APPLICATION OF BRAIDING SEQUENCES III: CONCORDANCE OF POSITIVE KNOTS

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Abstract. Applying the concept of braiding sequences and the inequality between the signature and number of roots of the Alexander polynomial on the unit circle, we prove that only finitely many special alternating knots are (even algebraically) concordant, in that their concordance class determines their Alexander polynomial. We discuss some extensions of this result to positive and almost positive knots, and links.

Keywords: special alternating knot, positive knot, genus, Alexander polynomial, concordance, polynomial root
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1 Introduction

Positive links are the links with diagrams where all crossings are positive (right-hand). These links seem to have drawn relevance not as much from the combinatorial property describing them, but from their relation to a series of different subjects, including dynamical systems [BW], algebraic curves [Ru, Ru2], and singularity theory [A, BoW, Mi]. The intersection of the class of positive and alternating links are the special alternating links studied extensively by Murasugi; see for example [Mu].

The concept of braiding sequences [Tr] of links was originally introduced with motivation from Vassiliev (finite degree) invariants [BL, BN, Va]. Braiding sequences were later related to positive and alternating knots [St4, STV, SV] by means of the fact that the set of knot diagrams on which the Seifert algorithm gives a surface of given genus decomposes into finitely many such sequences.

This paper is a continuation of the previous part of the work [St11]. Here we give further applications to properties of positive, in particular special alternating, links. We focus this time on concordance [Lv]. (One can work in the various categories: algebraic, topological[1] or smooth, depending on the circumstances.) Our main goal will be to use the methods we developed to address the following conjecture.

Conjecture 1.1 Any (algebraic) concordance class of knots contains only finitely many positive (or almost positive) ones.

[1] 'Topological' will for us always mean 'locally flat'; we will not discuss the PL case here.

We will discuss evidence for this conjecture below in §4.1, which somewhat varies with its several possible versions. (See also question 4.2.) In particular, it is important to keep distinction between the three levels of concordance: algebraic, topological, or smooth. (The results that separated these categories have taken a long time to prove, and have significant impact, also in our situation.)

First, §2 explains the background and main tools in the approach to conjecture 1.1. In §3 we deal with inequalities relating the signature and number of roots of the Alexander polynomial on the unit circle. We give a review of the background of this relationship. With a study of Tristram-Levine signatures, we prove then in §4 for a large class of positive knots that its intersection with any knot concordance class is finite. We state here our main advance towards conjecture 1.1.

Theorem 1.1 Any topological concordance class of knots (in fact, already any algebraic concordance class) contains only finitely many special alternating ones. That is, each special alternating knot is topologically (or algebraically) concordant to only finitely many special alternating knots. All these knots have the same Alexander polynomial.

More precisely, we will establish that a special alternating knot can share its Tristram-Levine signature jump function (30) with only finitely many others (see remark 4.1). In is worth emphasizing that in the smooth category, essentially the same proof gives a much stronger statement.

Corollary 1.1 In any infinite family of smoothly concordant positive knots, there is no special alternating one. In other words, each special alternating knot is smoothly concordant to only finitely many positive knots. Also, all have the same Alexander polynomial.

We will then make efforts to extend these results with focus on our conjecture, proving similar statements for some class of positive and almost positive knots (§4.3) and links (§4.4). We will give computational examples in §4.5, discuss further, verifiable and problematic, cases in §4.6, and conclude with some methodological remarks in §4.7.

At a very late stage the prerint [BDL] appeared. See remark 4.6 for its relation to this work.

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2 Preliminaries, Notations and Conventions

2.1 Generalities

The symbols $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ denote the integer, natural, rational, real and complex numbers, respectively. Let $\Re e$ and $\Im m$ denote the real and imaginary part of a complex number. We will also write $i = \sqrt{-1}$ for the imaginary unit, in situations where no confusion (with the usage as index) arises. Let us fix for $z \in \mathbf{C} \setminus \{0\}$ that the argument $\arg(z) = \Im m\,(\log(z))$ is taken in $[0, 2\pi)$.

For a set S, the expression $|S|$ denotes the cardinality of S. In the sequel the symbol '$\subset$' denotes a not necessarily proper inclusion.

We need next a few notations related to polynomials, which are understood in the broader sense as Laurent polynomials (i.e., variables are allowed to occur with negative exponents). Moreover, we will consider also Laurent polynomials with (only) half-integral powers, thus to save notation, let

$$\mathcal{L}[t] := \mathbf{Z}[t^{\pm 1}] \;\cup\; \sqrt{t}\,\mathbf{Z}[t^{\pm 1}]. \tag{1}$$

Let '$\doteq$' be equality up to units $\pm t^{\pm 1/2}$ in $\mathcal{L}[t]$. For a polynomial $X \in \mathcal{L}[t]$, and $a \in \mathbf{Z} \cup (\mathbf{Z}+1/2)$, let $[X]_{t^a} = [X]_a$ be the coefficient of t^a in X. For $X \neq 0$, let $\mathcal{C}_X = \{\, a \in \mathbf{Z} \cup (\mathbf{Z}+1/2) : [X]_a \neq 0 \,\}$ and

$$\min\deg X = \min\, \mathcal{C}_X\,, \quad \max\deg X = \max\, \mathcal{C}_X\,, \qquad \text{and} \qquad \operatorname{span} X = \max\deg X - \min\deg X$$

be the *minimal* and *maximal degree* and *span* (or breadth) of X, respectively. The *leading coefficient* of X is $\max\operatorname{cf} X := [X]_{\max\deg X}$. A similar nomenclature is deployed for 2-variable polynomials.

Let further for $X, X_1 \in \mathcal{L}[t]$ and $\xi \in \mathbf{C} \setminus \{0\}$, $\operatorname{mult}_\xi X$ be the *multiplicity* of ξ as a root (or zero), and $\operatorname{mult}_{X_1} X$ the multiplicity of X_1 as a divisor (or factor) of X. Set $\operatorname{mult} = 0$ if ξ resp. X is not a root resp. divisor. If $\operatorname{mult} > 1$, we call the root or divisor *multiple*. A polynomial with no multiple roots/divisors is *square-free*. (There is no ambiguity, as we will work only over separable fields.)

Of course, for irreducible X_1, we have

$$\operatorname{mult}_{X_1} X = \operatorname{mult}_\xi X\,, \tag{2}$$

for every zero ξ of X_1, and conversely, for any $\xi \in \mathbf{C} \setminus \{0\}$, the property (2) holds for the minimal polynomial X_1 of ξ.

We mention also that irreducibility is meant (because polynomial factorization can always be done) over $\mathbf{Z}$, by a lemma of Gauß (can be found, e.g., in [Se]).

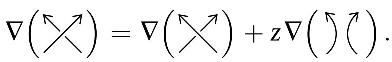
Some further notations will be introduced at an appropriate place in the text.

2.2 Conway-Alexander polynomial

The *Conway* [Co] and (1-variable) *Alexander polynomial* [Al] of a link L are regarded here as equivalent:

$$\Delta_L(t) \,=\, \nabla_L(t^{1/2} - t^{-1/2})\,. \tag{3}$$

They are defined to be 1 on the unknot. The skein relation for ∇ can be written



(4)

A *skein triple* D_+, D_-, D_0 is a triple of diagrams, or of their corresponding links L_+, L_-, L_0, equal except near one crossing, where they look like in (4) (from left to right). The replacement $L_\pm \to L_0$ is called *smoothing (out)* the crossing in $L_\pm$. The crossing in D_+ is called *positive*, the one in D_- *negative*. The sum of the signs of all crossings of D is called the *writhe* of D and will be written $w(D)$.

Let D be an oriented knot or link diagram. We denote by $c(D)$ the *crossing number* of D. The crossing number of a link is the minimal crossing number of all its diagrams. We use $n(D) = n(L)$ to designate the *number of components* of D or its link L. We write $s(D)$ for the *number of Seifert circles* of a diagram D (the loops obtained by smoothing out all crossings of D).

One can argue that $\min\deg \nabla_L(z) \geq n(L)-1$, i.e., the coefficient $[\nabla(L)]_j$ vanishes whenever $j < n(L)-1$. This in particular means that indeed $\nabla_L(z)$ is a genuine polynomial in z, and not a Laurent one, as (3) might suggest.

Throughout this treatise, Δ is thus normalized so that (3) holds. The word 'normalized' refers to comparison with other definitions of the Alexander polynomial (see (24) below), where one often leaves an ambiguity up to units in $\mathcal{L}[t]$.

Thus for *knots* K we will have

$$\Delta_K(1) = 1\,, \tag{5}$$

and for a general link L,

$$\Delta_L(1/t) = (-1)^{n(L)-1}\Delta(t) \tag{6}$$

(i.e., the sign is positive/negative for odd/even number of components). We will call property (5) *unimodularity*, and (6) *symmetry* (or *reciprocity*). It is well-known that for knots conditions (5) and (6) (with $n(L)=1$) exactly determine the Laurent polynomials occurring as Alexander polynomial. Note furthermore that

$$\Delta_L \in t^{(n(L)-1)/2}\mathbf{Z}[t^{\pm 1}]\,. \tag{7}$$

The reformulation of symmetry of Δ in terms of ∇ is that $\nabla_L(z)$ is an even/odd polynomial (i.e., has coefficients only in even/odd z-degree), when $n(L)$ is odd/even. The reformulation of unimodularity is that for knots $[\nabla]_0 \equiv 1$. More generally, $[\nabla(L)]_{n(L)-1}$ can be expressed in terms of component linking numbers; see [Ht].

It is important to note that for a knot K, because of (5) and (6), $\Delta_K(-1)$ is an odd integer, and hence

$$\Delta_K(\pm 1) \neq 0\,. \tag{8}$$

The value

$$\det(L) := \Delta_L(-1)$$

is sometimes called the *determinant* of L. For a link L (of more than one component) always $\Delta_L(1)=0$, but there are many links L with

$$\det(L) = \Delta_L(-1) \neq 0\,.$$

These include all (non-split) alternating links, by work of Crowell-Murasugi (related to theorem 2.2 given below).

2.3 Links and diagrams

Here, and in the sequel, for a knot or link K, we write $!K$ for its *obverse*, or mirror image. Similarly $!D$ is the mirror image of a link diagram D. If K is a *knot*, write $-K$ for its *inverse* (knot with opposite orientation). By $K_1\#K_2$ we denote the *connected sum* of K_1 and K_2.

For a few specific (prime) knots, we use the tables of [Ro, Appendix] up to 10 crossings, and for 11 to 16 crossings, the tables of [HT] (see also [HTW]), where we append non-alternating knots after alternating ones (of the same crossing number).

We say that a link diagram D is *l-almost positive* if it has exactly l negative crossings, that is, $w(D)=c(D)-2l$. A knot is l-almost positive if it has an l-almost positive diagram, but no $(l-1)$-almost positive one. Hereby, for both knots and diagrams, '0-almost positive' is called shorter *positive* and '1-almost positive' is *almost positive* [St7].

Note: There seems some division between knot theorists as to which links are to be called positive. In [Bu, MZ], the rather non-standard (and confusing) convention is used to call 'positive knots' the knots with positive braid representations (called 'positive braids' [Cr2], or better 'positive braid knots'). The convention here follows the now established standard, used in many publications, as [Cr, CM, MP, N, O, Ru, Ta, Yo, Zu], to call positive knots the (larger) class of knots with a positive diagram.

A link diagram D is called *split*, or disconnected, if it can be non-trivially separated by a simple closed curve in the plane. Otherwise we say the diagram is *non-split*, or *connected*. A split link is a link with a split diagram. Other links are said to be non-split. A crossing in a non-split diagram is *reducible*, if its smoothing gives a split diagram. A diagram is reducible if it has a reducible crossing, otherwise it is called *reduced*. To avoid confusion, unless otherwise stated, in the sequel all diagrams are assumed reduced, that is, with no nugatory crossings, and links are non-split.

A diagram D is called *composite*, if there is a closed curve γ intersecting (transversely) the curve of D in two points, such that both in- and exterior of γ contain crossings of D. Otherwise D is called *prime*. (Note in particular that prime implies reduced.) A link L is prime, if for every composite diagram D of L one of the in- or exterior of γ contain (only) an unknotted arc; otherwise L is composite.

Theorem 2.1 (Ozawa [O]; see also [Cr2]) If a positive diagram is prime, it depicts a prime link. In particular, prime factors of a positive link are positive.

A *region* of a link diagram D is a connected component of the complement of the plane curve of D. A region R of a diagram is called *Seifert circle region* (resp. non-Seifert circle, or *hole* region), if any two neighboring edges in its boundary (i.e., such sharing a crossing) are equally (resp. oppositely) oriented (between clockwise or counterclockwise) as seen from inside R. A diagram is called *special* iff all its regions are (either) Seifert circle regions or hole regions.

It is an easy combinatorial observation that for a connected diagram two of the properties alternating, positive and special imply the third. A diagram with these three properties is called *special alternating*. See, e.g., [Mu, Mu2]. A special alternating link is a link having a special alternating diagram. It can be described also (like in the introduction) as a link which is simultaneously positive and alternating. By definition such a link has a positive diagram, and an alternating diagram. That it has a diagram which enjoys simultaneously both properties was proved in [N, St6].

2.4 Genera

In the sequel we denote by $g(D)$ the *genus* of a diagram D, this being the genus of the surface coming from the Seifert algorithm applied on this diagram. More conveniently, if D is a link diagram, we use instead of $g(D)$ the notation $\chi(D)$ for the *Euler characteristic* of the Seifert surface given by the Seifert algorithm.

By $g(L)$ we will denote the genus and $\chi(L)$ the Euler characteristic of a link L, which are the minimal genus resp. maximal Euler characteristic of an orientable spanning (i.e., Seifert) surface for L. By $g_c(L)$ we denote the *canonical genus* of L, which is the minimal genus $g(D)$ of some diagram D of L. Similarly, $\chi_c(L)$, the *canonical Euler characteristic* of L, is the maximal $\chi(D)$ for all diagrams D of a link L. Further, for a link L we denote by $g_s(L)$ the *smooth 4-ball genus,* which is the minimal genus of a (smoothly) properly embedded surface in the 4-ball with boundary L. Finally, $\chi_s(L)$ is set to be the *smooth 4-ball Euler characteristic*.

A knot K is *slice* if $g_s(K)=0$. Two knots K_1 and K_2 are *(smoothly) concordant* if $-K_1\#!K_2$ is slice. Let us say that K_2 is *positively concordant* to K_1, if K_1 and K_2 are concordant and positive.

Theorem 2.2 (see [Cr, C, Mu2]) The Seifert algorithm applied on an alternating or positive diagram gives a minimal genus surface.

Thus the genus $g(L)$ of an alternating/positive link L coincides with the genus $g(D)$ of an alternating/positive diagram D of L, given by

$$g(D)=\frac{c(D)-s(D)+2-n(D)}{2}\,, \tag{9}$$

with $c(D)$, $s(D)$ and $n(D)=n(L)$ being the number of crossings, Seifert circles and components of D, resp. The preceding theorem implies that for alternating/positive links L,

$$g(L)=g_c(L)\,. \tag{10}$$

For positive links, we have additionally that

$$g(L)=g_s(L) \tag{11}$$

(see (18) below).

We recall two major ways of estimating genera of arbitrary knots from below. One comes from the Alexander/Conway polynomial. It is well-known that for split links $\nabla=0$, and (as partly stated already in §2.2) for a non-split link L, the coefficient $[\nabla(L)]_i$ is non-zero only if

$$n(L)-1\leq i\leq 1-\chi(L)\,,\ \text{ and }\ i-n(L) \text{ is odd.} \tag{12}$$

The right inequality is related to the description of Δ in terms of Seifert matrices (see below (24)). The parity condition is seen equivalent to the property (7) of Δ. Let us call an i satisfying (12) *admissible*, and in the same way the coefficient $[\nabla]_i$ admissible for such i.

The range of i in (12) means that (for $\nabla \neq 0$)

$$n(L)-1 \le \min\deg_z \nabla(L) \le \max\deg_z \nabla(L) \le 1-\chi(L)\,. \tag{13}$$

For many (non-split) links, including positive and alternating ones, the rightmost inequality is exact, i.e., an equality, and with (3) we can write

$$1-\chi(L) = 2\max\deg\Delta_L = -2\min\deg\Delta_L = \max\deg\nabla_L\,. \tag{14}$$

(In fact, this property is directly related to theorem 2.2.) More specifically, we have the following property.

Theorem 2.3 For positive and almost positive (non-split) links, ∇ is a *positive polynomial*, i.e., all its admissible coefficients are positive.

For the later applications of this theorem, it is enough that all admissible coefficients are nonnegative, and at least one is positive. This property is essentially proved in [Cr, Corollary 2.2]. For the clarification why no admissible coefficient is zero (not needed here) see, e.g., [St11] (for positive links) and then [St3, §4.1] (for how to extend it to almost positive links).

Note that for *knots* K, the rightmost inequality in (13) can be restated using (3) and $2g(K)=1-\chi(K)$ in the form

$$\max\deg\Delta(K) \le g(K)\,. \tag{15}$$

Again, equality holds for positive and alternating knots K.

For any knot K the left inequality in (13) is exact (and the constant term of ∇ is 1). Note also that (13) implies $2g(L) \ge \operatorname{span}\nabla(L)$, and equality occurs iff the leftmost and rightmost inequalities in (13) are both exact, in particular for positive links.

The other way of estimating genera comes from Bennequin's inequality [Be, theorem 3], and its subsequent improvements. We define the *Bennequin number* $r(D)$ of a diagram D of a link L to be

$$r(D) := \frac{1}{2}\big(w(D)-s(D)+1\big)\,. \tag{16}$$

Then it is known (see [Ru, He]) that

$$1-\chi(L) \ge 1-\chi_s(L) \ge 2r(D)\,, \tag{17}$$

which is called *slice Bennequin inequality*. We mainly require the following special case. For an l-almost positive diagram D of a knot K, by comparison of (9) (with $n(D)=1$) and (16), we have $r(D)=g(D)-l$, and (17) yields

$$g(D)-l = r(D) \le g_s(K) \le g(K) \le g_c(K) \le g(D)\,. \tag{18}$$

In particular, for positive diagrams ($l=0$), all inequalities become equalities (recall (10) and (11)). This amplifies, by addition of the slice genus, a special case of theorem 2.2.

There is a third estimation of genera, using Tristram-Levine signatures, which we explain next.

2.5 Tristram-Levine signatures

Here we introduce one further main technical tool appearing in the sequel. A more extensive discussion close to our course here can be found in [L, §8]. (We shall give a few additional references for further details.)

Set
$$S^1 := \{\, z \in \mathbf{C} : |z| = 1 \,\}, \quad \text{and} \quad S^1_+ := \{\, z \in S^1 : \Im m\, z > 0 \,\}. \tag{19}$$
Let M be a Seifert matrix of size $(1-\chi)\times(1-\chi)$ corresponding to a Seifert surface S of a link L of Euler characteristic χ. Note: $1-\chi < 0$ can occur (only) if S is disconnected; then $\Delta(L) = 0$, and such links will not be studied in this paper. We thus assume throughout that S is connected. Then for any $\xi \in S^1$ we define
$$M_\xi(L) := (1-\xi)M + (1-\bar{\xi})M^T, \tag{20}$$
where bar denotes conjugation and $\cdot^T$ transposition. This is a Hermitian matrix, and all eigenvalues are real. By $\sigma(M_\xi)$ we denote the *signature* (sum of signs of eigenvalues) and by $\nu(M_\xi)$ the *nullity* (dimension of the kernel, or number of zero eigenvalues) of M_ξ. They turn out to be independent of the surface and Seifert matrix, and become thus invariants of L. We obtain then a family, written by $\sigma_\bullet(L)$, of signatures $\sigma_\xi(L)$ for $\xi \in S^1$, called generalized or *Tristram-Levine signatures* [Ts, Le]. Similarly we have the *nullities* $\nu_\bullet(L)$. Clearly,
$$\sigma_\xi(L) = \sigma_{\bar{\xi}}(L), \tag{21}$$
which is why we will usually work only on S^1_+, while paying attention to $\xi = \pm 1$.

All the Tristram-Levine signatures satisfy for a skein triple $L_{\pm,0}$ the rules
$$\begin{array}{rcl} \sigma_\xi(L_+) - \sigma_\xi(L_-) & \in & \{0,1,2\}, \\ \sigma_\xi(L_\pm) - \sigma_\xi(L_0) & \in & \{-1,0,1\}. \end{array} \tag{22}$$
(Whether to have $\{0,1,2\}$ or $\{0,-1,-2\}$ in (22) is a matter of convention.) The behaviour of signatures under mirroring, inversion (for a knot L), and connected sum is likewise well-known:
$$\begin{array}{rcl} \sigma_\xi(L) & = & \sigma_\xi(-L) \;=\; -\sigma_\xi(!L), \\ \sigma_\xi(L\#K) & = & \sigma_\xi(L) + \sigma_\xi(K). \end{array} \tag{23}$$
For the purpose of this paper, we need another approach to the (normalized) Alexander polynomial, different from the skein property (3). The polynomial of a link L can be calculated from a Seifert matrix M of L by
$$\Delta_L(t) = t^{(\chi-1)/2}\det(M - tM^T). \tag{24}$$
(In this definition the unit $t^{(\chi-1)/2}$ is often omitted, but it is added here to conform to the normalization of §2.2.) In recalling the consequence (13), let us note that thus a link satisfies equality (14) exactly if it possesses a regular (non-zero determinant) Seifert matrix.

Tristram-Levine signatures are related to the 4-genus (*cf.* §2.4) via the Tristram–Murasugi inequality [Ts, Mu]: if ξ is a prime power root of unity,
$$\big|\sigma_\xi(L)\big| + \nu_\xi(L) \le 2g_s(L) + n(L) - 1. \tag{25}$$
This inequality holds also in the topological category. (Note: with 'g_s' replaced by 'g', for connected Seifert surfaces, the r.h.s. becomes $1-\chi(L)$, and (25) follows for all ξ by definition of $\sigma_\bullet$ and $\nu_\bullet$.) A consequence is that for a knot K,
$$|\sigma_\xi(K)| \le 2g_s(K) \ \text{ when } \Delta_K(\xi) \ne 0. \tag{26}$$
A further feature of $\sigma_\xi(L)$ for a link L is that
$$\text{when } \Delta_L(\xi) \ne 0, \text{ then } \sigma_\xi(L) - n(L) \text{ is odd.} \tag{27}$$
Moreover, in this case, for a knot K, the sign of $\Delta_K(\xi)$ (which is a real number when $|\xi| = 1$ and Δ_K is symmetric) determines σ_ξ mod 4 (see [St10]):
$$\begin{array}{rcl} \Delta_K(\xi) > 0 & \Longleftrightarrow & \sigma_\xi(K) \equiv 0 \bmod 4, \\ \Delta_K(\xi) < 0 & \Longleftrightarrow & \sigma_\xi(K) \equiv 2 \bmod 4. \end{array} \tag{28}$$

From the relationship (26) and properties (23) one easily deduces that $\sigma_\bullet(K)$ is a concordance invariant *outside* the zeros of the Alexander polynomial. Namely, if K_1 and K_2 are concordant, then $-K_1\#!K_2$ is slice, i.e., $g_s(-K_1\#!K_2)=0$. We have from (23) and (26) that $\sigma_\xi(-K_1\#!K_2)=\sigma_\xi(K_1)-\sigma(K_2)=0$ whenever $\Delta_{-K_1\#!K_2}(\xi)=\Delta_{K_1}(\xi)\cdot\Delta_{K_2}(\xi)\neq 0$. That is, if $K_{1,2}$ are concordant, and

$$\Delta_{K_1}(\xi)\neq 0\neq\Delta_{K_2}(\xi)\,, \tag{29}$$

then $\sigma_\xi(K_1)=\sigma_\xi(K_2)$.

To remove the restriction (29), one considers the *jump* of $\sigma_\bullet$ at ξ,

$$j_\xi(L):=\lim_{\varepsilon\searrow 0}\sigma_{\xi e^{i\varepsilon}}(L)-\lim_{\varepsilon\nearrow 0}\sigma_{\xi e^{i\varepsilon}}(L)\,, \tag{30}$$

which becomes a full concordance invariant for knots K. This property will be a key in our arguments. It is obvious that

$$j_\xi(L)\neq 0\quad\text{only if}\quad\Delta_L(\xi)=0\,. \tag{31}$$

Further, it follows from (21) that

$$j_\xi(L)\,=\,-j_{\bar\xi}(L)\,, \tag{32}$$

in particular the jumps $j_{\pm 1}(L)=0$, and are useless.

Here may be appropriate to emphasize that calculating $\sigma_\bullet$ and $j_\bullet$ using (large) Seifert matrices is not efficient in practice, regardless of being polynomial time, and requires a substantial algorithmical effort to be performed safely. We used a package for MATHEMATICA™ written by S. Orevkov. This method, though, is suitable only for limited scale computations, and thus we sought alternative means to evaluate these invariants. See, e.g., proposition 3.1.

The *(usual) signature* $\sigma=\sigma_{-1}$ of Murasugi [Mu] has a very distinguished role among the Tristram-Levine signatures. An important special case of (27) for a knot K is when $\xi=-1$. Because of (8), we have that $\sigma(K)=\sigma_{-1}(K)$ is always even for a knot. Moreover, (26) applies to give

$$|\sigma(K)|\leq 2g_s(K)\leq 2g(K)\,. \tag{33}$$

Murasugi first proved that these inequalities are exact (i.e., equalities) for special alternating knots.

More generally, for alternating knots there are nice combinatorial formulas available to calculate σ from an alternating diagram (see, e.g., [Mu, Kf, GL]). These formulas, together with the skein rule (22) ('1' does not occur on the right) and the property (28) (for $\xi=-1$) provide a tool for calculating σ, at least for knots, only using integer arithmetic. This is one of the methods with a huge practical advantage over diagonalizing M_{-1}, and was used for σ calculations throughout this paper.

We will need the following information on the signature of positive knots.

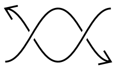
Theorem 2.4 (see [CGo, PT, St7, St9]) Let K be a positive knot of genus g and signature σ. Then

1. $\sigma>0$,
2. $\sigma=2$ if and only if $g=1$,
3. if $g=2$, then $\sigma=4$,
4. if $g=3$, then $\sigma\in\{4,6\}$,
5. if $g=4$, then $\sigma\in\{6,8\}$, except if $K=14_{45657}$ (where $\sigma=4$), and
6. if $g\geq 5$, then $2g\geq\sigma\geq 6$.

More recently Tristram-Levine signatures have been of some interest because of their relation to the classification of zero sets of algebraic functions on projective spaces [Or, F] and (a quantum version of) the Jones polynomial [G].

2.6 Braiding sequences and genus generators

Now let us recall, from [St4, St5], some basic facts concerning knot generators of given genus. We will set up some notations and conventions used below. This is discussed in much more detail in [St9]. Cromwell offers in his recent book [Cr3] (section 5.3) an introductory exposition on the subject.

We start by defining $\sim$-equivalence of crossings. A *reverse clasp* is, up to crossing changes, a tangle like . If exactly one strand has opposite orientation, we call the clasp *parallel*. We call a clasp *trivial* if both its crossings have opposite sign. Such a clasp can be eliminated by a Reidemeister II move.

Definition 2.1 Let D be a link diagram, and p and q be crossings. We call p and q $\sim$*-equivalent* and write $p \sim q$, if smoothing out one renders nugatory the other.

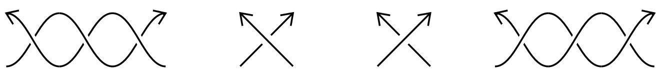
Another (and more commonly used elsewhere) way of saying this is that p and q can be made to form a reverse clasp after flypes. A minor argument will convince one that this is indeed an equivalence relation.

Definition 2.2 A $\sim$-equivalence class consisting of one crossing is called *trivial*, a class of more than one crossing *non-trivial*. A $\sim$-equivalence class is *reduced* if it has at most two crossings; otherwise it is *non-reduced*. A diagram is called *generating*, or a *generator*, if all its $\sim$-equivalence classes are reduced.

Let D be an oriented link diagram with crossings $c_1,\dots,c_n$. We explain now, following [St2], how to define a family of diagrams $\mathcal{D} = \mathcal{B}(D)$ called *braiding sequence* (or *series*). Consider the family of diagrams

$$\mathcal{D} \;=\; \{\; D(p_1,\dots,p_n) : \;\; p_1,\dots,p_n \in \mathbf{Z} \text{ odd } \;\}\,.$$

Herein the diagram $D(p_1,\dots,p_n)$ is obtained from D by replacing the crossing c_i by a tangle consisting of $|p_i|$ reverse half-twists of sign $\operatorname{sgn}(p_i)$:

 . (34)

$p_i = -3$ $\qquad$ $p_i = -1$ $\qquad$ $p_i = 1$ $\qquad$ $p_i = 3$

Following [St2], we will call $\mathcal{D}$ the *braiding sequence* $\mathcal{B}(D)$ associated to D. Note that $\mathcal{B}(D)$ does not in fact depend on how crossings in D are switched. In particular, we can, and will, assume without loss of generality that D is alternating.

We will use below the following simple technical argument from [St11].

Lemma 2.1 ([St11]) If
$$D_i = D(p_{1,i},\dots,p_{n,i}) \tag{35}$$
are infinitely many diagrams in one braiding sequence, then we may without loss of generality assume that there is a $1 \le d \le n$ such that $p_{d+1,i},\dots,p_{n,i}$ are constant, and

$$p_{k,i+1} > p_{k,i}$$

for all $i > 0$ and $k = 1,\dots,d$.

We call a *positive* (resp. *negative*) $\bar{t}_2'$ *twist* the replacement of the tangle for $p_i = 1$ (resp. $p_i = -1$) in (34) by the one for $p_i = 3$ (resp. $p_i = -3$).

This move does not change the canonical genus: when D' is obtained from D by a $\bar{t}_2'$ twist, then $g(D') = g(D)$. Thus $g(D') = g(D)$ is constant for all $D' \in \mathcal{B}(D)$. As it turns out, some kind of converse of this property is true for fixed $g(D)$, up to finite indeterminacy.

Theorem 2.5 (see [St4, St9]) The set of knot diagrams on which the Seifert algorithm gives a surface of given genus, regarded up to crossing changes and flypes, decomposes into a finite number of reverse braiding sequences $\mathcal{B}_i = \mathcal{B}(D_i)$ for generators D_i. The same is true for link diagrams of fixed number of components.

Usually one considers alternating diagrams D_i and their underlying knots K_i (which we also call generators). There are systematical ways to determine the generator sets $\{K_i\}$ for small g. The case $g = 1$ was done by hand in [St4] (and observed independently in [Ru]), and $g = 2, 3$ in [St5], already using substantial computation. For $g = 4$, the limit of the feasible, an account is given separately [St9]. The generator sets quickly become highly difficult, and each new set required an increasingly efficient algorithm to determine. Theorem 2.4 is to a large extent an application of this work.

3 Signature-zeros estimates

One further major tool used for our main result requires a longer treatise, and we devote this separate section to it. It should be noted that results in this section do not assume positivity (or almost positivity), except a few places where explicitly noted, and apply (for links) without *a priori* restricting to knots.

This section is organized thus. In §3.1 we discuss a relation between signature and number of zeros of the Alexander polynomial, which plays a central role in the proof of theorem 1.1 (and its later generalizations). The case of simple zeros in [St8] is recalled with some additions in §3.2. The subsection §3.3 is devoted to extensions of the signature-root number inequality, and some special cases and easy consequences, still without assumption of positivity.

3.1 The inequality of Alexander polynomial zeros and signature

In [St8] we studied, in a special case, a close relationship between the signature and the number of zeros of the Alexander polynomial on the (complex) unit circle. We formulate this relationship after an important clarification.

Definition 3.1 Let us in the following fix that zeros of a Laurent polynomial $X \in \mathcal{L}[t]$ over some complex domain S are always counted with multiplicity. In that sense, define for $S \subset \mathbf{C}$,

$$\zeta(X,S) := \sum_{\substack{\xi \in S \setminus \{0\} \\ X(\xi)=0}} \mathrm{mult}_\xi(X)\,.$$

Observe that

$$\zeta(X,\mathbf{C}) = \mathrm{span}\,X\,. \tag{36}$$

Moreover, there is the complex-analytic integral formula

$$2\pi i \cdot \zeta(X,S) = \oint_{\partial S} \frac{X'(z)}{X(z)}\,dz\,, \tag{37}$$

valid when S does not contain 0 and has at least piecewise smooth boundary ∂S (oriented counterclockwise) with no roots of X on it.

With (19), the mentioned relationship is stated as follows: for a link L with $\Delta_L \neq 0$,

$$\big|\,\sigma(L)\,\big| \le \zeta(\Delta_L, S^1)\,. \tag{38}$$

Clearly with any zero $z \in S^1$ of Δ, the conjugate $\bar{z}$ is also one. Moreover, for a *knot* K, there is no overlap because of (8). Thus

$$\zeta(\Delta_K, S^1_+) = \frac{1}{2}\zeta(\Delta_K, S^1), \tag{39}$$

and (38) can be paraphrased as

$$\zeta(\Delta_K, S^1_+) \;\geq\; \frac{1}{2}\big|\sigma(K)\big|. \tag{40}$$

We will first discuss several arguments for this special case, before turning to the subtleties of extending it to links.

Lemma 3.1 If K is a knot, then (40) holds.

Proof. This result follows from the more detailed property that for $\xi \in S^1$,

$$\big|\, j_\xi(K)\,\big| \leq 2\,\mathrm{mult}_\xi\,\Delta(K), \tag{41}$$

which can be also seen as a non-trivial extension of (31). One approach toward (41) goes back to the definition of Milnor's signature [Mi2], and its identification with the jump by Levine and Matumoto [Le, Ma]. Another approach, discussed in [G, St8], establishes that the branches of eigenvalues of M_ξ are smooth in ξ. It uses spectral theory and requires a result of Trotter [Tt] that a Seifert matrix can be regularized under S-equivalence. □

More background, albeit important, is too verbous to discuss here, in particular because the lemma is insufficient for us. It should be made clear that both Trotter's and Milnor's work make substantial use of unimodularity (5) (and Milnor further of the homological algebra of knot complements), and thus heavily restrict to knots. In Matumoto's proof, regularity of the Seifert matrix remains a basic assumption, and (8) is also used (albeit more tacitly), again leaving the case of links in limbo. This will be remedied with the more general arguments in §3.3, to which we prefer (and need) to assign more attention.

On a related matter, there is Shinohara's inequality [Sh, Theorem 3], valid for links when $\Delta \neq 0$:

$$|\sigma| \leq 2\max\deg\Delta = \mathrm{span}\,\Delta. \tag{42}$$

For knots, it is easily seen (from (36)) as a consequence of (40), but is proved in a purely algebraic (and thus more natural) way.

All consequences and special cases of (40) we need are taken from [St8] or reproduced below from our own tools (see, e.g., the remark below (42)). One such is stated here, to be used for theorem 1.1; we defer a proof to §3.3.

Corollary 3.1 If $\sigma(K) = 2g(K)$ for a knot K, then all zeros of Δ_K lie on S^1.

Murasugi's work (as quoted below (33)) shows that special alternating knots K satisfy the assumption of the corollary. Our main result of [St8] generalizes to links its conclusion, which reformulates to ∇ in the stated way from the relation (3).

Let in the following

$$\widetilde{\nabla}_L(z) := \nabla_L(\sqrt{z}). \tag{43}$$

Theorem 3.1 ([St8]) If L is a special alternating link, then any zero of $\widetilde{\nabla}_L$ (is real and) lies in the interval $[-4, 0]$, or equivalently, all zeros of Δ_L lie on the complex unit circle.

3.2 Signature jumps at simple roots

We will need some of the argument in [St8] for jumps at simple roots of Δ.

Lemma 3.2 ([St8]) Let K be a knot, such that Δ_K has no multiple zero. Then inequality (40) holds.

Proof. For a simple zero ξ_0 of Δ_K on S^1 the signature σ_ξ always changes by ± 2 in ξ_0, i.e.,

$$\big|\, j_{\xi_0}(K)\,\big| \,=\, 2 \ \text{ when } \ \text{mult}_{\xi_0}\Delta_K \,=\, 1\,. \tag{44}$$

This is essentially a consequence of the Implicit Function Theorem applied on $f(\xi,\alpha)=\det(M_\xi-\alpha\cdot Id)$. It allows one to bijectively (and smoothly) express the eigenvalue α close to 0 in terms of ξ close to ξ_0; in particular this (locally unique) eigenvalue must change sign. Inequality (40) is immediate. □

It will be useful to introduce the one-sided signature limits as follows (compare (30)):

$$\sigma_\xi^+(L) := \lim_{\varepsilon\searrow 0}\sigma_{\xi e^{i\varepsilon}}(L) \quad \text{ and } \quad \sigma_\xi^-(L) := \lim_{\varepsilon\nearrow 0}\sigma_{\xi e^{i\varepsilon}}(L)\,. \tag{45}$$

For knots, there is a way to bypass the Seifert matrix (as motivated below (32)) in testing the sign in (44) for some simple jumps, which will be helpful mostly for practical calculations (see §4.5).

Proposition 3.1 Assume ξ_0 is a simple zero of Δ_K for a knot K such that on the S^1–arc

$$\Xi(\xi_0) \,=\, \{\,e^{it}\,:\,0<t<\arg(\xi_0)\,\} \tag{46}$$

between 1 and ξ_0, all n zeros on Δ_K are also simple. Let K' be a knot obtained from K by changing a positive crossing to become negative. Then the following holds.

a) If $j_{\xi_0}(K)=2$, then $(-1)^n\Delta_{K'}(\xi_0)\ge 0$ for all K'.

b) If $j_{\xi_0}(K)=-2$, then $(-1)^n\Delta_{K'}(\xi_0)\le 0$ for all K'.

Proof. Because (44) applies for the n zeros on Δ_K on $\Xi(\xi_0)$, we have in (45)

$$\sigma_{\xi_0}^-(K)\equiv 2n \bmod 4\,.$$

If $\xi_{\xi_0}(K) \,=\, +2$ (similarly for -2), then $\sigma_{\xi_0}(K)\equiv 2n+1 \bmod 4$. Because of the skein property (22), we have $\sigma_{\xi_0}(K')\not\equiv 2n+2 \bmod 4$. If $\sigma_{\xi_0}(K')$ is odd, then $\Delta_{K'}(\xi_0)=0$. Otherwise, $\sigma_{\xi_0}(K')\equiv 2n \bmod 4$, and then use (28). □

Lemma 3.2 yields the part of theorem 3.1 for knots with square-free (i.e., without multiple zeros) Alexander polynomials.

Corollary 3.2 ([St8]) Let K be a special alternating knot, such that Δ_K has no multiple zero. Then all zeros of $\widetilde{\nabla}_K$ are real and lie in the interval $[-4,0]$.

Proof. We have $\sigma(K)=2g(K)$ by Murasugi (see §2.5), and can reinstate the chain of inequalities:

$$2g(K) \ \ge\ \zeta(\Delta_K,\mathbf{C}) \ \ge\ \zeta(\Delta_K,S^1) \ \ge\ \sigma(K) \ =\ 2g(K)\,, \tag{47}$$

which, complemented here by the property on the right, all become equalities. □

The proof of theorem 3.1 in [St8] proceeded by using some approximation result (of roots of link polynomials by roots of square-free knot polynomials), to deduce the full extent of the theorem from this special case.

3.3 Generalized signature-zeros estimate

Definition 3.2 Define (using the notation in §2.4) a knot or link L to be *regular* if equality (14) holds,

$$1-\chi(L) = 2\max\deg\Delta(L)\,,$$

and *irregular* otherwise.

Remember, that it follows from the the comment below (24) that L is regular if and only if L has a regular Seifert matrix.

Next, let us introduce the one-sided jumps (using (45), and comparing with (30)):

$$j^+_\xi(L) := \sigma^+_\xi(L) - \sigma_\xi(L)\,, \quad \text{and} \quad j^-_\xi(L) := \sigma^-_\xi(L) - \sigma_\xi(L)\,.$$

Then $j_\xi(L) = j^+_\xi(L) - j^-_\xi(L)$. Clearly only ξ with $\Delta_L(\xi) = 0$ are interesting, but these singular signatures $\sigma_\xi(L)$ have not been thoroughly studied. Matumoto's work (mentioned in the proof of lemma 3.1) gives some, but very partial, information. (Apart from the constraints discussed in §3.1, he needs that the Seifert matrix M is diagonalizable.)

Lemma 3.3 We have

$$\left| j^\pm_\xi(L) \right| \le \nu_\xi(L)\,.$$

Proof. We look at the forms M_ξ in (20) for a Seifert matrix M of L. Let us note that branches of eigenvalues of M_ξ being *continuous* in ξ (*cf.* proof of lemma 3.1) easily follows for example by using (37) on the characteristic polynomials of M_ξ, regarded as functions in the variable ξ. This implies the assertion of the lemma. □

It seems reasonable to believe that

$$\nu_\xi(L) \le \mathrm{mult}_\xi\,\Delta(L)\,, \tag{48}$$

which would lead to a refinement of (41),

$$\left| j^\pm_\xi(L) \right| \le \mathrm{mult}_\xi\,\Delta(L)\,. \tag{49}$$

Here is another useful observation (I am grateful to A. Ranicki for pointing this out):

Lemma 3.4 If L is regular, then (48), and therefore also (41), holds.

Proof. We have that (some) M in (20) is invertible, and then the Alexander polynomial mainly becomes the characteristic polynomial of $(M^T)^{-1}M$:

$$\chi_{(M^T)^{-1}M}(\xi) = \det(M)^{-1}\cdot\xi^{(1-\chi)/2}\cdot\Delta_L(\xi)\,.$$

Then (48) is just the formula relating eigenvalue multiplicity and eigenspace dimension. □

In particular, by lemma 3.4, if L is regular, we have (38) directly from (49). Using Trotter's work (see proof of lemma 3.1), one can obtain then (48) for all knots, and also derive (49) and (41) from it and lemma 3.3.

For irregular links this elegant short argument does not work. The previously discussed failure of all alternative proofs, though, strongly motivates an approach to (38) which applies to links and lifts all assumptions binding other methods (except, of course, $\Delta \ne 0$). Our partial effort was superseded by a proof, given very recently by P. Gilmer and C. Livingston, of (48) (and hence (38)) in the general case, which will be discussed elsewhere [GLv]. For another proof, the referee pointed to the Appendix of [Lc] (with the caveat there to use, in our notation, $j^\pm_{-1}$ instead of j_{-1}).

Corollary 3.3 Assume L is regular. Then

$$\zeta(\Delta_L, \mathbf{C} \setminus S^1) \leq 1 - \chi(L) - |\sigma(L)|\,. \tag{50}$$

Proof. Use (38) and $\zeta(\Delta_L, \mathbf{C}) = \operatorname{span}\Delta_L \leq 1 - \chi(L)$. □

Proof of corollary 3.1. Combining Shinohara's inequality (42) with (15) and our assumption, we see that both (42) (for $|\sigma(K)| = \sigma(K) > 0$) and (15) become equalities. Thus K is regular, along with (8) (for any knot), i.e.,

$$1 - \chi(K) - \operatorname{span}\Delta_K \,=\, \operatorname{mult}_{\pm 1}\Delta_K \,=\, 0\,. \tag{51}$$

Then (38) with (51), (15), (39), and our assumption, leads again to (47). □

Remark 3.1 For a positive or almost positive knot (as needed in the proof of theorem 1.1 and its later extensions), inequality (42) is not required. We have (51) from theorem 4.3, yielding again (47) (and (42) along with it), etc.

4 Concordance of positive knots

4.1 Finiteness conjecture

One can regard conjecture 1.1 (though it came up differently; *cf.* remark 4.5) as a follow-up to the fact that positive (or almost positive) knots have positive signature, and hence are not (algebraically) slice. This fact was first proved (for positive knots) by Cochran and Gompf [CGo, corollary 3.4]. Przytycki observed the result (also for almost positive knots) to be a consequence of Taniyama's work [Tn], but their account [PT] remained unpublished for a very long time. A proof of the positivity result of the signature, using similar methods to Przytycki-Taniyama, was written down in [St7].

The only previous results similar to conjecture 1.1 (though obtained with a different approach) appear to be about doubled knots, starting with Casson and Gordon [CG] and later extended in [Ki]. One can find very subtle particular examples where the detection of non-concordance is difficult [Ki2]. But efficient (non-abelian) concordance invariants seem hard to calculate in general. These invariants have so far become useful only in certain suitably constructed particular infinite families of knots. On the opposite hand, for any moderately large and natural class of knots, meaningful claims about concordance properties seem scarce. Among others, there are rather easy to identify infinite families of slice (even ribbon) knots, which are quasipositive, or l-almost positive at least for $l \geq 3$ (see example 4.2), or alternating (even rational). Thus the property in conjecture 1.1 must be in some way intrinsically linked with positivity.

Our main aim will be to settle a part of conjecture 1.1, first for special alternating knots (theorem 1.1), and later for a larger class of positive and almost positive knots satisfying a certain inequality on their signature and genus (theorem 4.5). In our approach concordance can be taken to be algebraic, except where we explicitly state otherwise (see also remark 4.1). See the update in remark 4.6.

A brief overview of this main section of the paper is as follows. The proof of theorem 1.1 is carried out in §4.2. In §4.3 we give versions of our main theorem for more general classes of positive and almost positive knots, and in §4.4 we discuss the case of links. In §4.5 we present some computations, discuss in §4.6 some related examples, and conclude in §4.7 with further comments about possible extensions and relations of our results.

4.2 Application to concordance

We first prove the statement given in the introduction. The proof will show how one can obtain some more general versions, which we will discuss later.

The following lemma is well-known, but we recapture an argument suitable for subsequent extensions.

Lemma 4.1 There are only finitely many positive knots with given Alexander polynomial.

Proof. Let D_i be infinitely many positive diagrams with given Δ. We need $g(D_i) = \max\deg\Delta$. Thus by theorem 2.5 and by lemma 2.1 we may assume (35). But it is easy to see from the skein relation of Δ under $\bar{t}_2'$ twists that

$$\max\mathrm{cf}\,\Delta(D(p_1,\dots,p_n)) > \max\mathrm{cf}\,\Delta(D(p'_1,\dots,p'_n)),$$

whenever $p'_i \geq p_i$ for all $1 \leq i \leq n$, and for at least one i strict inequality holds. This contradiction finishes the proof. □

Theorem 1.1 follows immediately from this lemma and the following statement.

Theorem 4.1 Let K, K_1 be algebraicly concordant special alternating knots. Then

$$\Delta(K_1) = \Delta(K). \tag{52}$$

Proof. Let K_1 be positive and concordant to K. Let $\Delta = \Delta(K)$ and $\Delta_1 = \Delta(K_1)$. For special alternating knots

$$\mathrm{span}\,\Delta = 2g = \sigma, \tag{53}$$

and hence we have

$$\max\deg\Delta = \max\deg\Delta_1. \tag{54}$$

From (53) for K and (41) it follows that all zeros ξ of $\Delta(K)$ lie on S^1 and satisfy equality in (41):

$$\big|\, j_\xi(K) \,\big| = 2\,\mathrm{mult}_\xi\,\Delta(K). \tag{55}$$

Now let X be an irreducible factor of $\Delta(K)$. Thus there is a zero ξ of X on S^1 with $|j_\xi(K)| = 2\,\mathrm{mult}_X\,\Delta(K)$.

Now look at $\Delta(K_1)$. Since by concordance $j_\xi(K_1) = j_\xi(K)$, we have from (41) for K_1 that $\mathrm{mult}_X\,\Delta(K_1) \geq \mathrm{mult}_X\,\Delta(K)$. Arguing over all X, we see that $\Delta \mid \Delta_1$. Then, comparing degrees in (54), and using unimodularity (5), we have $\Delta = \Delta_1$. This finishes the proof. □

Remark 4.1 We did not use more than the jump (Milnor's signature) function $j_\bullet$ as a concordance invariant, and thus it should be noted that in fact $j_\bullet$ identifies a special alternating knot up to finite ambiguity. In the same way, in most of our following statements (for knots) algebraic concordance can be weakened to equality of the jump function.

For corollary 1.1 we use lemma 4.1 and the following modification of theorem 4.1. We will establish the conclusion in several more situations, thus let us say in the following that for K *positive smooth concordance determinanes the Alexander polynomial*.

Theorem 4.2 Let K, K_1 be smoothly concordant positive knots and K be special alternating. Then (52) holds.

Proof. The fundamental reason for this extension is the availability, in the smooth setting, of the Bennequin-Rudolph machinery (17). (Regarding its failure in the topological category, see remark 4.3.) This leads to (11). Thus if K_1 smoothly concordant positive knots, (11) combines with (10) to give that $g(K_1) = g(K)$ is constant. Similarly by concordance $\sigma(K_1) = \sigma(K)$ is constant. If K is special alternating, then we obtain (53) for K_1 as well, and the proof of theorem 4.1 can be repeated. □

Summarizing the proof, we see that what we need is that the jump function $j_\bullet(K)$ detects in $\Delta(K_1)$ each irreducible factor of $\Delta(K)$ (with full multiplicity), and $\max\deg\Delta(K) = \max\deg\Delta(K_1)$. Several later extensions of theorem 1.1 base on this observation. The following proposition is mainly its formalization (with the obvious, and omitted, proof).

Proposition 4.1 Let K be a positive knot, such that for each irreducible factor $X \mid \Delta(K)$ there is at least one zero ξ of X which lies on S^1 and satisfies (55). If K_1 is positive and algebraicly concordant to K with $g(K_1) = g(K)$ (i.p., if K_1 is smoothly concordant to K), then (52) holds, in particular only finitely many K_1 occur. $\square$

This shows that it is necessary to look at just one zero of X, and in general releases us from keeping σ close to $2g$. Here is a small application.

Corollary 4.1 Let K be a connected sum of positive torus knots, and K_1 be positive and smoothly concordant to K. Then (52) holds, in particular only finitely many K_1 occur.

Proof. Let in the following X_n be the (irreducible) polynomial of the primitive n-th roots of unity. The formula for the Alexander polynomial of the torus knots $T_{p,q}$ is

$$\Delta(T_{p,q}) = t^{-(p-1)(q-1)/2}\,\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}\,. \tag{56}$$

It shows that

$$\mathrm{mult}_X(\Delta(T_{p,q})) = \begin{cases} 1 & \text{if } X = X_n \text{ for } n \mid pq \text{ but } n \nmid p,q, \\ 0 & \text{otherwise}. \end{cases} \tag{57}$$

It can be easily checked from the formula of the jump, due to Matumoto-Kearton (see [Ke2, §13]), that for each such p,q, the jump at the first root, $j_{\exp(2\pi i/n)}(T_{p,q}) = 2$. $\square$

For practical verification, we notice (taking into account (44)) two simple criteria that make the conditions in proposition 4.1 satisfied. Again, we should demand smooth concordance (or equality of the genera; see also remark 4.3 below).

Proposition 4.2 Let K be a positive knot. Assume that each irreducible factor X of $\Delta(K)$ has at least one zero on S^1 and
(a) X is simple, or
(b) (40) is an equality for K.
Then only finitely many positive knots K_1 are smoothly concordant to K. They satisfy (52). $\square$

Let us here make a helpful remark: the practical use of this proposition, as well as proposition 3.1, can be simplified by working with $\widetilde{\nabla}$ (in (43)). Surely, if $\widetilde{\nabla}$ factors, so does Δ. But Δ may factor even if $\widetilde{\nabla}$ is irreducible: e.g., stevedore's knot 6_1 has $\widetilde{\nabla} = 1-2z$ (obviously irreducible), while $\Delta(6_1) = -t^{-1}(2t-1)(t-2)$. Still, we observe why we will not mind below factorizing $\widetilde{\nabla}$ instead of Δ.

Lemma 4.2 If an irreducible factor $\tilde{X}(z)$ of $\widetilde{\nabla}(K)$ has at least one zero on $[-4,0]$, then the corresponding factor

$$X(t) = \tilde{X}(t-2+t^{-1})$$

of the Alexander polynomial

$$\Delta(K) = \widetilde{\nabla}(K)(t-2+t^{-1})$$

is irreducible (and has at least one zero on S^1).

Proof. Assume two mutually inverse roots ξ and ξ^{-1} of X have the same minimal polynomial Y. Then $Y(t) = ct^k \cdot Y(t^{-1})$ for some $c \in \mathbf{R}$, $c \neq 0$. By unimodularity (5), we have $Y(1) = \pm 1 \neq 0$, and thus $c = 1$, i.e., Y must be reciprocal. Then $Y = \tilde{Y}(t-2+t^{-1})$ would yield a factor $\tilde{Y}$ of $\tilde{X}$. Thus if $\tilde{X}$ is irreducible, but X is not, then X must factor as $cY(t)Y(t^{-1})$ for some (integer, not purportedly irreducible) polynomial Y. However, if $\tilde{X}(z)$ has a zero z on $[-4,0]$, then X has a zero ξ on S^1. But the minimal polynomial of ξ is real, thus such ξ has the same minimal polynomial as $\overline{\xi} = \xi^{-1}$, which we excluded. $\square$

4.3 Modifications and Extensions

4.3.1 Signature-root estimates for positive and almost positive links

Recall that for any (non-split) alternating link L, we have (14). In combination with theorem 2.2, we obtain thus in definition 3.2 that L is regular. More relevantly, we will need these properties here for positive [Cr] (as already mentioned) and almost positive links [St3].

Theorem 4.3 ([Cr, St3]) Positive links and almost positive links are regular.

Corollary 4.2 If K is a positive or almost positive knot, then inequality (40) holds.

Proof. Use again lemma 3.4 and theorem 4.3. □

Proposition 4.3 If L is a positive or almost positive link, then

$$\zeta(\Delta_L, \mathbf{C}\setminus S^1) \le 1-\chi(L)-\sigma(L)\,.$$

Proof. With theorem 4.3, use (50). We notice that $\sigma(L)\ge 0$, as L can be unlinked by switching positive crossings only. □

4.3.2 Positive large signature knots

Theorem 4.4 Let K be a positive or almost positive knot with $\sigma \ge 2g-2$. Then $j_\bullet(K)$ determines $\Delta(K)$ up to finite ambiguity.

Proof. Let X be an irreducible polynomial in $\mathbf{Z}[t]$. Let us assume X as a genuine polynomial (unlike Δ), and normalize it so that $\min\deg X = 0$ and $\max \mathrm{cf}\, X > 0$ (we will use X as a factor of some Alexander polynomial of a knot). Let for $X \ne t\pm 1$,

$$\text{j-mult}_X\,\Delta := \frac{1}{2}\max\{\,|j_\xi(K)|\,:\,X(\xi)=0\,,\ \ |\xi|=1\,\}\,,$$

when X has a zero ξ on S^1, and $\text{j-mult}_X\,\Delta = 0$ otherwise. The following is the part of Δ detected by $j_\bullet$:

$$\widetilde{\Delta} := \prod_{X|\Delta} X^{\text{j-mult}_X\,\Delta}. \tag{58}$$

(Note that because of (8), $X=t\pm 1$ does not occur in this product.) Then clearly $\widetilde{\Delta}$ is determined by $j_\bullet(K)$, and along the proof of theorem 4.1 we have

$$\widetilde{\Delta}\,|\,\Delta\,, \quad \operatorname{span}\Delta \ge \operatorname{span}\widetilde{\Delta} \ge \sigma\,.$$

From $\sigma \ge 2g-2$ we have then for $\Delta^* := \Delta/\widetilde{\Delta}$ that $\operatorname{span}\Delta^* \le 2$.

We remark that with $x\in\mathbf{C}$ also $\bar{x}, 1/\bar{x}$ and $1/x$ are zeros of Δ^*. Up to units in $\mathbf{Z}[t^{\pm 1}]$, thus again Δ^* is symmetric and unimodular, hence it admits a conversion to a Conway polynomial $\hat{\nabla}$ via (3), which is of the form

$$\nabla^*(z) = 1+bz^2\,. \tag{59}$$

If $\Delta^*=1$, then $j_\bullet$ determines $\Delta \doteq \widetilde{\Delta}$. Thus assume $\Delta^* \ne 1$. We claim then its both zeros are on S^1.

If $|x|\ne 1$ is a zero of Δ^*, in order the four numbers $x, \bar{x}, 1/\bar{x}$ and $1/x$ not to be all distinct (bar is conjugation in $\mathbf{C}$), we need that x is real. Since by reciprocity of Δ the product of norms of all zeros is 1, we cannot have exactly one zero off S^1. Thus we have a pair of zeros $x, 1/x$ for $x\in\mathbf{R}$.

We assumed that $x,1/x$ have no algebraic conjugates on S^1. This means that

$$\tilde{\Delta}^*(t) = a(x-t)(1/x-t) \in \mathbf{Z}[t]\,, \tag{60}$$

for some $a \in \mathbf{Z}$ must be a divisor of $\Delta(K)$. As such, $\Delta^*(t) = \tilde{\Delta}^*(t)/t$ is unimodular (up to sign; *cf*. below (6)), and from the form (60) it is seen to be symmetric. It is thus an Alexander polynomial of a knot (as an example, it occurs for a non-positive twist knot). Then Δ^* allows for a conversion via (3) to a polynomial ∇^*, which becomes of the form (59). It is easy to see that for real zeros of Δ^*, we need $b<0$.

When $b<0$ in (59), the polynomial ∇^* in (59) has a zero on the positive real line. Contrarily, the Conway polynomial of a positive knot or almost positive knot is positive (*cf.* theorem 2.3). Hence, $\nabla(K)$ cannot have such zero. Therefore, $\Delta^* \nmid \Delta(K)$.

It remains the possibility that ∇^* is one of the polynomials (59) with $b>0$. Because of (44), its Alexander polynomial Δ^* must have zeros already present in $\widetilde{\Delta}$, thus we must have $\Delta^* \mid \widetilde{\Delta}$. Since $\widetilde{\Delta}$ has only finitely many factors, this shows that only finitely many $\Delta = \Delta^* \cdot \widetilde{\Delta}$ occur. □

Theorem 4.5 Let K_i be (topologically or algebraically) concordant positive or almost positive knots with $\sigma(K_i) \geq 2g(K_i) - 2$. Then $\Delta(K_i)$ are constant, in particular only finitely many K_i occur.

Proof. To see that in fact Δ is unique, note that all possible Δ^* have simple zeros. However, Alexander polynomials of concordant knots $K_{1,2}$ have the Fox-Milnor property [FM]

$$f_1(t)f_1(1/t)\Delta(K_1) \,=\, f_2(t)f_2(1/t)\Delta(K_2) \tag{61}$$

(with $f_{1,2} \in \mathbf{Z}[t]$). It implies that the multiplicities of the same zero of $\Delta(K_1)$ and $\Delta(K_2)$ on S^1 have the same parity.

For the finiteness property in the almost positive case, we need an extension of lemma 4.1. For this see the argument in the proof of lemma 5.3 of [St7]. It shows that $[\nabla]_2$ increases under positive $\bar{t}_2'$ twists in an almost positive diagram unless the twists are trivial isotopies. Alternatively, use lemma 4.4 below. □

Definition 4.1 We call a knot *almost special alternating* if it has a diagram that differs by one crossing change from a special alternating one.

Corollary 4.3 Among almost special alternating knots only finitely many are (topologically or algebraically) concordant. □

In the spirit of corollary 1.1 (but with some more argument needed), we have the following.

Theorem 4.6 No infinite smooth concordance class of positive or almost positive knots contains a positive or almost positive knot K with $\sigma(K) \geq 2g(K) - 2$.

Proof. Let $\{K_i\}$ be smooth concordance class containing such a knot $K = K_1$.

Let again $\widetilde{\Delta}(K)$ and $\widetilde{\Delta}(K_i)$ be as in (58). Since $\widetilde{\Delta}$ is determined by $j_\bullet$, we have $\widetilde{\Delta}(K_i) = \widetilde{\Delta}(K) =: \widetilde{\Delta}$, and it divides $\Delta(K)$ and $\Delta(K_i)$. Let $\Delta^*(K) = \Delta(K)/\widetilde{\Delta}$ and $\Delta^*(K_i) = \Delta(K_i)/\widetilde{\Delta}$.

We know from the proof of theorem 4.4 that

$$\operatorname{span}\Delta^*(K) \leq 2\,, \tag{62}$$

all zeros of $\Delta^*(K)$ are on S^1 and $\Delta^*(K) \mid \widetilde{\Delta}(K) = \widetilde{\Delta}$. (For *knots* K, we cannot have $\operatorname{span}\Delta^*(K) = 1$ because of (8).) By the same argument, *if*

$$\operatorname{span}\Delta^*(K_i) \leq 2\,, \tag{63}$$

then all zeros of $\Delta^*(K_i)$ are on S^1 and $\Delta^*(K_i) \mid \widetilde{\Delta}$. This means that we are done (showing that only finitely many $\Delta(K_i)$, and hence K_i, occur) if we establish (63).

Keep in mind that $\operatorname{span}\Delta(K_i) = 2g(K_i)$ (theorem 4.3).

Case 1. If $g(K_i) < g(K)$, then $\widetilde{\Delta} = \Delta(K_i)$, and we are done.

Case 2. If $g(K_i) > g(K)+1$, then by (18) and almost positivity $g_s(K_i) \geq g(K_i) - 1 > g(K) \geq g_s(K)$, a contradiction. (Only at this place is smoothness of the concordance, but crucially, needed.)

Case 3. If $g(K_i) = g(K)$, then (63) follows from (62).

Case 4. Thus consider $g(K_i) = g(K)+1$. If $\widetilde{\Delta}(K) = \Delta(K)$, we have (63). Thus we assume that $\operatorname{span}\Delta^*(K) = 2$.

Take a zero ξ of $\Delta^*(K)$ on S^1; since $\xi \neq \pm 1$ from (8), and because of (62), it is a simple zero. Thus $\operatorname{mult}_\xi \Delta(K) - \operatorname{mult}_\xi(\widetilde{\Delta}) = 1$ is odd. Because of (61) and the remark after it, $\operatorname{mult}_\xi \Delta(K_i) - \operatorname{mult}_\xi \Delta(K)$ is even. Thus $\operatorname{mult}_\xi \Delta(K_i) > \operatorname{mult}_\xi \widetilde{\Delta}$ (keep in mind $\widetilde{\Delta} \mid \Delta(K_i)$).

Now let X be the minimal polynomial of ξ. Then for $\xi \neq \pm 1$, we have $X = \Delta^*(K)$. Thus

$$(\Delta(K) = \Delta^*(K) \cdot \widetilde{\Delta}) \mid \Delta(K_i)\,, \tag{64}$$

with $\operatorname{span}(\Delta(K_i)/\Delta(K)) = 2g(K_i) - 2g(K) = 2$, and then modifying the argument referred to directly before (63) by replacing $\widetilde{\Delta}$ by $\Delta(K)$, we obtain again $(\Delta(K_i)/\Delta(K)) \mid \Delta(K)$, and finitely many $\Delta(K_i)$. $\square$

4.4 Concordance of links

Here we will discuss ways of extending to links the proof of theorem 1.1 and its subsequent modifications in section 4.3. The stronger smooth concordance versions follow the reasoning for knots, and we highlight only additionally needed arguments.

We first address our main result. The below proof manifests the cornerstones of its extension.

Theorem 4.7 Only finitely many special alternating links are concordant. (More precisely, linking numbers and the signature jump function identify the Alexander polynomial of a special alternating link.)

Proof. First, concordance is a homology invariance, and preserves linking numbers. Thus it must map between split components, and we may just consider non-split (special alternating) links L. We use then the link case of theorem 3.1 to ascertain all roots of Δ_L to be on S^1.

Now we will need the fact that $j_\bullet$ is a link concordance invariant. This is less obvious than the case of knots, but is known (*cf.*, e.g., [CK, CF] or [F, Theorem 4.2]). Thus we can again employ signatures, and most of the proof of theorem 4.1 can be followed. The proof of lemma 4.1 remains valid for (positive) links.

The principal departure occurs in noticing that unimodularity fails for links. Evidently, each polynomial is the scalar multiple of a primitive polynomial (and shares its roots), but the option that $\Delta(K_i)$ are multiples of each other does exist for links. (It occurs for example for the reverse $(2,p)$ torus links.) One can get disposed of this problem thus.

Fix the number n of components of L. Hoste's formula (*cf.* remark below (7)) expresses

$$\lambda(L) := [\nabla(L)]_{n-1} = \sum_{\Gamma} \prod_{(i,j)\in\Gamma} l_{ij}$$

as a sum over spanning trees Γ of the linking graph $\Lambda(L)$ of L. This is the graph given by each vertex for a component $L_{[i]}$ of L and an edge labelled $l_{ij} = lk(L_{[i]}, L_{[j]})$ between vertices $L_{[i]}$ and $L_{[j]}$ if $l_{ij} \neq 0$. For a positive link, all $lk(L_{[i]},L_{[j]})\geq 0$, and if L is non-split, $\Lambda(L)$ is connected. Each summand in Hoste's formula is positive, and at least one is there. This shows

$$\lambda(L)>0 \tag{65}$$

(*cf.* below (14)). But since concordance preserves linking numbers, we need to have $\lambda(L)=\lambda(L')$ when L and and L' are concordant.

This removes the scaling ambiguity and determines Δ uniquely. □

For the extension to positive and almost positive links, we will use the work in §3.3, in particular corollary 3.3.

Theorem 4.8 If L_i are positive links with $\sigma(L_i)\geq -1-\chi(L_i)$, then only finitely many L_i are concordant.

Proof. We adapt the proof of theorem 4.4. First we need to provide for a possible zero $\xi=\pm 1$ of Δ by setting

$$\text{j-mult}_{t\pm 1}\,\Delta(L):=|j^{+}_{\mp 1}(L)|$$

for the definition (58) of $\widetilde{\Delta}(L)$. Then, using (50) and (41), we have again $\Delta(L_i)=\widetilde{\Delta}\cdot\Delta^*(L_i)$, with $\widetilde{\Delta}$ determined by $j_\bullet$ (and so equal for all L_i), and $\operatorname{span}\Delta^*(L_i)\leq 2$. Further note that, because of (65), $\Delta^*(L_i)$ are fixed under scaling with integers. We rewrite and normalize $\Delta^*(L_i)$ as

$$\hat{\Delta}_i(t):=\Delta^*(L_i)(t)\cdot t^{\max\deg\Delta(L_i)}=a\cdot\bigl(t-\bigl(x+1/x\bigr)+1/t\bigr)\in\mathbf{Z}[t^{\pm 1}]\,,$$

where $x\in\mathbf{R}$, $x\neq 0,\pm 1$ and $a\in\mathbf{Z}$, $a\neq 0$. (For links L_i, the additional possibilities $\Delta^*=t\pm 1$, $(t\pm 1)\cdot(t\pm 1)$ come in, but they do not affect any finiteness considerations.) Under (3), $\hat{\Delta}$ gives the polynomial

$$\hat{\nabla}(z)=az^2-a\,(x+1/x-2)\in\mathbf{Z}[z^2]\,. \tag{66}$$

A polynomial $\hat{\nabla}_i(z)$ of this type for $(a,x)=(a_i,x_i)$ is to divide each $\nabla(L_i)$ (with integer polynomial quotient):

$$\hat{\nabla}_i(z)\mid\nabla(L_i)\,. \tag{67}$$

This situation is managed by showing that only finitely many possible $\hat{\nabla}_i(z)$ occur, as follows.

If $x_i>0$, then the absolute term of $\hat{\nabla}_i/a_i$ in (66) is negative. Thus $\hat{\nabla}_i(\sqrt{z})$ has a real positive root, and hence cannot divide the positive polynomials $\nabla(L_i)(\sqrt{z})$ (as argued in the proof of theorem 4.4).

Let therefore $x_i<0$. Comparing the lowest coefficients in (67) gives

$$a_i\left(x_i+1/x_i-2\right)\mid\lambda$$

for $\lambda=\lambda(L_i)$ with (65), and it is crucial here that λ does not depend on i. *A priori*, for $x_i\in\mathbf{R}$ the parenthesis may represent rational numbers with arbitrarily large denominators. But we avoid this problem here by noticing that for $x_i<0$, we have

$$\left|x_i+1/x_i-2\right|>4\,.$$

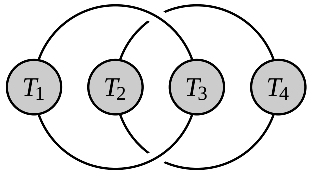
This bounds $|a_i|<\lambda/4$, and since $a_i\in\mathbf{Z}$, we have only finitely many possible values for a_i. Then the same is true for x_i+1/x_i-2, and hence for $\hat{\nabla}_i$ and $\hat{\Delta}$.

This finiteness argument puts though the proof of theorem 4.4. □

To deal with almost positive links, we first record the following observations.

Lemma 4.3 If L is almost positive (and non-split), then (65) holds.

Proof. Let D be an almost positive connected diagram representing L. The only way in which $\lambda(L)=0$, is if under the crossing switch from a positive diagram, one edge in $\Lambda(L)$ disappears and disconnects the graph. Then one easily sees that D must look like



(68)

(with all crossings in the tangles T_i positive), in which case L is a split positive link. □

Lemma 4.4 Let D be an almost positive connected diagram of n components and c a positive crossing of D. Set

$$\lambda'(D) \,=\, [\nabla(D)]_{n+1}\,.$$

Let D'_p be obtained from D by applying p times a $\overline{t'_2}$ twist at c. If

$$(\lambda(D),\lambda'(D)) = (\lambda(D'_p),\lambda'(D'_p))\,, \tag{69}$$

then c is nugatory after isotopy (i.e., the twists do not change the link type), or all links of D'_p are split.

Proof. We apply lemma 4.3 on the diagram D' obtained after smoothing out c. From (69) we have that $\lambda(D')=0$ (we need λ' in (69) for the case that c is a crossing of the same component of D). Then, reinstalling c, we see that either c is nugatory (if it connects two different T_i in (68)), or all links are split (if c occurs within some T_i). □

Theorem 4.9 Let L_i are almost positive links with $\sigma\ge -1-\chi$. Then only finitely many L_i are concordant.

Proof. The preceding lemmas provide the ingredients needed to modify the proof of theorem 4.5 (without claiming uniqueness of Δ). Lemma 4.4 shows that only finitely many almost positive links occur with given Alexander polynomial $\Delta\ne 0$. (Note again that we many consider only non-split links.) Lemma 4.3 eliminates the possiblily of $\Delta(L_i)$ being infinitely many multiples of the same polynomial (as needed to argue for theorem 4.7). □

Corollary 4.4 If L_i are almost special alternating links, then only finitely many L_i are concordant. □

Turning to smooth concordance, we need an extra caveat for the analogue of theorem 4.6.

Theorem 4.10
1. No infinite smooth concordance class of positive links contains a positive link L with $\sigma(L)\ge -1-\chi(L)$.
2. No infinite smooth concordance class of positive or almost positive links contains a positive or almost positive link L with $\sigma(L)\ge -\chi(L)$, or $\sigma(L)\ge -1-\chi(L)$ and $\det(L)\ne 0$.

Proof. Mostly we repeat the proof of theorem 4.6 (and use its notation). For links, we need to extend case 4 of that proof. This case occurs only for almost positive links (because of (11) for positive ones), whence the first statement in the theorem.

For the second statement, the goal is to show again (as in (64))

$$\Delta(L)\mid\Delta(L_i)\,. \tag{70}$$

Then again for $\hat\Delta(L_i)=\Delta(L_i)/\Delta(L)$ we have $\operatorname{span}\hat\Delta(L_i)\le 2$, and all zeros of $\hat\Delta(L_i)$ must be on S^1 and be ±1 or zeros of $\Delta(L)$. This leaves finitely many choices for $\hat\Delta(L_i)$ (with scaling by integers fixed), and $\Delta(L_i)$.

The argument in the proof of theorem 4.6 applies again if $\xi \neq \pm 1$. Notice the extension of (61) to links, due to Kawauchi [Kw, Theorem B]. Thus consider $\xi = \pm 1$. Then $X = t \mp 1$. If $X = \Delta^*(L)$, its zero is simple, and we argue as for theorem 4.6 that (70) holds. Otherwise, for $\operatorname{span}\Delta^*(L) \leq 2$, we have $\Delta^*(L) = (t \pm 1)^2$ or $\Delta^*(L) = t^2 - 1$ (keep in mind that Δ^* is reciprocal up to units in $\mathbf{Z}[t^{\pm 1}]$).

If $\Delta^*(L) = t^2 - 1$, the zeros are simple, and we have as before (70). If $\Delta^*(L) = (t-1)^2$, then (70) changes to $\Delta(L) \mid \Delta(L_i) \cdot (t-1)^2$. Now for n component links, because of (65), we have

$$\operatorname{mult}_1 \Delta = \min \deg \nabla = n - 1\,,$$

which is the same for L and L_i. Thus $\Delta(L) \mid \Delta(L_i) \cdot (t-1)^2$ implies (70). The situation $\Delta^*(L) = (t+1)^2$ cannot be dealt with any more, and the assumptions are made to exclude it. $\square$

Links can be covered also in some version of proposition 4.2, but we confined ourselves to knots only, intending the criterion as a technical tool for practical computations (see proposition 4.4).

4.5 Computational results

Our results allow computational verification on a large number of instances, thus drawing evidence for conjecture 1.1. For example, we have the following.

Corollary 4.5 If K_i are positive knots of genus at most 4, then only finitely many K_i are (topologically or algebraically) concordant. Moreover, only finitely many positive knots are smoothly concordant to any K_i.

Proof. The first part is an application of theorem 4.5. The compilation of generators (see theorem 2.4) allows one to verify that for genus $g \leq 4$ all positive knots have $\sigma \geq 2g - 2$, with one exception, the knot 14_{45657} (where $\sigma = g = 4$). For the smooth concordance, use theorem 4.6. $\square$

For low crossing knots, we focussed only on smooth concordance, as statements for topological concordance are trivial over a finite domain, and far more difficult over the same domain as the smooth version (see remark 4.3).

Proposition 4.4 If K is a positive knot of at most 16 crossings, then positive smooth concordance determines the Alexander polynomial (as in (52)). $\square$

We skip the proof, which is a lengthy computational check in the tables of KnotScape [HT] (for a proof of similar flavour, see theorem 4.11). Essentially, we ascertained using theorems 2.4 and 2.1 that we can invoke proposition 4.2 (resorting ourselves to $\widetilde{\nabla}$ with lemma 4.2), or if not, the slightly more general (but slightly more awkward to test) proposition 4.1.

Looking at the limits of our method, we see two major cases of failure of the proof of theorem 1.1:

a) When $\Delta(K)$ has an irreducible factor X with all zeros off S^1. A simple such factor could be $\Delta(6_3)$. (Alexander polynomials of non-positive twist knots can not occur; see the end of the proof of theorem 4.4.) (71)

b) When X is a (necessarily multiple) factor with $\sigma_\xi = 0$ for all roots ξ of X on S^1.

The practical effectiveness of proposition 4.1, observed in the proof of proposition 4.4, motivated us to exhibit a concrete instance on which the criterion fails to apply. These efforts, together with diverse other arguments, point to a noteworthy circumstance. Hereby, the proof of corollary 4.1 drew our attention to a particular choice of root. Let us below say that the *first root* ξ_0 of X on S^1 is the one of smallest $\arg(\xi_0)$, i.e., having $\Im m\, \xi_0 > 0$ and closest to 1 (so that X has no roots on $\Xi(\xi_0)$; see (46)).

Question 4.1 Let K be a positive knot. Is it true then that

i) all irreducible factors X of Δ_K have a root on S^1, and
ii) for the first root ξ_0 of X, we have $j_{\xi_0}(K) = 2\,\mathrm{mult}_X\,\Delta_K$?

This lends considerable further potential to proposition 4.1.

Note some pieces of observable evidence.

1) This is certainly true when $\Delta = X$ is irreducible. Then $\sigma > 0$ settles the first property i) (and together with lemma 4.2 shows that it is enough to test irreducibility of $\widetilde{\nabla}$). The second property ii) holds since $\sigma_\xi(K) \geq 0$ for all ξ (K has a positive unknotting sequence), and in (44) the absolute bars can be dropped.
2) Similarly one sees that part ii) follows from part i) in the case when (40) is an equality, and in that case also part (b) of proposition 4.2 applies.
3) It is enough to look at prime knots and prime (positive) diagrams by theorem 2.1.

We then also collected some amount of experimental support.

Theorem 4.11 The answer to question 4.1 is affirmative for

- positive knots with up to 16 crossings,
- knots with positive diagrams of up to 18 crossings, and
- positive braid representations on at most 5 strands with up to 25 crossings, and 6 strands up to 23 crossings.

In particular, for such knots any smoothly concordant positive knot has the same Alexander polynomial (as in (52)).

Proof. The first family was discussed in proposition 4.4 (which mainly triggered question 4.1). For the other two families, we started by extensively testing part i) of question 4.1 – again, using lemma 4.2 and working with $\widetilde{\nabla}$. (For braids, we obtained Δ from the Burau matrix.) By the above remark 3, we consider only prime diagrams. This test deals with (also part ii) for) knots when $\Delta = X$ is irreducible (by remark 1), or if (40) is an equality (by remark 2).

For the remaining knots, proposition 3.1 enabled us to test using values of $\widetilde{\nabla}(K')$ part ii) at least in the situation (which turned out to be rather generic) that ξ_0 (or X) is simple, and so are all 'preceding' roots of Δ on $\Xi(\xi_0)$.

The instances of failure of this test were collected, and duplications (of the underlying knots) partly removed using a functionality of KnotScape, leaving a list of about 280 entries. We found, for example, that (at least one of the two parts of) proposition 4.2 applies to all positive 17 crossing diagrams, and 18 crossing diagrams except for 5 *knots*.

To save some notation, let in the following for the trefoil 3_1,

$$\Pi_k(t) \,=\, \Delta(3_1)(t^k) = t^{-k} - 1 + t^k\,, \quad \text{and} \quad \Pi = \Pi_1\,. \tag{72}$$

All these remaining cases have the double factor $X = \Pi$, except for two, where $X = \Pi_2$. (No knots have more than one double factor, or one of multiplicity 3 or higher.) We checked (now much more slowly) using Orevkov's MATHEMATICA™ program (mentioned after (32), and partly assisted by Kearton's result (73) below for cables) that the jump at the first multiple root is 4, which finished the work.

Note that answering positively question 4.1 implies $\widetilde{\Delta} = \Delta$, thus along the way we can computationally extend the scope of proposition 4.4. In particular, after checking part i), when $\widetilde{\nabla}(K)$ (and hence $\Delta(K)$) has no repeated roots, we can use part (a) of proposition 4.2, and deduce the claim of proposition 4.4 without needing to apply the $\widetilde{\nabla}$ value test. $\square$

4.6 Some examples

However, we also became aware of the following examples.

Proposition 4.5 Case b) of (71) can occur. The answer to part ii) of question 4.1 is negative in general.

We do not know about a) of (71) (and part i) of question 4.1). See, though, also example 4.1 below.

Proof. If $K = T * P$ is a satellite knot with companion T and (zero-framed) pattern P of degree $n > 0$, then

$$j_\xi(K) \,=\, j_{\xi^n}(T) + j_\xi(P)\,. \tag{73}$$

This is a direct consequence of Kearton's satellite signature formula [Ke, Theorem], and suggests satellite constructions as a method in seeking (negative) examples.

Consider now the satellite knot $K = T * P$ around the (p,q)-torus knot $T = T_{p,q}$, with the (r,s)-torus knot pattern $P = T_{r,s}$, positioned as a closed r-braid in the solid torus. Applying (56) and Seifert's formula for the Alexander polynomial of a satellite knot, we have up to units in $\mathcal{L}[t]$,

$$\Delta(T_{p,q} * T_{r,s}) \doteq \Delta_{p,q,r} \cdot \Delta(T_{r,s})$$

(cf. (1) and below), with

$$\Delta_{p,q,r} \,=\, \Delta(T_{p,q})(t^r) \doteq \frac{(t^{pqr}-1)(t^r-1)}{(t^{pr}-1)(t^{qr}-1)}\,.$$

This polynomial decomposes, using (57), with

$$\mathrm{mult}_{X_n}(\Delta_{p,q,r}) \,=\, \begin{cases} 1 & \text{if } n \mid pqr \text{ but } n \nmid pr, qr, \\ 0 & \text{otherwise}\,. \end{cases}$$

Take $p = 2$, $q = 3$ (thus $T = T_{2,3} = 3_1$ is the trefoil) and $r = 5$. Then $X = X_6 = \Pi$ (from (72)) divides $\Delta_{2,3,5}$. If $X_6 \nmid \Delta(T_{r,s})$, then (73) gives

$$j_{e^{\pi i/3}}(T_{p,q} * T_{r,s}) \,=\, j_{e^{5\pi i/3}}(T_{p,q}) \,=\, -2\,,$$

and hence

$$j_{e^{5\pi i/3}}(T_{p,q} * T_{r,s}) \,=\, 2$$

(keep in mind (32)).

Take then some s not divisible by 6 (and 5), and let $s \geq 3 \cdot r = 15$ to ascertain that $T_{2,3} * T_{r,s}$ has a positive diagram (using the blackboard framing of the positive 3-crossing trefoil diagram). For example $s = 16$ will do. This gives a knot $K = T_{2,3} * T_{5,16}$ with a positive braid representation on 10 strands and 79 crossings, in particular $g(K) = 35$. It provides the negative answer to part ii) of question 4.1 in general.

To show that case b) of (71) can occur, consider $K\#3_1$. This 82 crossing knot is thus the simplest known potential element in a smooth counterexample family to conjecture 1.1. □

Example 4.1 For $K = 15_{253288}$, which occurred in the test for proposition 4.4, the Alexander polynomial factors (with (72)) as

$$\Delta(15_{253288}) \,=\, t^{-3} \cdot \Pi^3 \cdot (1 + 2t - 5t^3 + 2t^5 + t^6)\,.$$

The right factor X has 4 zeros off S^1, thus now two on S^1. Thus it cannot be the Alexander polynomial of a positive knot, since there is no positive knot of genus 3 with $\sigma = 2$. Thus we see that factors of polynomials of positive knots are not necessarily again polynomials of this type. This should serve as a warning regarding the limits of use of factorization.

Remark 4.2 A yet unsettled question is: if $\tilde{X} \mid \widetilde{\nabla}$ with $[\tilde{X}]_0 = +1$, is $\tilde{X}$ positive? Recall that by theorem 2.3 and the remark below it, $\widetilde{\nabla}(K)$ is a positive polynomial (more generally for positive links). Furthermore, we conjectured in [St8] (also for links) that $\widetilde{\nabla}(K)$ is *strictly log-concave*, i.e., $[\widetilde{\nabla}]_{k+1}[\widetilde{\nabla}]_{k-1} < [\widetilde{\nabla}]_k^2$ when $0 \leq k \leq \max\deg\widetilde{\nabla}$. We verified this property for all knots K in theorem 4.11. In fact, we also found that all irreducible factors $\tilde{X}$ of $\widetilde{\nabla}(K)$ (with $[\tilde{X}]_0 = +1$) are positive and strictly log-concave.

4.7 Further extensions and concluding remarks

We finish the treatment of concordance with several remarks that concern possible (and impossible) extensions of our results, and their relation to other work.

The difference between smooth and topological concordance seems extremely hard to understand. So far almost all of our knowledge centers around Freedman's deep result that all knots with trivial Alexander polynomial $\Delta = 1$ are topologically slice. Some are known to be not smoothly slice, the first example, the $(-3,5,7)$-pretzel knot, apparently due to Casson, using Donaldson's work. (Rudolph constructed later more examples; see remark 4.4.) Until recently, no other candidates for topologically but not smoothly slice knots have been confirmed.

Remark 4.3 If we consider the topological 4-genus g_t instead of the smooth one g_s, then Rudolph's slice Bennequin inequality (17) (along with its more recent invariant manifestations due to Ozsvath-Szabo and Rasmussen) fails. One can construct examples of positive (or almost positive) knots violating $g = g_t$ (resp. $g \le g_t + 1$), for instance, by using that the $(-3,5,7)$-pretzel knot, which is topologically slice, is strongly quasipositive. The inequality (26) still holds for g_t (as mentioned above it), but leads the problem to show that we can estimate g from above by σ (or some σ_ξ). For this see remark 4.6.

Remark 4.4 Many knots with trivial polynomial (i.e., $\Delta = 1$) are strongly quasipositive (see [Ru]). Therefore, for strongly quasipositive knots conjecture 1.1 fails in the topological category. This already provides some odds against a smooth category version (for strongly quasipositive knots). Our approach does lend some tangibility to the present formulation. In contrast, for a strongly quasipositive knot there seems no easy way to control g_c from g (although still $g = g_s$). Certainly, $g_c - g$ can be arbitrarily large. (One can construct such examples by taking iterated connected sums of the counterexamples to Morton's conjecture given in [St].) Thus, at the least, our approach would do very little towards a (smooth) strongly quasipositive knot version, an impasse which further discourages raising even conjecturally this more general case.

Example 4.2 For 2-almost positive knots, the family of twist knots studied by Casson and Gordon [CG] provides a counterexample to conjecture 1.1 at least in the algebraic category. (We do not know of further examples and about topological concordance.) For 3-almost positive one obtains many infinite families of (smoothly) concordant knots using the tangle surgery in [KL].

In a recent paper [Ba], K. Baker discusses a relevant related problem. He hinted (in private exchange) to a much stronger version of conjecture 1.1 (considered in a very special case by Rudolph, according to [Ba]). The reason I did not originally raise this quite striking point are the difficulties in dealing with concordance of (positive) mutants. (Certainly pairs $K\#K$, $K\#-K$ for non-invertible K are algebraically concordant.)

Question 4.2 Is there a (topological or smooth) concordance class containing more than one positive knot?

Remark 4.5 In [HU], Hirasawa and Uchida constructed infinite sets of knots, such that any two elements of a set have Gordian distance one, i.e., differ by a single crossing change. S. Baader (private communication) informed me that some of these families contain infinitely many quasipositive knots, and asked if one can find a family containing infinitely many positive knots. The work in this paper grew out of the attempt to prohibit (the existence of) such a family. Tristram-Levine signatures again give constraints, and one is left to rule out Gordian distance one between two knots with a positive diagram in the same braiding sequence (i.e., differing under iterated twisting), however, no easy tool seems to handle this situation. Note, contrarily, that such a family does exist for some Gordian distance bigger than one. (For distance 2, consider positive twist knots.)

Remark 4.6 The preprint [BDL] addresses a solution of conjecture 1.1, based on a solution of the signature bound problem (see remark 4.3). I have tried to incorporate some according changes here, but unfortunately, that preprint came out too late for another major revision. Just a brief remark is made. It appears that their proof in the smooth category does not require their signature bound, rather the latter upgrades the proof to the algebraic category. The proof can be easily extended to almost positive knots. The case of links leaves some argument to be discussed. Their appoach does not address (52), so that at least this aspect of the results on knots here remains untouched.

References

[A] N. A'Campo, *Generic immersions of curves, knots, monodromy and gordian number*, Inst. Hautes Études Sci. Publ. Math. **88** (1998), 151–169.

[Al] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), 275–306.

[BDL] S. Baader, P. Dehornoy and L. Liechti, *Signature and concordance of positive knots*, preprint `arXiv:1503.01946` (version 1).

[Ba] K. L. Baker, *A Note on the Concordance of Fibered Knots*, preprint.

[BN] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Soc. Math. de France, Astérisque **107-108** (1983), 87–161.

[BL] J. S. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225–270.

[BW] ——— " ——— and R. F. Williams, *Knotted periodic orbits in dynamical systems - I, Lorenz's equations*, Topology **22(1)** (1983), 47–82.

[BoW] M. Boileau and C. Weber, *Le problème de J. Milnor sur le nombre gordien des nœuds algébriques*, Enseign. Math. **30** (1984), 173–222.

[Bu] J. v. Buskirk, *Positive links have positive Conway polynomial*, Springer Lecture Notes in Math. **1144** (1983), 146–159.

[CG] A. Casson and C. McA. Gordon, *On slice knots in dimension 3*, "Geometric Topology" (R. J. Milgram, ed.), Proc. Symp. Pure Math. XXXII, AMS, Providence 1978, 39–53.

[CK] J. C. Cha and K. H. Ko, *Signatures of links in rational homology spheres*, Topology **41** (2002), 1161–1182.

[CF] D. Cimasoni and V. Florens, *Generalized Seifert Surfaces and Signatures of Colored Links*, Trans. Amer. Math. Soc. **360(3)** (2008), 1223–1264.

[CGo] T. D. Cochran and R. E. Gompf, *Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and Property P*, Topology **27(4)** (1988), 495–512.

[Co] J. H. Conway, *On enumeration of knots and links*, in "Computational Problems in abstract algebra" (J. Leech, ed.), 329–358, Pergamon Press, 1969.

[Cr] P. R. Cromwell, *Homogeneous links*, J. London Math. Soc. (series 2) **39** (1989), 535–552.

[Cr2] ——— " ———, *Positive braids are visually prime*, Proc. London Math. Soc. **67** (1993), 384–424.

[Cr3] ——— " ———, *Knots and Links*, Cambridge University Press, 2004.

[CM] ——— " ——— and H. R. Morton, *Positivity of knot polynomials on positive links*, J. Knot Theory Ramif. **1** (1992), 203–206.

[C] R. Crowell, *Genus of alternating link types*, Ann. of Math. **69(2)** (1959), 258–275.

[F] V. Florens, *Signatures of Colored Links With Application to Real Algebraic Curves*, J. Knot Theory Ramif. **14(7)** (2005), 883–918.

[FM] R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math. **3** (1966), 257–267.

[G] S. Garoufalidis, *Does the Jones polynomial determine the signature of a knot?*, preprint `math.GT/0310203`.

[GLv] P. Gilmer and C. Livingston, to appear.

[GL] C. McA. Gordon and R. A. Litherland, *On the signature of a link*, Invent. Math. **47 (1)** (1978), 53–69.

[He] M. Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, J. Knot Theory Ramifications **19(5)** (2010), 617–629.

[HU] M. Hirasawa and Y. Uchida, *The Gordian complex of knots*, Knots 2000 Korea, Vol. **1** (Yongpyong), J. Knot Theory Ramifications **11(3)** (2002), 363–368.

[Ht] J. Hoste, *The first coefficient of the Conway polynomial*, Proc. Amer. Math. Soc. **95(2)** (1985), 299–302.

[HT] ——— " ——— and M. Thistlethwaite, *KnotScape*, a knot polynomial calculation and table access program, available at `http://www.math.utk.edu/~morwen`.

[HTW] ——— " ———, ——— " ——— and J. Weeks, *The first 1,701,936 knots*, Math. Intell. **20 (4)** (1998), 33–48.

[Kf] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[Kw] A. Kawauchi, *On The Alexander Polynomials of Cobordant Links*, Osaka J. Math. **15** (1978), 151–159.

[Ke] C. Kearton, *The Milnor signatures of compound knots*, Proc. Amer. Math. Soc. **76(1)** (1979), 157–160.

[Ke2] ——— " ———, *Signatures of knots and the free differential calculus*, Quart. J. Math. Oxford Ser. (2) **30(118)** (1979), 157–182.

[Ki] S.-G. Kim, *Polynomial splittings of Casson-Gordon invariants*, preprint `math.GT/0102112`.

[Ki2] T. Kim, *An infinite family of non-concordant knots having the same Seifert form*, Comment. Math. Helv. **80(1)** (2005), 147–155.

[KL] R. Kirby and W. B. R. Lickorish, *Prime knots and concordance*, Math. Proc. Cambridge Philos. Soc. **86(3)** (1979), 437–441.

[Le] J. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

[L] W. B. R. Lickorish, *An Introduction to Knot Theory*, GTM 175, Springer-Verlag, New York, 1997.

[Lc] L. Liechti, *Signature, Positive Hopf Plumbing and the Coxeter Transformation*, preprint `arXiv:1401.5336`

[Lv] C. Livingston, *A survey of classical knot concordance*, in "The Handbook of Knot Theory", W. Menasco and M. Thistlethwaite (eds.), 319–348, Elsevier, 2005.

[Ma] T. Matumoto, *On the signature invariants of a non-singular complex sesquilinear form*, J. Math. Soc. Japan **29(1)** (1977), 67–71.

[MZ] W. W. Menasco and X. Zhang, *Positive knots and knots with braid index three have property P*, `math.GT/0010154`, Jour. Knot Theory Ramif. **12(4)** (2003), 427–444.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Math. Studies **61** (1968).

[Mi2] ——— " ———, *Infinite cyclic coverings*, 1968 Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967), 115–133, Prindle, Weber & Schmidt, Boston, Mass.

[Mu] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.

[Mu2] ——— " ———, *On the genus of the alternating knot*, J. Math. Soc. Japan **10** (1958), 94–105, 235–248.

[MP] ——— " ——— and J. Przytycki, *The skein polynomial of a planar star product of two links*, Math. Proc. Cambridge Philos. Soc. **106(2)** (1989), 273–276.

[N] T. Nakamura, *Positive alternating links are positively alternating*, J. Knot Theory Ramifications **9(1)** (2000), 107–112.

[Or] S. Yu. Orevkov, *Link theory and oval arrangements of real algebraic curves*, Topology **38(4)** (1999), 779–810.

[O] M. Ozawa, *Closed incompressible surfaces in complements of positive knots*, Comment. Math. Helv. **77** (2002), 235–243.

[PT] J. Przytycki and K. Taniyama, *Almost positive links have negative signature*, Abstr. AMS **12(3)** (1991), 327; `http://arxiv.org/abs/0904.4130`, J. Knot Theory Ramifications **19(2)** (2010), 187–289.

[Ro] D. Rolfsen, *Knots and links*, Publish or Parish, 1976.

[Ru] L. Rudolph, *Positive links are strongly quasipositive*, Geometry and Topology Monographs **2** (1999), Proceedings of the Kirbyfest, 555–562.

[Ru2] ——— " ———, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. **29** (1993), 51–59.

[Sh] Y. Shinohara, *On the signature of knots and links*, Trans. Amer. Math. Soc. **156** (1971), 273–285.

[Se] I. Stewart, *Galois theory*, Chapman & Hall, London, 1973.

[St] A. Stoimenow, *On the crossing number of positive knots and braids and braid index criteria of Jones and Morton-Williams-Franks*, Trans. Amer. Math. Soc. **354(10)** (2002), 3927–3954.

[St2] ——— ” ———, *Gauß sum invariants, Vassiliev invariants and braiding sequences*, J. Of Knot Theory and Its Ram. **9(2)** (2000), 221–269.

[St3] ——— ” ———, *Minimal genus and fibering of canonical surfaces*, to appear in New York J. Math.

[St4] ——— ” ———, *Knots of genus one*, Proc. Amer. Math. Soc. **129(7)** (2001), 2141–2156.

[St5] ——— ” ———, *Knots of (canonical) genus two*, `math.GT/0303012`, Fundamenta Math. **200(1)** (2008), 1–67.

[St6] ——— ” ———, *On some restrictions to the values of the Jones polynomial*, Indiana Univ. Math. J. **54 (2)** (2005), 557–574.

[St7] ——— ” ———, *Gauss sums on almost positive knots*, `math.GT/9803073`, Compositio Math. **140(1)** (2004), 228–254.

[St8] ——— ” ———, *Newton-like polynomials of links*, Enseign. Math. **51(3-4)** (2005), 211–230.

[St9] ——— ” ———, *Diagram genus, generators and applications*, preprint `arXiv:1101.3390`.

[St10] ——— ” ———, *Some applications of Tristram-Levine signatures*, Adv. Math. **194(2)** (2005), 463–484.

[St11] ——— ” ———, *Application of braiding sequences II: Polynomial and geometric invariants of positive knots*, preprint.

[STV] ——— ” ———, V. Tchernov and A. Vdovina, *The canonical genus of a classical and virtual knot*, Geometriae Dedicata **95(1)** (2002), 215–225.

[SV] ——— ” ——— and A. Vdovina, *Counting alternating knots by genus*, Math. Ann. **333** (2005), 1–27.

[Ta] T. Tanaka, *Maximal Bennequin numbers and Kauffman polynomials of positive links*, Proc. Amer. Math. Soc. **127(11)** (1999), 3427–3432.

[Tn] K. Taniyama, *A partial order of knots*, Tokyo J. Math. **12(1)** (1989), 205–229.

[Tr] R. Trapp, *Twist sequences and Vassiliev invariants*, Jour. of Knot Theory and its Ramifications **3(3)** (1994), 391–405.

[Ts] A. G. Tristram, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. **66** (1969), 251–264.

[Tt] H. F. Trotter, *Homology of group systems with applications to knot theory*, Ann. of Math. (2) **76** (1962), 464–498.

[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Yo] Y. Yokota, *Polynomial invariants of positive links*, Topology **31(4)** (1992), 805–811.

[Zu] L. Zulli, *The rank of the trip matrix of a positive knot diagram*, J. Knot Theory Ramif. **6(2)** (1997), 299–301.