GRID DIAGRAMS, LINK INDICES, AND QUASIPOSITIVITY

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Abstract. As the final of an extensive study of the relation between grid diagrams and braided surfaces of Euler characteristic 0, we consider the problem of quasipositivity of reverse 2-parallels and Whitehead doubles of knots K. Among many other results, we know that when K is alternating then these links are quasipositive if and only they are strongly so. However, in general we give examples that this equivalence fails. Our proofs involve a combination of our previously worked out approach via the ℓ and θ invariants, algorithmic development, Kauffman polynomial, Casson-Gordon invariants, and Vassiliev invariants.

Keywords: arc index, braid index, link polynomial, Thurston-Bennequin invariant, quasipositive, skein algorithm, Whitehead double, Casson-Gordon invariants, Vassiliev invariants.

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1 Introduction

The central focus of this paper is to uncover as much new knowledge as possible on the following question: when are reverse 2-parallels and Whitehead doubles of knots K quasipositive?

This is the third and final part of a longer project, initiated with G.T. Jin and H.J. Lee, to study the connection between strong quasipositivity and arc representations. The first two parts are available [JLS1, JLS2], and techniques therefrom will be extensively used and recalled.

The question we study here is heavily motivated by the answer for strong quasipositivity. As explained [JLS1], Rudolph's work reduces this question to determining the maximal Thurston-Bennequin invariant of K. This is a problem on its own. And while certain techniques are available in some cases, our own approach has developed new tools (recalled in §3). We will see this approach applied here quite beyond such singular estimations.

We discuss what previous results on strong quasipositivity can be extended to quasipositivity. Compare e.g. Theorem 5.25 with Corollary 2.20 below.

For many (companion) knots, incl. alternating ones, we establish that the quasipositivity and strong quasipositivity of knotted annuli are equivalent (for example, Corollary 5.6), but that in general they are not (Proposition 5.9).

For Whitehead doubles there is a strict dichotomy depending on the clasp.

For positive clasp, we have fairly general results, where both properties are equivalent (§5.2.1). This includes the complete result for alternating K (Theorem 5.22). However, we also know that some positive untwisted Whitehead doubles are quasipositive but not strongly quasipositive (Remark 5.11). This discord reveals related to a property we call λ-sliceness (Remark 5.14), which also emerges in a recent construction of Truol [Tr]. These phenomena remain to be further explored, and a complete clarification where the equivalence between quasipositivity and strong quasipositivity fails is possibly some way out (but perhaps also too difficult to attain).

For negative clasp, there is a tangible prediction that no non-trivial Whitehead double is quasipositive. Among others, we can confirm this for alternating knots (Corollary 5.29). Theorem 5.35 says the following: if K is amphicheiral, and its negatively clasped Whitehead double W is quasipositive, then W is untwisted and has trivial HOMFLY polynomial.

We also engage in a very substantial new computational development (Computation 5.37), the non-Alexander-variable truncated parallelized skein algorithm. It is an upgrade and porting to braids of the parallelized version (briefly summarized in §2.7) of the method introduced in [St3]. We call our new version also "braid-stack" algorithm, since it reorganizes the Millett-Ewing stack [EM] for braids. This allows one to *meaningfully* compute *pieces of* the HOMFLY polynomial in ranges up to about 400-500 crossings. At the end of this effort, we can rule out from being quasipositive negatively clasped Whitehead doubles of knots K (incl. composite ones) up to 16 crossings, with 14 exceptions remaining (Proposition 5.36).

The (theoretical) proofs extend previous methods developed for the strongly quasipositive case. It is inevitable to include detailed preliminaries from [JLS1, JLS2], since many of these techniques are not standard, and the paper needs to remain self-contained. But when strong quasipositivity is replaced by quasipositivity, then many considerations revolve around sliceness. The new proofs are also closely related to the problem of slicing Whitehead doubles. For Theorem 5.22, we have to additionally introduce (Gilmer's reinterpretation of) Casson-Gordon invariants, and for Theorem 5.35 we further use Vassiliev invariants.

The problems are also effectively resolved for torus knots K (§5.3) using our separate (quantum-group-related) work [MS+]. For torus K, we also simultaneously settle other issues, discussed in [JLS2], like braid indices, and the minimal string Bennequin surface.

2 Definitions and Preliminaries

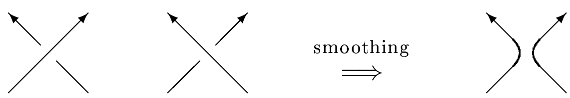
This section deals with preliminaries that existed mostly (though not entirely; see in particular the end of §2.10) prior to [JLS1, JLS2]. More specific details about these works will be compiled in the later sections.

2.1 Generalities

We say an inequality '$a \geq b$' is *sharp* or *exact* if $a = b$ and *strict* (or *unsharp*) if $a > b$. We use $\#E$ for the cardinality of a finite set E and $\lfloor x \rfloor$ for 'greatest integer' part of $x \in \mathbb{Q}$. We also afford a few standard abbreviations like 'l.h.s.' (for 'left hand-side'), 'w.r.t.' (for 'with respect to') and 'w.l.o.g.' (for 'without loss of generality').

2.2 Links and genera

All link diagrams and links are assumed oriented. Crossings in an oriented diagram D of a knot K are called as follows.



positive negative smoothed out

(2.1)

The *sign* of a positive/negative crossing is assigned to be ± 1 accordingly. Let $c_\pm(D)$ be the number of positive, respectively negative crossings of a link diagram D, so that the *crossing number* of D is $c(D) = c_+(D) + c_-(D)$ and its *writhe* is $w(D) = c_+(D) - c_-(D)$. We write $s(D)$ for the *number of Seifert circles* of D, which are the circles obtained after smoothing all crossings of D. We write $c(K)$ for the crossing number of a knot K, the minimal crossing number of all diagrams of K. The mirror image of K will be written $!K$, and the mirror image of diagram D (in the form obtained by switching all crossings of D) will be $!D$. If $K =!K$ (up to orientation), we call K *amphicheiral*. We use '$\bigcirc$' to denote the *unknot* (trivial knot) in formulas, and $T_{p,q}$ is used for the (p,q)-*torus knot*.

The symbol '#' is used for *connected sum* (when applied on links). The number of components of a link L is denoted by $\kappa(L)$.

The *(Seifert) genus* $g(L)$ resp. *Euler characteristic* $\chi(L)$ of a knot or link L is said to be the minimal genus resp. maximal Euler characteristic of a *Seifert surface* of L. We have

$$2g(L) = 2 - \kappa(L) - \chi(L)\,.$$

Similarly write $\chi_4(L)$ for the *smooth 4-ball* (maximal) Euler characteristic and

$$2g_4(L) = 2 - \kappa(L) - \chi_4(L)\,.$$

(In the following 4-ball genera and sliceness will always be understood smoothly.) A knot K is *slice* if $g_4(K) = 0$, or equivalently, $\chi_4(K) = 1$. We will refer to the following basic fact: if $\kappa(L) = 2$ and $\chi_4(L) = 2$, then both components of L must be slice (knots), and have linking number 0.

2.3 Knot tables

For notation from knot tables, we follow Rolfsen's [Ro, Appendix] numbering up to 10 crossings, except for the removal of the Perko duplication.

For 11 to 16 crossings we use the tables of [HT] (which for 11 to 13 crossing knots are now also on KnotInfo [LvM]), while appending non-alternating knots after alternating ones of the same crossing number. Thus, for instance, $11a[k] = 11_{[k]}$ for $1 \le [k] \le 367$, and $12n[k] = 12_{1288+[k]}$ for $1 \le [k] \le 888$. (For the origin of, and motivation for this numbering, please see the explanations in [St4].)

If it is relevant, mirror images will be distinguished on a case-by-case basis. Specifically, for the $(2,n)$-torus knots, we will say that the knot is *positively/negatively mirrored*.

The convention for 10_{132} is fixed in [JLS2] to be the *opposite* of Rolfsen's (that is, so that it has the HOMFY polynomial of the *positively* mirrored $(2,5)$-torus knot). The knot 10_{132} exhibits a certain property, of not being ℓ-sharp (see Definition 3.4), that has to be treated for higher crossing knots as well, but being the only Rolfsen knot with such status, it will merit special attention.

2.4 Braids and braided surfaces

We write B_n for the *braid group* on n *strands* or *strings*. The relations between the *Artin generators* σ_i, $i = 1, \ldots, n-1$ are given by

- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2$ and
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \le i < j-1 \le n-2$.

As in [JLS2], in diagrams we will orient braids left to right and number strings from top to bottom. (See §2.9 for an example.)

There is a *permutation homomorphism* $\pi : B_n \to S_n$, sending each σ_i to the transposition of i and $i+1$. By a *subbraid* of $\beta \in B_n$ we mean a braid obtained by taking only a subset $C \subset \{1, \ldots, n\}$ of the strands in β, which is invariant under the associated permutation $\pi(\beta)$ of β (i.e., C is a union of cycles of $\pi(\beta)$).

We define *band generators* $\sigma_{i,j}$ for $1 \le i < j \le n$ in B_n by

$$\sigma_{i,j} \;=\; \sigma_i \dots \sigma_{j-2}\sigma_{j-1}\sigma_{j-2}^{-1} \dots \sigma_i^{-1}\,, \tag{2.2}$$

for $j > i+1$, and set $\sigma_{i,i+1} = \sigma_i$.

A representation of a braid $\beta \in B_n$ in the form

$$\beta \;=\; \prod_{k=1}^{l} \sigma_{i_k,j_k}^{\pm 1}$$

is called a *band presentation*. (See e.g. [BKL].)

Usually, it will be more legible to use the symbol

$$[ij] = \sigma_{i,j}$$

when writing band generators in formulas. Similarly we use $-[ij] = \sigma_{i,j}^{-1}$. In certain cases, we even omit the brackets (see Definition 2.14 and Example 2.10). Also, when $j = i+1$, we often simply write i for σ_i and $-i$ for σ_i^{-1}, when no ambiguity arises.

The image of β under the abelianization $B_n \to \mathbb{Z}$ is the *writhe* (or exponent sum) of β, and is written $w(\beta)$.

A braid $\beta \in B_n$ whose *closure* $\hat\beta$ is the link L is a *braid representative* of L. Similarly a word for β gives a (braid closure) diagram $D = \hat\beta$ of L. When β is a word, then $w(\hat\beta) = w(\beta)$.

A representation of a braid β, and its closure link $L = \hat\beta$, as a word in $\sigma_{i,j}^{\pm 1}$ is called a *band representation*. A band representation of β spans naturally a Seifert surface S of the link L: one glues disks into the strands, and connects them by half-twisted bands along the $\sigma_{i,j}$. The resulting surface is called *braided Seifert surface* of L. In fact, a result of Rudolph [Ru] (later rediscovered independently by M. Hirasawa) says that any Seifert surface is of this form.

Rudolph [Ru] proves that every Seifert surface is a braided surface. If a braided surface is of minimal genus for L, it is called a *Bennequin surface* of L [BM2].

2.5 Quasipositivity and strong quasipositivity

A link is called *quasipositive* if it is the closure of a braid β of the form

$$\beta \;=\; \prod_{k=1}^{\mu} w_k \sigma_{i_k} w_k^{-1}, \tag{2.3}$$

where w_k is any braid word and σ_{i_k} is a (positive) standard Artin generator of the braid group (see [BO]). If the words $w_k \sigma_{i_k} w_k^{-1}$ are of the form σ_{i_k,j_k} in (2.2), so that

$$\beta \;=\; \prod_{k=1}^{\mu} \sigma_{i_k,j_k}\,, \tag{2.4}$$

then they can be regarded as embedded bands. Links which arise this way, i.e., such with *positive band presentations*, are called *strongly quasipositive links*. An overview of this topic can be found in [Ru4].

We write again $\chi(L)$ for the maximal Euler characteristic of L, as in §2.2, and $w(\beta)$ is the writhe/exponent sum of the braid β.

Theorem 2.1 (Bennequin [Be]) For $\hat\beta = L$, for $\beta \in B_n$, one has

$$-\chi(L) \ge w(\beta) - n\,. \tag{2.5}$$

Thus, Bennequin's work implies that any *strongly quasipositive Seifert surface*, a braided Seifert surface with only positive bands, is a Bennequin surface: if a link L has a strongly quasipositive Seifert surface S on r strands with μ bands, then $\chi(L) = \chi(S) = r - \mu$.

There is a similar version of the inequality (2.5) for $\chi_4(L)$, the *(smooth) slice Euler characteristic* of L,

$$-\chi_4(L) \geq w(\beta) - n\,. \tag{2.6}$$

This is sometimes called the *slice Bennequin inequality* (see e.g. [IS, St2]).

Corollary 2.2 Every strongly quasipositive surface is a Bennequin surface.

Baker-Motegi asked if every minimal genus surface of a strongly quasipositive link is strongly quasipositive. This is an open question. For some work on this question, see [St2] (for canonical surfaces) and [MS+] (for banded surfaces of torus links).

We also note that the aforementioned examples with Hirasawa [HS] are not strongly quasipositive. Thus one cannot use their Bennequin surfaces to address Question 2.8.

Definition 2.3 A link L is *Bennequin-sharp* if

$$-\chi(L) \;=\; \max\left\{\, w(\hat\beta) - n \;\mid\; \hat\beta = L,\ \beta \in B_n \right\}.$$

Similarly, it is *slice Bennequin-sharp* if the inequality (2.6) can be made into an equality for proper β.

Corollary 2.4 If L is strongly quasipositive, then L is Bennequin-sharp.

Problem 2.5 (Bennequin sharpness problem; see, e.g., [FLL, St2]) A link L is strongly quasipositive if and only if L is Bennequin-sharp.

For some related results below, see §5.2.5.

The author's effort to determine the quasipositivity of the (prime) 13 crossing knots [St4] also provides some evidence for a "4-ball" version of the Bennequin sharpness problem 2.5:

$$L \text{ is slice-Bennequin-sharp} \iff L \text{ is quasipositive}\,. \tag{2.7}$$

In practical terms, every proof of non-quasipositivity of a knot passes via showing that it is not slice-Bennequin-sharp.

2.6 Braid indices and minimal string surfaces

Definition 2.6
- Let $b(K)$ be the *braid index* of K, the minimal number of strings of a braid representative of K.
- Let $b_b(K)$ be the *Bennequin braid index* of K, the minimal number of strings to span a Bennequin surface of K.
- When K is strongly quasipositive, let $b_{sqp}(K)$ be *strongly quasipositive braid index* of K, which is the minimal number of strings to span a strongly quasipositive surface of K (only positive bands).
- Further, for a Seifert surface S, let $b(S)$ be the minimal string number on which S is spanned as a braided surface.
- If S is a strongly quasipositive surface, let $b_{sqp}(S)$ be the minimal string number on which S is spanned as such (i.e., arises from a positive band presentation).

We have then (with the right inequality only valid for strongly quasipositive K)

$$b(K) \leq b_b(K) \leq b_{sqp}(K)\,, \tag{2.8}$$

and by definition, with S being a Seifert surface of K,

$$b_b(K) = \min\{\, b(S)\,:\,\chi(S) = \chi(K)\,\}\,, \qquad b_{sqp}(K) = \min\{\, b_{sqp}(S)\,:\, S \text{ strongly quasipositive}\,\}\,. \tag{2.9}$$

We will further discuss these relations in §4.

We also feature the following result. It confirms an expectation originally formulated for $n = b(L)$ by Jones [J, end of §8] (later also referred to as the "weak" form) and subsequently extended by Kawamuro.

Theorem 2.7 (proof of the Jones-Kawamuro conjecture [DP, LaM]) For every link L, there is a number $w_{min}(L)$, so that every braid representative β of L on n strands of writhe w satisfies

$$|w - w_{min}(L)| \leq n - b(L)\,. \tag{2.10}$$

Let $b(K)$ be the braid index of K, the minimal number of strings of a braid representative of K. The Artin generators σ_i (see §2.4) are generalized to the band generators $\sigma_{i,j}$ in (2.2), so that $\sigma_i = \sigma_{i,i+1}$. We explained (§2.4) that a band presentation β naturally spans a Seifert surface of $L = \hat\beta$. Following Rudolph, we call this a braided surface of L.

A minimal genus braided Seifert surface S is called a Bennequin surface (see §2.4). If S is realizable as Bennequin surface on $b(L)$ strings, then S is called a *minimal string Bennequin surface* of L. By Bennequin and Birman-Menasco [BMe], a 3-braid link L has a minimal string Bennequin surface. However, by [HS], we know that this is not true for 4-braid knots already. For more work on minimal string Bennequin surfaces, see [St2].

A similar question arises for strongly quasipositive surfaces.

Question 2.8 (Rudolph; see [St11, Remark 8.3.3]) If L is strongly quasipositive, does L have a strongly quasipositive braid representative on $b(L)$ strands?

In the course of routine tabulation, the author has confirmed this for strongly quasipositive prime knots up to 16 crossings, for example. The contribution of [St11] was the resolution for (positive) alternating knots up to genus 4.

Generally speaking, we will use Theorem 2.7 to advance theoretical applications in our work, but for practical ones, another tool will be crucial, which we introduce next.

2.7 HOMFLY-PT and Kauffman polynomial

We use the *HOMFLY-PT polynomial* P [LiM], in the Morton [Mo] convention

$$P(\bigcirc) = 1\,, \qquad v^{-1}P_+ - vP_- = zP_0\,, \tag{2.11}$$

where P_+, P_- and P_0 refer to the polynomials of three links with diagrams equal except at one spot, where they contain the fragments of (2.1) from left to right. The right part of (2.11) is also called P's *skein relation*. We will use the suggestive notation $\min\deg_v P$ for minimal v-degree of (any monomial in) P, and similarly $\max\deg_v P$, and set $\mathrm{span}_v P = \max\deg_v P - \min\deg_v P$. We write $[P]_{z^k}$ for the coefficient of z^k in P, being a polynomial in v. Then, $[P]_{v^d}$ the coefficient of degree d in v (which is itself treated as a polynomial in z). Also set

$$\min\mathrm{cf}_v P = [P]_{v^{\min\deg_v P}} \tag{2.12}$$

to be the trailing (lowest degree) coefficient of P. The notation $P|_{z\geq k}$, $P|_{z\leq k}$, and $P|_{z\neq k}$ will mean (the polynomial consisting of) all terms in P of z-degree at least k, at most k, and different from k, respectively. The z-variable is left inside. Thus $[P]_{z^k}$ is a polynomial in v, while $P|_{z\geq k}$ is a polynomial in z,v. We occasionally refer to $P|_{z\leq k}$ as a *(z-)truncated polynomial*. We emphasize that much of the useful information of P can be obtained from truncations thereof (like (2.18)), which are much faster (subexponentially) to compute than the full polynomial. A program that calculates such truncations was introduced in [St3], and we will extensively apply it below. It builds on a variant of algorithm that was used by Millett-Ewing [EM]. This algorithm treats the handful of harder diagrams that do not admit local simplifications by creating needle-like fragments. For this reason, it will be referred to below as the *skein-needle algorithm*.

A CPU-parallelized upgrade of the truncated polynomial calculation was developed to settle the last 16 crossing prime knot standing to resolve for the below problem (4.1), $16_{1057125}$. This knot required a 4-cable polynomial; it has now its own description page on [St4]. The idea behind parallelizing (which will be relevant in Computation 5.37) is that different branches of the skein tree can be computed independently (and thus simultaneously on different CPUs), and later their contributions combined. It was realized that this splitting/adding can be easily performed by using the interrupt/restart facility of the program in [St3] (the idea for which was also initiated by Millett-Ewing).

Two further standard properties of P are that for a link L of $\kappa(L)$ components, $\min\deg_z P(L) = 1-\kappa(L)$, and $P(L)$ contains only monomials z^pv^q for p,q odd (resp. even) when $\kappa(L)$ is even (resp. odd). The mirroring behavior of P is (signed) v-conjugation:

$$P(!L)(v,z) = (-1)^{\kappa(L)-1}P(L)(v^{-1},z)\,. \tag{2.13}$$

We further use the identity (see [LiM, Proposition 21])

$$P(v,v^{-1}-v) = 1\,. \tag{2.14}$$

By the MFW [Mo, FW] inequalities, the writhe w of an n-string band presentation of L satisfies

$$w+n-1 \geq \max\deg_v P(L) \geq \min\deg_v P(L) \geq w-n+1\,, \tag{2.15}$$

thus

$$\mathrm{MFW}(L) := \frac{1}{2}\,\mathrm{span}_{\,v}P(L)+1 \leq b(L)\,, \tag{2.16}$$

where the left hand-side is the *MFW bound* for the braid index $b(L)$. If $\mathrm{MFW}(L) = b(L)$, we call L *MFW-sharp*.

When L is not MFW-sharp, there are ways to improve the braid index estimate using cables of L: when L' is a degree-c cable of L, then

$$\mathrm{MFW}(L') \leq b(L') \leq cb(L)\,,$$

thus

$$b(L) \geq \left\lceil \frac{1}{c}\,\mathrm{MFW}(L')\right\rceil\,. \tag{2.17}$$

The method is well explained in [MS] (certainly when $c=2$; some examples for $c=3,4$ can be found in [St3]). We refer to such estimates as the *cabled MFW*.

To relate this to the Jones-Kawamuro conjecture (Theorem 2.7), we point out that MFW plus cabled versions thereof is efficient to determine the braid index of most links. In some cases alternative methods apply, but for every link L whose braid index is decided so far, (2.17) is known give a sharp estimate at least for sufficiently large c. It is thus conjecturable that this is always the case (see [St4]):

Conjecture 2.9 For every link L there is a $c>0$ and a degree-c cable link L' of L making (2.17) sharp.

Obviously, when we can prove that a braid representative β of a link L is minimal, then we immediately also obtain $w_{min}(L) = w(\beta)$ in Theorem 2.7. However, it was also noticed in [St] that once (2.17) (for some c) gives a sharp estimate of $b(L)$, it proves along the way that $w_{min}(L) = w(\beta)$ is unique. (And it is not too hard to derive (2.10) either from that argument.) Thus Theorem 2.7 provides a theoretical underpinning, but is neither very practically helpful nor essential to determine $b(L)$ or $w_{min}(L)$ for a given L.

One main drawback of (2.17) is that in general the polynomial of a cable link L' is notoriously hard to calculate. But instead of the whole polynomial, we can use a truncation:

$$\mathrm{MFW}_d(L') = \frac{1}{2}\,\mathrm{span}_{\,v} P(L')|_{z^{\leq d}} + 1 \leq \mathrm{MFW}(L') \leq b(L')\,. \tag{2.18}$$

We refer below to such type of estimate of the braid index as *truncated (cabled) MFW*.

When quoting specific computations of P polynomials (see, e.g., (5.48)), the notation should be read thus. The first line contains the crossing number of the diagram the polynomial was computed from, an identifier, and $\min\deg_z P$ and $\max\deg_z P$. Then in each line follow $[P]_{z^d}$ for $\min\deg_z P \leq d \leq \max\deg_z P$ with $d - \min\deg_z P$ even. The line starts with $\min\deg_v [P]_{z^d}$, then $\max\deg_v [P]_{z^d}$, and then follow the coefficients $[P]_{z^d v^e}$ with $\min\deg_v [P]_{z^d} \leq e \leq \max\deg_v [P]_{z^d}$ and $e - \min\deg_v [P]_{z^d}$ even. These entries are aligned so that coefficients in the same v-degree are on the same left-right position.

Returning to surfaces, it follows from the right inequality in (2.15) that a Bennequin-sharp (in particular strongly quasipositive) link L satisfies

$$\min\deg_v P(L) \geq 1 - \chi(L)\,. \tag{2.19}$$

Morton also proves in [Mo] the *canonical genus inequality*, for any diagram D of L,

$$\max\deg_z P(L) \leq c(D) - s(D) + 1\,. \tag{2.20}$$

The *Conway polynomial* ∇ is given by

$$\nabla(L)(z) = P(L)(1, z)\,. \tag{2.21}$$

The *determinant* of a knot K can be defined by

$$\det(K) = |\nabla(2\sqrt{-1})|\,. \tag{2.22}$$

This is always an odd number (when K is a knot).

The *Kauffman polynomial* $F = F(a, z)(K)$ will be needed as well. We use the following well-known properties: for every link L,

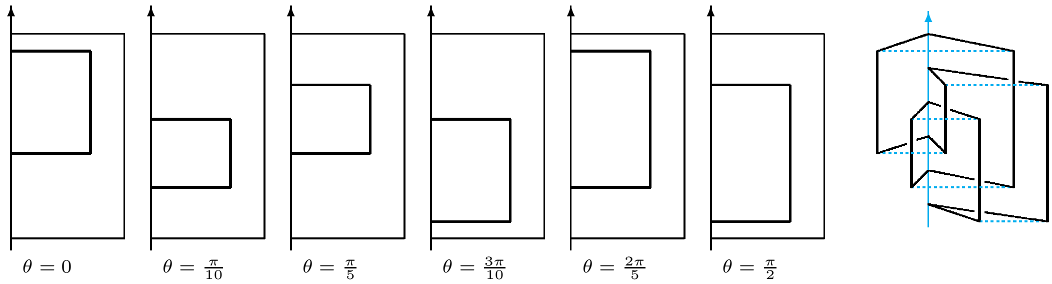
- $F(L)$ contains only monomials $a^p z^q$ for $p + q$ even.
- $$F(\sqrt{-1}, z)(L) = 1\,, \tag{2.23}$$
- and for a knot K,
$$[F(K)]_{z^0}(\sqrt{-1}v) = [P(K)]_{z^0}(v)\,. \tag{2.24}$$

We caution that our mirroring convention is so that the positive (right-hand) trefoil 3_1 has $\min\deg_a F(3_1) = 1$ and $\max\deg_a F(3_1) = 4$. (This convention is, e.g., opposite to [DM, Th], i.e., with a and a^{-1} interchanged. For course, still $F(\bigcirc) = 1$.)

The lowest degree term (2.24) has a similar property for links L, where z^0 is replaced by $z^{1-\kappa(L)}$. Also, there is a direct expression of this coefficient known in terms of $[P(K_i)]_{z^0}(v)$ for the components K_i of L and linking numbers. We use this relationship in the simplified variant (3.4) below.

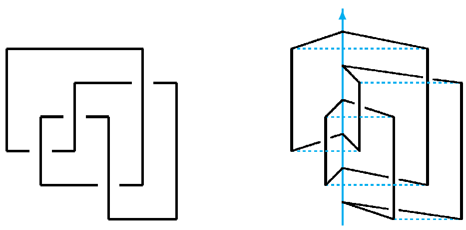
2.8 Grid diagrams, Arc index and Thurston-Bennequin invariant

An *arc presentation* of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc (e.g., [Cr2]).



A *grid diagram* (or, for simplicity simply called *grid* often below) is a knot or link diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges. We assume that horizontal/vertical positions of vertical/horizontal edges are pairwise distinct. In particular, away from crossings edges meet only at corners, and vertices are pairwise distinct.

It is a not hard to see standard fact that every knot admits a grid diagram. The figure below explains that arc presentations and grid diagrams correspond one-to-one.



(2.25)

We set the *size* $\mu(D)$ of a grid diagram to be the number of vertical *or* (equivalently) horizontal segments (but *not both* together). A grid (diagram) of size μ will also be shortly called a μ*-grid*.

In general, we will afford the sloppiness of abolishing the distinction between an ordinary and a grid diagram, whenever the grid structure is unnecessary. Thus, for instance, $c(D)$ can mean the crossing number of both an ordinary and grid diagram, whereas $\mu(D)$ would imperatively assume that D is given a grid shape.

Example 2.10 The default direction of reading a grid diagram will be from the *left*, with vertical positions numbered from bottom to top. Reading the grid diagram D in (2.25) gives the presentation

$$[14][35][24][36][15][26]\,, \tag{2.26}$$

which we often also write even omitting brackets, as

$$\texttt{14}\quad\texttt{35}\quad\texttt{24}\quad\texttt{36}\quad\texttt{15}\quad\texttt{26}\,. \tag{2.27}$$

Let $a(L)$ be the *arc index* of L, the minimal $\mu(D)$ over all grid diagrams D of L. It is the minimal number of pages among all arc presentations of a link L (e.g., see [MB, LT]).

Thus

$$a(K) \;=\; \min\{\,\mu(D)\,:\,D \text{ is a grid diagram of } K\,\}\,.$$

We say D is a *minimal* grid diagram of K if $\mu(D) = a(K)$.

There is a further correspondence: when a grid diagram of K is rotated by $\pi/4$, then it can be seen as a *Legendrian front diagram* of a Legendrian embedding of K. See [LvN, Ta, FT].

There is an invariant of Legendrian isotopy, called *Thurston-Bennequin invariant/number*. We define it here equivalently in terms of a grid diagram.

Definition 2.11 The *Thurston-Bennequin invariant* of a grid diagram D, written $TB(D)$, is defined as

$$TB(D) = -Z(D) + w(D)\,, \tag{2.28}$$

where $w(D)$ is the writhe of D (taken as a planar diagram of K) and $Z(D)$ the number of NW- or SE-corners of D.

For the following, see [Fe, Ma, Ta2].

Definition 2.12 The *maximal Thurston-Bennequin invariant* of a knot K, written $TB(K)$ is

$$TB(K) \;=\; \max\{\, TB(D) \,|\, D \text{ is a grid diagram of } K\,\}\,.$$

We note that the following was proved by Cromwell [Cr]. For two links L_1, L_2,

$$a(L_1\#L_2) = a(L_1) + a(L_2) - 2\,. \tag{2.29}$$

For knots L_i, it also follows from a relationship (2.42), derived by Dynnikov-Prasolov [DP], concerning the Thurston–Bennequin invariant (see end of §2.5 for further explanation), and the additivity of the invariant [EH, To].

2.9 Grid-band construction

The main topic of this sequel of works starts from the observation that a braided surface of Euler characteristic 0, which is a K-knotted annulus, is essentially a grid diagram of the underlying companion knot K.

Definition 2.13 Let for a knot K and integer t,

- $A(K,t)$ be the (link of the) t-framed K-knotted annulus,
- $W_+(K,t)$ and $W_-(K,t)$ the t-framed Whitehead doubles of K with positive and negative clasp, and
- $B(K,t)$ the t-framed Bing double of K.

We will usually abuse the distinction between the annulus and the link which is its boundary.

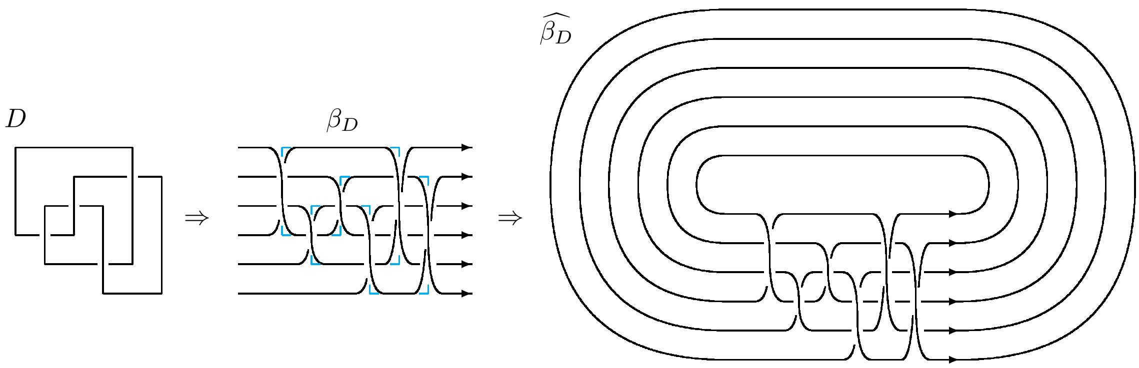
To disambiguate among different conventions for framing used elsewhere, we emphasize that t is here the linking number of the two components of $A(K,t)$. Thus, for example, $A(\bigcirc,1)$ is the positive (right-hand) Hopf link, and $A(\bigcirc,-1)$ the negative one. This definition of framing has the opposite sign to the one used by other authors (e.g., [DM]), where they take the writhe $w(D) = -t$ of a diagram D of K from which $A(K,t)$ is constructed as the blackboard-framed (reverse) 2-parallel.

Also, $W_+(\bigcirc,1)$ is the positive (right-hand) trefoil, and $W_+(\bigcirc,-1) = W_-(\bigcirc,1)$ the figure-8-knot. We can understand $W_+(K,t)$ resp. $W_-(K,t)$ as the result of plumbing a positive resp. negative Hopf band into $A(K,t)$ and taking the knot which is the boundary of the resulting Seifert surface. In a similar way, we can understand $B(K,t)$ as the 2-component link which is obtained by plumbing both a positive and a negative Hopf band into $A(K,t)$ and taking the boundary. Thus for instance $B(\bigcirc,0)$ is the 2-component unlink, and $B(\bigcirc,1)$ is the Whitehead link.

We next explain a construction, noted by Rudolph and later by Nutt, which will be referred to as *grid-band construction*.

Let D be a grid diagram of a knot K. Replacing each vertical segment with a half twisted band as shown below, we get a braid in band presentation, denoted by β_D. (Compare with [Ru3, Fig. 1] and [Nu, Theorem 3.1].) We refer to this process as the grid-band construction.

Then the closure $\widehat{\beta_D}$ bounds a twisted annulus. Therefore $\widehat{\beta_D} = A(K,t)$ for some t.



(2.30)

Definition 2.14 Also, we can use the band presentation of β_D to specify the grid diagram D itself (see Example 2.10). The mirroring of D is fixed by default by saying that β_D should be obtained when reading D from the left.

Reading the grid diagram D in (2.30) from the left gives the word (2.26) for β_D, where $[ij]$ now adopt the meaning of band generators $[ij] = \sigma_{i,j}$ of (2.2). Again, we will often use the bracket-less simplified notation (2.27). (The symbol $[ij]$ will thus be used for both a vertical segment, or a band generator, when it is evident from the context whether we work in a grid diagram or its band presentation.)

Since we deal with grids of size 10 or more, let us also already fix here that we use initial capital Latin letters `A,B,C,...` to denote two-digit integers $10, 11, 12, \dots$, so that for example, `4C` $= [4, 12] = \sigma_{4,12}$.

Consider the situation that the band presentation is positive. Then obviously $A(K,t)$ for the resulting framing t is strongly quasipositive. A question is what is the framing t, which we will write as

$$t = \lambda(D)\,, \tag{2.31}$$

in dependence of the diagram D, and how to read $\lambda(D)$ off D. This was explained in [JLS1], but was known to Rudolph [Ru3]:

$$\lambda(D) = -TB(D)\,. \tag{2.32}$$

2.10 Slice-torus invariants

We briefly recall Livingston's [Lv] formalism of "slice-torus invariant". An integer-valued invariant v on knots (i.e., not necessarily defined for multi-component links) is a *slice-torus invariant* if

- $v(K) = -v(!K)$, and $v(-K) = v(K)$, where $-K$ is K with the reverse orientation
- additivity under connected sum: $v(K_1 \# K_2) = v(K_1) + v(K_2)$
- crossing switch rule: $v(K_+) - v(K_-) \in \{0, 1\}$
- $v(K) \le g_4(K)$ (or equivalently $2v(K) \le 1 - \chi_4(K)$), and
- v satisfies Bennequin's inequality:
$$2v(K) \ge w - n + 1$$
for a braid representative of K on n strings and writhe w.

These properties are not minimal (i.e., some follow from special cases or combinations of others), but they are what we will use in §5. (We emphasize that it is not assumed v to be defined on multi-component links $\kappa(K) > 1$ in any way.)

There are two known instances of such an invariant, the Ozsváth-Szabó τ invariant, and (half of) Rasmussen's invariant s. Thus the concept was introduced mainly to give these two a uniform treatment.

The halved *signature* $\sigma/2$ satisfies the first four of the above five properties, but not the last. The same applies to *Tristram-Levine signatures* σ_ξ, that will enter later (via Gilmer's work) when we invoke Casson-Gordon invariants.

From the superposition of

$$2g(K) = 1 - \chi(K) \geq 1 - \chi_4(K) \geq 2v(K) \geq w - n + 1$$

with (2.5), it is straightforward that

$$\text{if } v(K) < g(K), \text{ then } K \text{ is not Bennequin-sharp}. \tag{2.33}$$

In relation, it follows that when a knot K is quasipositive, then

$$v(K) = g_4(K)\,, \tag{2.34}$$

and

$$\text{when } K \text{ is strongly so, then } v(K) = g_4(K) = g(K). \tag{2.35}$$

We will refer to these standard facts a few times below.

Turning to Whitehead doubles (see Definition 2.13 above), Ozsváth-Szabó defined a number $j_\tau(K)$, the *jump invariant* of τ, with

$$\tau(W_+(K,t)) = \left\{ \begin{array}{ll} 1 & t \geq j_\tau(K) \\ 0 & t < j_\tau(K) \end{array} \right. . \tag{2.36}$$

This jump number j_v can be similarly defined for any slice-torus invariant v [LvN]. Of course, when v is effectively computable, so is $j_v(K)$. But at least for $v = \tau$, there is a more closed expression. Hedden [He] has found that

$$j_\tau(K) = 1 - 2\tau(K)\,. \tag{2.37}$$

The picture for Rasmussen's invariant remains less clear.

This is further discussed in [JLS1], where we recovered the Livingston-Naik result [LvN]. (It also holds setting $\lambda(\bigcirc) = 1$.)

Proposition 2.15 For any slice-torus invariant v, and $K \neq \bigcirc$, we have

$$-\lambda(!K) < j_v(K) \leq \lambda(K)\,. \tag{2.38}$$

But we went further:

Corollary 2.16 ([JLS1]) Assume for a knot K there is a slice-torus invariant v so that the right inequality in (2.38) is sharp for K:

$$\lambda(K) = j_v(K)\,. \tag{2.39}$$

Specifically, for $v = \tau$, we can assume K satisfies

$$\lambda(K) = 1 - 2\tau(K)\,. \tag{2.40}$$

Then the Bennequin sharpness problem 2.5 is resolved for all $W_\pm(K,t)$.

This raises the question what knots satisfy (2.40).

Lemma 2.17 Every positive knot K satisfies (2.40).

However, we showed an example that strongly quasipositive knots in general do not.

A further series of instances satisfying (2.40), which will play a special role below, are slice knots K with (5.5). They can be suspected to be quasipositive (see Remark 5.4). But for (2.40) quasipositivity not necessary, as shows the below example.

Example 2.18 The knot $K = 12_{1628}$ has $\lambda = 1$ and $\tau = 0$ (see [LvM]), thus were it to be quasipositive, it would have $\tau = g_4 = 0$, so it would be slice. But this is easily ruled out from the Milnor-Fox condition; the determinant $\det(12_{1628}) = 17$ (see (2.22)) is not a square.

2.11 Strong quasipositivity of annuli and Whitehead doubles

Returning to $A(K,t)$ (recall Definition 2.13), set

$$\lambda(K) := \min\{\ t \mid A(K,t) \text{ is strongly quasipositive}\,\}\,.$$

The following was known to Rudolph, but its reproof in [JLS1] yields a fundamental insight into the new applications.

Theorem 2.19 (Rudolph; see [JLS1]) If $K \neq \bigcirc$ is non-trivial, then we have $\lambda(K) = -TB(K)$. Furthermore,

$$A(K,t) \text{ is strongly quasipositive} \iff t \geq -TB(K)\,. \tag{2.41}$$

This relationship originates from the grid-band construction in §2.9. Note that for the unknot $\lambda(\bigcirc) = 0$ but $-TB(\bigcirc) = 1$. We again prefer to use the symbol $\lambda(K)$ instead of $TB(K)$ in the sequel. (To obtain the statements about the maximal Thurston-Bennequin invariant, reverse signs. The unknot can always be handled *ad hoc*.)

Rudolph's work also easily implies the result on the Whitehead and Bing doubles (see Definition 2.13), which together with Theorem 2.19 is summarized thus.

Corollary 2.20 (Rudolph) Let K be a non-trivial knot. Then
(a) $W_+(K,t)$ is strongly quasipositive if and only if $t \geq \lambda(K)$, and
(b) $W_-(K,t)$ and $B(K,t)$ are never strongly quasipositive.

What is discussed in this paper is the quasipositivity of these links (with slightly less focus on the Bing doubles).

We should emphasize here the connection between $\lambda(K)$ and $a(K)$. When D is a grid diagram of a knot K, then one can regard the *mirror image* $!D$ as grid diagram of the mirrored knot $!K$, by changing all crossings and rotating by $\pm\pi/2$. With (2.28), set

$$\lambda(D) = -TB(D) = Z(D) - w(D)\,.$$

It is straightforward to observe

$$\lambda(D) + \lambda(!D) \;=\; \mu(D)\,.$$

Theorem 2.21 (Dynnikov-Prasolov [DP]) If D is a minimal grid diagram of $K (\neq \bigcirc)$, then $\lambda(D) = \lambda(K)$.

This yields the important "Matsuda-Dynnikov-Prasolov" relationship

$$\lambda(K) + \lambda(!K) = a(K)\,. \tag{2.42}$$

It had been conjectured previously [Ng], and Matsuda [Ma] had proved one of the inequalities,

$$a(K) \geq \lambda(K) + \lambda(!K)\,. \tag{2.43}$$

Here we reiterate our attitude towards Theorem 2.21. It is a very useful result. But it is developed in an approach different from what we create to work with. Simply invoking this result will often be limitedly impactful. Thus, while it subsumes pieces of certain statements we obtain along the way (e.g., Lemma 4.9), and we do not wish to pretend these are new, it is better avoided to be used frequently.

The analogy of (2.43) to the lower bounds of Morton-Beltrami [MB]

$$a(K) \geq \mathrm{MB}(K) := 2 + \operatorname{span}_a F(K)\,, \tag{2.44}$$

and the one for the maximal Thurston-Bennequin invariant [Fe, Ta, FT] (when $K \neq \bigcirc$)

$$\lambda(K) \geq \mathrm{FT}(K) := 1 - \min\deg_a F(K) \tag{2.45}$$

should be noted, that satisfy a similar relationship

$$\mathrm{MB}(K) = \mathrm{FT}(K) + \mathrm{FT}(!K)\,. \tag{2.46}$$

Tanaka proves for alternating knots [Ta] and positive knots K [Ta2] that (2.45) is an equality:

$$\lambda(K) = 1 - \min\deg_a F(K)\,. \tag{2.47}$$

3 The invariants ℓ and θ

3.1 Definition

We defined two invariants [JLS2], which are slightly rephrased here as follows.

Definition 3.1 ([JLS2]) Let K be a knot. Assume $t \in \mathbb{Z}$ is chosen so that

$$\min\deg_z P(A(K,t)) = 1 - 2m < 0, \qquad \max\deg_z P(A(K,t)) = 2n - 1 > 0, \tag{3.1}$$

for positive integers m, n. Then set

$$\ell(K) := \mathrm{MFW}(A(K,t)) = 1 + \tfrac{1}{2}\operatorname{span}_v P(A(K,t)) = m + n \tag{3.2}$$

and

$$\theta(K) := t + m\,. \tag{3.3}$$

Notice, by further remarks from §2.7, that for the 2-component link $A(K,t))$ the only monomials in $P(A(K,t))$ that occur are $z^p v^r$ with odd p, r.

Observe that therefore, $\min\deg_z P(A(K,t))$ and $\max\deg_z P(A(K,t))$ are always odd. Also

$$\min\deg_z P(A(K,t)) = -1\,,$$

and it is well known from skein theory [LiM] that

$$[P(A(K,t))]_{z^{-1}} = v^{2t}(v^{-1} - v)([P(K)]_{z^0})^2 \neq 0\,. \tag{3.4}$$

It is also easy to see from (3.4) that for every K there is a t with (3.1).

The problem of proving that ℓ is well-defined depends on very unlikely and not observable but potential and tricky cancellations. *In practical terms,* we often have

$$\ell(K) = \min_{t\in\mathbb{Z}} \mathrm{MFW}(A(K,t))\,. \tag{3.5}$$

We recall that t can be extracted from $P(A(K,t))$, and in fact in two independent ways. One is (3.4): this is possible because one can prove that for a knot K, when $[P(K)]_{z^0}$ is given up to units in $\mathbb{Z}[v,v^{-1}]$, then it is uniquely determined (see [Kn] and the proof of Theorem 5.35). But a much simpler way is

$$t = [P(1,z)]_z\,,$$

using that the Conway polynomial $\nabla(z) = P(1,z)$ contains the linking number. This was used in the much more complicated definition of $\theta(K)$ than (3.3), given in [JLS2], which does not assume (3.1). Note further the property

$$\theta(K) + \theta(!K) = \ell(K)\,, \tag{3.6}$$

in analogy to (2.42) and (2.45).

The definition (3.3) of θ fundamentally exploits the fact that when $A(K,t)$ is strongly quasipositive (and $K \neq \bigcirc$), then

$$\min\deg_v P(A(K,t)) > 0\,, \tag{3.7}$$

and that we have from the skein relation (2.11)

$$P(A(K,t)) = v^2 P(A(K,t-1)) + vz\,. \tag{3.8}$$

These invariants serve the following basic purpose.

Theorem 3.2 ([JLS2]) For every non-trivial knot K,

$$\ell(K) \le a(K) \quad \text{and} \quad \theta(K) \le \lambda(K)\,.$$

Remark 3.3 When K is an amphicheiral knot, $K = !K$, then $A(K,0)$ is an (orientedly) amphicheiral link. One can use this and (2.13) to conclude that in that case $\ell(K)$ is even (see also (3.6)). This is compatible with the fact that $a(K)$ is even through (2.42).

3.2 ℓ-sharp knots

Definition 3.4 ([JLS2]) A knot K is ℓ-sharp if $\ell(K) = a(K)$.

Example 3.5 ([JLS2]) Among the Rolfsen [Ro, Appendix] knots, $K = 10_{132}$ is the only one which is not ℓ-sharp. There are four ℓ-unsharp 11 crossing knots K (up to mirror images), i.e., such with

$$a(K) > \ell(K)\,, \tag{3.9}$$

and 21 further examples of 12 crossings.

Besides the torus knots (see [MS+]), one major motivation of dealing with this condition is:

Proposition 3.6 Every alternating knot K is ℓ-sharp, i.e., $\ell(K) = a(K) = c(K) + 2$.

The proof uses Rudolph's congruence [Ru5]. This is a powerful engine that will be invoked later substantially, and to provide ideas how to work with it, we repeat the proof here with some minor omissions.

Proof of Proposition 3.6. We follow the work of Diao and Morton [DM]. We stipulate *within this proof* that numeration of *theorems* refers to [DM], while propositions and equations to the present paper. In the case of notation, the default will be ours here, unless we specify otherwise. We also write

$$K_t = A(K,t)\,.$$

Beside the known fact

$$a(K) = c(K) + 2\,,$$

two main ingredients are needed. The first is Rudolph's congruence [Ru5] (Theorem 2.2), which relates $P(K_t)$ modulo 2 to $F(K)$. For the purpose of applying (2.16), write $\mathrm{MFW}_{\mathrm{mod}\ 2}(L)$ for the bound obtained from $\mathrm{MFW}(L)$ when the polynomial $P(L)$ has its coefficients reduced modulo 2, so that

$$\mathrm{MFW}_{\mathrm{mod}\ 2}(L) \le \mathrm{MFW}(L) \le b(L)\,.$$

(This notation is *not* to be confused with (2.18); we are not working with truncations here.)

Thistlethwaite has extended his proof of

$$\mathrm{span}_{\,a} F(K) = c(K) \tag{3.10}$$

in [Th], and the second ingredient is a refinement of Thistlethwaite's work, in the case of alternating links, due to Cromwell [Cr2] (Theorem 2.3). It exhibits (3.10) through explicit *odd* coefficients

$$[F(K)]_{z^{k_1}a^{l_1}} = [F(K)]_{z^{k_2}a^{l_2}} = 1 \quad \text{with} \quad l_2 - l_1 = c(K). \tag{3.11}$$

(Alternatively, see [Th, Corollary 1.1(iv)].)

When D is a reduced alternating diagram of K, and D is not a $(2,n)$-torus link diagram (we may assume n odd), then $k_1, k_2 > 1$. Via Theorem 2.2, this immediately implies that $\mathrm{MFW}(K_t) \ge c(K) + 2$ for all t. Because of (3.2) and Theorem 3.2, this shows our claim.

However, if D is a $(2,n)$-torus link diagram, then one of k_1, k_2 in (3.11) is 1, and there is exactly one framing t for which $\mathrm{MFW}_{\mathrm{mod}\ 2}(K_t) < c(K) + 2$. Diao and Morton then engage in a tricky calculation to show that still $\mathrm{MFW}(K_t) = c(K) + 2$ (see text preceding Proposition 3.7).

But we use (3.1). With (3.11), one can obviously choose a framing (with the notation of [DM]) f so that the r.h.s. of the congruence in Theorem 2.2, call it

$$R_f(v,z) = v^{2f}\left(-1 + \frac{v^{-1}+v}{z}\right) F(K)(v^2, z^2)$$

(keep in mind our opposite convention of both framing f and a-powers in F), has

$$\min \deg_v (R_f)_{\ \mathrm{mod}\ 2}(v,z) < 0 < \max \deg_v (R_f)_{\ \mathrm{mod}\ 2}(v,z)\,. \tag{3.12}$$

The congruence then implies (with our notation, but f retaining its role) that

$$\min \deg_v P(K_f) < 0 < \max \deg_v P(K_f)\,, \tag{3.13}$$

which with (3.10), (3.11) and (3.1) shows $\ell(K) \ge c(K) + 2$, as needed. □

We exhibited also, through skein algebra programming, the full panhandle for $(2,n)$-torus knots (of which Diao and Morton identified the "tip", and which later was generalized to arbitrary torus knots in [MS+]; see §5.3).

Proposition 3.7 For the $(2,n)$-torus knot $T_{2,n}$, the polynomial $P((T_{2,n})_{-n})$ is of the shape

$$P((T_{2,n})_{-n}) = \tilde{P}_n - zv^5\frac{v^{2n-4}-1}{v^2-1}\,, \tag{3.14}$$

where

$$-3 = \min\deg_v \tilde{P}_n \le \max\deg_v \tilde{P}_n = 3\,. \tag{3.15}$$

We have further the following "Matsuda-Dynnikov-Prasolov" type of relationship. This is implied by Theorem 2.21, but is needed for its extension to quasipositivity (Proposition 5.2). It uses Lemma 4.5.

Proposition 3.8 With the notation of §2.2 for mirror image,

$$\ell(K) \le \lambda(K) + \lambda(!K) \le 2a(K) - \ell(K)\,. \tag{3.16}$$

The property of being ℓ-sharp is useful because it allows to settle, among others, braid index questions. The following result yields much simpler formulations for alternating knots K, but we state (as we need later) its full form here.

Corollary 3.9 ([JLS2]) Assume a knot K is ℓ-sharp.

1. Assume $L = A(K,t)$ for some t. Then L has a minimal string Bennequin surface. Also, if L is strongly quasipositive, then L has a minimal string strongly quasipositive band presentation.

2. Let t' be so that
$$\max\deg_v P(A(K,t')) > 0$$
and assume
$$\max\mathrm{cf}_{\,v} P(A(K,t')) \ne \pm z^{-1}\,. \tag{3.17}$$
Assume $L = W_+(K,t)$ for some t. Then L has a minimal string Bennequin surface. Also, if L is strongly quasipositive, then, *without (3.17)*, L has a minimal string strongly quasipositive band presentation.

3. Let t' be so that
$$\min\deg_v P(A(K,t')) < 0$$
and assume
$$\min\mathrm{cf}_{\,v} P(A(K,t')) \ne \pm z^{-1}\,. \tag{3.18}$$
Assume $L = W_-(K,t)$ for some t. Then L has a minimal string Bennequin surface.

One can write down explicit formulas of braid indices. The one for $b(A(K,t))$ is simplest, and is an analogue of what Diao and Morton [DM] proved for the alternating knots. This was formulated in the below more general result. (It can also be restated in terms of $\lambda(!K)$ symbols using (2.42).)

Proposition 3.10 ([JLS2]) Assume K is a non-trivial ℓ-sharp knot. Then

$$b(A(K,t)) = \begin{cases} \lambda(K) - t & \text{if } t \le \lambda(K) - a(K)\,, \\ a(K) & \text{if } \lambda(K) - a(K) \le t \le \lambda(K)\,, \\ t - \lambda(K) + a(K) & \text{if } t \ge \lambda(K)\,. \end{cases} \tag{3.19}$$

3.3 Improvement over the classical bounds

We also discussed at length evidence that our new invariants perform better than their classical predecessors.

Question 3.11 Is
$$\mathrm{MB}(K) \leq \ell(K) \tag{3.20}$$
for all non-trivial knots K?

The argument based on Rudolph's congruence implies that the answer to Question 3.11 is affirmative if in the definition $\mathrm{MB}(K)$ in (2.44), coefficients of F are reduced modulo 2 (the waker form of inequality (2.44) previously discovered by Nutt).

We then engaged in extensive other invocations of Rudolph's congruence, which are summarized without repeating proofs. The argument for Proposition 3.6 also yields a more generalized version, which is a very practical way to test Question 3.11.

Corollary 3.12 Assume $F(K)$ has odd coefficients $[F(K)]_{a^{l_i}z^{m_i}}$, $i = 1, 2$, so that $l_1 = \min\deg_a F$ and $l_2 = \max\deg_a F$. Then (3.20) holds for K.

Computation 3.13 The inequality (3.20) holds for all prime knots K up to 16 crossings. Among knots up to 10 crossings, only 9_{46}, 10_{142}, and 10_{160} fail the premise of Corollary 3.12. This suggests that it filters potential counterexamples rather well. We explained that we settled the remaining knots by computing truncations of $P(K_t)$ (see §2.7), to estimate $\ell(K)$ from below.

We then treated an extended version of Question 3.11: is
$$\theta(K) \geq \mathrm{FT}(K) = 1 - \min\deg_a F(K)\,? \tag{3.21}$$

The following is the refinement of Corollary 3.12.

Proposition 3.14 Assume the premise of Corollary 3.12 is satisfied for $i = 1$. Then (3.21) holds for K.

Computation 3.15 The inequality (3.21), is true for prime knots K up to 16 crossings, as was verified by refining and extending Computation 3.13. (Here we need to treat mirror images separately.)

It is immediate, using (2.42), that a knot K if ℓ-sharp if and only if
$$\lambda(K) = \theta(K)$$
holds for K and $!K$. Thus (3.21) must hold for alternating knots K, and as
$$\mathrm{MB}(K) = \ell(K) = a(K)\,,$$
(3.21) must hold as an equality for such K. There is one further noteworthy special case to add besides alternating knots, that accords with [Ta2].

Corollary 3.16 For every positive knot K, the inequality (3.21) holds as an equality.

The abridged proof is as follows.

Proof. By Yokota's result [Yo], $\min\deg_a F(K) = \min\deg_v P(K) = 2g(K)$, and up to variable change, the minimal terms coincide. And by (2.14), the term
$$[P(K)]_{v^{2g(K)}z^{2g(K)}} = \pm 1 \tag{3.22}$$
is odd. This shows (3.21).

We recall that by the argument of Tanaka [Ta2] (see the proof of Lemma 2.17 above, as given in [JLS1]), for a positive knot K, we have (2.47), and thus (3.21) is an equality. $\square$

4 Braid indices and framing diagrams

4.1 Braid indices

We discuss some relation regarding the braid indices in Definition 2.6. (Remember the comparison with [Nu, Section 3.3] and [Ru3, Fig. 1].) As noticed, Bennequin's inequality (2.5) implies that a strongly quasipositive surface is a Bennequin surface, thus for K strongly quasipositive, we have (2.8). We know that $b_b(K) > b(K)$ is possible [HS], but the examples K known are not strongly quasipositive. Rudolph conjectures that

$$b_{sqp}(K) = b(K) \tag{4.1}$$

when K is strongly quasipositive, and this is true, among other families, if K is a prime knot of up to 16 crossings (see [St2]). By the proof of the Jones-Kawamuro conjecture (Theorem 2.7), a Bennequin surface of a strongly quasipositive link K on $b(K)$ strands is always strongly quasipositive, so that

$$b_b(K) = b(K) \tag{4.2}$$

implies (4.1) for strongly quasipositive knots K. The problem (4.2) is extensively studied in [St2].

Since a band presentation of $A(K,t)$ always comes from a grid diagram of K, and with a confirmative notice about the unknot, we have:

Corollary 4.1

$$\min\{b_b(A(K,t))\,:\,t\in\mathbb{Z}\} = a(K)\,. \tag{4.3}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the minimum.

Also, because choosing positive bands will give a band presentation of a strongly quasipositive annulus, we have with Theorem 2.19:

Corollary 4.2 $\min\{b_{sqp}(A(K,t))\,:\,t\geq\lambda(K)\} = a(K)\,.$ □

Forgetting the surface structure then yields an inequality of (ordinary) braid indices:

Corollary 4.3

$$\min\{b(A(K,t))\,:\,t\in\mathbb{Z}\}\leq\min\{b(A(K,t))\,:\,t\geq\lambda(K)\}\leq a(K) \tag{4.4}$$

Moreover, there are at least $a(K)+1$ consecutive integers t which realize the inequality $b(A(K,t))\leq a(K)$. □

We then went to discuss what (say, strongly quasipositive) framings λ are possible for given grid size μ, and in particular whether λ_{min}, the framing for a minimal (size $a(K)$) diagram (see Definition 4.4) is unique. This discussion and results appear in [JLS1, JLS2] and the section is an extraction of necessary material therefrom.

Since μ bounds the braid index of $A(K,t)$, and all have the same χ, Birman-Menasco [BM] imply that for given λ, only finitely many μ are possible. We will later give more precise statements (Finite-Cone-Theorem; see Propositions 4.16 and 4.17).

Definition 4.4 (see [JLS2]) For a knot K, define a number $\lambda_{min}(K)$ by the property that $A(K,\lambda_{min}(K))$ has a positive band representation on $a(K)$ strands.

We will use $\lambda_{min}(K)$ often in the following. Two caveats are in order regarding this notation. First, the 'min' refers to the minimum with respect to number of strings of the surface $A(K,t)$ (or horizontal segments in the grid diagram of K), *not* the framing t itself. And second, it is *not* assumed that λ_{min} is unique. At least for the unknot K,

$$\text{both } b(A(\bigcirc,0)) = b(A(\bigcirc,1)) = 2, \text{ thus } \lambda_{min}(\bigcirc) = 0,1. \tag{4.5}$$

This special behavior of unknot will require repeated attention.

The following is a direct consequence of (2.15) and the definition of $\lambda_{min}(K)$.

Lemma 4.5 For every knot K, there exists a strongly quasipositive framing $t = \lambda_{min}(K) \geq \lambda(K)$ of $A(K,t)$, so that

$$\min\deg_v P(A(K,t)) \geq 1\,, \quad \max\deg_v P(A(K,t)) \leq 2a(K) - 1\,. \tag{4.6}$$

For a non-trivial knot K, the uniqueness and minimality of $\lambda_{min}(K)$ was settled, as was discussed above; see Theorem 2.21. But we do not wish to exclude $K = \bigcirc$ consistently. We prefer to maintain the symbol $\lambda_{min}(K)$, stipulating that formulas involving $\lambda_{min}(K)$ are meant to hold whatever of either values (4.5) is chosen for $K = \bigcirc$. For $K \neq \bigcirc$, the reader may assume that

$$\lambda_{min}(K) = \lambda(K)\,. \tag{4.7}$$

Question 4.6 (a) Is $b(A(K,t)) \geq a(K)$ for any t?
(b) At least, is $b(A(K,t)) \geq a(K)$ for any strongly quasipositive $A(K,t)$?

If (b) fails, then it would give an example $A(K,t)$ answering negatively Rudolph's question 4.1. This question will be further treated in Remark 4.8, which provides evidence for its conjectured resolution.

Conjecture 4.7 For every knot K,

$$a(K) = \min_{t\in\mathbb{Z}} b(A(K,t)) \tag{4.8}$$

Remark 4.8 Using a special computation for $K = 10_{132}$, and the verification of (3.5) and $\ell(K) = a(K)$ (see Example 3.5) for all other prime knots K up to 10 crossings, we can conclude that the answer to (both parts of) Question 4.6 is affirmative for all these 249 knots.

One of the motivations for this conjecture was originally the following, which is subsumed by Theorem 2.21, but we need details from our entirely different technique.

Lemma 4.9 Assume (4.8) is true. Then (4.7) holds, in particular $\lambda_{min}(K)$ is unique.

4.2 Framing diagrams

We also specify a region which played an important role throughout [JLS2] and will be considered also in this paper.

Definition 4.10 We define the *framing diagram* $\Phi(K)$ of K as a subset of $\mathbb{R}^2$ by

$$\Phi(K) := \{\, (\mu,t) \,:\, A(K,t) \text{ has a strongly quasipositive band representation on } \mu \text{ strands} \,\}\,.$$

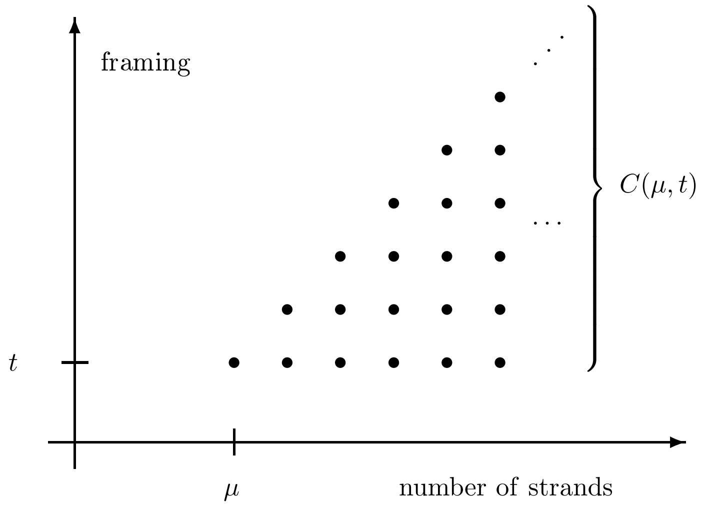
To formalize this topic better, we introduce notation relating to the two grid stabilizations[1]

(4.9)

Definition 4.11 We define the *cone* $C(\mu, t) \subset \mathbb{Z}_+ \times \mathbb{Z}$ by

$$C(\mu, t) = \{\ (s, \lambda) :\ s \geq \mu,\ t \leq \lambda \leq t + s - \mu\ \}.$$

We say (μ, t) is the *tip* of the cone.



The positive and negative grid stabilization (4.9) easily imply that $\Phi(K)$ is a union of cones.

Note that undoing the removal of disks connected by one band in a braided surface is, up to conjugacy, *positive braid stabilization*. The reverse of this operation, *destabilization*, also shows that

$$(\mu, t) \in \Phi(K) \Longrightarrow (\mu + 1, t) \in \Phi(K), \tag{4.10}$$

which equals the effect of the *negative grid* stabilization on the right of (4.9).

We studied the question of what is the minimal number of such cones needed.

Example 4.12 According to (4.5), we have

$$\Phi(\bigcirc) = C(2, 0) \cup C(2, 1)$$

being the union of two cones.

Question 4.13 If K is non-trivial, is $\Phi(K)$ a single cone? (This cone would have to be then $C(a(K), \lambda_{min}(K))$ with $\lambda_{min}(K) = \lambda(K)$.)

We proved [JLS2] the following extension of Lemma 4.9, which is not covered in Theorem 2.21.

[1] The edge labels correspond to the weights in the weight model of [JLS1].

Lemma 4.14 Conjecture 4.7 implies a positive answer to Question 4.13, that $\Phi(K)$ is a single cone

$$\Phi(K) = C(a(K), \lambda(K))\,.$$

A more general formulation of this causality is as follows.

Definition 4.15 Define the *defect* of K by

$$\delta(K) = a(K) - \min_{t\in\mathbb{Z}} b(A(K,t))$$

Then the argument for Lemma 4.14 modifies to show that an $a(K)$-band positive band presentation of $A(K,t)$ gives

$$\lambda(K) \le t \le \lambda(K) + \delta(K)\,, \tag{4.11}$$

and any positive band presentation of $A(K,t)$ on $s = a(K) + k$ strings will have

$$\lambda(K) \le t \le \lambda(K) + \delta(K) + k = \lambda(K) + \delta(K) + s - a(K)\,. \tag{4.12}$$

From this, we can conclude the following.

Proposition 4.16 For a non-trivial knot K, we have that $\Phi(K)$ is the union of at most $1 + \delta(K)$ cones.

Note that for $K = \bigcirc$, we have $\delta(K) = 0$, so that the claim is false due to the circumstance (4.5). (But, again, this case can be worked out separately: see Example 4.12.) In Remark 4.8 we have verified that $\delta(K) = 0$ for all prime knots K up to 10 crossings.

Obviously, from the definition,

$$\delta(K) \le a(K) - 2b(K)\,.$$

Thus in particular from (4.12), we have

$$\lambda(K) \le t \le \lambda(K) + s - 2b(K)$$

for any positive band presentation of $A(K,t)$ on $s \ge a(K)$ strings. Note also that, for computational purposes, one may replace '$1 + \delta(K)$' in Proposition 4.16 by '$1 + a(K) - \ell(K)$', with an analogous proof argument. (An analogous caveat regarding $K = \bigcirc$ is needed, where $a(K) = \ell(K) = 2$.) We thus obtain the following proposition.

Proposition 4.17 When K is a non-trivial knot, then $\Phi(K)$ is the union of at most $1 + a(K) - \ell(K)$ cones. $\square$

5 Quasipositivity

We have decided to strictly focus on strong quasipositivity of $W_\pm(K,t)$ and $A(K,t)$ in [JLS1, JLS2], essentially because of the direct relationship in Corollary 2.20 and its extensive consequences treated.

The problem of the *quasipositivity* of these links is far more obscure, but perhaps also very interesting. We put in advance an informal summary of what we know. The four families in Definition 2.13 show three kinds of behavior.

- *Some* $A(K,t)$ and $W_+(K,t)$ are strongly quasipositive (we know which), and *some more* are quasipositive. We prove restrictions on these occasional examples (though we cannot gain a complete description; *cf.* introduction).
- *No* $W_-(K,t)$ are strongly quasipositive and *apparently none* are quasipositive. (We exclude at least major cases.)
- *No* $B(K,t)$ are strongly quasipositive but *some* are quasipositive (see Example 5.12). We do not dwell upon this family here much (already for length reasons), but it may be worth returning to at a separate place.

5.1 Knotted annuli

Let us define

$$\lambda_q(K) := \min\{\ t :\ A(K,t) \text{ is quasipositive }\}\,. \tag{5.1}$$

Then we know that

$$\lambda_q(K) \le \lambda(K) = -TB(K)\,,$$

so the obvious question is: is $\lambda_q(K) = \lambda(K)$ for all K? Or, given (2.41), in other words: for every t, if is $A(K,t)$ quasipositive, is it strongly quasipositive? The answer is "no". We identify one major reason, which we call λ-sliceness (Definition 5.3), but it is possible what further, more peculiar, phenomena occur.

Question 5.1 For what knots is $\lambda_q(K) = \lambda(K)$?

Going further, we do not know if always

$$A(K,t) \text{ is quasipositive} \implies A(K,t+1) \text{ is quasipositive}\,. \tag{5.2}$$

That is, between the pans in [JLS2, (5.59)] (proof of Proposition 5.33 therein), the quasipositivity of the links can switch on and off a few times before, for long enough panhandle length, eventually strong quasipositivity settles in. This looks like an adventurous scenario, but see Corollary 5.10.

Since the (exact) connection of λ_q to the Thurston-Bennequin invariant remains elusive, so become certain methods focussing on the latter (like Theorem 2.21). But there are properties of λ_q that follow from our work above.

Many of the arguments based on the HOMFLY-PT polynomial do go through. In general, though, the preceding problems with the unknot extend here to K being slice. This changes (weakens) the inequalities a little, but to be precise, we either have to exclude sliceness, or work in cases.

Proposition 3.8 can be changed into the following form.

Proposition 5.2 We have

$$\begin{cases} \ell(K) \le \lambda_q(K) + \lambda_q(!K) \le 2a(K) - \ell(K) & \text{if } K \text{ is not slice} \\ \ell(K) - 1 \le \lambda_q(K) + \lambda_q(!K) \le 2a(K) - \ell(K) + 1 & \text{if } K \text{ is slice} \end{cases} \tag{5.3}$$

Proof. We only explain how to minorly modify the proof Proposition 3.8, given in [JLS2]; we cannot repeat this full proof here.

Lemma 4.5 holds regardless of K being slice or not. But (3.7) still holds for quasipositive $A(K,t)$ only if K is not slice, or $t \ne 0$. Thus, when K is slice, and $t = 0$, then we must allow, instead of (3.7), for

$$\min \deg_v P(A(K,0)) = -1\,, \tag{5.4}$$

since $\chi_4(A(K,0)) = -2$ can occur (see end of §2.2). Then the argument goes through, but the numerics changes slightly. □

The right inequalities in (5.3) are, as explained, not very interesting now (they follow from Matsuda's result (2.43)), but the left ones have some useful implications. To better formulate them, here we define a condition which will repeatedly play some role.

Definition 5.3 Call a knot K to be λ-*slice* if K is slice and

$$\lambda(K) = 1\,. \tag{5.5}$$

We say K to be λ-slice up to mirroring if one of K and $!K$ is λ-slice.

Remark 5.4 The condition (5.5) and Rudolph's version of (2.6) (see the proof of Theorem 1.5 in [He2]) then also imply that a λ-slice knot K is slice-Bennequin-sharp. The presumption (2.7) then possibly suggests that a λ-slice knot is quasipositive. (Example 2.18 shows that the τ invariant is insufficient to see this, though.)

Example 5.5 Among the Rolfsen knots, λ-slice (up to mirroring) are $K = 9_{46}$ and 10_{140}. As Remark 5.4 suggests, they are indeed quasipositive, and we fix here their quasipositive mirroring, which also satisfies

$$\min\deg_v P(K) = 0\,.$$

Corollary 5.6 Assume K is ℓ-sharp. Then either $\lambda_q(K) = \lambda(K)$, or K is λ-slice and

$$0 = \lambda_q(K) < \lambda(K) = 1\,. \tag{5.6}$$

Proof. Assume K is not slice. Then, (5.3) gives

$$\ell(K) \le \lambda_q(K) + \lambda_q(!K) \le \lambda(K) + \lambda(!K) \le 2a(K) - \ell(K)\,,$$

and $\ell(K) = a(K)$ implies that all inequalities are exact.

If K is slice, notice that K being ℓ-sharp makes (5.4) relevant only if $t = 0 = \lambda(K) - 1$, as can be seen thus.

Deal with $K = \bigcirc$ by a direct check. Then take a minimal grid diagram D of K. Because of ℓ-sharpness (or Theorem 2.21), we know that $\lambda(D) = \lambda_{min}(K) = \lambda(K)$ is unique. Obviously $t \ge \lambda_{min}(K)$ give strongly quasipositive $A(K,t)$, and are not interesting. So assume $t < \lambda_{min}(K)$. Then, $\ell(K) = a(K)$ also shows iteratively that

$$\min\deg_v P(A(K, \lambda(K) - k)) = 1 - 2k$$

for $k > 0$. Then $A(K, \lambda(K) - k)$ can be quasipositive (and (5.4) is relevant) only if $k = 1$. And then we need $\chi_4(A(K, \lambda(K) - 1)) = -2$, which requires $t = 0 = \lambda(K) - 1$, and means the condition (5.5). □

Corollary 5.6 lends further impetus to the study of $\ell(K)$. Among others, it and Proposition 3.6 show that for alternating links, Question 5.1 can be resolved using the present approach. This argument will be crucially used later also for the Whitehead doubles.

Lemma 5.7 If K is non-trivial and alternating, then K is not λ-slice, and hence $\lambda_q(K) = \lambda(K)$.

Proof. Assume K is non-trivial alternating and λ-slice. Because of sliceness $\sigma(K) = 0$, and because of (2.47), we have $\min\deg_a F(K) = 1 - \lambda(K) = 0$. Thus $\sigma(K) = \min\deg_a F(K)$. By the equality part of [St5, Proposition 4], we have that K is positive. But $\sigma(K) = 0$ implies then that K must be trivial, which we excluded. □

Remark 5.8 Notice that a λ-slice amphicheiral knot is trivial by (2.42). Thus in particular Corollary 5.6 holds for ℓ-sharp amphicheiral knots K. But for them (2.42) is not needed: Assume K is amphicheiral. Then we have from (5.3)

$$\ell(K) - 1 \le 2\lambda_q(K) \le 2\lambda(K) \le 2a(K) - \ell(K) + 1 \;=\; \ell(K) + 1\,,$$

but $\ell(K)$ is even by Remark 3.3. This is enough to see that the middle inequality is an equality.

The slice case continues to require special attention, as witnessed by the following explanation, which shows in more detail how to handle individual examples.

Proposition 5.9 If K is a prime knot of up to 10 crossings, then $\lambda_q(K) = \lambda(K)$, except (5.6) for $K = 9_{46}$ and $K = 10_{140}$.

The condition (5.5) also determines the mirroring of the knots, as fixed in Example 5.5.

Proof. First consider the case $\ell(K) < a(K)$. This applies only to $K = 10_{132}$ and $K = !10_{132}$.

Let $K = 10_{132}$. Then $\chi_4(A(K,t)) = 0$ for all t, and we computed that $\min\deg_v P(A(K,t)) > 0$ for $t \geq 0$. When $t \geq \lambda(K) = 1$, then $A(K,t)$ is already strongly quasipositive, so consider only $t = 0$. But we computed that $b(A(K,0)) = 8$, and the minimal writhe of a braid representative of $A(K,0)$ is 6. If $A(K,0)$ were to be quasipositive, then $\chi_4(A(K,0)) = -2$, which is not the case (as K is not slice). This finishes off $K = 10_{132}$.

Next let $K = !10_{132}$. Then again $\chi_4(A(K,t)) = 0$ for all t, and we computed that $\min\deg_v P(A(K,t)) > 0$ for $t \geq 8 = \lambda(K)$. But then again $A(K,t)$ is strongly quasipositive, so this case is done either.

Now let $a(K) = \ell(K)$. We need to consider only K being λ-slice (and K is non-trivial), so that only $K = 9_{46}$ and $K = 10_{140}$ remain (with the mirroring in Example 5.5). Their treatment continues in the argument below. □

This leads then to the following cautionary tale.

Corollary 5.10 For $K = 9_{46}$ and $K = 10_{140}$, we have (5.6).

Proof. We checked that, if $K = 9_{46}$ and 10_{140}, then

$$\min\deg_v P(A(K,0)) = -1\,,$$

which indeed leaves the opportunity that $L = A(K,0)$ is quasipositive. And it is easy to write down (minimal) braid representatives of L: use the grid diagrams given [J+], and make exactly one band negative (compare with Example 3.7 and Table 1 of [JLS2]).

Testing the quasipositivity of these braids is very difficult. But they satisfy, in (2.6), $w - n = -2 = \chi_4(L)$. Thus L are slice-Bennequin-sharp.

In attempting a solution, I consulted Stepan Orevkov. He indeed found the following (everything but self-evident) quasipositive form (2.3) for one of the braids for 9_{46} given below[2], where x^y stands for $y^{-1}xy$.

```
[36] [58] [27] [16] [48] -[37] [25] [14] =
[3 4 5 -4 -3   5 6 7 -6 -5   2 3 4 5 6 -5 -4 -3 -2
  1 2 3 4 5 -4 -3 -2 -1   4 5 6 7 -6 -5 -4   3 4 5 -6 -5 -4 -3
   2 3 4 -3 -2   1 2 3 -2 -1] =
4^[3 -4 3 2 -3 4 3 -4 3 2 -3 4 3 -4 3 2 1 -3 4 3 2 -3 -4
-5 4 3 -4 3 6 5 4 7 6 7 6 5 6 7]*
3^[2 -3 4 3 -4 3 2 -3 4 3 -4 3 2 -3 4 3 -4 3 2 1 -3 4 3
2 -3 -4 -5 4 3 -4 3 6 5 4 7 6 7 6 5 6 7]*
4^[3 3 5 5 4 6 5 6 7]*   2^[3]*   1^[2 3 -4]*  6
```

(He also argued that the other 7 braids obtained by making some of the other bands negative are quasi-positive as well.) This proves that $A(9_{46}, 0)$ is quasipositive, and $\lambda_q(9_{46}) = 0$. Orevkov also found quasipositive presentations for some of the similarly constructed 9-braids for $A(10_{140}, 0)$. □

Remark 5.11 By doubling a positive band (this preserves quasipositivity), we can also obtain a knot $L = W_+(9_{46}, 0)$ which is quasipositive, but not strongly so. A similar argument applies for $W_+(10_{140}, 0)$. As far as twisted positive clasped Whitehead doubles are concerned, we do not know whether the slicing construction (5.11) is exhaustive, but no further examples seem so far to be known. This, together with the proof of Theorem 5.22, will make clear that finding $W_+(K,t)$ for $t \neq 0$ which are quasipositive but not strongly so will be very hard, especially for ℓ-sharp knots K.

[2] Here the proper mirroring of 9_{46} is needed, and we chose to read the grid diagram in [J+] from the *bottom*, which gives this mirroring; *cf.* Example 2.10.

Example 5.12 By doubling both *one* positive band and *the* negative band, one has a band presentation of $B(K,0)$ (in Definition 2.13). Orevkov also found a quasipositive form for some of the thus obtained braids for $B(9_{46},0)$, for instance

```
[36] [58]^2 [27] (-[16])^2 [48] [37] [25] [14]=
3^[2 4 4 3 2 -4 -5 4 3 4 6 4 3 7 5 4 7 6 5 7 6]*
3^[2 4 4 3 2 1 -4 -5 4 3 1 6 5 4 7 6 5 7 6 5 7 7]*
4^[3 3 5 4 6 5 6]* 4^[3 3 4 5 6 -7]* 2^[1 3]* 2^[3 -4]
```

He also proved that the below braid for $B(10_{140},0)$ is quasipositive,

$$[47]^2[69][28](-[17])^2[59][48][36][25][13]\,.$$

(It can then be argued that all other braids obtained in a similar fashion from the brand presentations of $A(K,0)$ are quasipositive as well.) Thus $B(9_{46},0)$ and $B(10_{140},0)$ are quasipositive (while not strongly so; see Corollary 2.20).

The treatment of 10_{132} for Proposition 5.9 exemplifies why non-slice K are far easier to deal with. The special role of slice K is also underscored by the following fact, which can be proved similarly to Lemma 4.9, stated above, in [JLS2]. We do not like to repeat the previous proof here; one mainly has to replace χ by χ_4 in the relationship

$$\chi(A(K,t)) = \chi(A(K,t-1))\,, \tag{5.7}$$

that occurred in that proof. It is a more theoretical (and less practical) version of Corollary 5.6.

Proposition 5.13 If Conjecture 4.7 is true for K, then $\lambda_q(K) = \lambda(K)$, except if K is λ-slice. □

Note also that $A(K,t)$ for $K = 9_{46}$ (and other knots for which the condition (5.5) was relevant) were considered in the paper [Tr]. For a further occurrence of λ-slice knots, see Computation 5.37.

Remark 5.14 Before we leave this subsection, a relevant remark is in place that we cannot pursue further in the sequel. While obviously the failure $\lambda(K) \neq \lambda_q(K)$ appears related to λ-sliceness, we do not understand how close the connection is. In one direction, it would be useful to find systematic ways of exhibiting (beyond skillful computations, which offer little insight) quasipositive braid representations of $A(K,0)$ for λ-slice knots K. Starting with 9_{46} and 10_{140}, the family of pretzel knots (5.46) appears as suggestive. (As we will see, this family will also notoriously occur in Computation 5.37.) We may consider such question separately, but any outcome would anyway be decisively too elaborate to fit the volume of the present treatise...

5.2 Whitehead doubles

The quasipositivity problem of Whitehead doubles seems not to have been treated much in the literature. Mirroring is very relevant for quasipositivity, and the sign of the clasp plays a crucial role. Most negative clasped Whitehead doubles are still provably not quasipositive (see for example Computation 5.37). But the situation for positive clasped Whitehead doubles seems far less uniform.

5.2.1 Positively clasped ones

The only source I know where quasipositivity of Whitehead doubles was considered is the following.

Example 5.15 ([He, Examples, (5)]) The positive clasped Whitehead double $W_+(K,t)$ is not quasipositive when

$$t \le -\lambda(!K)\,, \tag{5.8}$$

and t is *not* of the form

$$t = -p(p-1)\,, \quad p \in \mathbb{N}\,. \tag{5.9}$$

The first condition (5.8) ascertains in Proposition 2.15 that $v(W_+(K,t)) = 0$. Then excluding the values (5.9) ensures that the determinant

$$\det(W_+(K,t)) = |1-4t| \tag{5.10}$$

is not a square, hence $W_+(K,t)$ is not slice by the Milnor-Fox property (compare with Example 2.18). Then $W_+(K,t)$ is neither quasipositive by (2.34).

The complication that $W_\pm(K,t)$ (while always having genus 1) can turn slice (i.e., 4-ball genus 0) becomes more subtle for Whitehead doubles. We only briefly digress on some of the history of this problem. (I am grateful to Chuck Livingston for some remarks. Under mirroring, let us fix to address only W_+.)

It is clear when $\chi_4(A(K,t)) = -2$, namely when $t = 0$ and K is slice, and then always $\chi_4(W_+(K,t)) = -1$, but these are not all: $W_+(\bigcirc,-2) = 6_1$, the stevedore knot, is slice as well. In fact, a slicing construction is known for $t \ne 0$ that also applies for some non-slice K. This was referenced in Remark 5.11, and is related to the proof of Theorem 5.22 (the curve α there represents a slice knot). Thus it is mentioned here (T stands again for a torus knot):

$$\text{when } K \text{ is concordant to } !T_{p-1,p},\ p>1, \text{ then } W_+(K,-p(p-1)) \text{ is slice.} \tag{5.11}$$

For $t = 0$, the ***proposed*** (and admittably optimistic) answer [Ki, Problem 1.38] is that $W_+(K,0)$ is slice (if and) only if K is such, but the problem remains unsettled even in this special case. When $K = \bigcirc$, the predicted constraint (that only $t = 0, -2$ occur) was confirmed in [CG]. This work is invoked for Theorem 5.22.

Not all these doubles are relevant for us, but this hinges on the next decision problem, when such knots are quasipositive. Even just considering the untwisted case $t = 0$, Remark 5.11 hints to extreme caution. More of the same is warranted by the following illustration.

Example 5.16 The knot 8_{20} is quasipositive and slice, and thus $W_+(8_{20},0)$ is slice. But it is not quasipositive – essentially this is the reason why 8_{20} falls out of the consideration in the proof of Proposition 5.9. However, if one takes the $+1/2$ twisted[3] *2-cable* of 8_{20}, i.e., the zero-framed cable with the pattern $\sigma_1 \in B_2$ (as lying in a solid torus), then it is both slice (since 8_{20} is so) and quasipositive (since 8_{20} is so, by a result of [St2]).

Remark 5.17 The provenance of 9_{46} and 10_{140} in Proposition 5.9 was from being λ-slice. Remark 5.4 then possibly suggests that $W_+(K,0)$ being quasipositive ***but not strongly so*** occurs only if K is quasipositive. But Example 5.16 shows that quasipositivity is not sufficient.

However, these examples raise the following question.

Question 5.18 Assume K is λ-slice. Is K quasipositive? Are $A(K,0)$ and $W_+(K,0)$ so?

Since Murasugi sum makes no sense in the 4-ball, there is no analogous relation for quasipositivity between $A(K,t)$ and $W_+(K,t)$, and Corollary 2.20 cannot be proved in this way for quasipositivity. A similar problem to (5.2),

$$W_+(K,t) \text{ is quasipositive} \implies W_+(K,t+1) \text{ is quasipositive}\,,$$

remains (generally) inaccessible, and we cannot extend Proposition 2.15 and Corollary 2.16 to λ_q. Obviously, one can modify (5.1) to define a number $\lambda_{q+}(K)$ etc., but instead of reiterating here a treatment analogous to $\lambda_q(K)$, it seems better to directly restrict the values of t for which $W_\pm(K,t)$ is quasipositive.

[3] We count the 2-cable framing by full-twist integers, so that a half-integer corresponds to a 2-cable knot, while an integer to a 2-component link.

Proposition 5.19 $W_+(K,t)$ is not quasipositive for

$$t \leq \lambda_{min}(K) - a(K) + \ell(K) - 2\,. \tag{5.12}$$

Proof. This is essentially the reasoning behind [JLS2, (5.8)], with the improvement coming from $\ell(K)$, as outlined in Remark 5.5 in [JLS2].

Namely, start with a grid diagram D of K with $\mu(D) = a(K)$ and $\lambda(D) = \lambda_{min}(K)$. Then apply the skein relation (3.8) backward, decreasing t, (at least) $a(K) - \ell(K) + 2$ times, and see that for t in (5.12), we have

$$\min\deg_v P(A(K,t)) \leq -3\,.$$

The skein relation (2.11) (as in the mirrored variant of (5.23)) then shows that

$$\min\deg_v P(W_+(K,t)) \leq -2\,,$$

so that $W_+(K,t)$ is not quasipositive. □

Remark 5.20 Even disregarding (5.9), the restriction (5.12) is generally a weaker condition than (5.8). For instance, using the left inequality of (3.16), and $\lambda_{min}(K) \geq \lambda(K)$, we have that

$$-\lambda(!K) \leq \lambda_{min}(K) - a(K) + \ell(K) - 2$$

definitely holds if

$$2\ell(K) \geq a(K) + 2\,,$$

which is practically always satisfied. (It certainly is for all ℓ-sharp knots, and it is not hard to check it either for all prime knots through 16 crossings, by using e.g. Computation 3.15.)

We give the following sample criterion for an equivalence between quasipositivity and strong quasipositivity of $W_+(K,t)$.

Proposition 5.21 Assume K is ℓ-sharp and $\lambda(K) > 1$. Then each $W_+(K,t)$ is quasipositive if and only if it is strongly quasipositive.

Proof. The proof is similar to Corollary 5.6. Assume K is ℓ-sharp. By applying the skein relation (2.11) at the positive clasp of $W_+(K,t)$, we see that $\min\deg_v P(W_+(K,t)) < 0$ for $t < \lambda(K) - 1$ and $\min\deg_v P(W_+(K,t)) \leq 0$ for $t = \lambda(K) - 1$. Again it follows that the stated equivalence can fail only if $t = \lambda(K) - 1$ and $W_+(K, \lambda(K) - 1)$ is slice.

But when $\lambda(K) > 1$, then $\lambda(K) - 1$ is positive, in contrast to the numbers (5.9), which are non-positive. □

Keep in mind that by Remark 5.11, there are ℓ-sharp examples with $\lambda(K) = 1$, for which the assertion fails. When we restrict to alternating knots K, we obtain a complete result.

Theorem 5.22 Assume K is alternating. Then $W_+(K,t)$ is quasipositive if and only if it is strongly quasipositive (for every t).

Proof. The case $K = \bigcirc$ can be handled *ad hoc*, so assume $K \neq \bigcirc$. By Proposition 3.6, and the argument in the proof of Proposition 5.21, we need to consider only

$$t = \lambda(K) - 1 \text{ so that } W_+(K,t) \text{ is slice.} \tag{5.13}$$

Because of (5.9), we have $\lambda(K) = -p(p-1) + 1$, and because of (2.47) for an alternating knot K [Ta], we have

$$\min\deg_a F(K) = p(p-1)\,,$$

so that

$$t = -p(p-1)\,. \tag{5.14}$$

Next, for an alternating knot K, it is known that $2\tau(K) = \sigma(K)$ is the *signature* (*cf*. §2.10), and

$$\sigma(K) \geq \min\deg_a F(K) = p(p-1)\,; \tag{5.15}$$

see [St5, Proposition 4].

Now, $W_+(K, -p(p-1))$ being slice implies that the jump number j_τ (§2.10) satisfies with (2.37)

$$-p(p-1) < j_\tau(K) = 1 - 2\tau(K) = 1 - \sigma(K) \leq 1 - p(p-1)\,,$$

which can only hold if in (5.15), $\sigma(K) = p(p-1) = \min\deg_a F(K)$. But then by the equality part of [St5, Proposition 4], we have that K is positive (special alternating). The genus follows then by Yokota's [Yo] result,

$$\min\deg_a F(K) = 2g(K)\,, \tag{5.16}$$

and t from (5.14).

Then K is positive (special alternating) of genus $g(K) = p(p-1)/2$ for some $p > 1$ and $t = -p(p-1)$, and it remains to prove that in that case $L = W_+(K,t)$ is not slice.

To deal with this final case, we use the work of Casson-Gordon [CG] and its reinterpretation due to Gilmer. The following description owes a lot to remarks I received from Chuck Livingston.

The genus-1 Seifert surface of L contains a curve α of framing 0, which represents the knot $\alpha = T_{p-1,p}\#K$. By Gilmer's work, the Casson-Gordon invariant of the q-fold cyclic branched cover of L can be expressed in terms of *Tristram-Levine signatures* $\sigma_\xi(\alpha)$ of α. The summation is given in Theorem 5 of [GL]. The convention for the index of σ is that $\sigma_0 = 0$, and $\sigma_{1/2} = \sigma$ is the classical signature, so that $\sigma_\xi = \sigma_{1-\xi}$ for $0 < \xi < 1/2$.

We have to take $m = p - 1 > 0$. It is enough to choose $q = 2$ (which was the original situation studied by Casson-Gordon), and a prime d dividing $(m+1)^2 - m^2 = 2m+1$ (which is $\sqrt{\det(W_+(K,t))}$ from (5.10)). Since that number is odd, we assume $d \geq 3$.

Now, $\alpha = T_{p-1,p}\#K$ is a positive (non-trivial) knot. Such a knot always has a positive crossing-changing sequence to the right-hand trefoil. This is a consequence of Taniyama's work [Tn] and was made more explicit in [PT], which was published long after being originally written. But the property can be derived independently (see [St6, Remark 6.6 and Exercise 6.4], or [IS]).

Considering the Tristram-Levine signatures of the right-hand trefoil, we find that then $\sigma_\xi(\alpha) > 0$ for $5/6 \geq \alpha \geq 1/6$, and $\sigma_\xi(\alpha) \geq 0$ otherwise. By letting in the summation given in Theorem 5 of [GL] s vary over all elements in $\mathbb{Z}_d^*$, we see that to show $L = W_+(K,t)$ is not slice, it is enough for every prime $d \geq 3$ to find an $0 < i < d$ so that $5/6 \geq i/d \geq 1/6$. But this is obvious. $\square$

Corollary 5.23 When K is a positive ℓ-sharp knot, then each $W_+(K,t)$ is quasipositive if and only if it is strongly quasipositive (for every t).

Proof. We need ℓ-sharpness to arrive, as in the proof of Proposition 5.21, at (5.13) and (5.14). Since $T_{p-1,p}\#K$ is a non-trivial positive knot (the case $K = \bigcirc$ can obviously be excluded), the Casson-Gordon-Gilmer part of the previous proof applies without change. $\square$

5.2.2 Negatively clasped ones

For $W_-(K,t)$, the reasonable expectation is that none are quasipositive(except $W_-(\bigcirc,0)$, which naturally illustrates some emerging constraints). We can almost completely verify this prediction in the following subsections, and here it is possible, and we make effort, to largely release ourselves of additional assumptions like ℓ-sharpness. First, we collect the following statements, that originate from previous results.

Lemma 5.24 If $W_-(K,t)$ is quasipositive, then it is slice. Also, for every slice-torus invariant v,

$$t > -j_v(!K)\,,$$

and in particular

$$t \ge -2\tau(K)\,,$$

and also

$$t = p(p-1)\,, \quad p \in \mathbb{N}\,. \tag{5.17}$$

Proof. Let v be a slice-torus invariant. Because $W_-(K,t)$ unknots by a negative crossing change, we have $v(W_-(K,t)) \le 0$. Were $W_-(K,t)$ quasipositive, (2.34) implies that $v(W_-(K,t)) = g_4(W_-(K,t)) \ge 0$.

This implies $g_4(W_-(K,t)) = 0$, i.e., that $W_-(K,t)$ is slice. The property (5.17) follows from the Milnor-Fox condition (5.9) under mirroring (the sign in (5.9) changes, since we changed the sign of the clasp).

But it also implies the equivalent conditions

$$\begin{aligned} v(W_-(K,t)) &= 0 \\ v(W_+(!K,-t)) &= 0 \\ -t &< j_v(!K) \\ t &> -j_v(!K)\,, \end{aligned}$$

and in particular with (2.37) also that

$$t > -j_\tau(!K) = -(1-2\tau(!K)) = -1 + 2\tau(!K) = -1 - 2\tau(K)\,. \qquad \square$$

The HOMFLY-PT polynomial does recover (albeit by entirely different means from the Murasugi sum) some parts of the complete result of Corollary 2.20 for quasipositivity. Obviously when $W_\pm(K,t)$ is strongly quasipositive, then it is also quasipositive, so here is what we obtain on the obstruction part. (Keep track of the case that $K = \bigcirc$, which is not excluded here, and which at least gives some hints to the limitations emerging.)

Theorem 5.25 For every knot K, there is at most one value t_0 so that $W_-(K,t_0)$ is quasipositive. If this value t_0 occurs, then the following hold.

1. We have (with (2.12))
$$\min \mathrm{cf}_{\,v}[P(K)]_{z^0} = \pm 1\,, \tag{5.18}$$

2. $$\min \deg_v [P(K)]_{z^0} = -p(p-1)\,, \quad p \in \mathbb{N}\,, \tag{5.19}$$
and

3. $$t_0 = -\min \deg_v [P(K)]_{z^0}\,. \tag{5.20}$$

4. Moreover, $t_0 \le \lambda_q(K)$, and equality can hold only if K is slice and $\lambda_q(K) = 0$.

Proof. If $W_-(K,t)$ is quasipositive, then $L = W_-(K,t)$ satisfies the variant of (2.19)

$$\min \deg_v P(L) \ge 1 - \chi_4(L)\,. \tag{5.21}$$

So in particular

$$\min \deg_v P(L) \ge 0\,. \tag{5.22}$$

It follows from applying the skein relation (2.11) that

$$P(W_-(K,t)) = v^{-2} - v^{-1} z P(A(K,t))\,. \tag{5.23}$$

Looking at the z^0-term in this formula (for the others, *cf.* Computation 5.37), and using (3.4), we see that

$$\min\deg_v [P(W_-(K,t))]_{z^0} \le -2\,,$$

disabling (5.22), unless

$$\min\deg_v (v^{2t}(v^{-1}-v)([P(K)]_{z^0})^2) = -1\,. \tag{5.24}$$

This occurs for exactly one $t = t_0$, namely the one in (5.20), and this t_0 must be of the form (5.17), giving (5.19). But beyond (5.24), we also need

$$\min\mathrm{cf}_{\,v}(v^{2t}(v^{-1}-v)([P(K)]_{z^0})^2) = 1$$

for a cancellation to occur, and this means that (5.18) must hold.

For Part 4 notice that since when $A(K,t)$ is quasipositive, then because of (5.21) we need

$$\min\deg_v P(A(K,t)) \ge -1\,. \tag{5.25}$$

By the skein relation (2.11), we have then

$$\min\deg_v P(A(K,t')) \ge 1$$

for $t' > t$, i.e., for each $t' \ge \lambda_q(K)+1$. Thus in (5.23), the second summand on the right will not cancel the first, and $\min\deg_v P(W_-(K,t')) < 0$ for each such t'. This shows $t_0 < \lambda_q(K)+1$.

Also, note that (5.25) being sharp requires $\chi_4(A(K,t)) = -2$, i.e., $A(K,t)$ bounds in the 4-ball two disjoint disks. This can occur only if K is slice and $t = 0$. This shows the equality property in Part 4. □

Remark 5.26 It can be concluded from the proof that if K is ℓ-sharp, then

$$t_0 = \lambda(K) - 1\,. \tag{5.26}$$

Remark 5.27 The inequality (5.21) and the mirroring property (2.13) also easily yield [He, Examples, (2)]: if both a knot K and its mirror image $!K$ are quasipositive, then K is slice. Together with (2.14), even more follows: we can see $P(K) = 1$. This tempts to suspect that the only possible K so that K and $!K$ are quasipositive is the unknot.

The restrictions on the singular value t_0 cannot so far eliminate it completely, but they do succeed in a series of rather self-contained special cases.

Corollary 5.28 If K is quasipositive and not slice (in particular non-trivial and strongly quasipositive), then no $W_-(K,t)$ is quasipositive.

Proof. Because of (5.21), we have that

$$\min\deg_v [P(K)]_{z^0} \ge \min\deg_v P(K) \ge 2g_4(K) > 0\,,$$

so that (5.19) cannot hold. □

Note that when K is *quasipositive and slice*, some of these knots yield in contrast the most difficult examples to resolve (see Computation 5.37).

The following completes the equivalence of quasipositivity and strong quasipositivity of Whitehead doubles of alternating knots.

Corollary 5.29 If $K \neq \bigcirc$ is alternating, then no $W_-(K,t)$ is quasipositive.

Proof. The single value t_0 that can make the term v^{-2} cancel on the right of (5.23) obviously needs $\min\deg_v P(A(K,t_0)) < 0$, so that $t_0 < \lambda(K)$ (because K is ℓ-sharp).

But then by the proof of Proposition 3.6, since $k_1 > 0$ in (3.11), there is a non-zero term in $P(A(K,t_0))$ of $z^b v^c$ with $b > 0$ and $c = \min\deg_v P(A(K,t_0))$. This means that on the right of (5.23) a lowest v-degree term survives the cancellation, and $\min\deg_v P(W_-(K,t_0)) < 0$. Thus $W_-(K,t_0)$ cannot be quasipositive. □

Corollary 5.30 If K is negative (i.e., $!K$ is positive) and non-trivial, then no $W_-(K,t)$ is quasipositive.

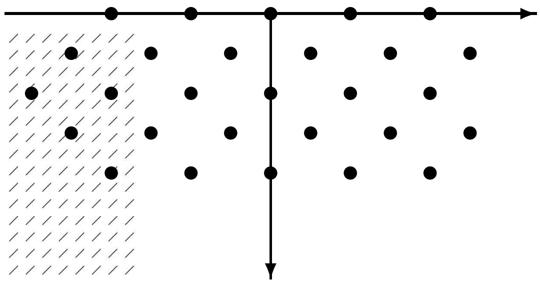
Proof. We have that $\max\deg_v P(K) = -2g(K) < 0$, and thus in (5.19), we need $p > 1$. But then again, the Casson-Gordon-Gilmer obstruction of the proof of Theorem 5.22 applies to show that $W_-(K,p(p-1))$ is not slice; just mirror the argument there: $!T_{p-1,p}\#K$ is a non-trivial negative knot. □

5.2.3 Composite and amphicheiral knots

The following condition is essentially behind (and more general than) Corollary 5.29, and is singled out since it provides a practical test (Computation 5.37) that is also useful in some further theoretical arguments (Theorem 5.35).

Theorem 5.31 Assume for K there is a t so that $W_-(K,t)$ is quasipositive. Then every coefficient of $F(K)$ of a monomial $a^{k_1}z^{k_2}$ for $k_2 \geq 1$ and $k_1 \leq \min\deg_a [F(K)]_{z^0}$ is even.

This schematic picture illustrates the area of even coefficients (with z-powers of monomials going top-down and a-powers left-right).



(5.27)

Note further that (5.18) can be also written via (2.24) as

$$\min \mathrm{cf}_a [F(K)]_{z^0} = \pm 1\,, \tag{5.28}$$

and this coefficient corresponds to the dot above the top right corner of the shaded region in (5.27).

Proof. Note the relationship (2.24). The claimed statement is proved similarly to Corollary 3.12 and Proposition 3.14 using Rudolph's congruence and (3.4). Assume that some of the designated coefficients of $F(K)$ is odd. If t is so that

$$\min\deg_v [P(A(K,t))]_{z^{-1}} = -1\,,$$

where the cancellation in (5.23) is possible, then either $\min\deg_v P(A(K,t)) < -1$, or $P(A(K,t))$ has terms of the form $v^{-1}z^{k_2}$ for $k_2 > -1$. This means that in (5.23), $\min\deg_v W_-(K,t) < 0$, and so $W_-(K,t)$ cannot be quasipositive. □

Remark 5.32 Now notice the following: because of the multiplicativity of F under connected sum of knots,

$$F(K_1\#K_2) = F(K_1)\cdot F(K_2)\,,$$

if the conjunction of (5.28) with the condition in Theorem 5.31 holds for $K_1\#K_2$, then it must hold for both K_1 and K_2.

A consequence of this observation, and the fact that Theorem 5.31 generalizes Corollary 5.29, is this:

Corollary 5.33 Assume K is a composite knot with an alternating connected sum factor. Then no $W_-(K,t)$ is quasipositive. □

For a positive knot K, we have more generally Corollary 5.28. However, the present consideration extends by Remark 5.32 to connected sums as well, which with the argument there is not possible.

Corollary 5.34 Assume K is a composite knot with a positive connected sum factor. Then no $W_-(K,t)$ is quasipositive.

Proof. As in the proof of Corollary 3.16, we use Yokota's result [Yo], which gives an odd term $[F(K)]_{z^{2g}a^{2g}}$ in (5.16) (with $g = g(K)$), with (3.22). □

The next application is similar to the foregoing corollaries of Theorem 5.25, but has a longer proof, and is given as a theorem.

Theorem 5.35 Assume K is amphicheiral and $W_-(K,t)$ is quasipositive. Then $t = 0$ and $P(W_-(K,t)) = 1$.

Thus this exhibits an analogous property to Remark 5.27 for $W_-(K,t)$ instead of K itself – albeit the argument there is far far simpler. The situation in our theorem is of course practically non-existent for any non-trivial K. Note in particular that for such K, we need

$$[F(K)]_{z^0} = [P(K)]_{z^0} = 1 \tag{5.29}$$

(because of (5.23), (2.24) and (3.4)) and

$$F(K) \equiv 1 \bmod 2 \tag{5.30}$$

(because of Theorem 5.31, as will become evident in the proof directly).

This is more than enough to see that there is no amphicheiral prime knot up to 16 crossings whose Kauffman polynomial fits the constraints. If one looks at low z-degree terms of F, the knot $K = 16_{378787}$ comes "closest". There the truncation $[F(K)]_{z\le 4}$ still fails, and $[F(K)]_{z\le 5}$ is needed to see its F polynomial depart from the required shape. However, K being alternating, this incoherence is also evident from Corollary 5.29. More generally, every semiadequate (amphicheiral) knot will fail (5.30).

This way of handling amphicheiral knots takes something of the magnitude of 10^{-7} of the effort (manual management plus digital complexity) of Computation 5.37 invested in verifying – still with a handful of unresolved cases – the entire up-to-16-crossing prime knot table.

Proof. We have that for amphicheiral K,

$$[P(K)]_{z^0}(v) = [P(K)]_{z^0}(v^{-1})\,. \tag{5.31}$$

If

$$\min\deg_v [P(K)]_{z^0} < -2\,, \tag{5.32}$$

then we need that $W_-(K, p(p-1))$ is slice for some $p > 2$, and can use Casson-Gordon-Gilmer again, noting that $\sigma_\xi(K) = 0$ for all ξ, but $!T_{p-1,p}$ is a non-trivial negative knot.

Now, it is known that $[P(K)]_{z^0}(v)$ is of the form $Y(v^2)(v - v^{-1})^2 + 1$ for some $Y \in \mathbb{Z}[v^{\pm 1}]$ (see [Kn]), and with (5.31) and (5.18), but without (5.32), the three polynomials $P_0 = [P(K)]_{z^0} = 1$, for $t = 0$ (which we call the "fake unknot"), and $P_0 = -v^{-2} + 3 - v^2$ ("fake 6_3") or $v^{-2} - 1 + v^2$ ("fake 4_1"), for $t = 2$, are the only options.

Consider first the fake unknot with $t=0$ and (5.29). The property (5.30) follows at once from Theorem 5.31 and

$$F(K)(a^{-1},z)=F(K)(a,z)\,, \tag{5.33}$$

which is true for every amphicheiral knot K.

It is then also clear from (3.4) that

$$[P(A(K,0))]_{z^{-1}}=v^{-1}-v\,.$$

Note also that (see Remark 3.3)

$$P(A(K,0))(v^{-1},z)=-P(A(K,0))(v,z) \tag{5.34}$$

If now $P(A(K,0))$ had terms in negative v-degree except v^{-1}, then with a similar argument to the proof of Theorem 5.31, we would have $\min\deg_v P(W_-(K,0))<0$. This means that such terms do not exist for negative v-degree, and because of (5.34) for positive v-degree either. Thus $P(A(K,0))=[P(A(K,0))]_{z^{-1}}=v^{-1}-v$, and the claim follows from (5.23).

It remains to rule out the cases $t=2$.

Consider the fake 4_1. Because of (2.24) we have

$$[F(K)]_{z^0}=-a^{-2}-1-a^2\,. \tag{5.35}$$

Because of Theorem 5.31, we can assume that $F(K)$ has odd coefficients only in a-degrees $e=-1,0$ and 1 for z-degree $d>0$. Also note further that when e is even, and hence z-degree d is even, too, then (2.23) and (5.33) imply that $[F(K)]_{z^da^0}$ is also even (it is even divisible by 4).

Thus $[F(K)]_{z^{\ge 1}}$ mod 2 is of the form

$$[F(K)]_{z^{\ge 1}} \bmod 2=\sum_{i=1}^{n}(a+a^{-1})z^{d_i}\,,$$

where $d_i>0$ are odd. To see that such a polynomial $F(K)$ cannot occur, consider Vassiliev invariants. The Kauffman polynomial gives rise to Vassiliev invariants

$$F_{i,j}(K):=\sqrt{-1}^{\,i+j}\,\frac{d^j}{da^j}\Big|_{a=\sqrt{-1}}[F(K)]_{z^i}\,, \tag{5.36}$$

as used in [St7, Section 3.3]. These invariants be more conveniently normalized to

$$\begin{aligned}\tilde F_{d,1}&=F_{d,1}\\ \tilde F_{d,2}&=F_{d,2}+F_{d,1}\\ \tilde F_{d,3}&=F_{d,3}+3F_{d,2}\\ \tilde F_{d,4}&=F_{d,4}+6F_{d,3}-6F_{d,1}\end{aligned}$$

so that they become symmetric or antisymmetric, i.e.,

$$\tilde F_{i,j}(K)=(-1)^{i+j}\tilde F_{i,j}(!K)\,.$$

With the above normalization, we gave in [St7, Section 3.7.2] the relation

$$31\tilde F_{0,2}+5\tilde F_{0,4}-16\tilde F_{1,3}+16\tilde F_{2,2}-4\tilde F_{0,2}^2=0\,, \tag{5.37}$$

which originates from the dimensional overrepresentation of Vassiliev invariants of degree 4.

Since $[F(K)]_{z^0}$ is known from (5.35), we can directly calculate

$$\tilde F_{0,2}=F_{0,2}=8\,,\quad F_{0,4}=120\,,\quad \tilde F_{0,4}=-24\,. \tag{5.38}$$

Now if we write

$$[F(K)]_{z^1} = \sum_{k>0\,\text{odd}} c_k(a^k + a^{-k})\,,$$

then

$$\tilde F_{1,3}(K) = \sum_{k>0\,\text{odd}} 2c_k(k-1)k(k+1)\,,$$

with

$$32 \mid \tilde F_{1,3}(K)\,, \tag{5.39}$$

since c_k is even for $k>1$ odd. Likewise, if we set

$$[F(K)]_{z^2} = b_0 + \sum_{k>0\,\text{even}} b_k(a^k + a^{-k})\,,$$

then

$$\tilde F_{2,2}(K) = \sum_{k>0\,\text{even}} 2b_k\cdot k^2\,,$$

with

$$16 \mid \tilde F_{1,3}(K)\,, \tag{5.40}$$

since all b_k are even.

Now putting (5.38), (5.39) and (5.40) into (5.37), we see that

$$256 \mid 31\cdot 8 + 5\cdot(-24) - 4\cdot 64 \;=\; -128\,,$$

which does not hold. This gives the sought contradiction. Note that the r.h.s. is still divisible by 128, and with good reasons: since it is only a parity argument that enters, it is clear that the contradiction can succeed at most by one factor of 2, and the real $K=4_1$ demonstrates that.

The calculation for the fake 6_3 is essentially the same, just reverse the signs of $\tilde F_{0,2}$ and $\tilde F_{0,4}$. This finishes the proof. □

5.2.4 (Companion) knots up to 16 crossings

In this section we report on how efficient are the previously developed tests to rule out that $W_-(K,t)$ is quasipositive in practice, by using the tables of prime knots in [HT]. (Note that, here, the result for prime knots does *not* straightforwardly imply the like for their connected sums.) The summary of the effort is this.

Proposition 5.36 For K a knot up to 16 crossings (incl. mirror images), the only $W_-(K,t)$ which may be quasipositive are for $t=0$ and K being one of the following 14 (prime, slice) knots

$$\begin{array}{ccccccc} 12_{1870}\star & 13_{8401}\star & 14_{37748}\star & 15_{197742}\star & 15_{223505} & 15_{229315} & 15_{241091} \\ 16_{439229} & 16_{959746} & 16_{1055696}\star & 16_{1193665} & 16_{1231515} & 16_{1301673} & 16_{1307407} \end{array} \tag{5.41}$$

always mirrored so that $\min\deg_a F(K)=0$ (and (5.5) holds).

The details are given in the below computation. (The role of the symbol '$\star$' will also be explained.)

Computation 5.37 This test went over a number of separate stages and developed into a long and very difficult project. It motivated a number of implementational developments that can be used fully independently of this particular application, but are only outlined. Some more details on the algorithms are presented in my online lecture series [St10].

Part 1. Consider the tables of [HT] of prime knots K up to 16 crossings. By using $F(K)$ and testing (5.19) via (2.24) and (5.28), and then applying the condition in Theorem 5.31 on either mirror images, we are left with 2174 (non-alternating) knots. (The lowest crossing knots passing through this test are 10_{128} and 10_{140}.)

Part 2. Using Remark 5.32 (i.e., (5.28) and Theorem 5.31, without (5.19)) on the three non-alternating prime 8 crossing knots shows that the only composite knot K of up to 16 crossings which needs to be further treated (for quasipositivity of some $W_-(K,t)$) is

$$K = !8_{19}\#!8_{19}\,, \tag{5.42}$$

negatively mirrored, for $p=5$ in (5.19).

Part 3. Then we can use the (non-)cancellation argument in the proof of Theorem 5.31 from some truncation of $P(W_-(K,t))$.

This restricts

$$\min\deg_v P(W_-(K,t_0)) \geq 0$$

for only 69 knots K, the simplest of which is $K = 10_{140}$ (thus the knot $K = 10_{140}$ reappears). For all these knots $t_0 = 0$ and (5.5) (compare with (5.26)). Also, the composite knot (5.42) falls out (or see Corollary 5.30; so we deal further only with prime knots).

Example 5.38 Consider $L = W_-(10_{140},0)$. Since L is indeed slice, this constraint does not yield anything here. But the case can be handled thus. We have

$$\min\deg_v P(L) = 0\,, \quad \max\deg_v P(L) = 14\,. \tag{5.43}$$

Now, we have a 9-braid representative of L of writhe 6, which can be obtained by switching one positive band in a presentation of $A(10_{140},1)$ coming from a grid diagram of size 9 (since $a(10_{140}) = 9$ and $\lambda(10_{140}) = 1$), and doubling the negative band. This 9-braid representative of $W_-(10_{140},0)$ is not slice-Bennequin-sharp.

If quasipositive, L would have a slice-Bennequin-sharp braid representative, and by the argument based on Theorem 2.7 (see the proof of Lemma 4.9 in [JLS2]), every minimal braid representative of L would be such. Thus L must have braid index $b(L) \leq 8$, less than $a(10_{140}) = 9$. But this can be ruled out by a truncated 2-cable HOMFLY-PT polynomial calculation. (Compare with Computation 5.23 in [JLS2].)

The other (68) examples K almost identically replicate $K = 10_{140}$. We were able to confirm using our old compilations in [St5, Appendix] that all are slice. While we did not check that all are quasipositive, we expect them to be so (see Question 5.18)[4]. By any means, $\ell(K) = \mathrm{MB}(K)$ (recall (2.44)), which first suggests us (and practically requires us, to keep calculations within limits) to confirm that they are ℓ-sharp.

We confirmed ℓ-sharpness of all 69 knots using the the arc indices of the (non-alternating prime) 14 crossing knots we simultaneously determined (see Remark 5.28 in [JLS2]), which resolved 12 knots, and also the previous tabulations of [JKLM] for $a(K) \leq 12$, resolving further 13. This left 44 knots of 15,16 crossings.

There are 8 of these 44 with $a(K) = 14, 15$, which were confirmed using the knot-spoke/binding circle method ([JP]). For the 36 knots with $a(K) = 13$, we used the newly developed grid scanning tool, introduced in [J+2, §4 end]. For the 36th knot, 16_{968387}, which took particularly long to find, we further used the characteristic tuple (explained by the author separately in [St9]) to narrow down the list of diagrams to check.

Part 4. Once we know $a(K)$, we find that in all instances of

$$L = W_-(K,0)\,,$$

[4] We certainly know (following [St4]) quasipositivity for the 6 knots up to 13 crossings, which are 10_{140}, 11_{506}, 12_{1870}, 13_{8401}, 13_{9392} and 13_{9537}.

the HOMFLY-PT degrees (5.43) generalize to

$$\min \deg_v P(L) = 0\,, \quad \max \deg_v P(L) = 2a(K) - 4\,. \tag{5.44}$$

Then, again we need a 2-cable HOMFLY-PT polynomial calculation to show

$$b(L) > a(K) - 1\,, \tag{5.45}$$

with the method explained in [St8] (see in particular (51) and Example 9.2 therein).

Of course, here we need to replace the HOMFLY-PT polynomial by a(n as small as possible z-degree) truncation. This 2-cable of L is a degree-4 satellite of K. Not only is every crossing of K 16-folded, but there are extra (4-folded) crossings needed to undo the (blackboard) framing of the Whitehead double of K. The resulting diagrams have from about 200 to 340 crossings. Some of them additionally required the parallelized upgrade of the truncated skein algorithm (see §2.7), and were run on a server machine using between 40 and 120 CPUs over several weeks.

Part 5. There were two substantial improvements made along the way in the truncated skein algorithm of [St3]:

- Taking into account the general version of (3.4) (for more than two components) in case only the lowest z-term of a skein node needs to be computed. This means that all mixed crossings can be completely eliminated in that case, just by introducing a correction coming from the (total) linking number. Especially through this improvement, the lowest z-degree term becomes about 200-300 times faster to obtain, the second still about 60 times, and the third about 20 times.
- A finer complexity test of arcs which to shrink to a needle (see §2.7). The crossings on the arc that need to be switched and smoothed turn out to cost more when they are self-crossings of the component. Thus, in sufficiently complicated diagrams, arcs are chosen by minimizing first the number of self-crossings, only next the number of all crossings, to alter.

We always attempted connected (i.e, half-integer framed[5]) 2-cables of L. One example, which took about 25 sec with a single CPU on an old 2013 desktop, is $K = 15_{213108}$. It has the following z^0-term of the $-8\frac{1}{2}$ twisted 2-cable of $!L = !W_-(15_{213108}, 0) = W_+(!15_{213108}, 0)$, represented in a 289 crossing diagram.

```
289 213108     0     0
 -48  -14      -12    186  -1214   4126  -6164  -7558  62986 -171270 298261 -376232 359888 -265738 151650 -66040  21272  -4706    574     -8
```

We leave it as an exercise to the reader to infer from that polynomial, and (5.44) with $a(K) = \ell(K) = 12$, as well as (5.5), that the braid index of $L = W_-(15_{213108}, 0)$ is $b(L) = a(K) = 12$, in confirmation of (5.45).

Part 6. Within 6 months, we computed the z^0- and, if necessary, z^2-truncations of connected 2-cables of L for these knots K. We were able to resolve the problem (5.45) for 51 out the 69 knots, with 12_{1870}, the pretzel knot (5.46)

$$\textit{pretzel knot } \mathcal{P}(k, 3, -3) \tag{5.46}$$

for $k = 6$, being the simplest instance failing within the (complexity) reach of our method. (We also found the z^4-truncation for 4 knots, $K = 12_{1870}$, 13_{8401}, 15_{229315}, 15_{241091}, which stretched this phase to about $9\frac{1}{2}$ months, but to no avail[6].) In general, the pretzel knots (5.46) (for $k \geq 6$) provide one family especially difficult to handle (see for this also the remarks after the proof). For orientation, these knots are labelled with a '$\star$' in (5.41).

Part 7. Unsatisfied with the relatively large number (18) of remaining cases, I realized that a new method is needed to compute truncations more efficiently, in that v-variable truncations are helpful. This can be

[5] See footnote on p. 27.

[6] The (parallelized) CPU time required for the last knot, 15_{241091}, was about $18\frac{1}{2}$ years. Compare this with 15_{213108} mentioned earlier.

seen in the instance from the polynomial of 15_{213108} of Step 5: only one v-term in this polynomial is really needed to deduce (5.45).

One insight gained from parallelizing the z-truncated computation (§2.7) was that the moves used in a skein-needle type of algorithm augment the number of Seifert circles to a surpassing extent. This can be inferred from the partial polynomials from individual branches of the skein trees that were computed in Part 6: they contain v-powers much higher than the end result (i.e., their sum). The question is then whether or not all high-degree v-terms cancel out in the sum. However, discarding low degree v-terms does not make such verification (much) easier.

This suggested as rather useless in trying to implement v-variable truncations using Millett-Ewing type of moves. Instead, we need to design a skein tree so that nodes have as few as possible Seifert circles. This would allow to cut out large portions of the tree by using (2.15), and prevent (mostly cancelling) high v-degree terms from occurring at first place. Such approach immediately puts to mind (closed) braids.

Developing and testing *the braid-skein algorithm* took about 3 months. We used L-moves (a variant of destabilization after crossing changes), and had to implement bigon reductions to further save running time. (Of course all the previous features like z-truncations and parallelizing were transferred to the new algorithm.) For technical reasons we designed the program so as to cut off low (rather than high) degree v-terms. Thus here we had again to work with mirror images.

We take a minimal grid diagram of K, and the band representation from the grid-band construction. Then we either (i) make all bands negative except one (positive), which we double, or (ii) make all bands positive except one (negative), which we double, and then take the mirror image of the resulting band presentation (so that bands now go behind disks). This gives a to-be-proved minimal braid representative β of $!L = W_+(!K, 0)$.

We have checked that K was mirrored properly (i.e., the correct option between (i) and (ii) was chosen) by testing (see (5.44))

$$\max\deg_v P(!L) = 0\,.$$

For example for $K = 16_{1168697}$, with $a(K) = 14$, we used option (ii) for the band representation

```
95  95  8C  BE  2D  1A  9C  7E  6B  8D  5A  47  36  24  13
```

where $[ji]$ for $j > i$ strands for $\sigma_{i,j}^{-1}$. This gives a word β of 14 strands and length 125 in $\sigma_i^{\pm 1}$,

```
14 125 1168697 [-5 -6 -7 8 7 6 5 -5 -6 -7 8 7 6 5 ... -2 -3 2 -1 -2 1]
```

The length of this braid (word) β is proportional to the *sum of vertical segment lengths* of D. For some knots we had to change the grid diagram D so as to reduce this parameter. This was accomplished with the grid scanning tool of [J+2] in conjunction with filtering using the characteristic tuple introduced in [St9] (see Step 3).

Then we had to 2-cable β using

$$\sigma_i \quad \mapsto \quad \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}\,.$$

This gives a braid representative β_2, on $2a(K)$ strings, of the (integer) $(3 - a(K))$-framed (disconnected) 2-cable L_2 of $!L$. This braid β_2 usually has (say, about 25%) more crossings than the 2-cabled diagrams of a (different, and connected) 2-cable of $!L$ we used the skein-needle algorithm on. But this turns out to be a very moderate price to pay for the increased efficiency of computing.

One can deduce from (2.15) that

$$\max\deg_v P(L_2) = \max\deg_v P(\hat{\beta}_2) \le 11 - 2a(K)\,,$$

and (5.45) would follow if

$$\max\deg_v P(L_2) \ge 9 - 2a(L)\,.$$

Thus, we need

$$v\text{-truncation in degree} \ge 9 - 2a(K)\,, \tag{5.47}$$

and only the two highest possible v-degree terms need to be computed. Even for this simplification, the full computation would still be too complex, but the point is that now we can go with z-truncations much higher than we could in the skein-needle algorithm[7].

The result of the followed example, for z-truncation 9, is

```
 500 1168697      5      9
  -19  -19       -28
  -19  -19     -6430
  -19  -19   -239414
total CPU secs: 19052757
```
(5.48)

which (evidently) took about 220½ CPU-days, and was done parallelized in about 52½h on 105 CPUs (with about 96% of capacity usage).

One (and probably the only one) minor drawback of the v-truncation is that the substitution $v = 1$ and using the Conway polynomial (via the cabling formula) as a "checksum" is not available. But there are other clues, e.g., only the lower (degree $9 - 2a(K)$) v-term appears, and no z-degree terms of degree < 3 (which is very plausible given that no such terms were observed for a connected 2-cable using the skein-needle method).

We were able to make the computation for (5.47) and z-truncation 9 (although, for the successful cases, in retrospect much less was needed). Along the 18 knots left over by Part 6, the result (was not 0 and) ruled out 4 more of them: 15_{96825}, 16_{752876}, $16_{1156937}$ and (as evidenced) $16_{1168697}$.

We attempted also z-truncation 11 for the rest 14 knots, given in (5.41), but only 0 came out. We also tried 15_{96825}, which correctly reproduced (while adding one more to) the (non-zero) terms previously obtained for z-truncation 9. This computation, which took about 20.3 years time in total on 120 CPUs, is about the limit of the reasonably feasible with this current state of technology. □

Experience has shown that distribution of coefficients of a knot polynomial like $P(!L)$ and its cable polynomial $P(L_2)$, when plotting v and z powers as coordinates in the plane, usually follow similar "geometric" patters. This suggests as very likely that, for (most if not) all of the 14 knots K in (5.41), the polynomial $P(L_2)$ will lack terms of sufficiently high v-degree, and thus $\mathrm{MFW}(L_2)$ is still not enough to resolve $b(L)$. This may be investigated later for some K using quantum algebra with Vivek Singh.

Conjecture 2.9, on the page of $16_{1057125}$ on [St4] (see §2.7), has a "companion" guess that for every bounded $c \le C$, there is a knot K so that c-cables do not suffice to determine $b(K)$. The given computation suggests that we may have at hand at least candidates for a sequence: the simplest suspects (of our making) would be

$$K = W_-(\mathcal{P}(k, 3, -3), 0)$$

in (5.46) (for increasing k).

More generally, all above considerations strongly suggest that maybe no (non-trivial) negatively clasped Whitehead double is quasipositive. But in fact, there is a more general question, which at least passed initial verification (crossing changes in minimal diagrams of prime knots up to 16 crossings).

Question 5.39 If a knot is quasipositive (and slice), can it unknot by switching a negative crossing to positive?

5.2.5 Bennequin sharpness

We remark here that the preceding discussion is also useful to resolve the Bennequin sharpness problem 2.5 for many Whitehead doubles. Compare the following with Lemma 2.17.

[7] Keep in mind that now L_2 is a 2-component link, so only odd powers of z and v make sense.

Proposition 5.40 Let K be ℓ-sharp. Then $A(K,t)$ and $W_+(K,t)$ is strongly quasipositive iff it is Bennequin-sharp.

Proof. Note that when K is ℓ-sharp and $t < \lambda(K)$ (the only non-trivial case), then $\min\deg_v P(A(K,t)) < 0$. Thus $A(K,t)$ is not Bennequin-sharp unless $K = \bigcirc$ and $t = 0$ because of the right inequality in (2.15).

In a similar way, $\min\deg_v P(W_+(K,t)) \le 0$, since applying the skein relation at the positive crossing of the clasp does not lead to a (relevant) cancellation. □

For $W_-(K,t)$, clarifying the status of Bennequin-sharpness can avoid most complications that occur for slice-Bennequin-sharpness.

Proposition 5.41 No $W_-(K,t)$ is Bennequin-sharp unless $K = \bigcirc$ and $t = 0$.

Proof. It is enough to see (again because of the right inequality in (2.15)) that $\min\deg_v P(W_-(K,t)) \le 0$ for every K and t. Except for the singular value $t = t_0$ in Theorem 5.25 this is obvious, since then $\min\deg_v P(W_-(K,t)) \le -2$.

When $t = t_0$ and (5.18) holds (which we can assume by the argument there, *unlike* (5.19)), then notice that because of (3.4) the second coefficient of $[P(A(K,t))]_{z^{-1}}$, in v-degree 1, is odd. This means that $[P(W_-(K,t))]_{z^0}$ will have an odd v^0-term. □

5.3 Torus knots

Torus knots have turned out to archetype a class on which all previous tools apply well to settle the variety of proposed geometric questions.

Mironov, Sati, and Singh have proved the Panhandle Conjecture for the torus knots. It implies a result of Etnyre-Honda about the arc index of the torus knot [EH2]

$$a(T_{p,q}) = p + q \tag{5.49}$$

which, with (2.47) and (5.16), and

$$(p-1)(q-1) = 2g(T_{p,q})$$

also yields

$$\lambda(T_{p,q}) = -pq + p + q\,, \quad \lambda(!T_{p,q}) = pq\,. \tag{5.50}$$

The new proof uses tools (quantum algebra and the Rosso-Jones formula) which are not helpful to discuss here, but we quote the following parts of their work. We assume below that

$$(p,q) = 1 \text{ and } q > p.$$

Theorem 5.42 (Mironov-Sati-Singh [MS+]) For $t' = (1-p)q$, and $K = T_{p,q}$,

$$\max \mathrm{cf}_{\,v} P(A(K,t')) = (1-p)z\,, \quad \max\deg_v P(A(K,t')) = 2q-1\,. \tag{5.51}$$

Remark 5.43 Before we continue applying this theorem, we make a clarification regarding causality. A more detailed form of Theorem 5.42 is proved in [MS+] using representation theory and a simplified version of Lemma 5.45. But that simplified version is reproduced there, and thus the proof of Theorem 5.42 in [MS+] is self-contained. (In fact, Theorem 5.42 is the partial statement that does not require the lemma.) Then this theorem is applied here, using the preceding definition of the ℓ and θ invariants. In turn, a small piece of the results in this section is (reviewed and) used in the back sections of [MS+] for the partial generalizations to torus links. Even so, the explanation here should clarify that there is no ring inference, i.e., nowhere does a statement enter into it own proof. There was no clearer way to split the treatise, also because the work was partially done individually, and partially with two different groups of coauthors.

Lemma 5.44 We have $\min\deg_v P(A(K,t')) = 1-2p$. In particular, torus knots are ℓ-sharp, i.e.,

$$\ell(T_{p,q}) = p+q\,.$$

(For comparison to the Morton-Beltrami bound, see (5.57).)

Proof. Since it is easy to construct a $p+q$-arc presentation of $K = T_{p,q}$, we have $\ell(T_{p,q}) \le p+q$, and hence with the right equality of (5.51), that

$$\min\deg_v P(A(K,t')) \ge 1-2p\,.$$

To prove equality, it is enough to exhibit a non-zero coefficient $[P]_{z^r v^s}$ of $P(A(K,t'))$ in v-degree $s = 1-2p$. For this we consider z degree $r=-1$. We use (3.4). Here $t = t' = (1-p)q$. We claim that

$$\min\deg_v [P(A(K,t))]_{z^{-1}} = (p-1)(q-1)\,. \tag{5.52}$$

Then counting degrees in (3.4) we have

$$\min\deg_v [P(A(K,t))]_{z^{-1}} = 2(1-p)q - 1 + 2(p-1)(q-1) = 1-2p\,.$$

For (5.52) see (5.53). □

Notice that for $K = T_{p,q}$ we have

$$\min\deg_v P(K) = (p-1)(q-1) = 2g(K)\,,$$

since K is positive.

Lemma 5.45

$$\big|\,[P(T_{p,q})]_{z^0 v^{(p-1)(q-1)}}\,\big| \;>\; 1\,. \tag{5.53}$$

Proof. For this we use Nakamura's observation [Na].

Assume that $\beta \in B_n$ is a positive braid word. Then for $\beta_{[i]} = \beta\sigma_j^i$ and $j = 1,\dots,n-1$ fixed, and any $r,s \in \mathbb{Z}$, we have for $L_i = \hat\beta_{[i]}$, and $L = L_0$,

$$\big|\,[P(L_2)]_{z^r v^{s+2}}\,\big| \;=\; \big|\,[P(L_1)]_{z^{r-1} v^{s+1}}\,\big| \;+\; \big|\,[P(L)]_{z^r v^s}\,\big|\,. \tag{5.54}$$

Let $n(L)$ be the number of components of a link L. Now in (5.54) we set $r = 1-n(L)$, $s = 1-\chi(L)$, and $n = p$.

Let us write $\beta_1 \succ \beta_2$ if (a) $\beta_1 = \beta_2\sigma_{p-1}^2$ for $\beta_2 \in B_{p-1}$ via the braid group inclusion $B_{p-1} \subset B_p$, or (b) $\beta_1 \in \{\alpha, \alpha\sigma_j\}$ and $\beta_2 = \alpha\sigma_j^2$ for a positive braid word α. We further allow to permute letters cyclically in the β_i. Transitively expand this relation $\succ$ to become a partial order.

Assume that β is a positive braid so that there is a sequence $(\beta = \beta_k, \beta_{k-1}, \dots, \beta_0)$ with $\hat\beta_0 = T_{p,q}$, and $\beta_{i-1} \succ \beta_i$. Then for $L = \hat\beta$ and $n(L)$ components,

$$\big|\,[P(T_{p,q})]_{z^0 v^{(p-1)(q-1)}}\,\big| \;\ge\; \big|\,[P(L)]_{z^{1-n(L)} v^{1-\chi(L)}}\,\big|\,. \tag{5.55}$$

For option (b) it follows from (5.54). Note that when $n(L_1) = n(L)-1$ (and not $n(L)+1$), then the first term on the right of (5.54) is zero. For option (a) it follows because $\hat\beta_2 = \hat\beta_1 \# H$ for the positive Hopf link H, and its polynomial is straightforwardly checked.

We start with $\beta_0 = \tau_{p,q} := (\sigma_1 \dots \sigma_{p-1})^q$. This braid contains the center twist $\Delta_p^2 = (\sigma_1 \dots \sigma_{p-1})^p$, and it is easy to see that for every positive words δ, γ, we have $\Delta_p^2\delta\gamma \succ \Delta_p^2\gamma$. This is because one can write $\Delta_p^2 = \alpha\sigma_j$ with a positive word α for every $j = 1,\dots,p-1$. Thus

$$\tau_{p,q} \succ \tau_{p,p+1}\,,\text{‘} \tag{5.56}$$

and we can restrict ourselves to $q = p+1$. Then we simplify further $\tau_{p,p+1} \succ \sigma_1\tau_{p,p} = \sigma_1\Delta_p^2$.

Now we write

$$\sigma_1\Delta_p^2 = \sigma_1\Delta_{p-1}^2(\sigma_{p-1}\dots\sigma_1)(\sigma_1\dots\sigma_{p-1})\,,$$

where $\Delta_{p-1}^2 \in B_p$ is meant via the inclusion $B_{p-1} \subset B_p$. By definition part (b) of $\succ$, and successively removing letters σ_i^2 for $i = 1,\dots,p-2$ in the last two factors, we have

$$\sigma_1\Delta_p^2 \succ \sigma_1\Delta_{p-1}^2\sigma_{p-1}^2 \succ \sigma_1\Delta_{p-1}^2\,,$$

where the last step is the option (a) in the definition of $\succ$. This means that for (5.55) it is enough to check L being the closure of $\beta_k = \sigma_1\Delta_2^2 = \sigma_1^3 \in B_2$, which is the trefoil. There it is straightforward to calculate that the value on the right is 2, and (5.53) follows. □

Remark 5.46 By using instead of (5.56) that

$$\tau_{p,q} \succ \tau_{p,p} = \Delta_p^2\,,$$

and the general version of (3.4) for a link $T_{p,p}$ of more than 2 components, it can be easily deduced that the polynomial

$$\Gamma(p,q) := [P(T_{p,q})(\sqrt{-1}v, \sqrt{-1}z)]_{z^0}(v) = [F(T_{p,q})(v,z)]_{z^0}(v)$$

satisfies beyond (5.53), giving

$$\min\deg_v \Gamma(p,q) = (p-1)(q-1)\,,$$

also that

$$\max\deg_v \Gamma(p,q) = (p-1)(q+1)\,.$$

Together with the formula of Yokota-Labastida-Perez [Yo2, LP] of $F(T_{p,q})$, one easily obtains that

$$\mathrm{MB}(T(p,q)) = \begin{cases} 2p & p \text{ odd}\,, \\ p+q & p \text{ even}\,. \end{cases} \tag{5.57}$$

Corollary 5.47 Assume $K = T_{p,q}$ or mirror image. Then

1. each $A(K,t)$ is quasipositive iff it is strongly quasipositive,
2. each $W_+(K,t)$ is quasipositive iff it is strongly quasipositive, and
3. no $W_-(K,t)$ is quasipositive.

Proof.

1. See Corollary 5.6 and K is not slice (and hence not λ-slice).
2. When $K = T_{p,q}$ then use Corollary 5.23. When $K = !T_{p,q}$, then $\lambda(K) > 1$ (because of (2.45) and $\min\deg_a F(K) < 0$), and Proposition 5.21 applies.
3. Use Corollary 5.28 for $K = T_{p,q}$ and Corollary 5.30 for $!K = T_{p,q}$. □

We may point out that, beside a general *knot* K, even for the simplest non-trivial torus *link*, the Hopf link, the two first equivalences are false (see [MS+]).

Corollary 5.48 Assume $K = T_{p,q}$ or mirror image. Then each $A(K,t)$ and $W_\pm(K,t)$ is strongly quasipositive iff it is Bennequin-sharp.

Proof. Use Propositions 5.40 and 5.41. $\square$

Corollary 5.49 Assume $K = T_{p,q}$ or mirror image. Then each $A(K,t)$ and $W_\pm(K,t)$ has a minimal string Bennequin surface, and if strongly quasipositive, a minimal string strongly quasipositive surface.

Proof. Use Corollary 3.9. For $A(K,t)$ we are done.

To use Corollary 3.9 for $W_\pm(K,t)$ (and K being either of $K = T_{p,q}$ or mirror image), we need to show that there is a t' so that

$$\min\deg_v P(A(K,t')) < 0 < \max\deg vP(A(K,t'))$$

and

$$\min\mathrm{cf}\,_v P(A(K,t')), \max\mathrm{cf}\,_v P(A(K,t')) \neq \pm z^{-1}\,. \tag{5.58}$$

Again the point behind these conditions is to show that terms in negative v-degree do not cancel on the right of (5.23).

Obviously we can use (5.51) with $t' = (1-p)q$. For the maximal coefficient we are done for (5.58), since there is a term in positive z-degree.

It remains to look at the minimal coefficient. Now, when we use (3.4), then it is enough to show (5.53). But this was done in Lemma 5.45. So we are done. $\square$

Remark 5.50 We obtain from this calculation that, similarly (and additionally) to (5.58),

$$\min\mathrm{cf}\,_v P(A(K,t')), \max\mathrm{cf}\,_v P(A(K,t')) \neq \pm z\,.$$

(When $p = 2$, see Proposition 3.7.) This is sufficient to conclude that when $K = T_{p,q}$, then the MFW inequality (2.16) is exact on all $A(K,t)$ and $W_\pm(K,t)$. This of course includes the special case of (3.19) for torus knots (with (5.49) and (5.50)).

5.4 Framing diagrams

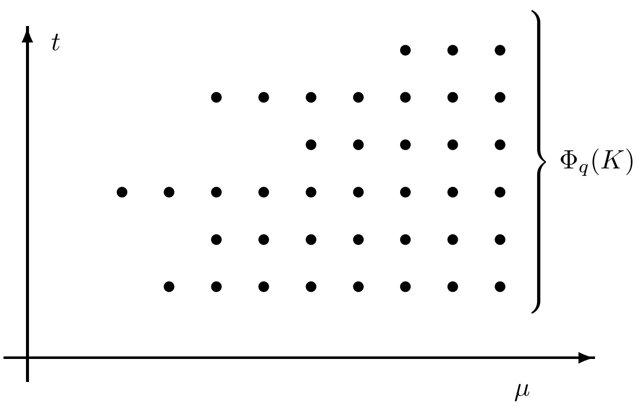
The structure of the analogue of Definition 4.10,

$$\Phi_q(K) := \{\, (\mu,t) \,:\, A(K,t) \text{ has a quasipositive braid representative on } \mu \text{ strands} \,\}\,,$$

remains less clear. Positive *braid* stabilization still implies, as in (4.10), that

$$(\mu,t) \in \Phi_q(K) \Longrightarrow (\mu+1,t) \in \Phi_q(K)\,,$$

which gives a, much poorer, *ray structure* on $\Phi_q(K)$. But the cone structure argument obviously fails so far: in a quasipositive braid representation, the grid is not evident, and grid stabilization makes no sense. That is, we cannot exclude the type of shape:



Nonetheless, some partial information on $\Phi_q(K)$ can be recovered.

Proposition 5.51 When K is any knot, then $\Phi_q(K)$ is *contained in* the union of at most $2+\delta(K)$ cones, and in at most $1+\delta(K)$ cones if K is not slice.

Proof. This is obtained by the same reasoning as for Proposition 4.16. But when K is slice (and $t=0$), then we must allow for (5.4) instead of (3.7) . Since $\Phi_q(K)$ has no cone structure, we can only claim that $\Phi_q(K)$ is contained in a union of cones. □

Finally, we obtain the following way to reestablish the expected shape of $\Phi_q(K)$.

Corollary 5.52 If K is not λ-slice and $\delta(K)=0$, then $\Phi_q(K)=\Phi(K)$ is a single cone.

Proof. Assume K is not slice. Obviously $\Phi_q(K)\supset\Phi(K)$, and from Proposition 4.16 we have

$$\Phi_q(K)\supset\Phi(K)=C(a(K),\lambda_{min}(K))\,,$$

so that $\Phi_q(K)$ contains a cone of the form $C=C(a(K),t)$. On the opposite end, from Proposition 5.51 we know that $\Phi_q(K)$ is contained in a single cone $C(\mu,\lambda_{min}(K))$, and we must have $\mu\geq a(K)$ because $\delta(K)=0$. This is only possible if $\Phi_q(K)=\Phi(K)=C$.

Now if K is slice, but not λ-slice, then the condition $\delta(K)=0$ enables us to use, instead of HOMFLY-PT as in Corollary 5.6, the proved Jones-Kawamuro conjecture (Theorem 2.7), as for Lemma 4.14. It is the type of argument that allows us to state λ-slice in Proposition 5.13 (and was outlined above it). □

In conclusion, we note that it was explained in [Ha] from the work in [LaM] that every quasipositive link has at least one quasipositive minimal braid representative. Thus at least a problem like (4.1) is off the table, and an analogue of much of the discussion in [JLS2, §6.2], for instance, is not very interesting for quasipositivity. But Orevkov's question still stands whether in fact *all* minimal braid representatives of a quasipositive link are quasipositive – an assertion which is obviously false for strong quasipositivity.

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