{-2}\sigma_1^{-1}\sigma_3^{-1}$	$\sigma_3\sigma_2\sigma_1\sigma_3$	1	$\varnothing$	$*$
7	$\sigma_1^{-1}\sigma_3^{-2}\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}$	$\sigma_3\sigma_2^{-1}$	1	$-1,0$	$\varnothing$
8	$\sigma_1^{-1}\sigma_3$	$\sigma_3^2\sigma_2\sigma_1\sigma_3\sigma_2$	1	$*$	$*$
9	$\sigma_1^{-1}\sigma_3\sigma_2\sigma_1^2\sigma_3^{-1}$	$\sigma_3\sigma_2$	2	$*$	$*$
10	$\sigma_1^{-3}\sigma_3\sigma_2^{-1}\sigma_1$	$\sigma_3\sigma_2^{-1}$	1	-1	$*$
11	$\sigma_3^{-1}\sigma_2^{-2}\sigma_1^{-1}\sigma_3^{-2}$	$\sigma_3^{-1}\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_3$	1	$\varnothing$	$\varnothing$
12	$\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_1\sigma_3^{-2}\sigma_2$	$\sigma_3^{-1}\sigma_2^{-1}$	2	0	$\varnothing$!
13	$\sigma_2^{-1}\sigma_1\sigma_3\sigma_2^2$	$\sigma_3\sigma_2^{-1}\sigma_1\sigma_3$	2	$\varnothing$	$\varnothing$
14	$\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-2}$	$\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-2}$	1	$*$	$\varnothing$
15	$\sigma_1^{-1}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^2\sigma_1\sigma_3^{-1}$	$\sigma_3\sigma_2^{-2}\sigma_3$	1	$\varnothing$	0
16	$\sigma_1^{-1}\sigma_2\sigma_3\sigma_2^{-2}\sigma_1^{-1}\sigma_3\sigma_2$	$\sigma_3\sigma_2^{-2}\sigma_3$	1	$*$	0
17	$\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-3}\sigma_2^{-1}$	$\sigma_3^{-1}\sigma_2^{-1}$	1	0	0

TABLE 1. Conjugacy properties for some 5-braid examples.

Note that the symmetry pattern allows for translations along the integers (since in indexing the sequence (b_m) there is no canonical choice of $m=0$), thus the value μ is mostly relevant through its parity – but either occurs.

We expect conjugacy '$*$', like in examples 1,2,4, to extend to all m (beyond (33); see also Observation 6.4). Among common instances of *sporadic conjugacy*, $\Xi \neq *, \varnothing$, examples 3 and 7 show slightly more complicated behavior. We do not know if conjugacy always occurs for all m when it does for at least three different m. Example 10 shows, though, that sporadic conjugacy is not bound to the 'simplest' (closest to $\mu/2$) pair(s) (m,m').

Examples 4,5,15,16 do not satisfy (32), which suggests that the symmetry pattern may depend less on that condition. Also, thus far no braids were found where two different μ occur (in the sense of (35), and except for the obvious failures (23)). This leaves some possibility open that the assumption of Proposition 6.1 may be relaxable (although not easily, as shows [29, Example 6.3]).

In examples 8,9,10,14 the closure $\hat{b}$ is a knot (trefoil, unknot, 5_2, and 8_5, resp.). Note that this scenario depends on $n=5$ being odd, and from (30) we need $\pi(b)^3(5)=4$ (keep in mind the remark below (32)), which we checked to be consistently satisfied. Despite this condition, the conjugacy pattern does thus occur for knot closures as well.

The last column gives the set Θ of m for which b_m is conjugate to the word-reverse $\mathrm{rev}(b_{m'})$ of $b_{m'}$ (as defined in §2.1). Let us then say that b_m is *reverse-conjugate* to $b_{m'}$. Note that under word-reversal, most calculable (orientation non-sensitive) link invariants v yield

$$v(L_{b_m}) = v(L_{\mathrm{rev}(b_m)})\,, \tag{34}$$

and in Proposition 6.1 (as well as in Theorem 3.6) conjugacy can be extended by allowing for reverse-conjugacy. A necessary condition for reverse-conjugacy is that $\hat{b}$ is an invertible link.

Despite this constraint, and the insight of Observation 6.4, at least in low-crossing ranges so far examined, reverse-conjugate $(b_m, b_{m'})$ strikingly outnumber conjugate ones. We illustrate this with one figure, about length-15 exchangeable words b_m we generated in the form (18) with (33) and selected after the test (36) mentioned below. About 13% of the pairs (m,m') matching χ are conjugate, while approximately (but, of course, not complementarily) 87% are reverse-conjugate.

Example 6 is, therefore, far more typical than example 14. (In [29, Example 6.3] with $n=6$, we have $\mu=-1$ and $\Theta=*$ as well, while $\Xi=\{-1\}$.) Example 7 displays the largest *finite* number yet (two) of m for which b_m and $b_{m'}$ are found conjugate but not reverse-conjugate.

Moreover, sporadic reverse-conjugacy (i.e., $\Theta \neq *,\varnothing$), like in examples 15,16,17, is much more rare and harder to find than sporadic conjugacy. It does sometimes happen, though, for $\Theta=\varnothing$ and μ even that $b_{\mu/2}$ is conjugate to its reverse, which is indicated by the '!' in the entry for example 12. For this see also Observation 6.4.

We came to notice (34) in trying to explain non-conjugacy when we found $v(L_{b_m}) = v(L_{b_{m'}})$ for various v. In addition to this failure to detect non-conjugacy, we know that the symmetry pattern in Remark 6.2 appears in several conjugacy invariants of b_m also when the braids are non-conjugate even up to word reversal. (This category of phenomenon seems quite common.)

Consider examples 11 and 13 (where $\hat{b}$ has two components). They satisfy (32) and easily fall into the first alternative on the r.h.s. of (29) (after '$1,n$' is changed to '$n-1,n$'). Of course, when $\Xi=\Theta=\varnothing$, then the definition

$$\mu = m+m' \ \text{ for } b_m \text{ (reverse-)conjugate to } b_{m'} \tag{35}$$

(and $m\neq m'$) does not really make sense. To remedy this, we specify that we determined coincidences of the characteristic polynomial $\chi(\psi_5(b_m))$ of the Burau matrix. So far observed, if (24) holds, then

$$\chi(\psi_5(b_m)) = \chi(\psi_5(b_{m'})) \tag{36}$$

occurs for one (unique) value of $m+m'$, and we can then understand μ as this value. (See, though, Remark 6.5 for $n>5$.)

It can be inferred from (46) below (and $\psi_n(\delta_n^2) = t^n \cdot Id_{n-1}$), that $\chi(\psi_n(b))$ for $b\in B_n$ is equivalent to $\nabla(L_b^*)$, where L_b^* is the set of all satellite links of L_b in which the axis component is cabled (with some braid pattern, say, and arbitrary cable degree allowed), but without cabling the component(s) of

$\hat{b}$. (See, e.g., also [23].) With a look at the proof of Proposition 6.1 in [29], we can then also see that $\nabla(L_b^*)$ does not determine ∇ of sublinks of L_b.

In examples 11 and 13, we found that L_{b_m} and $L_{b_{m'}}$ have the same (not only ∇ but also) skein P polynomial. But they have different hyperbolic volume and Kauffman polynomial, thus confirming that b_m and $b_{m'}$ are not conjugate (even up to word reversal).

Prompted by example 14, we record the following.

Observation 6.4. When $\beta = \beta'$, then for $\mu = 1$ we have $\Xi = *$. If $\beta = \text{rev}(\beta')$, then all b_m are conjugate to $\text{rev}(b_m)$. In particular, if $\beta = \text{rev}(\beta) = \beta'$, then for $\mu = 1$ we have $\Xi = \Theta = *$.

It is, of course, a (very) partial statement. It leaves unexplained, among others, sporadic conjugacy or reverse-conjugacy (or their frequency), and even values of μ. We cannot even ascertain that μ is unique, unless we impose (32) (or at least the condition of Proposition 6.6 below). However, it clearly shows that (33) is not a relevant restriction.

Remark 6.5. When $n > 5$, more care will be needed in making sense of μ via a condition like (36) when $\Xi = \Theta = \varnothing$. By checking that Bigelow's 5-braid $\phi_5 \in \ker \psi_5 \subset B_5$ [4] does not commute with $\delta^2_{[1,4]}$, and setting $\beta = \phi_5 \in B_{n-5,n-1} \subset B_n$ in (18), one can make $\psi_n(b_m)$ constant in m.

Next we explain how to (largely) remedy this issue in theory.

6.2. **Extension of non-conjugacy properties.** It should be made clear that, as far as non-conjugacy between the b_m is concerned, the argument in Remark 6.2 has very practical extensions. We have the following version of Proposition 6.1.

Proposition 6.6. *Assume some $\mathbb{Q}$-Vassiliev braid conjugacy invariant υ distinguishes some b_{m_1} and b_{m_2} (for some $m_1 \neq m_2$). Then* (29) *still holds for all but finitely many m, i.e., there is an $M = M(\upsilon)$ so that* (29) *holds when replacing on the left $\nu(\mathcal{B})$ by $\nu(\mathbb{Z} \setminus [-M,M], \mathcal{B})$. More exactly, μ in* (35) *is unique (if existent) when defined for $|m|, |m'| > M$.*

Remark 6.7.

1. A $\mathbb{Q}$-Vassiliev conjugacy invariant is meant to be a conjugacy invariant of n-braids which is a $\mathbb{Q}$-valued Vassiliev invariant of braids. Since such an invariant is determined by its values on finitely many braids (see [3, 37]), $\mathbb{Q}$-valued is equivalent to $\mathbb{Z}$-valued.
2. The invariants of [29] operating under (19), and yielding Proposition 6.1, can be argued to lie in this class. Polynomial invariants of L_b, as well as $\nabla(L_b^*)$ or its equivalent $\chi(\psi_n(b))$, can be understood as infinite collections of such υ.
3. Reverse-conjugacy in (35) should not be considered in this context, unless υ is orientation non-sensitive, i.e., $\upsilon(\text{rev}(b)) = \upsilon(b)$. But it is not assumed to relate m_1, m_2 to μ in any way.
4. While (24) remains the most general assumption, it is clear that the one of Proposition 6.6 is, at least in any practical sense, equivalent (with a more specific assertion). Cases like Remark 6.5 remain fairly special, and still leave a large repertoire of other applicable υ.

Proof of Proposition 6.6. W.l.o.g. let (up to scaling) υ be $\mathbb{Z}$-valued. Then again $Y(m) = \upsilon(b_m)$ is a (non-constant) polynomial $Y : \mathbb{Z} \to \mathbb{Z}$ by [37], thus up to scaling, we may assume $Y \in \mathbb{Z}[m]$.

The case that $k = \deg Y$ is odd is obvious. (It continues comprising the first alternative on the right of (29), which continues holding because its condition implies (19).)

Thus assume $k > 0$ is even. Choose N so that $|Y^{-1}(c)| \in \{0,2\}$ for all $|c| > N$ and (first)

$$M = \max\{\,|x| : |Y(x)| \le N\,\}.$$

(We will take later, as usual, the freedom to augment M a finite number of times independently of m.) This already suffices for the claimed estimate on $\nu(\mathbb{Z} \setminus [-M,M], \mathcal{B})$. For the uniqueness of μ, we include the following argument about the behavior of polynomials on integers.

There remains to prove that if Y is not even up to translation, i.e., no $s \in \mathbb{R}$ with

$$Y \in \mathbb{Z}[(x+s)^2] \tag{37}$$

exists, then

$$\lambda + \tilde{\lambda} : \{\lambda, \tilde{\lambda}\} = Y^{-1}(c) \cap \mathbb{Z},\ |c| > N$$

(with $\lambda \ne \tilde{\lambda}$) is unique for proper N. If (37) holds, then uniqueness is obvious.

We prove now that for some N

$$Y^{-1}(c) \text{ for } |c| > N \text{ does not contain more than one integer.} \tag{38}$$

We have

$$Y(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots \quad \text{with all } a_i \in \mathbb{Z} \text{ and } k \ge 2. \tag{39}$$

W.l.o.g. $a_k > 0$. By considering

$$x = x' - \frac{a_{k-1}}{k\,a_k},$$

we remove the x^{k-1} term.

We claim for $Y(x')$ that $Y^{-1}(c)$ contains at most one

$$x' = \frac{x''}{k\,a_k} \in \frac{\mathbb{Z}}{k\,a_k}.$$

By scaling

$$\tilde{Y}(x'') = Y\left(\frac{x''}{k\,a_k}\right) \cdot (k\,a_k)^k \in \mathbb{Z}[x''],$$

thus it is enough to prove (38) w.l.o.g. when $a_{k-1} = 0$ in (39).

Set in (39) that $Y = Y_e + Y_o$, where Y_e is the even part and Y_o is the odd part. We have $\deg Y_e = \deg Y = k$ and $\deg Y_o \le k-3$ but, by excluding (37), that $Y_o \ne 0$.

Choose M so that $Y(x)$ decreases for $x < 1-M$ and increases for $x > M-1$. Assume $\lambda \in \mathbb{Z}$ is a root of $Y(x) = c$ (so $c = Y(\lambda)$).

Now

$$\begin{aligned} X_1(\lambda) &= Y_e(1-\lambda) - Y_e(-\lambda) \quad \text{and} \\ X_2(\lambda) &= Y_e(-\lambda) - Y_e(-1-\lambda) \end{aligned}$$

are polynomials of λ of degree $k-1 > k-3 \ge \deg Y_o$. So when $|\lambda| > M$, then

$$\begin{aligned} |X_1(\lambda)| &> |Y_o(-\lambda)| + |Y_o(1-\lambda)| \quad \text{and} \\ |X_2(\lambda)| &> |Y_o(-\lambda)| + |Y_o(-1-\lambda)|, \end{aligned}$$

which by monotonicity implies that the other root $\tilde{\lambda}$ of $Y(x) = c$ will be in $\tilde{\lambda} \in (-1-\lambda, 1-\lambda)$, but since $Y_o(-\lambda) \neq 0$ for $|\lambda|$ large, $\tilde{\lambda} \neq -\lambda$. So $\tilde{\lambda}$ is not an integer. $\square$

In conclusion, we formulate here the most optimistic (and simplest) expectation regarding the (non-)conjugacy of b_m, which combines Ito's (weakest) assumption and our (strongest) assertion, and which is supported by some (and not yet refuted by any) both theoretical and computational evidence.

Conjecture 6.8. Assume an exchange move is non-degenerate. Then there is at most one value of $m+m'$ such that b_m is conjugate to $b_{m'}$ (and $m \neq m'$).

7. Braid index and MWF

The standard tool for estimating the braid index remains the MWF inequality [20, 10]. We use the skein (HOMFLY-PT) polynomial as in §2.3. When

$$P(L) = \sum_{t,s} a_{ts} l^t m^s ,$$

then we set for the m-coefficient in degree s

$$P_s = [P]_{m^s} = \sum_t a_{ts} l^t , \tag{40}$$

and similarly $[P]_{l^t}$. Also $\min\deg_l P = \min\{\, t : [P]_{l^t} \neq 0 \,\}$ and $\operatorname{span}_l P = \max\deg_l P - \min\deg_l P$.

It is well-known that if L is a link of $n(L)$ components, then $P_s(L) = [P(L)]_{m^s} \neq 0$ only if $s \geq 1 - n(L)$ and $s + n(L)$ is odd. In this sense, we can speak of the k-th (m-)term of $P(L)$ being $P_{2k-1-n(L)}(L)$. Using the implementation of [35], one can obtain the low-degree m-terms of $P(L)$ much faster than the entire polynomial. We will make substantial use of this opportunity.

If a link L admits an n-braid β of exponent sum $e = e(\beta)$, then

$$\max\deg_l P - n + 1 \leq e \leq \min\deg_l P + n - 1 , \tag{41}$$

whence in particular

$$b(L) \geq \frac{\operatorname{span}_l P(L) + 2}{2} .$$

We will call the r.h.s. the *MWF bound* for the braid index, writing it still as MWF(L), and all e satisfying (41) *admissible*. Observe that, in this setting, one can always replace P by any of its m-degree truncations.

When MWF(L) fails to determine $b(L)$, a suggestive idea was to apply P on cables of L (see e.g. [24]). For a knot K, a 2-cable K_p is given by some integer framing p. (In this scaling, the cable is *connected*, i.e., a knot, if p is odd.) Then (for the ceiling function)

$$b(K) \geq \left\lceil \frac{\operatorname{span}_l (P(K_p)) + 2}{4} \right\rceil . \tag{42}$$

We call this the '2-cabled MWF' and write shortly '2cMWF', also identifying it abusively with the estimate it provides. Whenever we talk of 2cMWF we will mean it for a suitable framing p.

In all computations we did for 2cMWF, we took a blackboard framed connected 2-cable knot $K' = K_{2w-1}$ of some diagram D of K, usually the one in [15], with writhe $w(D) = w$ and one negative half-twist.

More generally, when $K' = K * \beta$ is an n-cable for pattern $\beta \in B_n$ in the solid torus being the complement of its axis, then

$$b(K) \geq \left\lceil \frac{\mathrm{span}_l(P(K'))+2}{2n} \right\rceil . \tag{43}$$

We will use the blackboard framed disconnected 3-cable $K' = K * \delta_3^{2w}$ (from (4)) a few times.

MWF+ 2cMWF does give the correct braid index (i.e., estimates it sharply) for ≤ 12 crossing non-alternating knots [33]. (MWF is sharp for alternating knots up to 17 crossings.) On 98 of the 1126 non-alternating prime knots of ≤ 12 crossings does 2cMWF improve the MWF bound, as it does for 366 (all non-alternating) of the 13 crossing prime knots.

For non-alternating prime knots there is the knot 13_{9684} where $b(K) = 5$ requires 3-cable MWF [35]. Systematic check showed that there are 3 more 13 crossing knots K where it was apparent that $b(K) = 5$, but MWF gives $b(K) \geq 4$, and 2cMWF does not improve upon that bound:

$$13_{6586},\ 13_{7417} \text{ and } 13_{7647}. \tag{44}$$

No method to exclude $b(K) = 4$ appears available, except determining some portion of the 3-cable P polynomial. It in the end established that $b(K) = 5$, but the potential complexity discouraged me from attempting this calculation, until after most of what follows below was done (Example 9.1). Thus the 3 knots in (44) were carried along in some (redundant) tests, but with consistent results.

8. Jones' test for 4-braids

8.1. **Jones' test up to Markov equivalence.** It is better to focus on braid index 4. With the three knots in (44) later excluded, there are 4073 prime knots of ≤ 13 crossings with $b(K) = 4$. (When MWF and 2cMWF give bound 3, all knots were found as closed 3-braids. In [32] it was proved that MWF+2cMWF will always estimate $b(K) = 3$ sharply. See below Example 9.2, though.) On 135 of these 4073 knots 2cMWF bound 4 improves the one of MWF.

Example 8.1. Extensive check of non-braid 4 Seifert circle diagrams up to 17 crossings left a list of 1899 knots from these 4077 (incl. (44) and 13_{9684}, which I decided to carry along) not found in such diagrams. This list is led by the 22 knots of [28, §5], which indeed resisted exhibition as (4-braid) exchange move admitting.

Jones [17, (12.8)] outlines a test for an exchange move admitting 4-braid, based on the evaluation of the Hecke algebra at $t = e^{\pi i/5}$. We state a minorly improved version of his result, writing $V_L = V(L)$ for the Jones polynomial of a link L and $\Delta_L = \Delta(L)$ for its Alexander polynomial (as defined in §2.3).

Theorem 8.2 (Jones)**.** *The image of the Burau representation $\psi_3(B_3)(e^{\pi i/5})$ in $GL(2,\mathbb{C})$ is finite. Thus so is the set*

$$\left\{ \left(V_L(e^{\pi i/5}), \Delta_L(e^{\pi i/5})\right) \,:\, L \text{ admits an exchangeable 4-braid} \right\}.$$

The proof, and its later application, will consist in explicitly determining this image. For computational purposes, it is better to avoid crunching with complex numbers and floating point numbers, and use integers. Thus we prefer to take the attitude that we mod out in $\mathbb{Z}[t^{\pm 1}]$ by the ideal of

$$Z = t^4 - t^3 + t^2 - t + 1,$$

the minimal polynomial of that value. We will write by ψ_n^* for the n-strand Burau representation (of dimension $n-1$) with entries reduced modulo Z. Also $\bar{\psi}_3^*$ is ψ_3^* composed with the homomorphism $B_4 \to B_3$ given by $\sigma_1 \mapsto \sigma_1$, $\sigma_2 \mapsto \sigma_2$, $\sigma_3 \mapsto \sigma_1$.

Jones uses a somewhat indirect argument, so it would be better to have concrete and simple examples at hand.

It is also better to keep the Alexander polynomial along, since it is determined by Burau [17, (7.4)], and can be tested at very little extra cost. Also, the normalization of Δ that is included in this formula is needed. Shifting by units does not behave well mod Z, so we would lose information.

The formulas for the Jones and the Alexander polynomial are given in [32, (7.13)] and [17, (7.4)]; for the former it is

$$V_{\hat{\beta}}(t) = \left(-\sqrt{t}\right)^{e-3}\left[\frac{t(1-t^3)}{1-t^2}\,\mathrm{tr}\,\psi_4 + \frac{t^2}{1+t}\,\mathrm{tr}\,\bar{\psi}_3 + \frac{1-t^5}{1-t^2}\right]. \tag{45}$$

And for the Alexander polynomial it is the special case for $n=4$ of the formula

$$\left(-\sqrt{t}\right)^{e-n+1}\Delta_{\hat{\beta}}(t)\frac{1-t^n}{1-t} = \det(Id_{n-1}-\psi_n(\beta)). \tag{46}$$

Note that the multiplier of Δ on the left is invertible mod Z, but organizing the formula thus one saves the calculation of the inverse. Formulas (45) and (46) admit a generalization to $P(\hat{\beta})$ (for $n=4$), but we will only allude to it below Examples 8.4 and 8.8.

In both (45) and (46), notice also the presence of the square root of t, which is (only) a 20-th root of unity. This will account for a sign ambiguity later.

Proof of Theorem 8.2. The test centers around the assertion that the factored 3-strand Burau, and hence 4-strand Burau on

$$<\sigma_1,\sigma_2>= B_3 \subset B_4 \tag{47}$$

has finite image. Jones claims (essentially) that it 'could easily be written down' [17, p 269, l-3], but the result strongly justifies the appeal to a computer.

First one needs to compile the image of the product of Burau with the parity $-\psi_4^*$. (For the sign see [17, Note 5.7].) From now on, we will always understand that

$$-\psi_4^*(\gamma) \text{ means } (-1)^{e(\gamma)}\psi_4^*(\gamma). \tag{48}$$

We need to generate the 4-braid Burau on (47) modulo Z. Since one checks that $\psi_4^*(\sigma_i^{10})$ is trivial, it is enough to consider positive words (in σ_1,σ_2). Then we used word extension, and check modulo redundancies by braid relations and discarded every word (and its extensions) whose matrix was already recorded. Note that $\det(-\psi_4^*(\sigma_i)) = t$, which is a 10-th root of unity, so $-\psi_4^*(\beta)$ captures $\bar{e}(\beta) = e(\beta)$ mod 10. It is then more practical to record matrices in groups $W_{\bar{e}}$ indexed by $\bar{e} = e$ mod $10 \in \{0,\dots,9\}$ and to check coincidences of $-\psi_4^*(\beta)$ for some positive word β in $\sigma_{1,2}$ only within $W_{\bar{e}(\beta)}$. This method yielded the 1200 matrices in about 1 sec, divided into 10 groups of 120. In the process still some words needed to be processed of length about 480.

The success of this procedure thus in particular gives a computational proof of Jones' finiteness property. But some other rudimentary attempts were far less successful.

For example, it is clear that the kernel of the factored Burau $-\psi_4^*$ has finite weight; it is normally generated by words of length $|\text{image}| = 1200$ in any set of generators of B_3. But a quest for a reasonable (small) set of normal generators of the kernel led to little (after much computation time). However, there are elements therein like σ_i^{10}, whose exponent sum is not divisible by 20, which leads to a sign ambiguity in our test. Also, elements like $(\sigma_1^3\sigma_2^3)^{10}$ have a non-trivial permutation, so that trying to account for strand permutations in the α_i of (49) does not bring anything.

Now, when $e, V(K), \Delta(K)$ are given for some knot (or link) K, one can calculate first $-\psi_4^*(\beta)$ and $-\bar{\psi}_3^*(\beta)$ of a potential braid representative β of K with $e(\beta) = e$ and being of the form

$$\alpha_1\sigma_3\alpha_2\sigma_3^{-1}\,. \tag{49}$$

Obviously we need $e(\alpha_1) + e(\alpha_2) = e(\beta)$. Since one can move σ_1 between the α's, we can w.l.o.g. assume $\overline{e}(\alpha_1) = 0$ and $\overline{e}(\alpha_2) = \overline{e}(\beta)$.

Note that under the inclusion (47), the 3-strand Burau manifests itself as the main 2×2 minor of the 4-strand Burau. The extra diagonal entry is a parity. (This theory is adequately explained by Jones.)

So then take a matrix in $M_1 \in W_0$ and one $M_2 \in W_{\overline{e}}$, and build

$$B_1 = M_1 \cdot (-\psi_4^*(\sigma_3)) \cdot M_2(-\psi_4^*(\sigma_3^{-1})) \qquad \overline{B}_1 = \overline{M}_1 \cdot (-\bar{\psi}_3^*(\sigma_1)) \cdot \overline{M}_2(-\bar{\psi}_3^*(\sigma_1^{-1}))\,,$$

where $\overline{M}_i$ is the 2×2 minor of M_i, and keeping (48) (and its analogue for $-\bar{\psi}_3^*$) in mind.

Using these matrices, calculate (46) and (45) modulo Z by replacing $-\psi_4^*(\beta)$ by B_1 and $-\bar{\psi}_3^*(\beta)$ by $\overline{B}_1$. The square root of t in (46) and (45) gives to the 10th power only -1, which means that the moduli

$$(V^*, \Delta^*) = (V, \Delta) \bmod Z$$

of V, Δ must be tested up to sign; all four combinations in $(\pm V^*, \pm\Delta^*)$ must be considered. But both moduli should be tested for the same choice (M_1, M_2). If no choice matches both V, Δ moduli (up to sign), a braid form like (49) is excluded. Obviously there are $|W_0| \times |W_{\overline{e}}| = 14,400$ choices (M_1, M_2) to test. □

The test, we call below *Jones-Alexander test*, thus requires as input $V(K)$, $\Delta(K)$ (modulo Z) and e (modulo 10). If used on individual knots, it can be performed in about 2-3 seconds per knot K, and initially ran on the 1899 knots in slightly over an hour (on my modest Linux laptop). For longer input, it can be further sped up: the 14,400 choices (M_1, M_2) need to be processed only once for $\overline{e}_1 = 0$ and each $\overline{e}_2 = \overline{e} \in \{1,3,5,7,9\}$. (For knots e must be odd.) Also, duplications of (V^*, Δ^*) can be discarded and the remaining pairs sorted for logarithmic-time search.

Example 8.3. The Jones-Alexander test excludes 6 of the 22 Rolfsen knot candidates from Example 8.1:

$$9_{29},\ 9_{46},\ 9_{47},\ 10_{98},\ 10_{113} \text{ and } 10_{145}. \tag{50}$$

Thus every 4 Seifert circle diagram of these knots must be a braid diagram. (Also they will have no special diagram of 5 Seifert circles admitting an MPC move, as will be made clearer elsewhere.) It excludes 33 of the 96 knots of 11 crossings, 76/260 for 12, 568/1521 for 13 crossing knots, so total 683/1899. Thus the success ratio is about 1/3 and appears to slightly improve when crossing number goes up.

It should be added that when abandoning the Alexander polynomial test Δ^*, 627 of the 683 knots work with V (and e) alone. The simplest example where V alone does not work is 10_{145}. As an aside, the Jones-Alexander test does not give any restricted exclusion of (this special type of) 4-braid on 13_{9684} or the knots (44).

Now the Jones-Alexander test uses the writhe as restricted from P. In some cases when the MWF bound is improved by 2cMWF, it gives additional restriction on a 4-braid writhe, which may pay off.

Among the 1899 knots to test for exchangeable 4-braid, there are 34 knots K for which 2cMWF improves MWF.

If K has a 4-braid writhe e, then $K' = K_{2w-1}$ from §7 would have an 8-braid writhe $2e+2w-1$. Using MWF inequalities shows then that

$$\frac{\min\deg_l P(K') + 8 - 2w}{2} \leq e \leq \frac{\max\deg_l P(K') - 6 - 2w}{2} \tag{51}$$

(In practice it is better to work with low-degree m-truncations of $P(K')$ to save computation time.) Now the hand-sides of the estimate are integers, and will allow for an integer e when

$$\operatorname{span}_l P(K') \leq 14, \tag{52}$$

which is the 2cMWF in its original form (42). But there is the additional remark that (for a knot) e must be odd. It is possible that both inequalities in (51) are exact and determine an even integer e. This will exclude $b(K) = 4$ despite (52). However, it does occur only for 14 crossing knots (see [32, Example 7.6] and compare with Example 9.2 below). But nonetheless (51) determines otherwise for ≤ 13 crossing knots a unique (odd) writhe of a 4-braid. This information can be combined with the Jones-Alexander test to bring its success.

Example 8.4. For example, for 12_{2099} MWF allows for 4-braid writhe 3 and 5. The Jones-Alexander test is successful excluding $e = 5$, but fails for $e = 3$. However, $e = 3$ is excluded from 2cMWF, successfully completing the exchange move test. The knots 13_{8838} (-9 excluded by test, -7 by 2cMWF) and 13_{8884} (-5 excluded by test, -3 by 2cMWF) are similar examples. This brings the total number of prime ≤ 13 crossing 4-braid knots with excluded exchange move to 686.

The Burau spectral radius test rules out 15 more knots, starting from 12 crossings (along with several for which Jones-Alexander works as well). This method will receive its own separate account [39].

A question would be if using a modulus of P instead of V^* and Δ^* would improve the test. We did not attempt this, since an advance looks practically impossible. Now, it is known that there are 4-braid knots with equal V, Δ but not equal P; the simplest pair is $14_{41516}, 13_{7369}$. But by [32, Proposition 7.3] we do not have a pair admitting an equal 4-braid writhe.

Remark 8.5. Testing the form $\alpha_1(\sigma_3\sigma_2^2\sigma_3)^k$ for $\alpha_1 \in B_3$ and k from Definition 5.4 could obstruct to a link admitting a degenerate exchange move on a 4-braid, thus showing ICP. (Several choices of $k = \pm\lambda_U/2$ may be needed for different U.) We did not consider this in §5.3, in particular because the low-crossing links listed there did exhibit a degenerate exchange move on a 4-braid. This does not yet make ICP fail, because the links may admit other non-degenerate exchange moves on a 4-braid as well.

8.2. **A 3-braid test.** As noted in [17], one can also, much simpler, use the finiteness of the Z-factored ψ_3 as a 3-braid test. Again it is better to take e, again using (41), into account and test with V and Δ moduli using the set of 120 Burau matrices corresponding to e mod 10. This test is now much smaller, since 120 3-braid Burau matrices instead of 14,400 4-braid ones need to be evaluated. One has to use a similar version of (45) (see [32, (7.15)]) and $n = 3$ in (46). This test can exclude from $b(K) = 3$ the known examples $K = 9_{42}$ and 10_{150}. It fails on 9_{49}, but the other two 10 crossing knots with unsharp MWF, 10_{132} and 10_{156}, are "excused": they have the skein polynomial of (the closed 3-braids) 5_1 and 8_{16} resp. (See notes after Jones' table [17, 15.9].) This test can exclude also four 11 crossing knots and 11 knots of 12 crossings, 72 knots of 13 crossings and 104 knots of 14 crossings with MWF-bound ≤ 3.

8.3. **Jones' test up to conjugacy.** One can replace V^* and Δ^* by $\operatorname{tr}(\psi_4^*)$ and $\operatorname{tr}(\bar{\psi}_3^*)$, and gains a conjugacy test to a non-exchangeable 4-braid based on $(e, \operatorname{tr}(\psi_4^*), \operatorname{tr}(\bar{\psi}_3^*))$. (Note that with this information, there is no gain at looking at the other coefficients of the characteristic polynomial of ψ_4 and $\bar{\psi}_3$.)

Example 8.6. We were able to establish thus that

9_{38} `-1 -1 -2 -2 -3 -3 -2 1 -2 3 -2` and
9_{48} `-1 -1 -2 -3 2 1 -3 2 -1 -3 2`

admit non-exchangeable 4-braids up to conjugacy, plus the following knots of 10 crossings, in the complement of the list (50), but within the one in [28]: 10_{95}, 10_{121}, 10_{122}, 10_{136}, 10_{146}, 10_{147}.

Example 8.7. The following knots K (with Perko's duplication discarded) were found in non-exchangeable 4-braids of 13 crossings, but have exchangeable 4-braids from [28]:

```
9    33     -1 -2 -2 -3 2 -1 -3 2 2 2 -1 3 2
10   84     -1 -1 -2 -2 -2 3 2 2 -1 2 -3 -3 2
10   163    -1 -2 -2 3 -2 1 -2 -3 2 -3 2 -1 2
```

or because $b(K) = 3$:

```
10   104    -1 -1 -2 -2 -3 2 -1 2 2 -3 2 2 3
10   148    -1 -2 -2 -2 3 -2 1 -2 3 2 2 -1 -3
10   155    -1 -1 -2 3 2 -1 2 2 -3 -3 2 -1 2
10   159    -1 -1 -2 -2 -3 2 -1 -3 -2 1 3 3 -2
```

along with 10_{109}, 10_{149}, 10_{157}. As already discussed in §1, this provides strong evidence that there are no meaningful simplifications of an "exchange-Markov theorem without stabilization" like in [9].

We conclude this treatment of 4-braids with a remark on another related application of Theorem 8.2. Its finiteness property can also be used as an irreducibility test for a 4-braid (recall §2.1). Note that the condition depends on the sign of stabilization, and sometimes the other sign can be ruled out from (41). We, though, limited such experiments. One main reason is the Burau spectral test [39], which identifies many low crossing examples, and applies to general braid groups. Rather, we did try to look for a new quality of example.

Example 8.8. Let us say that an exchangeable braid b (with a particular form in Figure 1) is *totally irreducible* if all of the braids b_m shown in Figure 2 are irreducible. It is easy to see that if the Jones test can detect 5 consecutive $b_m \in B_4$ as irreducible, it will do for all m. After extensive (and optimized) search, we do not know if a *non-minimal* exchangeable 4-braid b can thus be exposed as totally irreducible. (At most 3 rest classes m mod 5 were ruled out from being reducible.)

However, we found many such (minimal) b with an unsharp MWF. Except possibly 9_{49}, at least 4 of the 5 Rolfsen knots occur (compare §8.2); for reference, here are two such braids for

9_{42}	`-2 1 -2 1 -2 -2 3 2 -1 2 -3` and
10_{156}	`-2 1 -2 -1 -1 -2 -2 3 2 -1 2 2 -3`

(We took the form of the exchange move to conjugate by powers of σ_1^2 one of the subwords between the $\sigma_3^{\pm 1}$, as clarified in (18).)

Similar examples arise from the Burau spectral test. With the skein polynomial $P(\hat{b})$ for $b \in B_4$ being determined by Burau matrices, the presence of knots like 10_{132} and 10_{156} is particularly noteworthy, since they duplicate polynomials of 3-braid knots. Thus $\psi_n(b)$ may contain extra information about, i.a., minimality of a braid b. At this stage it appears encoded in way eluding understanding even for $n \leq 4$, though.

We will record more worthwhile output from the Burau spectral in [39]. For instance, it can prove (when b is non-minimal) the set of m with b_m irreducible to be asymptotically dense. Such a statement is beyond the capacity of the Jones test.

9. DECIDING BRAID INDEX 3 OR AT LEAST 5

Among non-alternating ≤ 13 crossing knots, 6179 knots K have $b(K) = 5$, comprised of 6175 with $b(K) = 5$ coming from MWF+2cMWF, the knot 13_{9684}, and the 3 knots in (44) (where $b(K) = 4$ was only later excluded). All 6179 have 5-braid representatives and all but 26 have such admitting an exchange move. The exceptions exclude 13_{9684} and the 3 knots in (44); an example is given in Figure 3.

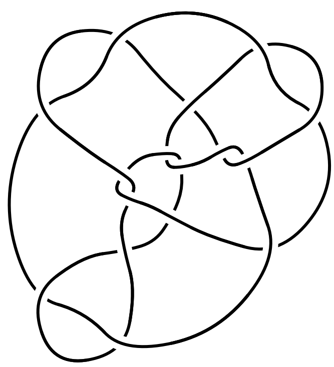


FIGURE 3. The knot 13_{9684} in a diagram showing an exchangeable 5-braid using the right equivalence in (2).

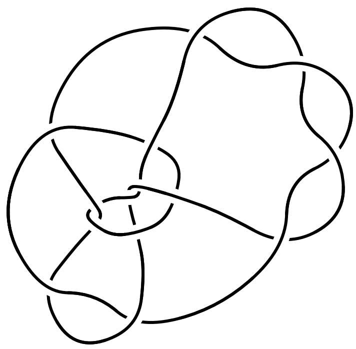


FIGURE 4. The knot 14_{35855}.

There are 2285 braid index 6 and 213 braid index 7 knots. For all is the braid index given by MWF+2cMWF, and all have a minimal braid representative admitting an exchange move. It thus appears that most knots have infinitely many minimal braid conjugacy classes, if their crossing number is not too high compared to the braid index.

Example 9.1. To complete the braid index table of 13 crossing knots, a final examination of the knots in (44) was performed. We took the disconnected blackboard-framed 3 parallel of the KnotScape [15] diagrams (with 117 crossings). The third term of the 3-cable skein polynomial, $P_2 = [P]_{m^2}$ (in (40)), showed (with 13 l-monomials) that $b(K) = 5$, and took 5-11 minutes CPU time per knot with the algorithm in [35], a less-than-expected painful computation. The result was then confirmed by computing the next term, P_4, which took 1:50~2:20h of CPU time per knot (closer to the expected complexity). In later attempts to obtain the list of $b(K) = 4$ prime knots of 14 crossings, there were about 30 knots where 3cMWF had to be used (to rule out a 4-braid representative).

There is some evidence (from [36] and the later proof by Menasco-Dynnikov of the Jones conjecture) that MWF on sufficiently high cables of P will yield the exact braid index, though increasingly complicated examples obviously appear. One such example is worth mentioning, which came up in attempts to compile the list of $b(K) \leq 3$ knots from the tables in [15].

Example 9.2. The knot $K = 14_{35855}$ has MWF$= 3$ and, in the 2-cable framing 17 (shown in Table 2), 2cMWF$= 3$, but a 3-braid can be ruled out by the test of Murakami [25, Corollary 10.5] (and also Kanenobu [18, Theorem 2]). In [32] it was proved that if $b(K) = 3$, then $\mathrm{MWF}(K) = 3$ or $2\mathrm{cMWF}(K) = 3$, at least for some framing of the 2-cable. This surprising example shows that the converse is not true, at least if we are loose about which 2-cable framing we choose. (It was also known from [34] that $b(L) = 3$ is implied by $\mathrm{MWF}(L) = 3$ for alternating links L and from [32] by $\mathrm{MWF}(L) = 2\mathrm{cMWF}(L) = 3$ for positive braid links L.) Murakami-Kanenobu's test is, as observed before, the most efficient practical 3-braid test. (Not a single instance of failure is yet known!) Its outperformance of 2cMWF was not previously observed, though.

By uncabled MWF, K would (if a closed 3-braid) have a 3-braid writhe of $e = -8$. Now consider the 2-cable, for which we use the notation K_p as appearing in (42). The first polynomial in Table 2 is for one negative half-twist in the -8 KnotScape diagram blackboard framing; thus it is $P(K_{-17})$. (Its

```
57 35855      0     32
 -36   -26          -4        10       103       178        94         4
 -36   -26         142       612       150     -1654     -1602      -208
 -36   -26        -495     -4493     -6349      6989     13540      2976
 -36   -26          71     12809     33716    -16691    -69066    -20103
 -36   -26        2291    -17695    -91123     24385    230793     77571
 -36   -26       -4826      9227    151280    -22764   -530575   -188302
 -36   -26        4756      6788   -165835     13934    865803    304166
 -36   -26       -2687    -14899    124074     -5647  -1022827   -338433
 -36   -26         917     11638    -64149      1500    885183    264936
 -36   -26        -187     -5226     22844      -251   -564554   -147566
 -36   -26          21      1456     -5488        24    265492     58587
 -36   -26          -1      -249       848        -1    -91517    -16429
 -34   -26                    24       -76         0     22782      3175
 -34   -26                    -1         3         0     -3979      -402
 -28   -26                                                 462        30
 -28   -26                                                 -32        -1
 -28   -28                                                   1
57 35855      0     30
  24    34          -4       -78      -138       -70        -1         4
  24    34         174      1212      1130      -222      -454       -88
  24    36       -2195     -9289     -4051      4759      3038       438        64
  24    36       13114     42844      8120    -20955     -7739      -884      -316
  24    36      -44540   -128685     -9830     47645      9168       879       627
  24    36       94361    263794      7449    -66379     -3602      -459      -626
  24    36     -131511   -379952     -3596     60466     -3347       128       342
  24    36      124459    391213      1103    -36943      5190       -18      -103
  24    36      -81440   -290639      -208     15187     -3113         1        16
  24    36       37104    156271        22     -4131      1045         0        -1
  24    32      -11711    -60592        -1       711      -205
  24    32        2507     16728         0       -70        22
  24    32        -347     -3201         0         3        -1
  24    26          28       403
  24    26          -1       -30
  26    26                     1
```

TABLE 2. The polynomials $P((14_{35855})_{-17})$ and $P((!14_{35855})_{15})$

calculation uses the implementation of [35], without truncation, and the format explained in [33].) The knot K_{-17} would admit a 6-braid writhe -33, which contradicts in (41) the maximal degree $\max\deg_l P = -26 > -33+6-1$.

That $2\text{cMWF} = 4$ is seen in the second polynomial, which is for a negative half-twist in the $+8$ blackboard framing of the mirror image $!K$, i.e., it is the polynomial

$$P((!K)_{15})(l,m) = P(K_{-15})(l^{-1},m)\,.$$

(The cable $(!K)_{15}$ would have a 6-braid writhe of 31, and $\min\deg_l P < 31-5$ contradicts in (41).) This example is extraordinarily peculiar. It seems the first knot showing that 2cMWF (at least in its naive form (42), without the parity argument for e below (51)), can depend on the 2-cable framing p. We did not seriously consider this possibility, so we confirmed the above skein calculation with the program of [24], after finding a 4-braid representative of K. This should serve as a sufficiently serious caveat to an easy-handed use of 2cMWF.

10. Outlook

Our original intention was also to explore the right equivalence in (2), and especially Lemma 3.7, in a less computational and more systematic way. For example, we aim to show using it that there are infinitely many non-conjugate minimal braids (if $b(L) \geq 4$) for every 2-bridge knot (which is likely true) and link L (which would need some more work). But this topic does seem to require its own separate discussion. It may find its place in a sequel to this paper.

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