EXCHANGED BRAIDS FOR ALTERNATING LINKS

A. Stoimenow

Dongguk University, WISE campus,

123, Dongdae-ro, Gyeongju-si,

Gyeongsangbuk-do 38066, Republic of Korea

e-mail: `stoimeno@stoimenov.net`,

WWW: `http://stoimenov.net/stoimeno/homepage/`

Abstract. We prove that if L is a two-bridge link and $n \geq \max(4, b(L))$, then L admits infinitely many non-conjugate n-braid representatives. More precisely, it admits an exchangeable n-braid, all whose iterated positive exchanges are pairwise non-conjugate. We give similar results for alternating pretzel links and alternating knots of given genus. The proofs exploit a connection between algebraic properties of the exchange move and combinatorial features of Seifert circles in link diagrams, which originates from the Vogel move. Another ingredient is the completion of the algebraic non-conjugacy result for iterated exchanged braids.

Keywords: braid group, exchange move, braid index, Seifert circle, alternating link, flype, Vassiliev invariant, pretzel link, linking number.

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1 Introduction and results

Originating from Markov's theorem (§2.1), it is known that n-braid representatives of a given link decompose into conjugacy classes. The question how many such classes occur began to be studied by Birman-Menasco [BM3], and was intrinsically linked to the *exchange move* (see in particular Theorem 2.4). The conjugacy classes for $n \leq 3$ have been determined in [BM, Mu2], and are finite. Until very recently, constructions of infinitely many non-conjugate braid representatives have been given only in very special cases, notably for the unknot and $n = 4$ (see [Fi]). The work in this paper will imply the complete infiniteness result for 2-bridge links. Here $b(L)$ stands for the minimal n for which an n-braid representative of L occurs and is called *braid index*; see §2.1.

Theorem 1.1 If L is a two-bridge link and $n \geq \max(4, b(L))$, then L admits infinitely many non-conjugate n-braid representatives. More precisely, it admits an exchangeable n-braid, all whose iterated positive exchanges are pairwise non-conjugate.

One reason for proposing the result is that its proof (which mostly occupies §4–6) is almost completely different from what one may expect. It has very little to do with braids. Many reviewed facts about braids are needed only for motivating and explaining the statements. Indeed, no braid has to be written down in the proofs. (The one in (49) provided only for consistency.) The argument is nearly entirely combinatorial and relies on exhibiting a specific type of diagram of L (Lemma 5.3). The whole connection to the properties of the exchange move will transpire in §6 and appears to be one of the first broader theoretical payoffs specific to the Vogel move [Vo]. Thus the spirit of the paper is to emphasize that this algebraic/geometric problem has also a combinatorial viewpoint, which is worthwhile sometimes.

It should be readily cautioned that this infiniteness property is far from automatic. For $n > b(L)$, in large generality, infinitely many classes occur [St5] (see Remark 6.7). However, the 4-braid knot 9_{29} has no exchangeable 4-braid [St, Example 8.3], and hence by Birman-Menasco (see Theorem 2.4), it does not admit infinitely many 4-braid conjugacy classes. This is one known lowest crossing example, but among very many. A new insight from the Burau matrix [St2] makes practically clear (and provable!) that a "generic" closed n-braid will have a similar status. Thus the property is (very) false when $n = b(L)$, even under strong restrictions on the link L. This certainly applies to alternating braid links, and thus also fibered alternating links or alternating links of given braid index (at least 4) [Mu].

The two-bridge links are thus among the few meaningfully general classes, for which the claim is true (without exceptions at least). In order not to isolate our study to them, in §7 and §8 we examine our method in the generators-series setting of [St9], and we briefly discuss a relation to the Graph Index Conjecture. In §9 we upgrade our approach to handle the alternating (pure) pretzel links. The below is a simplification of the outcome summarized in Theorem 9.1.

Theorem 1.2 If L is an alternating pure pretzel link, and $n \geq \max(5, b(L))$, then the conclusion of Theorem 1.1 holds.

We also complete, with [SS2, St6], an algebraic analogue of Ito's theorem [I] in §10. It allows some stronger non-conjugacy statement of iterated exchanged braids under generic – and easy to test – (linking number) assumptions, and the proof of Theorem 9.1 will demonstrate its applicability.

2 Link-braid theory

To see the connection between algebra and combinatorics clearly at the end, a number of details must be explained. While we do not carefully discern what is known from what is new, we will gradually move from the former to the latter.

2.1 Braids and braid closures

Most of the terminology on braids follows [SS1, SS2, St, St6], and one may consult there for additional details.

The Artin *braid group* B_n [Ar, Ar2] on n strands can be defined by

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i & |i-j| = 1 \end{array} \right\rangle .$$

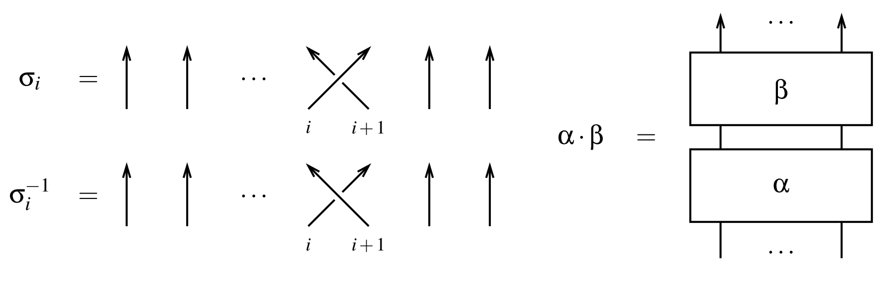
Herein σ_i are called the *Artin standard generators*. An element $\beta \in B_n$ is an *n-braid*.

A braid β has an *associated permutation* $\pi(\beta)$, given by the homomorphism

$$\pi : B_n \to S_n , \qquad \pi(\sigma_i) = \tau_i = (i\; i+1) .$$

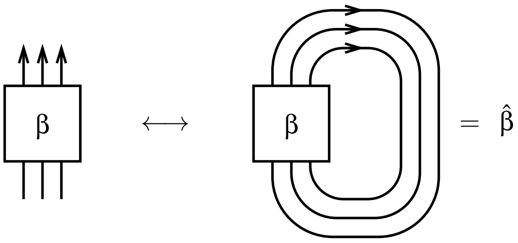
If $\pi(b) = Id$, we call b a *pure* braid. We write $P_n \subset B_n$ for the pure braid group. For convenience, we will often identify a cycle of $\pi(b)$ with its elements (being a subset of $\{1, \dots, n\}$).

There is a graphical calculus of diagrams representing braids, wherein for σ_i (resp. σ_i^{-1}) strands i and $i+1$ cross *positively* (resp. *negatively*), and multiplication is given by stacking:



(1)

Every braid β has a *closure* $\hat{\beta}$, given by joining strands on top and bottom:



(2)

This closure is a *knot* $S^1 \hookrightarrow S^3$ or (more generally) *link* $S^1 \cup \ldots \cup S^1 \hookrightarrow S^3$.

Note that the number of *components* of $\hat{\beta}$, i.e., number of embedded S^1, is equal to the number of cycles of the associated permutation $\pi(\beta)$.

Theorem 2.1 (Alexander '23 [Al]) Any link L is the closure of a braid β.

Definition 2.2 $b(L) := \min\left\{ n \in \mathbb{N} : \text{there is a } \beta \in B_n \text{ with } \hat{\beta} = L \right\}$ is the *braid index* of L.

We call β a *braid representative* of L. If $n = b(L)$ then β is called *minimal*. The computation of $b(L)$ is not trivial in general, however, some inequalities are known [FW, Mo, Oh, St4], and for two-bridge links (and some other classes) it has been determined by Murasugi [Mu].

Theorem 2.3 (Markov '35, Birman '76 [B]) If $\hat{\beta}_1 = \hat{\beta}_2$, then $\beta_{1,2}$ are related by a sequence of

1. conjugacies in the braid group $\beta \longmapsto \alpha\beta\alpha^{-1}$
2. (de)stabilizations $B_n \ni \beta \longleftrightarrow \beta\sigma_n^{\pm 1} \in B_{n+1}$ (where on the right β is of course understood via the obvious embedding $B_n \hookrightarrow B_{n+1}$).

The study of braid closures and Markov's theorem experienced a dramatic increase in importance in the 80's through the construction of link invariants following Jones [J].

The first Markov move implies that $\{\beta \in B_n : \hat{\beta} = L\}$ is a union of conjugacy classes. Our question is whether for given n and L this number is finite or not. There are ways to handle the conjugacy problem, at least on individual examples, on a group-theoretic footing (e.g., [Ga, BKL]). The difficult part in Markov's theorem is the second move, which leads to the question: how does it relate different conjugacy classes?

We will write below for $\alpha, \beta \in B_n$

$$\alpha \sim \beta \quad \text{for} \quad \alpha \text{ is conjugate to } \beta\,.$$

See also Remark 6.7 on the role of [St5] when $n > b(L)$.

As a final remark in this section, nothing goes wrong in Theorem 1.1 under the mirroring automorphism of B_n, exchanging $\sigma_i^{\pm 1}$ in braid representatives. Neither does it under the anti-automorphism of B_n of word reversal. This means that we can freely mirror the link L and reverse orientation, *when orientation of all components is simultaneously changed* (for a link; compare with §3.1).

2.2 Subbraids and linking numbers

Let $b \in B_n$. We number in b strands $1, \ldots, n$ at the bottom from left to right in the explained graphical calculus (1). Strands are propagated bottom-up and assumed exchanged at crossings. (Thus they do not appear in order $1, \ldots, n$ at every horizontal section of the braid.)

For every $C\subset\{1,\dots,n\}$, there is a *subbraid* $b_{[C]}$ obtained by deleting in b strands outside C. We have the properties

$$(\sigma_i^{\pm1})_{[C]}=\begin{cases}\sigma_{i'}^{\pm1} & \text{for } i'=|C\cap\{1,\dots,i\}| \text{ if } i,i+1\in C\\ Id & \text{otherwise}\end{cases}$$

and

$$(\alpha\beta)_{[C]}=\alpha_{[C]}\beta_{[\pi(\alpha)(C)]}\,,$$

which suffice to define it formally. This is a map $\cdot_{[C]}:B_n\to B_{|C|}$ which is not a homomorphism (but one if restricted onto P_n).

We call a crossing in (1) for σ_i *positive*, and one for σ_i^{-1} *negative* (*cf.* Definition 3.1). Then for every $1\le i<j\le n$, there is a *linking number* $lk_{i,j}=lk_{i,j}(b)$ defined as half the sum of the signs of all crossings (exponents of Artin generators) involving braid strands i,j. Again this can be easily formalized algebraically:

$$lk_{i,j}(\sigma_k^{\pm1})=\begin{cases}\pm1/2 & \text{if } \{i,j\}=\{k,k+1\}\\ 0 & \text{otherwise}\end{cases}$$

and

$$lk_{i,j}(\alpha\beta)=lk_{i,j}(\alpha)+lk_{\pi(\alpha)(i),\pi(\alpha)(j)}(\beta)\,.$$

If $(\pi(b)(i)-\pi(b)(j))(i-j)>0$, the result is an integer, otherwise a half-integer.

Similarly set for $C_{1,2}\subset\{1,\dots,n\}$ (with $i\notin C_1$ and $C_1\cap C_2=\varnothing$)

$$lk_{i,C_1}=\sum_{j\in C_1}lk_{i,j}\,,\qquad lk_{C_1,C_2}=\sum_{i\in C_1}lk_{i,C_2}\,.$$

In particular when $C_{1,2}$ are cycles of $\pi(b)$, then lk_{C_1,C_2} is the linking number of the components $L_j=\hat b_{[C_j]}$ in the link $\hat b$ (see §3.1).

2.3 Exchange move

To simplify the problems of stabilization, and in particular not to switch between different B_n, Birman-Menasco in a series of papers [BM, BM2, BM3, BM4, BM5, BM6] extensively investigated the *exchange move*. The exchange move was apparently discovered by Markov in an earlier version of his theorem, but later showed a consequence of his other two moves. Birman and Menasco's work then, however, restored much of its due prominence.

There are a few (equivalent) versions of the move. The one given here is different from [SS1, SS2] (thus the rewording of Theorem 6.1 below, for instance).

Let $n\ge4$. We say that $b\in B_n$ *admits an exchange move* or is *exchangeable*, if b is of the form

$$b=b(\alpha,\beta)=\sigma_1^{-1}\alpha\sigma_1\beta\,,\tag{3}$$

where α,β are words that do not involve a letter $\sigma_1^{\pm1}$.

The *exchangeable crossings* will be those corresponding to $\sigma_1^{\pm1}$ in (3).

An (iterated) *exchange move* is the transformation between the braid b and the braids

$$b_m=b_m(\alpha,\beta)=b(\delta^m\alpha\delta^{-m},\beta)=\sigma_1^{-1}\delta^m\alpha\delta^{-m}\sigma_1\beta\,,$$

where $\delta=(\sigma_3\cdots\sigma_{n-1})^{n-2}$ is the "full twist" on strands $3,\dots,n$. Here m is some non-zero integer. We can set $b_0=b$.

Of course, no non-trivial braid on 2 strands admits an exchange move, and all exchange moves on 3 strands are trivial, so that *we will naturally assume $n\ge4$ throughout*.

It is easy to observe that $\hat{b}_m = \hat{b}$. There are a few minor caveats like why b_m does not depend on how b is written as in (3), but they can be settled without too much trouble; see for this the formalization of an *exchangeable structure* in [St, St6].

The alternative version (4) of exchangeability is

$$b = \alpha'\beta', \quad \text{where } \beta' \text{ involves no } \sigma_1^{\pm 1} \text{ and } \alpha' \text{ no } \sigma_{n-1}^{\pm 1}. \tag{4}$$

By conjugating $\alpha' = \gamma^{-1}\alpha''\gamma$ for $\gamma = \sigma_1 \cdots \sigma_{n-1}$, and cyclically permuting letters, one can see the form (3).

In this second variant, which occurred in [SS1, SS2], the exchange move takes the form $b_m = \alpha'\delta'^{-m}\beta'\delta'^m$, where $\delta' = (\sigma_2 \cdots \sigma_{n-2})^{n-2}$ (full-)twists strands 2 to $n-1$ (incl.) There are no exchangeable crossings.

This second variant will will not be deployed much here, except for §10 and its application. But it does also have the noteworthy consequence (which will be used; see Remark 6.7) that *stabilized* braids, i.e., such obtained by the second Markov move, are exchangeable.

Theorem 2.4 (Birman-Menasco [BM3]) The n-braid representatives of a given link decompose into a finite number of classes under the combination of exchange moves and conjugacy.

We can say that the *positive* resp. *negative* exchange move is the one changing b to b_1 resp. b_{-1}. Note also the relationship

$$(b_m)_{m'} = b_{m+m'}\,. \tag{5}$$

By observing the formula

$$b_m(\alpha,\beta) \sim b_{1-m}(\beta,\alpha)\,, \tag{6}$$

one sees that even among *conjugate* exchangeable braids, the sign of m cannot be unambiguously specified. However, it can when the exchangeable braid is *fixed*, and this is pertained to in the abstract claim (we will return to in Lemma 6.5).

We will be concerned with the injectivity of (restrictions of) the map

$$m \quad \mapsto \quad (\text{conjugacy class of } b_m). \tag{7}$$

The property of *subsymmetry (SS)* of an exchangeable braid b was found in [SS2], and can be formalized by saying

$$\text{there is a } \mu \in \mathbb{Z} \text{ such that whenever } b_m \sim b_{m'} \text{ (for } m \neq m'\text{), then } m + m' = \mu. \tag{8}$$

This includes the situation that all b_m are pairwise non-conjugate, i.e., the map (7) is injective, in which case any μ will do. Otherwise, of course μ is unique.

The condition (8) does not appear very natural at first, but there are more than technical reasons why it emerges. As a minor indication, put $\alpha = \beta$ in the conjugacy (6); this gives examples for $\mu = 1$. Translating m over $\mathbb{Z}$ will yield any other odd μ. There are computational examples for even μ as well, but their occurrence for large n is unclear so far. While the above construction shows that in general subsymmetry cannot be further improved, it is very possible that except for trivial (Ito's *degenerate*) cases, the property is universal ([St, Conjecture 6.8]). See [St, St6] for further discussion.

We will crucially return to this property in §6 with Theorem 6.1.

2.4 Conujugacy Vassiliev invariants of braids

For technical reasons (§10) we recap also the weaker version of subsymmetry called *quasi-subsymmetry (QSS)*, which is defined by the existence of a finite set $S \subset \mathbb{Z}$, so that (8) holds whenever $m, m' \notin S$.

We call $\upsilon : B_n \to \mathbb{Z}$ a *conjugacy Vassiliev invariant* of n-braids, if υ is both a conjugacy invariant (i.e., coincides on conjugate braids), and is a Vassiliev invariant of braids, in the sense of [BN]. The following standard argument will be used.

Proposition 2.5 ([St, Proposition 6.5]) If there is a conjugacy Vassiliev invariant υ that distinguishes two b_m, then b is QSS.

Here we add a clarification (which needs to complement the discussion of Proposition 2.5, as given in [St]). If we aboslish conjugacy, it is known (due to Kohno) that *non-equal* pure braids will be distinguished by some (potentially *conjugacy-sensitive*) Vassiliev invariant. Bar-Natan [BN] explains this completeness of Vassiliev invariants of pure braids, and proves that they all come from $gl(N)$ (and that his approach works for non-pure braids as well). These invariants are related to the HOMFLY polynomial and its cables of the braid closure[1]. But at any event, since b_m have the same closure, this viewpoint cannot be very helpful for us.

If we consider (pure) braids up to conjugacy (i.e., in the solid torus), completeness of (conjugacy) Vassiliev invariants is not clear, as far I know. Among the impact of such a result would be that Vassiliev invariants can distinguish the orientation of some axis addition links (see [SS1, SS2] or §9.2 below). More relevantly here, Ito's theorem (as mentioned below Theorem 1.2) would be readily upgraded by Proposition 2.5 to QSS, comprising Theorem 10.3, and yielding a "quasi"-confirmation of the brought up Conjecture 6.8 in [St]. But a very different approach is likely needed, since the combed form in [BN] (which resurfaces here in some simple variant in (65), for instance) is very uncontrollable under conjugacy.

It should be remarked that all the invariants of [SS1, SS2, St6] we used to distinguish b_m, as well as later in §10, are (interconvertible to conjgacy) Vassiliev invariants. Proposition 2.5 has thus a broad practical merit. But the efforts in these references have protruded that and why properly finding and evaluating such an invariant is not straightforward.

3 Links and diagrams

3.1 Link diagrams and linking numbers

A *link diagram* is a collection of planar curves with transverse intersections, *crossings*, at each of which one crossing strand (*overcrossing*) is distinguished. (This can be very easily formalized in terms of planar 4-valent graphs. Multiple edges must be allowed, and loop edges can, but will not be relevant to us.) We will generally assume that curves are *oriented*.

Links can be understood as diagrams modulo *Reidemeister moves*. (We refer to [Li] for their definition.) The nomenclature follows the tables in [Ro, appendix].

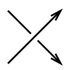
A diagram is *alternating* if every curve passes crossings alternatingly over-under. An alternating link is a link with such a diagram.

We write $c(D)$ for the *number of crossings* of a link D, and $c(L)$ for the *crossing number* of the link L, which is the minimal crossing number of all its diagrams.

A *region* of a diagram D is a connected component of the complement of D in the plane. Diagrams are considered up to homeomorphisms of the plane. Also unless we specify otherwise, the region at infinity is not fixed, thus diagrams are effectively considered up to moves in S^2.

A *component* a diagram (or link) is one of the embedded S^1 (depicted by the diagram). A link is a *knot* if it has one component.

We call a crossing of a component with itself a *self-crossing* and a crossing of two different components a *mixed crossing*. For instance, the diagram (47) has 2 components; each has one self-crossing, and there are 10 mixed crossings.

Definition 3.1 A crossing in an oriented diagram looking like  is called *positive*, and  is a *negative* crossing. This dichotomy is called also *(skein) sign*.

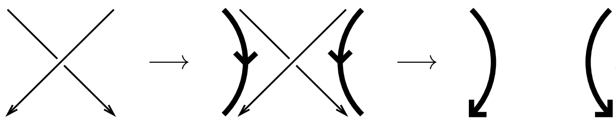
[1] but obviously not determined by them, as stated imprecisely in Corollary 4.1 or [BN]. There must be conjugacy-sensitive Vassiliev invariants, which cannot come from any closures. I.e., conjugate (or Markov equivalent) braids cannot be distinguished by any invariant of the closure.

Let $L_{1,2}$ be two components of a link (diagram). There is a *linking number* $lk_{L_1,L_2} = lk_{L_1,L_2}$ defined as half the sum of the signs of all mixed crossings involving components L_1, L_2. This is an integer.

We can define the *linking graph* of L by a vertex for each component L_j of L, and an edge with label lk_{L_i,L_j} connecting vertices L_i and L_j. An edge not drawn is set to have label $lk_{L_i,L_j} = 0$.

3.2 Seifert circles

When in an oriented link diagram every crossing is *smoothed out*

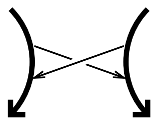
 , (9)

one obtains a collection of oriented planar loops called *Seifert circles*. Our convention will be to draw Seifert circles through a thicker (than the link diagram) solid line. Note the Seifert circles naturally inherit an orientation from the underlying diagram.

We write $s(D)$ for the *number of Seifert circles* of D.

A particular but for us later relevant example is given in (48). It is a 2-component link diagram with 4 Seifert circles and 10 crossings.

While often on the right of (9) the crossings smoothed out can be fully removed (as in (48)), it is sometimes useful to keep track of their location. That is, we can modify the right of (9) to

 (10)

In such terms, we can speak of a crossing *connected* to a Seifert circle. When a crossing p is connected to Seifert circles a, b, we say that *a is (immediately) attached* to b (and vice-versa).

An example of the style in (10) with 4 Seifert circles and 2 crossings (although somewhat disproportionally drawn) is given in (16).

Crossings attached to a Seifert circle split the Seifert circle into *Seifert circle arcs*, each of which belongs to some component of the link. (Seifert circle arcs are thus *not* the same as Seifert arcs, as will be encountered in §3.4.)

A crossing never connects a Seifert circle with itself; in fact, connection has a certain *bipartacy* property. Define the *Seifert graph* $\Gamma(D)$ by a vertex for every Seifert circle and an edge for every crossing. This graph is then planar and bipartite. Note that we must allow multiple (parallel) edges.

The collection of Seifert circles of a diagram and their location in the plane will be called *Seifert picture*. Note that the Seifert picture does not depend on how crossings of the diagram are switched.

3.3 Conway polynomial

We only briefly introduce the *Conway polynomial*. See [SS1, SS2] for more extensive treatise in our context. Every (oriented) n-component link L has an integer polynomial $\nabla(L) = \nabla(L)(z)$, with terms z^d occurring only if $z \geq n-1$ and $z-n$ is odd. Let $[\,.\,]_{z^d}$ denote the coefficient.

We write for later reference for a braid β,

$$\nabla_d(\beta) = [\nabla(\hat{\beta})]_{z^d}\,. \tag{11}$$

This can be regarded as a conjugacy Vassiliev invariant of degree d of braids β.

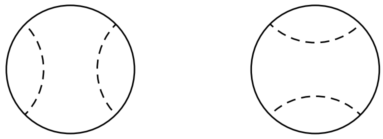
We need the following formula of Crowell-Murasugi ([Mu3]): if D is an alternating non-split link diagram, then

$$\max\deg_z \nabla(D) = 1 - s(D) + c(D)\,. \tag{12}$$

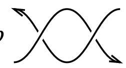
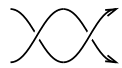
3.4 Tangles

Definition 3.2 A *tangle* Y is a set of two *arcs* or *strands*, properly embedded in a ball $B(Y)$. Tangles are considered up to homeomorphisms of $B(Y)$ that keep fixed its boundary.

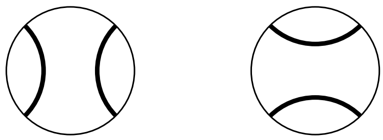
It is not necessary here to allow closed components inside the tangle. A tangle will be suggestively depicted as a planar circle with strands connecting four points in the boundary. When the circle is evident from the four points, it will be omitted. Our convention will also be that a dotted line indicates the *connectivity* of the tangle, i.e., which pair of endpoints are connected by the same tangle strand.



(13)

A *clasp* is a tangle with 2 crossings bounding a 2-gon region. In an oriented diagram a clasp is called *positive*, *negative* or *trivial*, if both crossings are positive/negative, respectively of different sign. Depending on the orientation of the involved strands we distinguish between a *reverse clasp*  and a *parallel clasp* . So a clasp is reverse if it contains a full Seifert circle, and parallel otherwise.

The construction of Seifert circles can be done within tangle diagrams as well. We again depict a Seifert circle inside a tangle by a thick line. The main difference to link diagrams is that there will be two arcs that connect the tangle endpoints that do not close off to circles. There are again two choices in which these *Seifert arcs* can connect.



(14)

These options are in general, of course, different from the (dashed) connectivity arcs (13). Thus one must distinguish between the tangle's arcs (or strands) and its Seifert arcs.

3.5 Braid diagrams and braid algorithms

Under closure (2), braids give links and braid *words* give link *diagrams*. In most situations, it will not be needed to separate between the two types of correspondence.

However, here we are in particular interested in the diagrams $\hat{\beta}$ that can be obtained by the operation (2). We call these diagrams *(closed) braid diagrams*. It will be important below to understand well what kind of link diagrams arise this way, when diagrams are considered up to moves in S^2 (changes of the region at infinity). This requires some thought on the nesting of Seifert circles.

It will be helpful here to fix the infinite region of D. Accordingly, every Seifert circle has an *interior* and *exterior*. We say that b,b' *lie on the same side of* a, if both b,b' lie in the interior of a or both b,b' lie in the exterior of a.

We say two Seifert circles a,b are *coherent*, if they bound a common region R and are oriented oppositely w.r.t. R's boundary, i.e., one is oriented with the induced orientation, and one is oriented against it. (Region is again understood in the complement of the Seifert circles.)

Definition 3.3 Let a,b be Seifert circles of a link diagram.

1. When b lies in the interior of a, we say *a contains b*. When b lies in the exterior of a, we say b is *outside a*.

2. We say that a Seifert circle is *empty* or *innermost* if it contains no other Seifert circle.

3. We say that a Seifert circle is *separating* if it contains other Seifert circles in both interior and exterior, and otherwise *non-separating*.

4. We say it is *outermost* no other Seifert circle contains it. (This is *not* the same as saying that its exterior is empty.)

5. We say that *a immediately contains b* if a contains b and there is no Seifert circle c such that a contains c and c contains b.

6. We say that a is *braid-like* if the following is true. Let b be contained in a or $b = a$. Then either b is empty, or b immediately contains exactly one Seifert circle c, and this Seifert circle c is coherent with b.

7. We say that a is *maximal braid-like* if it is braid-like and it is not contained in a braid-like Seifert circle.

A diagram is a (closed) braid diagram if and only if all Seifert circles are braid-like, and there is one outermost (or maximal braid-like) Seifert circle, or two outermost Seifert circles, which are coherent.

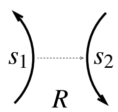
A simpler way of saying this is that there are exactly two Seifert circles which have either empty interior or exterior, i.e., are non-separating. (This form is used in Proposition 7.4.)

Write $s_s(D)$ is *the number of separating Seifert circles*. Thus a diagram D is a braid diagram if $s_s(D) = s(D) - 2$. If $s_s(D) = 0$, we call D *special*.

It is necessary to review the (now likely most common) algorithms that transform a general link diagram into a (closed) braid diagram.

The *Vogel move* [Vo] is a Reidemeister II move of a special type. It affects two edges $e_{1,2}$ in the boundary of a region R, which have the *same orientation* as seen from inside R, and belong to *distinct Seifert circles* $s_1 \neq s_2$.

'Same orientation' should be henceforth understood so that *none or both* of them coincide with the induced orientation on the boundary of R. Let us call $e_{1,2}$, regarded as arcs of the Seifert circles s_1, s_2, as well as s_1, s_2 themselves, *locally incoherent*. The Vogel move then creates a reverse trivial clasp inside R (while preserving the number of Seifert circles). We will indicate the move by a dashed arrow (lying within R) connecting part of $e_{1,2}$.



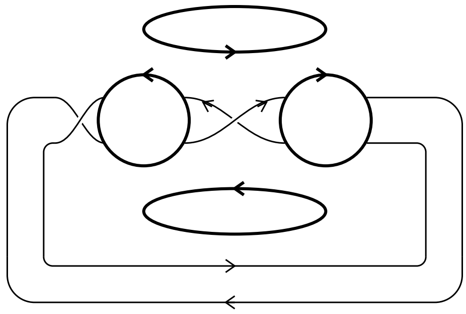
(15)

An example of such a move is indicated in (48).

Vogel explains that D is a closed braid diagram if and only if D has no pair of locally incoherent Seifert circles. He proves that a sequence of such moves always transforms any link diagram into a closed braid diagram (of the same number of Seifert circles). This gives an elegant proof of Theorem 2.1.

Another way that braid diagrams are commonly specified is through the absence of incoherent Seifert circles. Let us call two *distinct* Seifert circles of a diagram D to be *incoherent*, if they bound a common region *R in the complement of the union of the Seifert circles of D* and if their orientation is the same w.r.t. R.

There is a subtlety here: locally incoherent Seifert circles are incoherent, but if two Seifert circles are incoherent, they may not be locally incoherent, since they may not bound a common region of D. An example is:

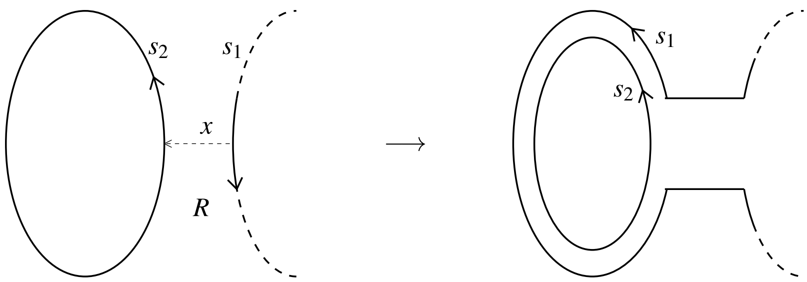


(16)

But it is very easy to see that the *existence* of either type of 'incoherent' pairs of Seifert circles is equivalent.

Another braid algorithm was previously found by Yamada [Y].

Let (s_1, s_2) be a locally incoherent pair, connected by an arc x directed from s_1 to s_2 in a region R of the diagram. We will use the Yamada move [Y], by threading inside R a small piece of s_1 along x to become close to s_2, and laying it outside along s_2:



(17)

Yamada's algorithm was the first algorithm that rendered a braid diagram *without changing the number of Seifert circles*. We will use the Yamada move (in combination with Vogel's) for proving Proposition 8.1 and related arguments. Its important feature here is that it does not alter the interior of s_2.

3.6 Two-bridge links

Rational or *two-bridge* links are a well-known class of links that have been extensively studied. Standard works like [Li, Ro] offer their own treatise. We will reduce ours here to what we need to define and use them.

Figure 1 shows the elementary tangles, tangle operations and notation, mainly leaning on Conway.

For two tangles Y_1 and Y_2 we write $Y_1 + Y_2$ for the *tangle sum*. This is a tangle obtained by identifying the NE end of Y_1 with the NW end of Y_2, and the SE end of Y_1 with the SW end of Y_2. The *closure* of a tangle Y is a link obtained by identifying the NE end of Y with its NW end, and the SE end with the SW end.

The closure of $Y_1 + Y_2$ is called *join* $Y_1 \cup Y_2$ of Y_1 and Y_2. Note that representing a diagram D as a join $Y_1 \cup Y_2$ is equivalent to specifying a loop in the plane intersecting D in 4 edges; $Y_{1,2}$ are obtained from the interior and exterior of the loop.

Definition 3.4 A *rational* tangle diagram is the one that can be obtained from the primitive Conway tangle diagrams by iterated left-associative product in the way displayed in Figure 1. (A simple but typical example of is shown in the figure.)

Note that tangle 'sum' is not commutative, and 'product' is neither commutative nor associative. The default associativity of the product is thus set left. (In §9 we will consider the result of right-associative product.)

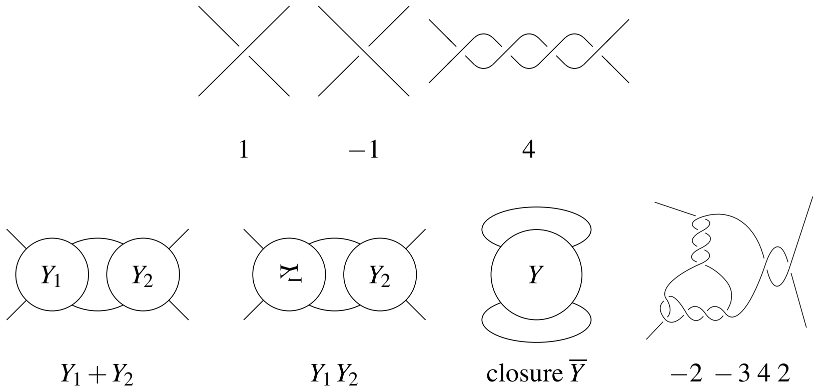


Figure 1: Conway's primitive tangles and operations with them.

Crossings that belong to the same letter in the Conway notation will form a *twist*.

For well-known reasons, for a rational link (*not* generally a tangle) one can restrict oneself to even integers. This will be very important below, so let us adopt this standpoint.

Accordingly, we describe first rational tangle we need by a sequence

$$T(a_p, a_{p-1}, \dots, a_2, a_1) \tag{18}$$

of even non-zero integers a_i. For simplicity, we will abbreviate a repeated entry by

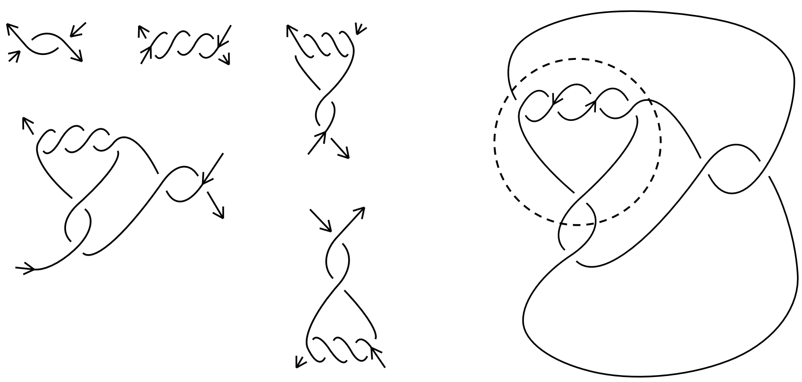
$$(\dots, \underbrace{a, a, \dots, a}_{k \text{ times}}, \dots) = (\dots, a^{[k]}, \dots)\,.$$

Our convention is hereby that

- twists are added from right to left in the vector $(a_i) = (a_p, \dots, a_1)$,
- the sign of a_i reflects the sign of the crossings in the twist (as per Definition 3.1), and
- strand orientation is always so that the twists are reverse (i.e., form reverse clasps).

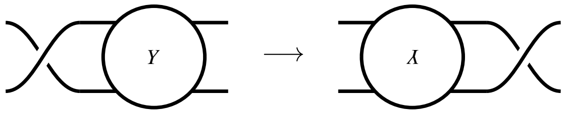
This determines the orientation, up to simultaneous change of orientation of all strands. For the reasons outlined at the end of §2.1, this ambiguity poses no problem.

A few examples are shown below.


(19)

On the left we have $T(-2)$, $T(4)$, $T(-4,-2)$ and $T(4,-2,2)$.

A notice is added here. Under *flypes*, there is a freedom of moving twists added on bottom to the top; and similarly for left and right.



(20)

Hereby the effect on Y the (spatial) π-rotation along the horizontal axis in the plane, not a reflection along it. (We refer to [Li] for more treatise on the flype.)

This is why the left of (19) features two different versions of $T(-4,-2)$, so related. Our understanding will be, therefore, that (a_i) determines $T(a_i)$, but its diagram only up to flypes.

A *rational (or 2-bridge) link*

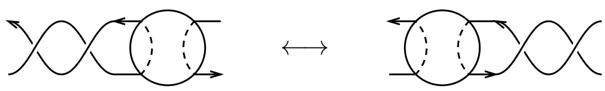
$$L(a_i) = L(a_p,\dots,a_1) \tag{21}$$

is the closure of a rational tangle $T(a_i)$. (Closure could be performed so that the twist a_1 cannot be undone.) We call a twist of self-crossings a *self-twist* and a twist of mixed crossings a *mixed twist*.

The right of (19) shows $L(4,-2,2)$, the closure of one of the tangles on the left. For further examples, $L(2,-2,2,-2)$ is the knot 8_{12}, and $L(2^{[k]})$ is the $(2,k+1)$-torus link (parallely oriented).

It is very well known that all two-bridge links arise this way. (One can also use the P move of §4.1 to see that all they are alternating.) The link is a knot if p is even and has 2 components when p is odd. Note also that in the latter case the twist of a_i is a self-twist precisely when i is even.

The freedom to perform flypes on $T(a_i)$ will be very important in the following. We will mainly apply it at a self-twist a_i for 2-component links; note in particular the connectivity of the tangle $T(a_{i-1},\dots,a_1)$ in this case:



(22)

Next observe that for a 2-component link, every tangle $T(a_i,\dots,a_1)$ features both components of $L(a_p,\dots,a_1)$ in its two tangle strands.

The useful feature of the effect of the flype (22) at a self-twist a_i is then that, for a 2-component link, it changes the self-crossing component.

Reversal of the sequence gives the same link, and negating all entries its mirror image, the distinction of which for the purpose of Theorem 1.1 (and all its related claims) can well be ignored (see the end of §2.1).

It will be helpful to use that a *split* of the vector (a_i), which we will indicate by a vertical bar, will result in a decomposition of the diagram of $L(a_i)$ as a join of two rational tangles. For example,

$$L(-4,2|,-2,2) \tag{23}$$

separates the 10 crossing diagram as a join of tangles with 6 and 4 crossings.

It is possible, and will be helpful, to separate in this decomposition crossings in the same twist. E.g., the dashed loop on the right of (19) defines the join decomposition

$$L(4,-1|-1,2). \tag{24}$$

Something that should be noted and kept in mind is that in any tangle $T(a_k,\dots,a_1)$, as well as in any of the tangles of the decompositions of the type (23) and (24), *both tangle strands belong to the two different components of the link* $L = L(a_p,\dots,a_1)$ (when not a knot).

4 Seifert circle reduction process of two-bridge links

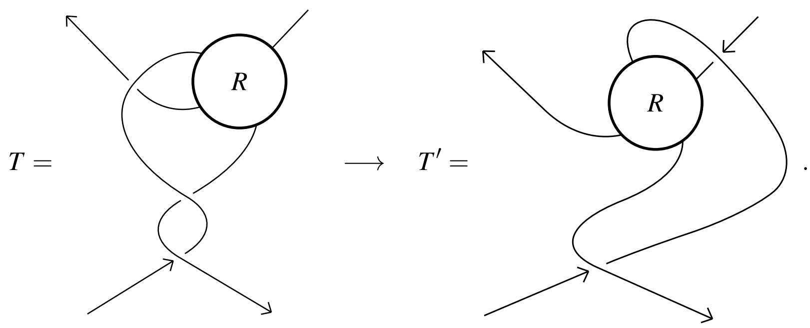
It should be noted that when the Seifert circles of a diagram $L(a_i)$ are drawn, there is one Seifert circle which intersects every tangle decomposition loop like in (23) but *not* like in (24). We call this the *big Seifert circle*.

The braid index of the link $L = L(a_i)$ was known to Murasugi [Mu], and we describe now a procedure we will use to generate a minimal (number of) Seifert circle diagram of L, starting from $L(a_i)$.

4.1 pass move

Definition 4.1 Let us say that a pair (a_i, a_{i+1}) for $1 \leq i < p$ in (21) is *alternating* if $a_i a_{i+1} < 0$ and *non-alternating* otherwise.

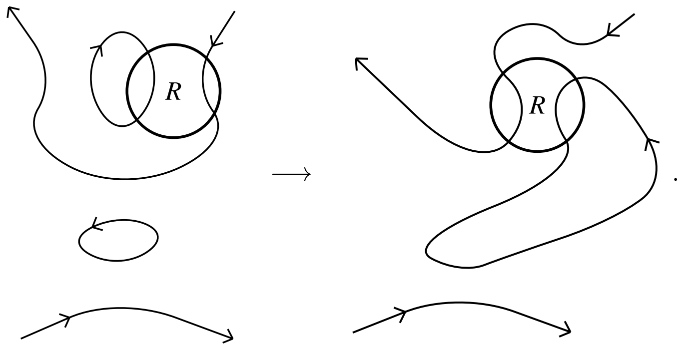
The *P move (pass move)* acts on each non-alternating pair as follows:



(25)

The twist on the left of R is drawn with only one crossing. There is at least one other crossing, which is absorbed into R. We apply P moves *recursively from right to left* in the notation (21), so that the crossing absorbed into R could have been removed by a previously applied instance of the P move.

The absorption of this crossing also fixes the connectivity of the Seifert arcs in R, and the Seifert picture becomes



(26)

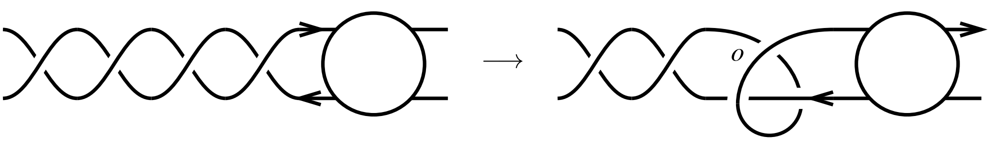
Obviously one Seifert circle is reduced. It will be quite important to observe and keep in mind below that *the P move does not change the connectivity of the Seifert arcs*.

Note also that these properties do not change when the lowermost crossing in (25) and its smoothing in (26) are removed. Except for this removal, all following P (or OT) moves do not affect the Seifert picture inside the tangle T' of (25).

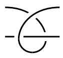
When we use tangle decomposition of L, we prefer the form (24), leaving at least one crossing of a twist on either side of the loop. This will allow us to perform all necessary P moves within both joined tangles separately.

4.2 overtwist move

The other type of move we need is for twists of $|a_i| \geq 4$ crossings. We call this an *OT-move (overtwist move)*



They should be applied on pairs of crossings so that at least one crossing of $|a_i|$ remains, and is left- or rightmost (see (35) for some variants of the move).

The newly created crossing o will be called *overtwist crossing (OTC)*; it connects a newly formed Seifert circle within , which we call the *overtwist Seifert circle (OTS)*. It is clear that, like the P move, the OT move also keeps the connectivity of the Seifert arcs.

Of course, tangle diagram moves (or tangle isotopies) preserve the connectivity of the tangle arcs as well. (For us tangle diagram moves will be flypes, OT move and P moves.) In particular this means that there is a natural identification of arcs between tangle diagrams differing by tangle diagram moves. This identification will be assumed for instance in Lemma 5.1.

4.3 Seifert circle count

For every twist of a_i, there are $|a_i|-1$ Seifert circles added to the big Seifert circle in the diagram (21).

Thus

$$s(L(a_i)) = 1+\sum_{i=1}^{p}|a_i|-1\,. \tag{27}$$

We have

$$\sum_{i=1}^{p}\frac{|a_i|-2}{2} \tag{28}$$

OT moves to apply, and

$$\big|\{\,1\le i<p\,:\,a_ia_{i+1}>0\,\}\big| \tag{29}$$

P moves. Thus the number of Seifert circles at the end is

$$b(L) = (27)-(28)-(29) \,=\, 1-\big|\{\,1\le i<p\,:\,a_ia_{i+1}>0\,\}\big|+\sum_{i=1}^{p}\frac{|a_i|}{2}\,, \tag{30}$$

which is Murasugi's formula for the braid index of L.

To conclude this part, we observe from (30) that $b(L)\ge 4$ if and only if one of the following five scenarios occur, up to mirroring. (We use Definition 4.1.)

1. There are two *disjoint* alternating pairs (a_{i_1},a_{i_1+1}) and (a_{i_2},a_{i_2+1}), for $i_2>i_1+1$.

2. There is an alternating pair (a_i,a_{i+1}) and a j with $|a_j|\ge 4$ (where $j\in\{i,i+1\}$ is possible).

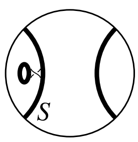
3. There are two j with $|a_j|\ge 4$, where w.l.o.g. we may assume that all a_i have the same sign.

4. There is one j with $|a_j|\ge 6$ (and all a_i having the same sign).

5. $(a_p,\dots,a_1)=(2^{[k]},-2,2^{[l]})$.

5 Constructing specific link diagrams

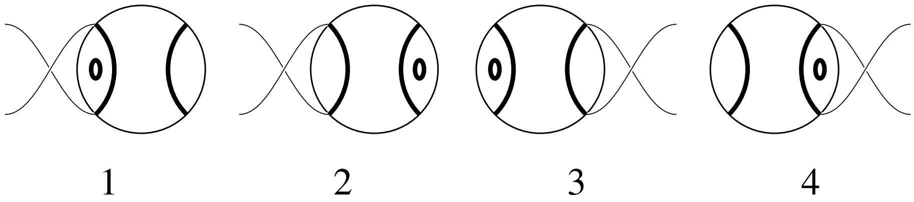
We can obviously say that a Seifert circle in a tangle Seifert picture is empty. It is evident what means that c is directly attached to a Seifert arc S. We say that c is directly outside attached if it is contained in the region of the complement of S (within the disk) not containing the other Seifert arc.



(31)

In th below lemma we will separate the crossing which may be affected by a following P move, and draw the rest of the tangle in a disk though its Seifert picture.

We will seek the existence of an OT Seifert circle directly attached outside to a(n obviously unique) Seifert arc. There are four options (where for simplicity we do not draw the attaching crossing(s)):



(32)

Lemma 5.1 Assume $T = T(a_k, \dots, a_1)$ and some $|a_i| \geq 4$. Assume also that

$$a_j a_i > 0 \text{ for all } j > i. \tag{33}$$

Then it is possible to perform flypes in T and apply the maximal number of OT and R moves so that there is an empty OT Seifert circle attached outside to any given of the two Seifert arcs. By distinguishing the crossing in T that may be consumed by the following P move, we can realize all 4 options in (32).

Moreover, in options 1 & 4 jointly, one can choose the OT crossing to belong to any given of the two arcs of T.

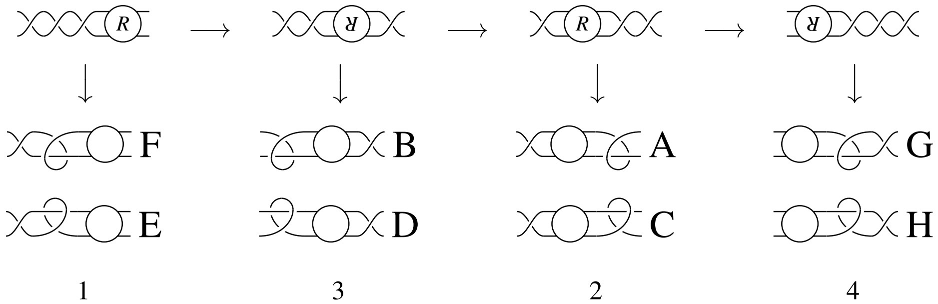
Similarly either arc can be chosen to host this crossing in options 2 & 3 jointly.

Proof. By choosing the largest i, we may in addition to (33) assume

$$|a_j| = 2 \text{ for all } j > i. \tag{34}$$

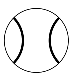
We do induction over $k - i$.

To start the induction, assume $i = k$ and $|a_i| \geq 4$. Consider the below diagram of options to flype and perform an OT move. The last row indicates the type in (32).

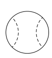


(35)

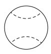
We draw only 3 crossings in the twists, as one may have been consumed by a previous P move inside the tangle $T(a_{k-1}, \dots, a_1)$ of the disk.

Since such P moves would not change the connectivity of the Seifert arcs, the Seifert arc connectivity is always . In all transformations we must take care that a crossing of $|a_i|$ remains on the left or right, so a possible following P move can be applied.

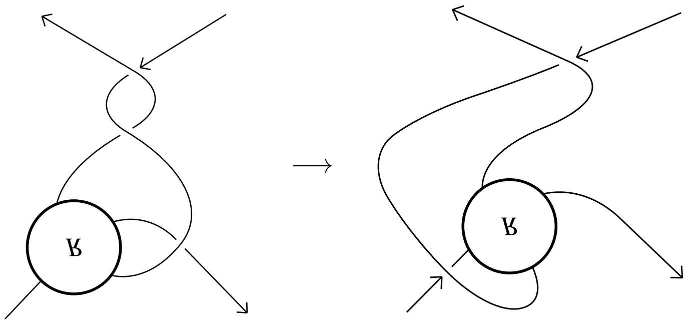
There are two cases.

Case 1. The twist of a_i is a self-twist. The connectivity in T is . In his case the choices are given by

choice in (35)	arc of OT crossing	type in (32)
A	2	2
B	1	3
E	1	1
G	2	4

Case 2. The twist of a_i is a mixed twist. The connectivity of T is . In this case the choices A-H realize any of the types in (32) with any of the two arcs having a twist crossing.

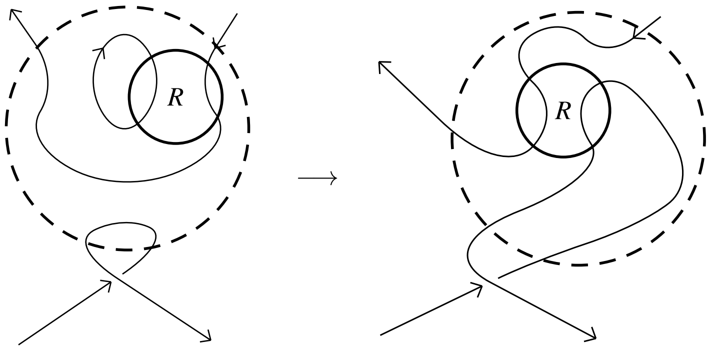
Now consider induction step. Because of (33) we have to apply a P move. We will use (25) together with its following variant.



. (36)

(Note that this is *not* exactly the π-rotation of (25).)

By taking out the crossing in (32), we have the following version of (26). The disk in (32) is drawn dashed.



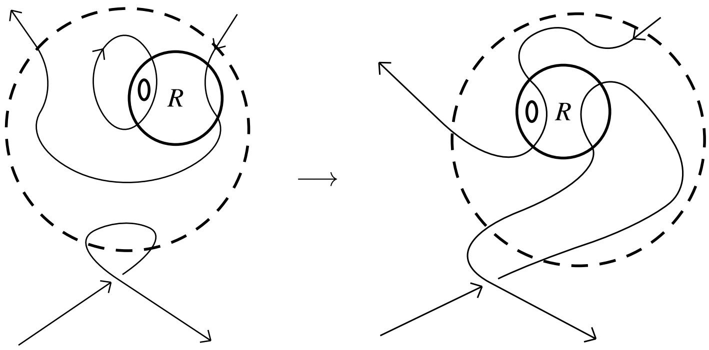
. (37)

Let us fix that $-\pi/4$ is he angle under which we rotate (37) to superpose with (32) to identify type.

If now

$$R = T(a_{k-1} - \mathrm{sgn}(a_{k-1}), a_{k-2}, \ldots, a_1)$$

with its crossing on the left in T of (25) is of type 1 in (32), then (37) refines to



, (38)

thus $T = T(a_k, \dots, a_1)$ is of type 2. If R is of type 2, then T is of type 1. (Keep in mind that we have assumed (34).) To have types 3 and 4 handled (and exchanged between R and T), use the variant (36). □

Lemma 5.2 For every Seifert arc S of R in (25), one can flype T in (25) *outside* R, so that S is contained in a Seifert arc of T'.

Proof. The form (25) will work for the left Seifert arc S of R on the left, as can be seen in (26). For the right Seifert arc of R, use the form (36) and the (omitted) equivalent of (26). As cautioned, passing between (25) and (36) is not a rotation (in the plane; its effect on R is one along the horizontal axis). Thus left and right Seifert arcs of R are not exchanged.

Also note that while (25) and (36) operate under (34), one can easily modify this argument to $|a_j| > 2$, by flyping all the crossings in the a_j twist to one side. □

Lemma 5.3 Let $L \neq L(2^{[k]}, -2, 2^{[l]})$ be a 2-bridge link of $b(L) \geq 4$. Then L admits a minimal Seifert circle diagram, which is not a (closed) braid diagram, and all components of L have a self-crossing.

This lemma accomplishes the key part of the work. In §6 it will become (fully) clear why the lemma proves Theorem 1.1.

Proof. First, let us say a word about the knots (p even). They will be largely disregarded in the proof. For them the same procedure applies as for the (2-component) links. Only the self-crossing condition is trivial and does not need to be taken care of. This simplifies the argument considerably, and we leave it to the reader to go through the below steps and omit the unnecessary parts for the knot case.

Our attitude will be to show that one can flype two tangles in a tangle decomposition like (24) of L so that after applying the maximal number of P and OT moves, the resulting (minimal Seifert circle) diagram is not a braid diagram and has self-crossings of both components.

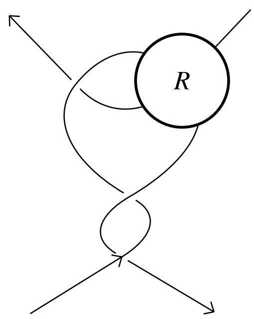
We will now work off the cases at the end of §4.3, except the last, which will be featured in §6.

Case 1. We consider a tangle decomposition $T_1 \cup T_2$ in (23) (T_i are left and right of the bar), in which both tangles are of the form (18), where for each tangle not all a_i are of the same sign.

We consider one of the tangles T_1 in (23). Now consider an alternating pair (a_i, a_{i+1}) of maximal i. So

$$a_j a_{i+1} > 0 \text{ for } j > i+1. \tag{39}$$

Now the left of (25) looks like



. (40)

Since only crossings are switched, the left of (37) remains the same.

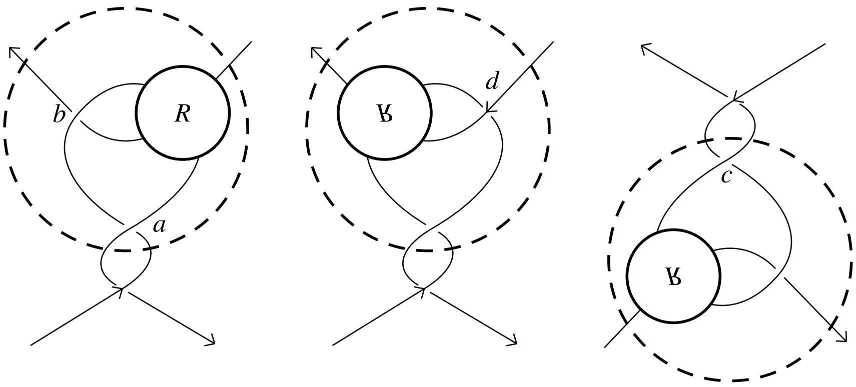
We consider three forms of the tangle

$$T = T(a_{i+1}, \dots, a_1)$$

with

$$R = T(a_i - \mathrm{sgn}(a_i), a_{i-1}, \dots, a_1)\,,$$

two of which are, up to crossings changes, those on the left in (25), (36).

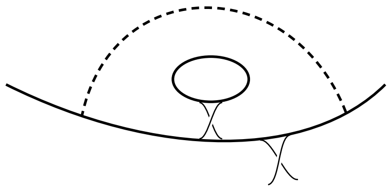


(41)

The reason we need to choose between these three is that we have a self-crossing condition mandated. (This will become irrelevant for two-bridge knots.)

We have to ascertain that either arc of T can be chosen to have a self-crossing. If a is a self-crossing, then (see remarks below (21)) b is mixed, and c is a self-crossing of the other component. If a is mixed, then b is self-crossing and d is self-crossing of the other component.

Note that any following moves do not change the Seifert picture inside the dashed circle. By looking at its upper Seifert arc in the first two pictures in (41) and the lower Seifert arc in the third, we see inside the dashed circle a pattern like



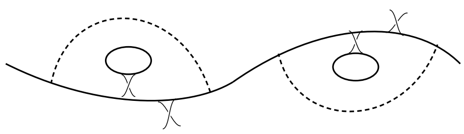
(42)

It will be important to maintain the indicated part of the dashed circle, which does not intersect anything. The Seifert circle drawn empty is not necessarily so, but this will not be relevant.

Iteratedly apply Lemma 5.2 (flypes and P, OT moves). We need a P move for every j because of (39). We can then make the arc in (42) become part of one of the Seifert arcs of T_1.

Similarly can be done for T_2. Now the Seifert arcs of T_1 and T_2 join to a (big) Seifert circle.

This means that we obtained a (minimal Seifert circle) diagram of $L = T_1 \cup T_2$ with a (big) Seifert circle having two (disjoint) spots of the form (42). One of the two ways in which they can connect is



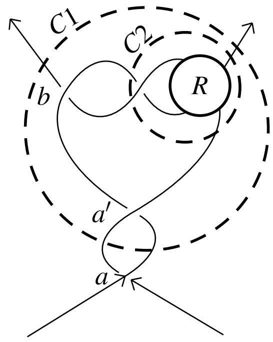
It is easy to see that then the big Seifert circle has at least 3 other Seifert circles attached and thus the diagram is not a braid diagram. (The fewest possible attached Seifert circle scenario is to have the dashed arcs in the two instances of (42) to be on the same side of the big Seifert circle, and the lower crossings to connect the same third one.)

We can also choose both components of L to have a self-crossing by choosing which arcs in T_1 and T_2 to have one.

The reason why this argument does not work when the alternating pairs are not disjoint is that when using (24) instead of (23), the Seifert arcs of T_i do not join to the (same) big Seifert circle. This situation will be handled in §6.

Case 2. We have an alternating pair (a_{k-1}, a_k) and some $|a_i| > 2$. Up to reversal, we may assume $i \le k-1$, and by choosing such i maximal, we also assume $a_i, \dots, a_{k-1}$ have the same sign.

Consider the following picture of $T = T(a_k, \dots, a_1)$:



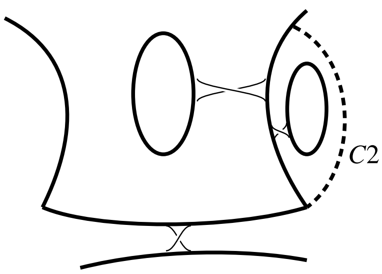
. (43)

Case 2.1. Assume crossing a (and a') is mixed. Then b is a self-crossing, and b remains; it is not deleted by a P move.

If the twist of b is at least 4, i.e., $i = k-1$, then in the forms of $T(a_{k-1}, \dots, a_1)$, given by A and C in (35),

(44)

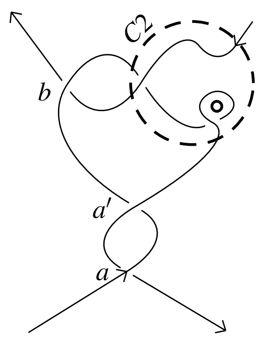
there will be a self-crossing of the component of L different from the one crossing at b. The Seifert picture inside $C1$ of (43), with an appropriate piece of the circle $C2$ drawn, is



. (45)

We drew a piece of $C2$ not intersecting any Seifert circle or crossing. The diagram (45) makes clear that there is a Seifert arc with at least 3 Seifert circles attached. The Seifert picture (45) inside $C1$ is not changed by following OT or P moves. (The drawn Seifert circles may not be empty, but this is not relevant.) This can not be part of any braid diagram.

If the twist of b is 2, i.e., $i < k-1$, then apply Lemma 5.1 on R. Choose type 1 or 4 in (32) with a self-crossing component different from the one at b. The picture (43) after replacing R by one of its type diagrams in (32) is:



(46)

The Seifert picture inside $C1$ becomes again (45), and again there is a Seifert arc with at least 3 Seifert circles attached. (The other type for R gives something equivalent.)

Case 2.2. Now assume a in (43) is a self-crossing. Then b is mixed, and a' is a self-crossing of the same component as a, which remains after a P move.

If the twist of b is at least 4, i.e., $i = k-1$, then choose in (44) the option in which the OT crossing involves the component different from the one at a (and a').

If the twist of b is 2, i.e., $i < k-1$, then in type 1 or 4 of Lemma 5.1 choose the component with the self-crossing to be different from the one at a. The pictures (46) and (45) and the non-braid diagram argument based on them remain the same.

From now on we can assume there is *no alternating pair*, i.e., all a_i have the same sign.

Case 3. If $|a_{i_1}|, |a_{i_2}| \geq 4$, then one can write $L = T_1 \cup T_2$ with T_i subjectable to Lemma 5.1. Then again we obtain a diagram with a big Seifert circle having two empty Seifert circles attached. It is not a braid diagram, unless it has these 3 Seifert circles, which we excluded. Also we can choose self-crossings of arcs in T_i to be in the two components of L.

Case 4. If $|a_{i_1}| \geq 6$, then the straightforward modification of Lemma 5.1 shows that within one T_i, we can have two empty (OT) Seifert circles attached to a Seifert arc, with OT crossings belonging to different arcs of T_i. The rest of the argument is as above.

This finishes the proof of Lemma 5.3. □

6 Conclusion of proofs

One major piece that connects our previous work with our goal is the following theorem (resuming the notation of §2.3).

Theorem 6.1 ([SS2]) Assume b is exchangeable as in (3), and $\pi(b)(i) \neq i$, $i = 1,2$. Then b is SS.

A similar result was proved by Ito [I] under a weaker (in fact, the weakest possible) assumption on b using some dilitation bound. His condition was discussed in more detail in [St]; here it will be implicit in some arguments in §10, and will be shortly mentioned. (See Remarks 10.2 and 10.4.) The conclusion drawn from this assumption will be sufficient for the first statement in Theorem 1.1. However, the map of (7) is proved only to be finite-to-one, and the identification which b_m are not conjugate is not very practical. The stronger property of Lemma 6.5 seems at least so far beyond the geometric method. To circumvent this problem, with Theorem 9.4 and Lemma 10.1 below we will enable enough of the algebra to work in all situations we need. (SS implies (7) to be at-most-two-to-one, among others.)

The other major connecting piece we need lies in the following observation.

Lemma 6.2 Let L be a link, and assume L has an n Seifert circle diagram which is not a (closed) braid diagram, and all components have self-crossings. Then L has an exchangeable n-braid b with $\pi(b)$ having no fix-element.

Proof. Apply Vogel moves. The last move, which exists, since there is at least one, will give an exchangeable braid. (This observation in Example 7.1 of [St3] was discussed in [St].) Vogel moves will not delete component self-crossings, so any subbraid of b for cycles of $\pi(b)$ will have self-crossings, and $\pi(b)$ cannot have a fix-element. □

In very few cases, we need a slightly more precise version of Lemma 6.2, which is the observation what exchangeable braids rendered by the Vogel move *precisely* fail the assumption of Theorem 6.1. See [St, Lemma 3.7].

Lemma 6.3 Let L have a non-braid diagram D of $n = s(D)$ Seifert circles transformed into a braid diagram by Vogel moves. If the braid obtained fails the assumption of Theorem 6.1, then the last Vogel move involves strands of two *distinct* components, at least one of which is a 0-crossing circle. □

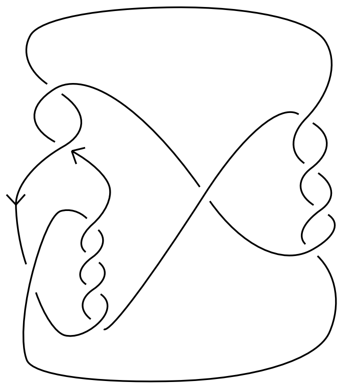
Theorem 6.4 If L is a two-bridge link and $n \geq \max(4, b(L))$, then L admits a SS exchangeable n-braid representative.

Proof. If $b(L) \leq 3$, take a 3-braid; at least one component has a self-crossing (since there are at most two components). Thus stabilize so that the other component has a self-crossing, too. A stabilized braid is exchangeable (in the form (4)) and (the corresponding version of) Theorem 6.1 can be applied.

Above we proved in Lemma 5.3 that if $b(L) \geq 4$, and $n = b(L)$, then there exists a non-braid n Seifert circle diagram of L in which both components have a self-crossing. Thus by Lemma 6.2 and Theorem 6.1, the exchangeable braid obtained by the Vogel algorithm is SS.

It remains to look at $L = (2^{[k]}, -2, 2^{[l]})$ (and $b(L) = 4$). When $k + l$ is odd, we have a knot, and the situation is rather obvious.

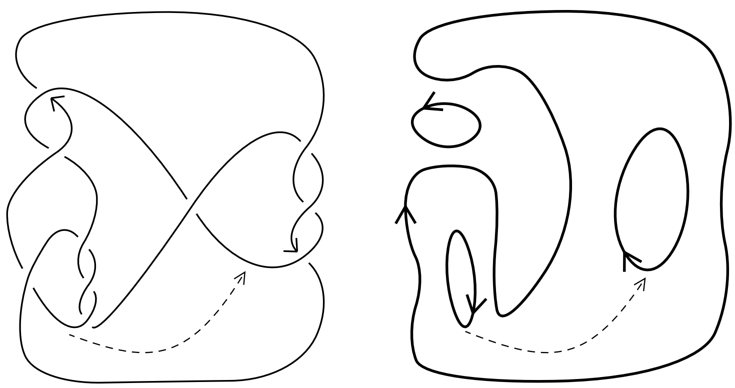
If k, l are even, then by the method of §4, we find the following non-braid diagram of 4 Seifert circles (drawn exemplarily for $k = l = 4$),



(47)

Both of its components have self-crossings. Thus Theorem 6.1 can be applied again.

If k, l are both odd, then indeed (47) has a component with no self-crossing. However, applying the Vogel move as indicated (when $k = l = 3$)



, (48)

one observes that it involves arcs of the same component. Thus by Lemma 6.3, the braid b will have $\pi(b)(i) \neq i$ for $i = 1, 2$. Explicitly, one can read off the braid (up to reversal)

$$b = \sigma_1^{-1}\sigma_2^k\sigma_3^{-1}\sigma_2\sigma_1\sigma_2^l\sigma_3^{-1}\sigma_2, \tag{49}$$

where $\pi(b)$ fixes $3 \neq 1, 2$. Thus one can use Theorem 6.1 again.

For $n > \max(4, b(L))$, just stabilize arbitrarily (self-crossings of components will not disappear). $\square$

Once a SS exchangeable braid b is found, the property from the abstract is straightforward; in fact, one can simultaneously have it for 'positive' and 'negative' iterated exchanges altogether.

Lemma 6.5 If L admits a SS exchangeable braid b, then L admits an exchangeable braid $\tilde{b}$ on which all positive exchange moves $\{\tilde{b}_m : m > 0\}$ give pairwise non-conjugate braids, i.e., $\tilde{b}_m \not\sim \tilde{b}_{m'}$ for $m > m' > 0$, and similarly do all negative exchange moves (i.e., $\tilde{b}_m \not\sim \tilde{b}_{m'}$ for $m < m' < 0$).

Proof. Subsymmetry implies that μ in (8) exists. Then take $\tilde{b} = b_{\lceil \mu/2 \rceil}$ and use (5). $\square$

From this also Theorem 1.1 becomes completely clear.

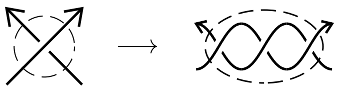
Remark 6.6 Our proof here results in a very explicit method how to obtain such non-conjugate n-braid representatives for every two-bridge link L. Apply Vogel's algorithm on the diagram we constructed. The conjugacy invariant behind Theorem 6.1 is a quadratic polynomial in m, and can be tested from 3 different values of m (to determine the vertex of the parabola). For the geometric inequality approach, see the remark below Theorem 6.1.

Remark 6.7 It should be noted that, although [St5] gives the general (non-finiteness) answer for $n > \max(3, b(L))$, the method there is totally non-constructive. Ito's theorem (as discussed below Theorem 6.1) implies that result, but again the method here turns out to mostly give the outcome with the added property of Lemma 6.5. (See the argument starting the proof of Theorem 6.4, or the one ending the proof of Proposition 9.3.)

7 Alternating knots by genus

While, as outlined, the infinite non-conjugacy property is not very generic at least for minimal braids, a few sporadic other realization results are possible.

For *generators*, *series* and $\bar{t}'_2$ *twists*, see [St9, St10], for example. Let D be an oriented *alternating* link diagram. For each crossing in D, there is a local move, which are call $\bar{t}'_2$, and is shown on (50). (Note that the strand orientation at the crossing is essential.) A diagram D is *generating* if after flypes it admits no tangle as on the right of (50). The *series* $\langle D \rangle$ of D is the set of diagrams obtained from D under arbitrarily many $\bar{t}'_2$ moves.

 (50)

This definition can be made for non-alternating diagrams as well, and was used thus elsewhere, but here we restrict ourselves to alternating ones. This is the much more important that we need to retain some control on the braid index.

Definition 7.1 To simplify language, set the braid index of a diagram to be the braid index of its link. We call a series $\langle D \rangle$ *regular* if for any $D' \in \langle D \rangle$, we have

$$b(D') = b(D) + \frac{c(D') - c(D)}{2}.$$

I.e., the braid index of the diagram is equal to the braid index of the generator plus the number of $\bar{t}'_2$ moves applied.

In [St9] regularity was studied in detail. This has also some relation to the *graph index* $\operatorname{ind}(D)$. Only some essentials can be repeated here from the long exposition about this topic.

We can specify $\operatorname{ind}(D)$ in the simplest way, following Traczyk [Tr, Tr2], by the maximal size of an *independent* set S of edges of the Seifert graph $\Gamma(D)$. We call S independent if in each cycle C with $|C| = 2m$ we have $|C \cap S| < m$. (Keep in mind that $\Gamma(D)$ is bipartite, see §3.2, so cycles are even.) In particular, $\operatorname{ind}(D) = 0$ is equivalent to D having no simple edge. (Pairs of parallel edges are of course regarded as a length-2 cycles.)

There is a Graph Index Conjecture by Murasugi-Przytycki, which we only briefly highlight, but which also here finds its relevance.

Conjecture 7.2 (Murasugi-Przytycki) If D is an alternating diagram of a link L, then $b(L) = s(D) - \operatorname{ind}(D)$.

The following was proved in [St9, Corollary 7.4.5]. It requires the *Morton-Williams-Franks* (MWF) inequality [FW, Mo], which we do not discuss here.

Lemma 7.3 If D is a special generator, the intersection of all maximal independent sets of D is empty, MWF is sharp on D, and D satisfies Conjecture 7.2, then $\langle D\rangle$ is regular.

The main reason for introducing diagram series was the study of the *genus* $g(L)$ of an alternating (for simplicity) knot L. This allows one, for alternating knots, to prove some limitations on the exceptions to Theorem 1.1.

The following test for exchangeable braid representatives returns to the use of the number of Seifert circles $s(D)$ and the number of separating ones $s_s(D)$, as introduced in §3.5.

Proposition 7.4 Assume a link has a diagram D with

$$s(D) - \operatorname{ind}(D) = n > 3 \tag{51}$$

and $\operatorname{ind}(D) + s_s(D) < n-2$. Then L has an exchangeable braid representative on n strands.

Proof. Apply $\operatorname{ind}(D)$ Chalcraft-Murasugi-Przytycki (CMP) moves. Each such move augments the number of separating Seifert circles by at most 1. At the end one arrives at a diagram with $s(D) - \operatorname{ind}(D)$ Seifert circles and at most $\operatorname{ind}(D) + s_s(D)$ separating. This is not a braid diagram, and apply the Vogel algorithm. □

Remark 7.5 In particular, a diagram with $\operatorname{ind}(D) = 0$ and $s(D) = n$ which is not a braid diagram will do. (Again, it really makes sense to consider $n = b(L)$ only. So one could have $s(D) - \operatorname{ind}(D) \le n$ in (51), but this will not add any further worthwhile cases.)

We will only give a short explanation for the below statements. If L is an *alternating* link, and the claim of Theorem 1.1 (along with its more precise specification in Lemma 6.5) fails for L and some $n \ge \max(4, b(L))$, we call L an *exception*. We write $c(L)$ for the crossing number of L (see §3.1).

Theorem 7.6 If L is an exception *knot*, then $n = b(L)$, and the following holds.

1. If L is of genus at most two, the only possible exception is $L = 9_{38}$ (with $n = 4$).

2. There are only finitely many exceptions for genus 3, 4.

3. If Conjecture 7.2 is true, then there are only finitely many exceptions for any fixed genus.

4. The exceptions for fixed genus g are not asymptotically dense over *bounded* increasing crossing number. I.e.,

$$\limsup_{c\to\infty} \frac{\left|\{\, L \text{ exception } \mid c(L) \le c,\, g(L) = g \,\}\right|}{\left|\{\, L \text{ alternating} \mid c(L) \le c,\, g(L) = g \,\}\right|} < 1\,. \tag{52}$$

For now 9_{38} is undecided. (We do know from [St] that it admits 4-braids non-exchangeable up to conjugacy. It could, though, admit exchangeable ones as well.) The case $g = 3$ comprises 9_{29} (from the introduction), so it is confirmably false in completeness. Given that exceptions are well-spread, it will be likely very hard to establish more self-contained properties, even for knots. (See Remark 7.7 for links.)

We defer Proposition 9.2 and the argument for it to its more suitable place, although needed a few times.

Proof. When $n > b(L)$, we can use Remark 6.7. Thus we can assume $n = b(L)$.

1. Genus 1 readily follows from Theorem 1.1 (with $p = 2$ for (21)) and Proposition 9.2 (with $p = 3$); see [St7]. From the classification in [St8] it is not too hard to check genus 2.

   The series of 5_1 is Proposition 9.2 with $p = 5$, and 8_{12} follows from Theorem 1.1 with $p = 4$.

   We will use Proposition 7.4 in an example. Note that genus 2 series are all regular (as per Definition 7.1; for this see [St9] and part 3).

   The generator 12_{1202} has one separating Seifert circle; $b(12_{1202}) = 7$ and the (unique) alternating diagram has 9 Seifert circles. By using 2 CMP moves, we get a 7 Seifert circle diagram with at most 3 separating Seifert circles, which is thus not a braid diagram. Likewise can be argued with knots obtained after $\bar{t}'_2$ twists. (One OT move is needed per twist.)

   With similar arguments (and using the table in [SS1]) we can deal with all other generators. It is enough to see that there is a minimal Seifert circle diagram which is not a braid diagram. The Seifert circles that come from the $\bar{t}'_2$ twist make it easier to create a non-braid diagram nesting. The deals with all knots, except 9_{38} (and $n = 4$).

2. This follows from part 3, since Conjecture 7.2 was checked in [St9] for genus 3, 4.

3. This is basically the generalization of the proof of part 1. It was explained in [St9] that the behavior of $\operatorname{ind}(D)$ stabilizes under $\bar{t}'_2$ twists. (See in particular [St9, Lemma 7.4.4].)

4. In [St10, Lemma 5.16] we proved that Conjecture 7.2 holds on the series of at least one maximal generator for every genus. (This generator is always odd, so that so far one can replace '$c(L) \leq c$' with '$c(L) = c$' in (52) only for odd crossing number.) □

Remark 7.7 Most of Theorem 7.6 will hold for links. However, some checks for link generators require more volume, so complications will go even somewhat beyond the self-crossing condition. Exceptions – or potential ones – however few, will increase and will not be pleasant to describe. (Among others, Remark 6.7 does not readily apply to three or more components.)

8 A further exception-less family

We next present one more moderately self-contained class we found sharing the virtue of being free of exceptions to the assertions of Theorem 1.1.

Proposition 8.1 If D is a special alternating generating diagram with $\operatorname{ind}(D) = 0$ and at most two components, then no exception occurs on the series $\langle D \rangle$ of D.

The proof will introduce some further techniques, which will be used for the pretzel links in §9; thus it is given in some detail (which is later abbreviated). Proposition 9.2 is a (very) special case of Proposition 8.1 (where D is the $(2, p)$-torus link diagram). Example 9.5 provides inconclusive instances for more than 2 components. Note also that $\operatorname{ind}(D) = 0$ implies that D is generating.

Proof. Consider first knots. In the case of a special generator D with $\operatorname{ind}(D) = 0$, the test of Lemma 7.3 reduces to D itself. That this succeeds follows by the (proof of the) result in [DHL].

Then it is clear how to construct minimal Seifert circle diagrams for elements in the series of D: apply OT moves at the crossings created by $\bar{t}'_2$ twists. It is easy to see that the nesting of Seifert circles can avoid a braid diagram (if at least 4 Seifert circles remain).

There is left to examine the diagram D itself, but we can use Remark 7.5. Since we assumed D is special, it is not a braid diagram either, unless $s(D) = 2$ (and we are in the case of Proposition 9.2). But we treat only $n \geq 4$, and if $n > b(L)$, we can use Remark 6.7.

Now consider 2-component links. The argument for minimality of Seifert circles remains the same: [DHL] works for links, as does [St9]. (Note that, while [St9] was chiefly focusing on knots, this assumption was not relevant for Lemma 7.3 and the arguments behind it.)

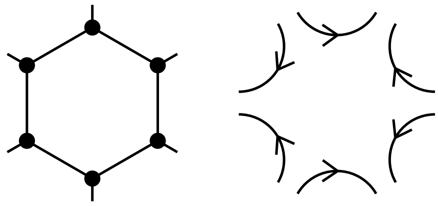
So consider a 2-component special alternating diagram $D' \in \langle D \rangle$.

Case 1. First consider $D' = D$ itself.

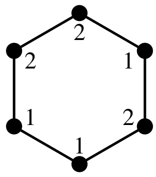
Lemma 8.2 Unless D is the $(2, p)$-torus link diagram, there exist locally incoherent arcs of the same component, or both components have self-crossings.

Proof. D is determined by its Seifert graph $\Gamma(D)$ because D is special. W.l.o.g. assume the ∞ region is not a Seifert circle region, so all Seifert circles are empty.

Every non-Seifert circle region of D gives a face of $\Gamma(D)$. Cycles are even-length as Γ is bipartite.

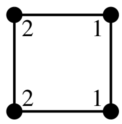


locally incoherent arcs belong to Seifert circles which are even-distance vertices in a face of $\Gamma(D)$. Label each piece of a Seifert circle between crossings with "1" or "2" depending on the component. This corresponds to a marking in the corner of a face of $\Gamma(D)$.



(53)

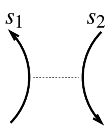
If all (face) cycles of $\Gamma(D)$ are length-2, then D is the $(2, p)$-torus link diagram (which will be handled in Proposition 9.2). Thus there exists a face cycle of length ≥ 4. Then the only way no incoherent pair has the same component is a 4-cycle face with



.

But if corners of adjacent vertices have the same label, the edge is a self-crossing of that component. Thus both components have self-crossings. $\square$

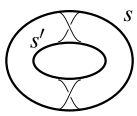
If both components have self-crossings, then use Lemma 6.2. So assume locally incoherent arcs have the same component. Let s_1, s_2 be the Seifert circles in D to which the arcs belong. (Keep in mind that the Seifert circles are empty.)



(54)

Let us say the *weight* of a Seifert circle is the number of Seifert circles it contains plus itself.

Applying a Vogel move on s_1, s_2 gives a Seifert circle s of weight 2, containing an empty Seifert circle s' attached via (two) self-crossings (of opposite sign).



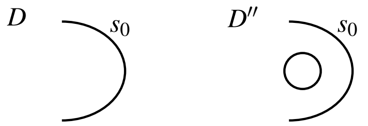
(55)

If we apply further Vogel moves, the interior of s may be changed. But we can use Yamada moves (17) instead. Note that Yamada is also given by a pair of locally incoherent arcs, (15), but unlike Vogel, the result depends on the direction of the arrow. We can choose the arrow pointing towards s, so that the Yamada move never changes the interior of s. (If none of the Seifert circles in the pair is s, then it does not matter anyway.) This process will give a braid diagram with the interior of s unchanged. This will give the exchangeable crossings (as specified in the proof of Proposition 9.3) to be self-crossings of the same component. Thus by Lemma 6.3, we are done.

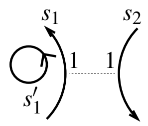
This deals with $D' = D$ itself.

Case 2. Now consider $D' \in \langle D \rangle$ of a link L', $D' \neq D$. A minimal Seifert circle diagram D'' of L' is obtained by attaching OTS inside some Seifert circle of D.

Let first D' be obtained by one $\bar{t}_2'$ move. So there is exactly one OTS inside some Seifert circle s_0 of D.

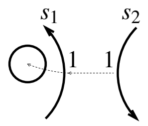


If this Seifert circle $s_0 \neq s_1, s_2$ from (54), then the argument as for D applies. So let $s_0 = s_1$. Let us say the component of the locally incoherent arcs of s_1, s_2 is 1. Let s_1' be the (empty) OTS inside s_1.



Case 2.1. s_1' contains an arc of component 2. By the nature of OTS, it is possible to flype the OT tangle, so that s_1' is attached a self-crossing of component 2.

Now apply the argument as for D, except that we start with a *pair* of Vogel moves involving the arc of s_2:



. (56)

The first move will create a self-crossing of component 1, so that at the end both components will have self-crossings.

Case 2.2. s_1' contains only arcs of component 1. The second of the Vogel moves (56) will create exchangeable braid self-crossings of component 1, as in (55).

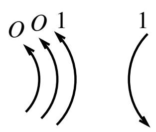
This finishes the case that D' is obtained from D by one $\bar{t}_2'$ move.

Case 3. More than one $\bar{t}_2'$ move is applied to obtain D' from D. Note that by the freedom to choose which crossing components to put the OT crossing at, we can obtain a minimal Seifert circle diagram of self-crossings of both components, unless (which we then assume) all $\bar{t}_2'$ moves are applied at self-crossings of the same component $O \in \{1,2\}$.

In that case, all OTS are attached (inside some Seifert circles of D) by self-crossings of O, and contain only arcs of O.

We can again find a locally incoherent pair of, say, component 1, and can again assume one of s_1, s_2 from (54) contains OTS inside.

First, use Vogel moves inside s_1, s_2 to make them braid-like. This adds inside s_i only self-crossings of component O. So all crossings inside s_i are self-crossings of component O (and there are such crossings at least for s_1). Then a part of the Seifert picture with arcs labeled by components is



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Now let s_i have weight w_i. In a generalization of (56), by $w_1 w_2$ Vogel moves, one can transform s_1, s_2 into a braid-like Seifert circle with weight $w_1 + w_2$. If $O = 1$, the last move will create a pair of self-crossings of component 1, which (by following Yamada moves) can be made exchangeable braid crossings.

If $O = 2$, then the first move will create self-crossings of component 1, while there are self-crossings of component $O = 2$ inside some s_i. Thus at the end both components will have self-crossings. $\square$

9 Alternating pretzel links

In this section, we treat most of the alternating *pretzel links*.

The pretzel (link) $L = P(x_1, \dots, x_p)$, for $x_i \in \mathbb{Z} \setminus \{0\}$ and $p > 2$, can be specified, as an unoriented link, in terms of Conway's calculus of Figure 1 by the tangle closure of the right-associative product $x_1(x_2(\dots(x_p\ 0))\dots)$. The vector $(x_1, \dots, x_p)$ can be dihedrally permuted, which does not change the link type.

The diagram is obviously alternating when $x_i > 0$, which will be assumed throughout. To simplify matters, we also mostly consider *pure* alternating pretzels $P(x_1, \dots, x_p)$, where $x_i > 1$.

The case of multiple components is obviously more challenging, but orientation issues for pretzel links in their entirety are a general complication entering before even putting our approach to work. (Also, the determination of $b(L)$ is not fully clear even for non-pure alternating ones.) We use boldface to indicate that the twists are parallel.

The below summarizes (most of) what is proved in the three following subsections (Propositions 9.2, 9.3, 9.6 and 9.7).

Theorem 9.1 The claim of Theorem 1.1 holds for the alternating pure pretzel link L, except if $n = 4$ and $L = 6^3_1$ with the reverse orientation, 8^4_1 with the parallel orientation, or a link $L = P(\mathbf{x}_1, 2, \mathbf{x}_2, 2)$ with $\mathbf{x}_j > \mathbf{1}$ arbitrary.

9.1 reverse pretzels

We start with the alternating pretzel links, in which *all twists are reverse*. We write $L = P(x_1, \dots, x_p)$ where all $x_i > 0$ and either all are odd or all are even. (Here letting some $x_i = 1$ is no problem.) We can also assume $p \geq 3$ (otherwise we have reverse $(2, x_1 + x_2)$-torus links).

Thus the even pretzels are the series of $P(2, 2, \dots, 2) = P(2^{[p]})$, while the odd ones are the ones of $P(1, \dots, 1)$, which is the $(2, p)$-torus link diagram.

The odd case is easier and can be proved by using the OT move alone. (These are again knots and 2-component links, depending on the parity of p. See also Proposition 8.1.)

Proposition 9.2 The claim of Theorem 1.1 holds for the alternating pretzel links $L = P(x_1, \dots, x_p)$ ($p \geq 3$) with $x_i > 0$ odd.

Proof. Every twist of x_k needs $(x_k - 1)/2$ OT moves. If $n > b(L)$ (and in particular $b(L) < 4$), use Remark 6.7. Remember that L has at most two components.

Thus assume henceforth that $n = b(L) \geq 4$. We need at least two $x_j \geq 3$ or one $x_j \geq 5$. It is easy to see that after OT moves the nesting of a braid diagram can be avoided. For the link case (p even), all crossings are mixed, so with (at least) two OT moves, one can choose OT crossings to belong to different components. $\square$

The even twist case is more interesting, also because it involves an unlimited number of components.

Proposition 9.3 The claim of Theorem 1.1 holds for the alternating pretzel links $L = P(x_1, \dots, x_p)$ ($p \geq 3$) with $x_i > 0$ even, except possibly $P(2, 2, 2) = 6^3_1$ and $n = 4$.

For this it is not sufficient to use Theorem 6.1. However, in [St6] we proved an extension of Theorem 6.1, which operates under $\pi(b)(1) = 1$, $\pi(b)(2) \neq 2$.

Theorem 9.4 ([St6]) Assume b is exchangeable and $\pi(b)(1) = 1$, $\pi(b)(2) \neq 2$. Let $C_i \not\ni 1$ be the cycles of $\pi(b)$. Assume there is no $\lambda \in \mathbb{Z}$ so that $lk_{1,C_i} = \lambda \cdot |C_i \setminus \{2\}|$ for all i. Then b is subsymmetric.

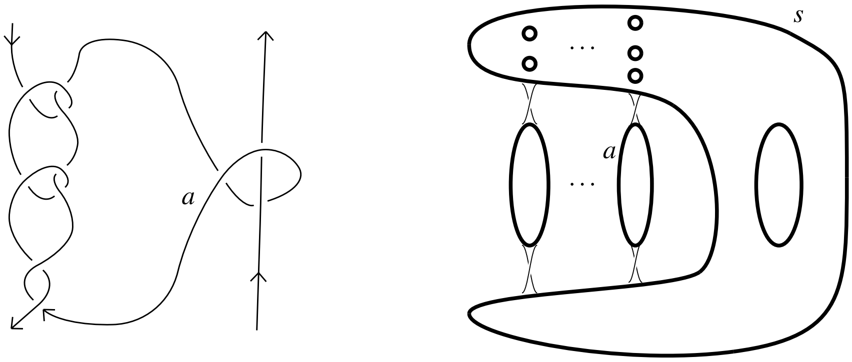
Proof of Proposition 9.3. Fix p throughout. It is not too hard to make direct calculation to see that the assumptions of Lemma 7.3 are satisfied on $D = P(2^{[p]})$ (with $\mathrm{ind}(D) = 1$). The series of $D = P(2^{[p]})$ is thus regular.

If $D' \in \langle D \rangle$ belongs to some link L', then it is clear how to obtain a minimal Seifert circle diagram for L'. Apply OT moves at the crossings created by $\bar{t}'_2$ twists *and* on one clasp of the generator, which we call the *extra* OT move.

Let us thus first deal with $n = b(L')$. It is easy to see that the nesting of Seifert circles is not that of a braid diagram (see (57)).

If all $x_i > 2$, then it is easy to see that one can perform OT moves so as to create a self-crossing of every component. This is sufficient for Lemma 6.2.

Thus let w.l.o.g. $x_p = 2$, and assume the extra OT move is applied at x_p.



(57)

Now note that the arc of the Seifert circle on the right labeled by a belongs to the component a of the self-crossing of the extra OT move (which is always present), as indicated on the left.

One can use Vogel moves inside s to make it braid-like (recall Definition 3.3).

Similarly one can use $(p-1)(p-2)/2$ Vogel moves to make a braid-like Seifert circle out of those $p-1$ Seifert circles outside s. Hereby we start from the left Seifert circles, so that the last of the Vogel moves involves the arc a.

This will give the exchangeable crossings (as defined in §2.3), in which one of the components, a, has self-crossings.

This is not sufficient for Theorem 6.1, if the other component, a', has no self-crossings. However, we can use Theorem 9.4. It implies that in the case of failure of subsymmetry, this second exchangeable crossing component a' either has zero linking number with all other components, or non-zero linking number with all of them.

This immediately reduces the argument to $p = 3$ (and still $x_3 = 2$), which is rather easy to complete.

If two $\bar{t}'_2$ moves, so (non-extra) OT moves, are applied, then one can put OT self-crossings into every of the 3 components. There remain to check $P(4,2,2)$, which is easy, and $P(2,2,2)$ which (as per [St, §5.3]) remains unclear.

This argument will deal with minimal braid representatives $n = b(L')$. For non-minimal ones, note that a minimal braid always has a self-crossing component, which we can lay into strand 1, and then stabilize with σ_{n-1}. Then both strands 1 and n belong to non-trivial cycles of $\pi(b)$ of an exchangeable braid b in the alternative variant (4). And (the corresponding variant of) Theorem 6.1 can be used. $\square$

9.2 parallel pretzels

Alternating parallel (pure) pretzels are given by an even number $p \geq 4$ of arbitrary positive integers $\mathbf{x}_i > \mathbf{1}$. As stipulated, we use boldface to indicate that the twists are parallel (for distinction purposes later in §9.3, however, not thoroughly in formulas where not ambiguous).

Example 9.5 To emphasize the subtlety of the situation for links, consider the pretzel links $L = P(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ with all $\mathbf{x}_i > \mathbf{1}$ and at most one odd, and twists within $\mathbf{x}_i$ parallel. It is already a simple family (of 3 or 4 component links with $b(L) = 4$) where Theorem 6.1 apparently cannot be used.

However, the following is provable with the upgraded toolkit.

Proposition 9.6 Consider the alternating pretzel links $L = P(\mathbf{x}_1, \dots, \mathbf{x}_p)$ with all twists parallel, with $p \geq 4$ even and all $\mathbf{x}_i > \mathbf{1}$. Then the claim of Theorem 1.1 holds for L except if $n = 4$ and $L = 8^4_1$ (with the suitable orientation).

Proof. First assume at least one $\mathbf{x}_i$ is odd. The argument for Proposition 8.1 can be easily modified.

We apply again [DHL] for the Seifert circle minimality $b(L) = p$. Let first $n = b(L)$. We need the corner marking (53), the use of the Yamada move to preserve (55), and the mentioned Theorem 9.4 (in its full form).

Say $\mathbf{x}_1$ is odd. What Theorem 9.4 leaves behind to check is that $n = p = 4$ and $x_2 = x_3 = x_4$ are all even (and equal). This can be done by an explicit calculation of the type of invariant used in [SS1, SS2].

Recall the Conway polynomial $\nabla(z)$ of §3.3. Let $\Lambda(b)$ be the *axis link* of $b \in B_n$, given by the closure of $\sigma_n \cdots \sigma_2 \sigma_1^2 \sigma_2 \cdots \sigma_n \cdot b \in B_{n+1}$.

Let $k = \mathbf{x}_1$ be odd, and $l = \mathbf{x}_2$ even, regarded as variables, and $d > 0$ fixed. By a standard Vassiliev invariants argument ([St3]), the (conjugacy invariant) map for exchanged braids for $P(\mathbf{k}, \mathbf{l}, \mathbf{l}, \mathbf{l})$,

$$(\mathbf{k}, \mathbf{l}, m) \mapsto [\nabla(\ \Lambda(\ b_m(\sigma_2^k \sigma_3 \sigma_2^l, \sigma_2^l \sigma_3^{-1} \sigma_2^l)\ )\ )]_{z^d}\,, \tag{58}$$

is a polynomial of degree at most d in $\mathbf{k}, \mathbf{l}, m$. (Degree is counted with respect to all three variables together, i.e., monomials of $k^x l^y m^z$ occur only for $x + y + z \leq d$, not $x, y, z \leq d$.)

This polynomial can be explicitly calculated with MATHEMATICA™ in a few minutes from a handful of particular values. For $d = 5$, it is linear in m with linear term $-l/2 + l^2/4$. Since by mirroring (this is equivalent to choosing orientation on the braid axis component of b_m in $\Lambda(b_m)$), we may assume w.l.o.g. $\mathbf{l} < \mathbf{0}$, we are done. This finishes the verification for $n = b(L)$.

When $n > b(L)$, use the last paragraph of the proof of Proposition 9.3.

This finishes the case that at least one $\mathbf{x}_i$ is odd.

Far more interesting is the case that all $\mathbf{x}_i$ are even. For this situation, all minimal braids are pure, and neither Theorem 6.1 nor Theorem 9.4 can be used. We will advance the algebraic technology for

$$\pi(b)(i) = i, \quad i = 1, 2 \tag{59}$$

in §10. We need to use Lemma 10.1 here, though.

Now return to the link $L = P(\mathbf{x}_1, \dots, \mathbf{x}_p)$ for all $\mathbf{x}_i$ even and $p \geq 4$.

Let first $p = 4$. Any of two pairs of components with 0 linking number can be chosen to form the exchangeable crossings of an exchangeable minimal braid of L. Lemma 10.1 will leave behind the case $P(\mathbf{k}, \mathbf{l}, \mathbf{l}, \mathbf{l})$.

Setting $\mathbf{k}$ even in (58), and using even $d \geq 6$ (one has to change the parity of d, as so does the number of components of $L = P(\mathbf{k}, \mathbf{l}, \mathbf{l}, \mathbf{l})$), one can obtain the claim. Assume again w.l.o.g. $\mathbf{l} < \mathbf{0}$. For $k \neq l$ use $d = 6$, which will give a leading linear term in m, of coefficient

$$-\frac{l(k-l)}{4} + \frac{l^2(k-l)}{8}\,.$$

For $\mathbf{k} = \mathbf{l}$ use $d = 8$. Now the polynomial (58) is quadratic in m, and the leading coefficient is $l^2/4 - l^4/16$. This will fail if $\mathbf{k} = \mathbf{l} = \pm\mathbf{2}$, which leads (up to mirroring) to the stated exception $P(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}) = 8^4_1$.

For $n > p = b(L)$, use Theorem 9.4. Any component L_1 of L will not have equal linking number with all other components. Choosing L_1 to form strand 1 of a minimal braid representative of L, and (if needed iteratedly) stabilizing strand n (at any other component) will give a braid on which Theorem 9.4 can be used. (Keep in mind that the form of the exchange move has to be switched.)

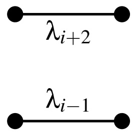
Let now $p > 4$. While one can use Theorem 10.3, much of its underlying algebraic complexity in the definition of υ^* is redundant, and the subbraid argument to adapt Lemma 10.1 can be simplified.

Let us say that $\mathbf{x}_i$ is a twist between component L_i and L_{i+1} of L. One can always find an exchangeable braid representative with exchangeable crossings between components L_i and L_{i+2} of L. (Indices should be taken modulo p.) In Lemma 10.1, we exhibited a conjugacy invariant υ of B_4 so that $m \mapsto \upsilon(b_m) = \nabla_d(\delta^{-u}\overline{b}_m)$ is a non-constant polynomial in m, where δ^2 is central and d,u do not depend on m.

Now let $b \in B_n$ for $n > 4$ be a pure braid. One can define a conjugacy invariant υ^* of b by

$$\upsilon^*(b) = \sum_C \upsilon(b_{[C]}), \tag{60}$$

where the sum runs over all 4-element subsets C of $\{1,\dots,n\}$ whose subbraids $b_{[C]}$ of b have an induced linking subgraph isomorphic to



, (61)

and

$$\lambda_i = \mathbf{x}_i/2. \tag{62}$$

We recall that the reason for this summation is that υ^* must be defined by a condition not involving any particular choice of components. Potential conjugacies between the b_m could, in principle, exchange cycles of $\pi(b)$. The goal is, though, to choose the summation condition so that the sum can be effectively disposed of.

Note that the exchange move becomes a conjugacy when strand 1 or n is deleted. Thus in the sum (60) for $\upsilon^*(b_m)$ only $C \ni 1,n$ will contribute something non-constant in m. The linking subgraph (61) was chosen so that when strand $1 \in C$ of b_m belongs to component L_i of L, and strand $n \in C$ to component L_{i+2}, there is only one choice for the remaining two elements in C to match this linking subgraph. This means that the sum (60) effectively reduces to one term.

Also note that υ in (67) depends on C and two parameters u,d from (71). To "clean up" the definition (60), one can calculate u,d for the unique C we are interested in (which does not depend on m), and formally fix them for every C. Their application on any other C is provably fictitious. (Alternatively, one can consider the $\mathbb{Z} \times \mathbb{Z}_+$-indexed array of invariants (67) for all u,d, a more wasteful option.)

The rest of the argument (also when $n > b(L)$) remains the same. □

Note that Lemma 10.1 comes with a minor disadvantage: under QSS one can have pairwise non-conjugacy under arbitrary positive exchange moves, but for positive and negative altogether, as noted above Lemma 6.5, one would have to exclude a finite number on one side. Anyway, the spirit of Lemma 10.1 was to avoid some unwieldy calculations (even if in B_3), for which this small manco seems a price worth paying. And the process remains completely constructive.

9.3 mixed pretzels

The (pure) *mixed pretzels* $P(x_1,\dots,x_p)$ are given by an even number $p_1 > 0$ of $\mathbf{x}_j > \mathbf{1}$ of arbitrary parity, standing for parallel twists, and an arbitrary number $p_2 > 0$ of $x_i > 1$ of even parity, standing for reverse twists. We assume again $p = p_1 + p_2 \geq 3$. We have then the following outcome for this most general type of pretzel.

Proposition 9.7 A mixed alternating pretzel link $P(x_1,\dots,x_p)$ satisfies the claim of Theorem 1.1, unless $L = P(\mathbf{x}_1, 2, \mathbf{x}_2, 2)$ with $\mathbf{x}_j > \mathbf{1}$ arbitrary, and $n = 4$.

Proof. Note that for an alternating pretzel, the sign of twists is determined by their type (parallel or reverse). Thus $\bar{t}_2'$ twists are applied only at crossings of the same (say, negative) sign. Then it is possible to adapt the cited arguments for [St9, Lemma 7.4.4] and [St9, Corollary 7.4.5] to find the braid index, and minimal Seifert circle diagrams for L.

The rest of the proof is a straightforward combination of the arguments used for the reverse and parallel pretzels. With a little bit of (component moving) care, one can see that Throems 6.1 and 9.4 apply almost always. This combinatorial argument works unless

- $L = P(\mathbf{x}_1, 2, \mathbf{x}_2, 2)$ with $\mathbf{x}_j > \mathbf{1}$ arbitrary, and $n = 4$, or
- all $\mathbf{x}_j$ are even and all $x_i = 2$, and $n = p$, or
- $L = P(\mathbf{k}, \mathbf{k}, 4)$ for $\mathbf{k}$ even and $n = 4$.

The third series can be checked using a Vassiliev invariant test as in (58).

Now we deal with the second series, while excluding the (overlap with) the first.

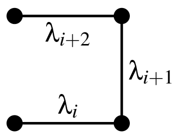
If $p = 4$, we can use Lemma 10.1. There is always a component whose non-zero linking numbers with other components have different sign.

Now let $p > 4$. We again use a subbraid sum as in (60). By a straightforward combinatorial argument, we can always find exchangeable crossings belonging to components of distance 2 or 3 in the linking graph of L. Note that this graph is a cycle of length p with non-zero edge labels λ_i. To specify λ_i, (62) is complemented by

$$\lambda_i = -x_i/2 \quad \text{for reverse twists } x_i.$$

Also, if $p = 6$, one can always achieve distance 2 (this is needed to avoid undesired terms entering into (60) by ambiguity).

Then for distance 2 use the sum (60) with the condition (61) on C, while for distance 3 between components L_i and L_{i+3} (indices taken modulo p) we modify this condition to



.

This will again ascertain that only the desired one term (of Lemma 10.1) enters the sum (60).

The argument for $n > b(L) = p$ remains, too. □

More interesting is the first family, for many of these links have *mutants* that admit (SS) exchangeable 4-braids. Prohibiting themselves from having such could thus be a difficult task. We will, however, not discuss here the obstruction theory for exchangeable braids; see [St] for the (Burau-)Jones and [St2] for the spectral test.

10 Quasi-subsymmetry of iterated exchanged braids

We will, as outlined, complete here the algebraic version of the pairwise non-conjugacy statements of iterated exchanged braids in the missing case (59).

In this section, we use the alternative version (4) of exchangeability. Then (59) must be rewritten as

$$\pi(b)(i) = i, \quad i = 1, n. \tag{63}$$

Recall §2.2 for linking numbers and the end of §2.3 for QSS.

Lemma 10.1 Let $b = \alpha'\beta'$ be an exchangeable *pure* 4-braid. If $lk_{1,2} \neq lk_{1,3}$ and $lk_{2,4} \neq lk_{3,4}$, then b is QSS.

Proof. We will construct a $\mathbb{Z}$-valued conjugacy Vassiliev invariant ω on B_4 so that

$$m \mapsto \omega(b_m) \tag{64}$$

is a non-constant polynomial in m. By Proposition 2.5, the claim of the lemma will follow.

Let the "winding braids" be (see [St6], but beware of our modification for $\kappa_{2,4}$)

$$\kappa_{1,2}=\sigma_1^2,\ \ \kappa_{1,3}=\sigma_1\sigma_2^2\sigma_1,\ \ \kappa_{2,3}=\sigma_2^2,\ \ \kappa_{2,4}=\sigma_3\sigma_2^2\sigma_3,\ \ \kappa_{3,4}=\sigma_3^2\,. \tag{65}$$

By a combed normal form argument, we can write

$$b_m=V_1\sigma_2^{m+m_1}V_2\sigma_2^{-m+m_2}\,, \tag{66}$$

where V_1 is a word in $\kappa_{1,2},\kappa_{1,3}$ containing some $\kappa_{1,2}^{\pm 1}$, and V_2 is a word in $\kappa_{2,4},\kappa_{3,4}$ containing some $\kappa_{3,4}^{\pm 1}$.

Apply the homomorphism $\bar{\cdot}: B_4\to B_3$ given by $\bar{\sigma}_{1,2,3}=\sigma_{1,2,1}$.

We are thus led to examine the conjugacy in B_3 of elements

$$\overline{b}_m=\gamma_m=X_1\sigma_2^{m+m_1}X_2\sigma_2^{-m+m_2}\,,$$

where $X_k=\overline{V}_k$ are words in $\kappa_{1,2},\kappa_{1,3}$ with at least one $\kappa_{1,2}^{\pm 1}$. We will find a conjugacy invariant υ on B_3 so that $m\mapsto\upsilon(\overline{b}_m)$ is a non-constant polynomial in m, and set

$$\omega(b_m)=\upsilon(\overline{b}_m)\,. \tag{67}$$

Let

$$\delta=\sigma_1\sigma_2\sigma_1\,, \tag{68}$$

so that δ^2 generates the center of B_3. By factoring out a power of δ^2 (which does not depend on m), we can replace $\kappa_{1,3}$ by $\kappa_{2,3}^{-1}$. Thus we assume now

$$\gamma_m=\delta^{u'}W_1\sigma_2^mW_2\sigma_2^{-m}\,, \tag{69}$$

u' is even and W_k are words in $\kappa_{1,2}=\sigma_1^2$, $\kappa_{2,3}=\sigma_2^2$ with at least one $\kappa_{1,2}$.

Now there is a well-known procedure, due to Schreier (see [BM, §7]), of writing any pure $\gamma\in B_3$ as $\gamma=\delta^u\gamma'$, where u is even and γ' is alternating. (Alternating means that no σ_1^{-1},σ_2 or no σ_1,σ_2^{-1} occur in γ'. We keep in mind that γ_m are pure and regard powers of σ_1 as alternating.) To accomplish this, iteratedly substitute $\delta^uV\sigma_1\sigma_2W$ by $\delta^{u+1}V'\sigma_1^{-1}W$, where V' is V with $\sigma_{1,2}$ exchanged. (One has to see that the number of applications of this rule is even.)

Apply Schreier's procedure to γ_m (incorporating the central factor on the right of (69)) to represent

$$\gamma_m=\delta^{u_m}\gamma'_m\,. \tag{70}$$

It is important to notice that $u=u_m$ are constant for large m and the alternating braids γ'_m have two syllables of length growing with m (and opposite exponent sign).

We have u_m are even. By factoring this (constant in m) power of δ^2, we have alternating braids γ'_m of growing length $l(\gamma'_m)$. By Crowell-Murasugi's properties of ∇ of alternating links (12), using the alternating link diagram $D=\hat{\gamma}'_m$,

$$\max\deg_z\nabla(\widehat{\delta^{-u}\gamma_m})\;=\;l(\gamma'_m)-2,$$

so that the polynomials are not constant in m (while u is). Thus some Vassiliev invariant (∇-coefficient) of $\hat{\gamma}'_m$, which is also a conjugacy Vassiliev invariant υ of $\gamma_m=\overline{b}_m$,

$$\nabla_d(\delta^{-u}\gamma_m)=\nabla_d(\gamma'_m)=\upsilon(\gamma_m) \tag{71}$$

(with (11)), will distinguish two γ_m. (By a component count, $d\geq 2$ must be even.)

We have thus found an invariant (67) so that (64) is a non-constant polynomial in m. This concludes the proof of Lemma 10.1. □

Remark 10.2 That neither W_k in (69) become powers of σ_2 is, for $n = 4$, precisely captured by Ito's [I] non-degeneracy condition. (Compare the discussion in [St] and below Theorem 6.1.) One can, with minor extra care, release oneself of (63), and fully recover his result for $n = 4$, with the better QSS conclusion.

While this lemma was sufficient for our application, it does not take too much to get a far more general version of it, which provides the sought extension of Theorem 9.4 to the case (63).

Theorem 10.3 Let $b = \alpha'\beta'$ be an exchangeable n-braid with (63). Assume that $lk_{1,i}$ for $i = 2,\dots,n-1$ are not all equal, and neither are all of $lk_{n,i}$. Then b is QSS.

Proof. We fix throughout a power $p > 0$ so that b^p is pure. Let $C_q \not\ni 1,n$ for $q = 1,\dots,\nu$ be the cycles of $\pi(b)$. Let for $i = 1,n$, and $1 < j < n$ with $j \in C_q$,

$$\lambda_{i,j} = p \cdot \frac{lk_{i,C_q}(b)}{|C_q|}\,.$$

Then

$$lk_{i,j}(b_m^p) = \lambda_{i,j}\,.$$

By a trivial observation from the theorem assumption, there are $1 < i < j < n$ so that

$$lk_{1,i}(b) \neq lk_{1,j}(b) \text{ and } lk_{n,i}(b) \neq lk_{n,j}(b). \tag{72}$$

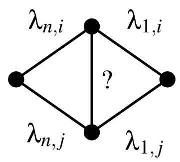
For now these i, j are fixed.

We consider the (B_n-)conjugacy invariant on P_n

$$\upsilon^*(\beta) = \sum_C \sum_{d,u} \big(\upsilon_{d,u}(\beta_{[C]})\big)^2, \tag{73}$$

which we seek to apply for $\beta = b_m^p$.

Here the sum runs over all 4-element subsets C of $\{1,\dots,n\}$ whose subbraids $\beta_{[C]}$ of β have an induced linking subgraph isomorphic to



, (74)

where '?' may stand for any integer, and if any edge label is 0, the edge can be assumed deleted. With u,d even and d positive, $\upsilon_{d,u}$ should be an invariant on B_4 of the type

$$\upsilon_{d,u}(\gamma) = \nabla_d(\delta^{-u}\bar{\gamma}) \tag{75}$$

from the proof of Lemma 10.1, where δ is as in (68) and we must ascertain that d,u are chosen independently of m (though they may depend on b).

When we evaluate $\upsilon^*(b_m^p)$, there is, by design, a matching of (74) in (73) for

$$C = \{1,i,j,n\}\,. \tag{76}$$

The braid $(b_m^p)_{[C]}$ is basically the concatenation of p copies of (66), where V_k and m_k may vary with the copy.

Thus (69) modifies to

$$\gamma_m = \overline{(b_m^p)_{[C]}} = \delta^{u'}(W_{1,1}\sigma_2^m W_{1,2}\sigma_2^{-m})(W_{2,1}\sigma_2^m W_{2,2}\sigma_2^{-m})\cdot\ldots\cdot(W_{p,1}\sigma_2^m W_{p,2}\sigma_2^{-m})\,. \tag{77}$$

By absorbing $W_{l,k}$ being powers of σ_2, one can decrease in (77) the number of factors to $p' < p$, and assume all $W_{l,k}$ contain at least one σ_1^2-syllable.

Because of (72), at least $W_{1,1}$ and $W_{1,2}$ remain unabsorbed, so $p' > 0$. When m is large enough, the Schreier process of finding the form (70) for (77) will be similar to the one in the proof of Lemma 10.1. The alternating parts γ'_m will grow with m while the central exponent $u_m = u$ will become constant in m.

One can then find a proper d, and for these u,d, the invariant defined in (75)

$$\upsilon_{d,u}(b_m^p) \tag{78}$$

will be a non-constant polynomial in m.

Now obviously (76) may not be the unique choice matching (74). There is no way to avoid this for a general braid b under our assumptions.

To remedy this problem, we can find all matchings C in b^p (which do not depend on m, and again only with $1,n \in C$), calculate by the above argument the corresponding d,u for $(b_m^p)_{[C]}$ for large enough m (so that they do not depend on m either), and then sum for these particular d,u in (73) the squares of $\upsilon_{d,u}$ to avoid cancellations.

Since for every C we can find some d,u where the square of (78) grows unboundedly with m, in fact one can determine d,u for one C, and the rest summands in (73) will be polynomials in m which will not spoil non-cancellation. (The sum over several d,u in (73) is not really needed thus.)

The rest of the argument is then the same as for Lemma 10.1. □

Remark 10.4 As in Remark 10.2, to ascertain that after (77), neither $W_{1,l}$ becomes a power of σ_2 (so we have $p' > 0$), the assumption is enough that $b_{[C]}$ is non-degenerate for some C as in (76). (This condition is more relaxed because of (72).) One can also use *certain* $\alpha'_{[C]}\beta'_{[C']}$ when considering $W_{k,2}$ instead of $W_{1,2}$.

The 'upgrade' to SS seems to cost a disproportionate amount of further effort, though. (This is related to the caveat formulated at the end of §9.2.)

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