GENERALIZED BANDED DIAGRAMS AND STRONG QUASIPOSITIVITY OF LINKS

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Abstract. From Seifert's algorithm, one can view a link diagram as a diagram consisting of twisted bands and disks representing a Seifert surface. We develop a theory of generalized banded diagram that generalizes this disk-and-band decomposition illustration of Seifert surface. We prove generalization of Alexander-Yamada's theorem; under some additional assumptions, generalized banded diagram can be converted to a closed braid diagram preserving its disk-and-band decomposition structure of Seifert surface. As an application, we prove strongly quasipositive property for various cases, including almost positive links.

1. Introduction

By the famous Seifert algorithm, from a diagram D of a link L one can construct a Seifert surface of L called the *canonical Seifert surface*. Although this algorithm is easy and intuitive, it is a useful tool to investigate knots and links from their diagrams. The canonical Seifert surface often attains the minimum genus, thus it gives a direct connection between diagrams and surfaces.

In a slightly different prospect, from a (closed) braid representative of a link L over the band generators one naturally obtains a Seifert surface called the *braided surface*. This plays an important role due to a close connection to contact geometry.

As we will review in Section 2, as a natural framework to treat canonical Seifert surfaces and braided surfaces in a unified manner, a notion of *banded diagram* has appeared (without the name, or, in a slightly different form) in several places (see [HIK, Section 6], or [St2, Section 3] for example). Roughly speaking, a banded diagram is a diagram that consists of mutually disjoint oriented circles and bands depicted by signed arcs connecting circles. This represents a Seifert surface rather than a link itself.

The aim of this paper is to develop a theory of *generalized banded diagrams*. Roughly speaking, a generalized banded diagram $\mathbb{D}$ is a diagram consisting of circles and signed arcs that indicate twisted bands, allowing more general configurations and intersections of circles and arcs.

As in the usual link diagrams and banded diagrams, we can assign the oriented surface $S_{\mathbb{D}}$ which we call the *canonical surface* $S_{\mathbb{D}}$ of $\mathbb{D}$. In general, such a surface is not embedded, it is immersed with ribbon singularities. We will mainly treat the case where $S_{\mathbb{D}}$ is embedded, hence gives rise to a Seifert surface, which we call *geometric*.

We will introduce generalized banded diagrams in Section 3, and develop a notion of marking, a method to relate usual link diagrams and generalized banded diagrams

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in Section 4. As we will see there, unlike the canonical Seifert surface construction, where a minimum genus Seifert surface cannot appear for many knots, our marking method can construct a minimum genus Seifert surface.

Our main theorem proven in Section 5 is the following.

Theorem 5.12. *Let $\mathbb{D}$ be a geometric generalized banded diagram. If $\mathbb{D}$ is simple and admissible, then by finitely many Y-moves, B-moves, and A-moves, we can make $\mathbb{D}$ into an ordinary banded diagram $\mathbb{D}'$ such that all the Seifert circles of $\mathbb{D}'$ are coherent (i.e., $\mathbb{D}'$ is a closed braid diagram).*

The main theorem says that under additional assumptions which we call simple and admissible, one can convert a generalized banded diagram $\mathbb{D}$ into a closed braid diagram, by particular operations which we call Y-moves, B-moves, and A-moves.

Theorem 5.12 can be understood as a strengthened version of Alexander's theorem and Yamada's refinement [Ya]. Alexander's theorem says that every link diagram can be converted to a closed braid diagram. Yamada's refinement shows that such a conversion can be done preserving the number of Seifert circles. The moves in Theorem 5.12 preserve the disk-and-band decomposition (i.e., handle decomposition) structure of the canonical surface $S_{\mathbb{D}}$. Thus Theorem 5.12 says that a conversion to a closed braid diagram can be done preserving the Seifert surfaces determined by the diagrams, together with their disk-and-band decomposition structures.

This allows us to study the braid index, the strong quasipositivity, and the maximum euler characteristic from generalized banded diagrams. Thus the generalized banded diagram method can be understood as an enhancement of the classical and naive diagrammatic approach.

As applications of generalized banded diagram techniques, we will prove strong quasipositivity for various classes of diagrams.

Theorem 6.4. *A successively almost positive braid link is strongly quasipositive.*

Theorem 6.11.[FLL, Theorem A] *An almost positive link is strongly quasipositive.*

As we will see and discuss, our proof based on generalized banded diagram is constructive and gives additional insight on the braid index of strongly quasipositive braid representatives. Furthermore, our argument can be used to show the strong quasipositivity for many other cases.

The generalized banded diagram technique give a supporting evidence for the assertion that weakly successively almost positive links [IS], a generalization of positive links, are strongly quasipositive.

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2. Banded diagrams

Throughout the paper, we always treat knots or links in $\mathbb{R}^3$, and diagrams in $\mathbb{R}^2$, though we sometimes use an isotopy in S^3 or S^2 to make the embeddings or diagrams simpler.

By abuse of notation, we often view an object in $\mathbb{R}^2$ as an object in $\mathbb{R}^3$ by taking a suitable lift. For example, for a circle s in $\mathbb{R}^2$, by the same symbol s we often mean a circle $s \times \{h\} \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ where the height h is suitably chosen.

2.1. **Relations of two circles.** We summarize our terminologies of relative positions of disjoint oriented circles in $\mathbb{R}^2$ (see Figure 1).

A circle s in $\mathbb{R}^2$ bounds a disk D_s in $\mathbb{R}^2$. We will often say that a subset Y of $\mathbb{R}^2$ lies in the *inside* (resp. *outside*) of s if $Y \subset D_s$ (resp. $Y \subset \mathbb{R}^2 \setminus D_s$).

Definition 2.1. Let s and s' be disjoint circles in $\mathbb{R}^2$.

- We say that s' is *contained* in s if $s' \subset D_s$, namely, the circle s' lies in the inside of s.
- We say that s and s' are *nested* if $D_s \cap D_{s'}$ are not empty.

Thus s and s' are non-nested if and only if $D_s \cap D_{s'} = \emptyset$. More generally, we say that a family of mutually disjoint circles $\mathcal{S} = \{s_1, \dots, s_m\}$ in $\mathbb{R}^2$ are *non-nested* if $D_{s_i} \cap D_{s_j} = \emptyset$ for $i \neq j$.

Definition 2.2 (Coherent circles)**.** Let s and s' be disjoint, oriented circles in $\mathbb{R}^2$. By viewing s and s' as oriented circles in $S^2 = \mathbb{R}^2 \cup \{\infty\}$, $s \cup s'$ cuts an annulus A from S^2. We say that s and s' are *coherent* (resp. *incoherent*) s and s' are homologous (resp. non-homologous) in A.

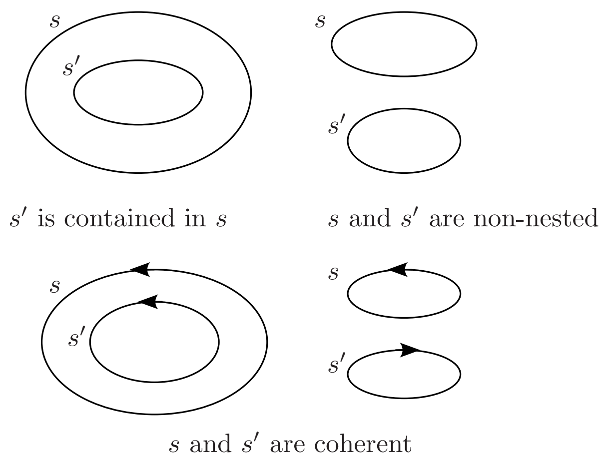


Figure 1. Relative positions of two (oriented) circles in $\mathbb{R}^2$

2.2. **Banded diagram.** By Seifert's algorithm the link diagram D in $\mathbb{R}^2$ is decomposed as a disjoint union of circles (called *Seifert circles*) and positively or negatively twisted bands (crossings of D). We denote by $s(D)$ the number of Seifert circles of D.

By expressing a band by a signed arc, we may view the link diagram D as a union of circles and (signed) arcs. This arc-and-circle illustration is useful because it describes not only a link itself, but also its Seifert surface together with its specified handle decomposition.

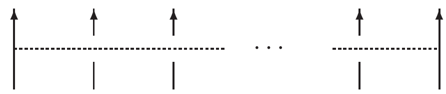
The *banded diagram* is a framework to treat arc-and-circle illustrations of link diagrams.

Definition 2.3 (Banded diagram)**.** A *banded diagram* $\mathbb{D}$ is a pair $(\mathcal{S}, \mathcal{A})$ such that

- $\mathcal{S} = \mathcal{S}(\mathbb{D})$ is a disjoint union of oriented circles in $\mathbb{R}^2$.
- $\mathcal{A} = \mathcal{A}(\mathbb{D})$ is a disjoint union of signed arcs in $\mathbb{R}^2$ connecting two circles $s, s' \in \mathcal{S}$.

We call a circle in $\mathcal{S}$ a *Seifert circle*, and an arc in $\mathcal{A}$ a *band*. The bands are allowed to transversely intersect Seifert circles in their interiors, but they satisfy the following coherency assumption.

(Coherency): For each band a, The signs of intersections of Seifert circles are the same.



The coherency assumption implies that

- For each Seifert circle s and band a, the number of intersections of a and s is at most one.
- If a band a intersects with two distinct Seifert circles s and s', then s and s' are coherent.

As in the usual diagram case, we denote by $s(\mathbb{D}) := \#\mathcal{S}$ the number of Seifert circles of $\mathbb{D}$. As we will see shortly, we will regard a band a as lying above a Seifert circle s, thus we usually write a band a as an overarc.

As a natural generalization of Seifert's algorithm, from a banded diagram we obtain a surface as follows.

Definition 2.4 (Canonical Seifert surface of banded diagram)**.** We take a disjoint union of disks $D_s \subset \mathbb{R}^3$ bounded by Seifert circles $s \in \mathcal{S}$ having constant z-coordinate (height) h_s. The heights $\{h_s\}_{s \in \mathcal{S}}$ are taken so that they satisfy

$$h_{s'} > h_s \quad \text{if } s' \text{ is contained in } s. \tag{2.1}$$

Along each (signed) arc $a \in \mathcal{A}$ we attach a positively or negatively twisted band so that

$$\text{the band } a \text{ lies above of disks } D_s\,. \tag{2.2}$$

We call the surface $S_{\mathbb{D}}$ obtained in this manner the *canonical Seifert surface* of $\mathbb{D}$. We call the link $\partial S_{\mathbb{D}}$ the link represented by $\mathbb{D}$.

This construction is a common generalization of two natural constructions of Seifert surfaces.

Example 2.5 (Canonical Seifert surface of link diagram)**.** According to Seifert's algorithm, a link diagram D in $\mathbb{R}^2$ is naturally regarded as a banded diagram $\mathbb{D} = \mathbb{D}(D)$. The canonical Seifert surface $S_{\mathbb{D}}$ is the Seifert surface obtained by Seifert's algorithm.

Remark 2.6. We point out that when we use diagrams in S^2, the condition of nested Seifert circles does not make sense, thus neither does (2.1). Keeping track of nesting of Seifert circles is one main technical reason we mostly work with diagrams in $\mathbb{R}^2$. Furthermore, there are other choices for the heights of disks when one applies Seifert's algorithm. In particular, there can be several non-isotopic Seifert surfaces obtained by Seifert's algorithm from the same diagram.

In some papers, 'canonical Seifert surface' means a Seifert surface obtained by Seifert's algorithm from a diagram allowing such a kind of freedom of disk height choices. We stipulate that in our paper, we regard as canonical only the Seifert surface coming with the height property (2.1).

Example 2.7 (Braided surface from band generators)**.** Let β be an n-braid expressed as a product of band generators $\{a_{i,j}^{\pm 1}\}_{1\leq i<j\leq n}$. Here the band generator is a braid defined by

$$a_{i,j} = (\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j-1})^{-1}\sigma_i(\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j-1}),$$

where σ_i denotes the standard generator of the braid group B_n (see Figure 2).

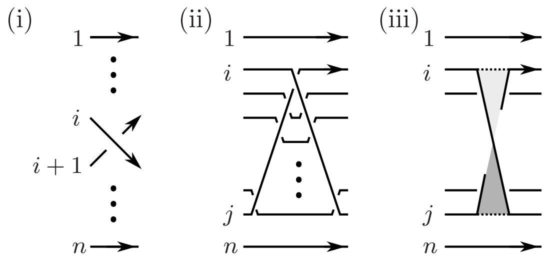


FIGURE 2. (i) The standard generator σ_i. (ii) The band generator $a_{i,j}$. (iii) $a_{i,j}$ viewed as a boundary of a twisted band

The band generator $a_{i,j}^{\pm 1}$ is viewed as a boundary of positively or negatively twisted band attached to the i-th and j-th strand. Consequently, the closed braid $\widehat{\beta}$ has a natural Seifert surface, called the *braided surface* S_β, consisting of n disks bounded by braid strands and twisted bands that correspond to each $a_{i,j}$ in β. We may view the diagram D_β as a banded diagram $\mathbb{D} = \mathbb{D}(\beta)$ so that its canonical Seifert surface $S_{\mathbb{D}}$ is the same as S_β.

Using the terminologies of banded diagrams, braided surfaces and quasipositive Seifert surfaces are defined as follows.

Definition 2.8 (Braided surface)**.** Let $\mathbb{D}$ be a banded diagram such that all the Seifert circles are coherent (hence $\partial S_{\mathbb{D}}$ is a closed braid diagram). We say that a Seifert surface S of a link L is a *braided surface* if it is isotopic to the canonical Seifert surface $S_{\mathbb{D}}$.

Definition 2.9 (Quasipositive Seifert surface)**.** A Seifert surface S is *quasipositive* if S is isotopic to a braided surface having only positive bands. A link L is *strongly quasipositive* if L bounds a quasipositive Seifert surface (i.e., L is represented by a strongly quasipositive braid, a braid which is a product of positive band generators $\{a_{i,j}\}$).

2.3. **Y-move.** In [Ya] Yamada showed that a knot diagram D can be converted to a closed braid diagram by particular types of isotopy which he called bunching operations. A remarkable feature is that his operation preserves the number of Seifert circles. Consequently, the braid index $b(L)$ of the link is given by

$$b(L) = \min\{s(D) \mid D \text{ is a diagram of } L\} \tag{2.3}$$

([Ya, Theorem 3]).

The *Y-move* is Yamada's bunching operation adapted to banded diagrams.

Definition 2.10 (Y-move)**.** Let $\mathbb{D}$ be a banded diagram, and s and s' be its Seifert circles. Assume that there exists an oriented simple arc γ from a point on s and a point on s' such that the interior of γ is disjoint from $\mathbb{D}$.

We replace a Seifert circle s with a new Seifert circle s_γ, which is the band sum of s and s' along γ, to get a new generalized banded diagram $\mathbb{D}_\gamma$ (see Figure 3).

We say that $\mathbb{D}_\gamma$ is obtained from $\mathbb{D}$ by *Y-move along* γ.

According to the positions of s and s', there are three cases of Y-moves.

(i) s and s' are non-nested.
(ii) s' is contained in s.
(iii) s is contained in s'.

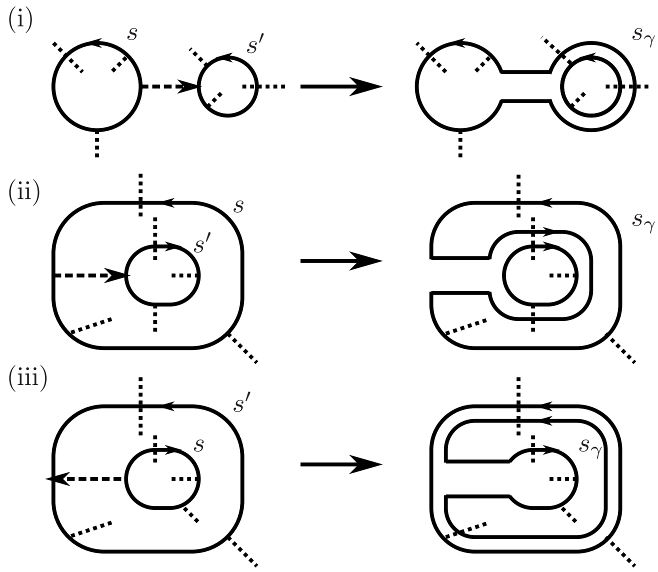


FIGURE 3. Y-move along an arc γ (dashed arrow). (i) the case s and s' are non-nested. (ii) the case s' is contained in s. (iii) the case s is contained in s'.

A diagram D is a *closed braid diagram* if all the Seifert circles are coherent. The Y-move makes $\mathbb{D}$ closer to a closed braid diagram since after the Y-move the two Seifert circles s and s' are coherent.

The advantage of the banded diagram point of view is that the canonical Seifert surface $S_{\mathbb{D}_\gamma}$ is isotopic to $S_{\mathbb{D}}$. Therefore, a Y-move preserves not only the number of Seifert circles and underlying links, but also the isotopy classes of canonical Seifert surfaces and their disk-and-band decomposition structure.

Using the banded diagram terminology, one can generalize Yamada's theorem [Ya] in the following form.

Theorem 2.11. [HIK, Theorem 6.4], [St2, Lemma 3.3] *For every banded diagram $\mathbb{D}$, by applying Y-moves finitely many times, one can make all the Seifert circles of $\mathbb{D}$ coherent. In particular, the canonical Seifert surface of a banded diagram is isotopic to a braided surface having the same number of disks and (positively and negatively twisted) bands.*

Consequently, we get a generalization of Yamada's equality (2.3).

Corollary 2.12.

$$b(L) = \min\{s(\mathbb{D}) \mid \mathbb{D} \textit{ is a banded diagram of } L\}\,,$$

where $s(\mathbb{D})$ is the number of Seifert circles of $\mathbb{D}$.

Furthermore, it recovers a famous result that positive links are strongly quasipositive.

Corollary 2.13. [Na, Rud3] *If all the signs of bands of a banded diagram $\mathbb{D}$ are positive, then $S_{\mathbb{D}}$ is quasipositive on $s(\mathbb{D})$ strands. In particular, the canonical Seifert surface of a positive link is quasipositive, so a positive link is strongly quasipositive.*

2.4. **Self-linking number.** A closed n-braid diagram D_β is regarded as a not only an oriented link, but as a *transverse link* $\mathcal{T}$ in the standard contact S^3.

The self-linking number, the most fundamental invariant of a transverse link, is given by the following formula of Bennequin [Be]:

$$sl(\mathcal{T}) = -n + w(D_\beta) = -n + c_+(D_\beta) - c_-(D_\beta), \tag{2.4}$$

where $c_\pm(D_\beta)$ denotes the number of positive and negative crossings of D_β and

$$w(D_\beta) = c_+(D_\beta) - c_-(D_\beta)$$

is the writhe of the diagram D_β.

The *maximum self-linking number* $\overline{sl}(L)$ of a link L is defined as the maximum of the self-linking number of a transverse link which is topologically isotopic to L.

Theorem 2.14 (Bennequin's inequality [Be])**.** *Let $\chi(L)$ be the maximum euler characteristic of Seifert surfaces of L. Then*

$$\overline{sl}(L) \leq -\chi(L)\,.$$

As a natural extension of Bennequin's formula (2.4) we define the self-linking number of banded diagram as follows.

Definition 2.15 (Self-linking number of banded diagram)**.** Let $a_\pm(\mathbb{D})$ be the number of positive and negative bands of a banded diagram $\mathbb{D}$. The *self-linking number* of a banded diagram is defined by

$$sl(\mathbb{D}) = -s(\mathbb{D}) + a_+(\mathbb{D}) - a_-(\mathbb{D})\,.$$

Unlike the closed braid diagrams, banded diagrams no longer represent transverse links in a canonical manner. However, since the Y-move of banded diagrams preserves the self-linking number, Theorem 2.11 justifies the definition, and leads to the following.

Corollary 2.16. *For a link L,*

$$\overline{sl}(L) = \max\{sl(\mathbb{D}) \mid \mathbb{D} \textit{ is a banded diagram of } L\}\,.$$

3. Generalized banded diagram

When we try to determine the genus, the braid index, or the maximum self-linking number, a common strategy is as follows. First we look for a candidate that potentially attains the minimum or maximum. Then we confirm that the candidate attains the minimum or maximum, by using inequalities concerning computable invariants.

From this point of view, banded diagrams are useful because they give more chance to find a candidate attaining the minimum or maximum. In this section, we introduce a generalized banded diagram, a general form of the banded diagram machinery.

3.1. **Generalized banded diagram.**

We extend banded diagrams by relaxing the condition that Seifert circles are mutually disjoint.

Definition 3.1 (Abstract generalized banded diagram)**.** An *abstract generalized banded diagram* $\mathbb{D}$ is a pair $(\mathcal{S}, \mathcal{A})$ such that

- $\mathcal{S} = \mathcal{S}(\mathbb{D})$ is a set of finitely many oriented circles in $\mathbb{R}^2$, which we call *Seifert circles*. We allow that two Seifert circles transversely intersect, forming double point singularities.
- $\mathcal{A} = \mathcal{A}(\mathbb{D})$ is a disjoint union of signed arcs in $\mathbb{R}^2$ connecting two distinct Seifert circles. We call an element of $\mathcal{A}$ a *band*. Each band a is transverse to Seifert circles forming double points and it satisfies the coherency property.
  (Coherency): For each arc a, the signs of intersections of a and Seifert circles are the same.

Note that in particular coherency implies that an arc does not intersect a Seifert circle more than once. A banded diagram is naturally regarded as an abstract generalized banded diagram. We denote by $s(\mathbb{D})$ the number of Seifert circles of $\mathbb{D}$.

As in the case of banded diagram, we define the *self-linking number* of abstract generalized banded diagram by

$$sl(\mathbb{D}) = -s(\mathbb{D}) + a_+(\mathbb{D}) - a_-(\mathbb{D}),$$

where $a_\pm(\mathbb{D})$ is the number of positive and negative bands.

In the construction of the canonical Seifert surface of a banded diagram $\mathbb{D}$, the heights of disks and bands are determined from the diagram $\mathbb{D}$ by the conditions (2.1) and (2.2).

To apply an analogous construction for abstract generalized banded diagrams, we add additional information, the height of disks and bands.

Definition 3.2 (Height assignment)**.** Let $X(\mathbb{D})$ be the set of intersection points of the interior of arcs and Seifert circles of $\mathbb{D}$. A *height assignment* $H = (h, h')$ of $\mathbb{D}$ is a pair of two functions (h, h') such that

$$h : \mathcal{S} \to \mathbb{R}$$

is an injection and

$$h' : X(\mathbb{D}) \to \{o, u\}.$$

(Here o represents 'over' and u represents 'under'.)

Definition 3.3 (Generalized banded diagram)**.** A *generalized banded diagram* is an abstract generalized banded diagram $\mathbb{D}$ with height assignment (h, h').

In the following, by abuse of notation, we use the same symbol $\mathbb{D}$ to represent both an abstract generalized banded diagram and a generalized banded diagram. We remark that both $s(\mathbb{D})$ and $sl(\mathbb{D})$ do not depend on the height assignment.

We express the height assignment by overarc-underarc notation[1]. Near the double point intersection of Seifert circles s and s', we draw s as an overarc if $h(s) > h(s')$. Similarly, near the intersection point x of the intersection of a band a and Seifert circle s, we draw a as an overarc (resp. underarc) if $h'(x) = o$ (resp. $h'(x) = u$).

One can try to construct a surface from a generalized banded diagram $\mathbb{D}$ in the following manner. For each Seifert circle s we take a disk D_s in $\mathbb{R}^3$ bounded by s having constant z-coordinate $h(s)$. For each arc $a \in \mathcal{A}$ we attach a twisted band b_a, so that at each crossing point x of a and Seifert circle s, the band b_a lies above (resp. below) of the disk D_s if $h(x) = o$ (resp. $h(x) = u$).

Definition 3.4 (Canonical surface)**.** We call the (possibly immersed) surface $S_{\mathbb{D}}$ obtained by this procedure the *canonical surface* of $\mathbb{D}$.

We call $\partial S_{\mathbb{D}}$ the *link represented by the generalized banded diagram* $\mathbb{D}$. We often view $\mathbb{D}$ as a usual link diagram in an obvious way. See Figure 4 for an (abstract) generalized banded diagram and its canonical surface.

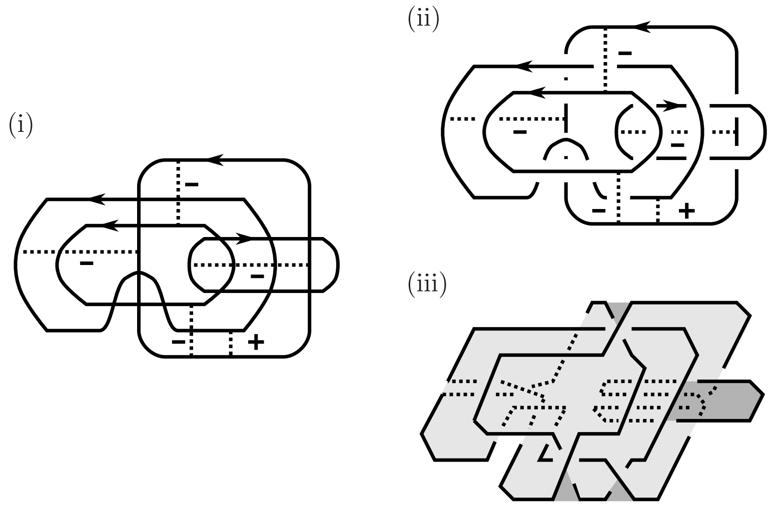


FIGURE 4. (i) Abstract generalized banded diagram . (ii) Generalized banded diagram, where the height assignment is depicted by over-arc/under-arc notation. (iii) The canonical surface of the generalized banded diagram (ii).

The canonical surface $S_{\mathbb{D}}$ is not necessarily embedded, but it is always a ribbon surface (see Figure 5).

Thus a generalized banded diagram provides the bound of slice genus.

Proposition 3.5. *If $\mathbb{D}$ is a generalized banded diagram representing a link L, $g_4(L) \le g(\mathbb{D})$.*

[1]However, we remark that the height assignment may not be completely determined from the overarc-underarc notation.

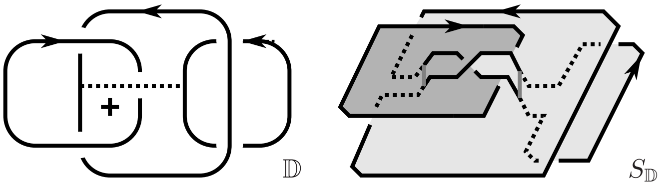


FIGURE 5. A generalized banded diagram whose canonical surface is immersed. Its height assignment forces bands and disks to intersect.

Although a non-geometric canonical surface is still useful and deserves to be studied thanks to Proposition 3.5, in the rest of the paper we will mainly treat the following class of generalized banded diagram.

Definition 3.6 (Geometric generalized banded diagram)**.** A generalized banded diagram $\mathbb{D}$ is *geometric* if $S_{\mathbb{D}}$ is embedded. In this case we call a canonical Surface $S_{\mathbb{D}}$ the *canonical Seifert surface* to emphasize it is a Seifert surface of the link represented by $\mathbb{D}$.

3.2. **Characterization of ordinary banded diagram.** Throughout the rest of the paper, we call a banded diagram $\mathbb{D}$ as defined in Definition 2.3 *ordinary banded diagram*, to distinguish from generalized banded diagrams. An ordinary banded diagram is regarded as a generalized banded diagram by taking the height assignment $H = (h, h')$ so that

(i) $h(s) > h(s')$ if s is contained in s'
(ii) $h'(x) = o$ for all $x \in X(\mathbb{D})$ (i.e., a band always lies above of disks).

Since one of the major differences of generalized and ordinary banded diagrams is that a generalized banded diagram allows for the intersections of Seifert circles, the following quantity serves as a measure of to what extent a generalized banded diagram $\mathbb{D}$ is far from ordinary banded diagrams.

Definition 3.7. The *Seifert circle crossing number* $sc(\mathbb{D})$ of an (abstract) generalized banded diagram $\mathbb{D}$ is the number of crossings of Seifert circles.

The ordinary banded diagram is characterized as follows.

Proposition 3.8 (Characterization of ordinary banded diagram)**.** *A generalized banded diagram $\mathbb{D}$ is an ordinary banded diagram if and only if it is geometric, $sc(\mathbb{D}) = 0$, and its height assignment $H = (h, h')$ satisfies*

$$h(s) > h(s') \text{ if } s \text{ is contained in } s'.$$

Proof. Since 'only if' direction is obvious, we show that $\mathbb{D}$ is ordinary banded diagram under these three conditions. To see this, it is sufficient to see that $h'(x) = o$ for every crossing x of a band a and a Seifert circle s. Assume, to the contrary, that there is a band a connecting Seifert circles s_0 and s_1 and a Seifert circle s such that a and s form a crossing x with $h'(x) = u$. With no loss of generality, we assume that s_0 is contained in s, thus by assumption $h(s_0) > h(s)$. This implies that the band a must intersect with the disk D_s (see Figure 6), which contradicts the assumption that $\mathbb{D}$ is geometric. $\square$

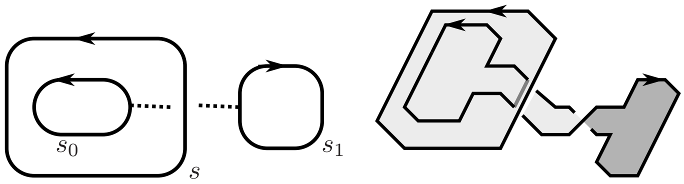


FIGURE 6. The condition $h(s_0) > h(s)$ and band a passing below of s implies that the canonical surface is not embedded.

4. Marking of diagrams

As a framework to relate link diagrams and generalized banded diagrams we introduce the following notions.

Definition 4.1 (Marking and associated generalized banded diagram)**.** A subset C of the set of crossings of link diagram D is a *marking* if smoothing the crossings of D except C gives rise to a union of circles $\mathcal{S}$ that can be regarded as a generalized banded diagram with appropriate height assignment (i.e., $\mathcal{S}$ is a union of circles with distinct heights).

We assign a generalized banded diagram $\mathbb{D}(D, C)$ by viewing the crossings other than C as a band. We call $\mathbb{D}(D, C)$ the generalized banded diagram associated to the marking C.

A marking C contains the same number of positive and negative crossings. When $C = \emptyset$, its associated generalized banded diagram $\mathbb{D}(D, \emptyset)$ is nothing but an ordinary banded diagram in Example 2.5.

For a general marking, its associated banded diagram is not always geometric, as the next example shows.

Example 4.2. Let D be a link diagram in Figure 7 left, and let C be the set of shaded crossings. The associated generalized banded diagram (Figure 7 right) is not geometric.

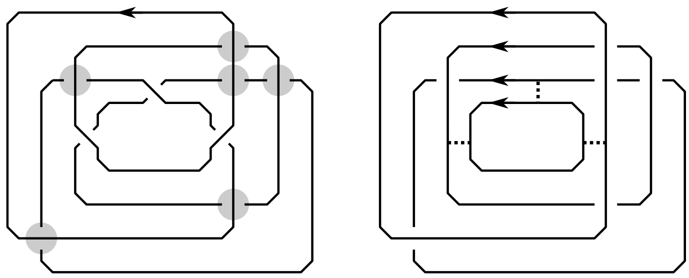


FIGURE 7. Marking of a diagram D and associated generalized banded diagram $\mathbb{D}(D, C)$ (the signs of bands are omitted).

As in the previous section, we will mainly treat a *geometric marking*, a marking whose associated generalized banded diagram $\mathbb{D}(D, C)$ is geometric.

The canonical Seifert surface $S_{\mathbb{D}(D,C)}$ is a generalization of the canonical Seifert surface construction. It gives a useful technique to explore the genus of knots from its diagrams.

The *canonical genus* of K is the minimum genus of a canonical Seifert surface $S_D := S_{\mathbb{D}(D,\emptyset)}$ of a diagram of K. Clearly we have the inequality

$$g_c(K) := \min\{g(S_D) \mid D \text{ is a diagram of } K\} \geq g(K)\,. \tag{4.1}$$

It is known that the inequality is strict in general. Indeed, the difference $g_c(K) - g(K)$ can be arbitrary large as the Whitehead double of a suitable knot shows.

On the other hand, when we utilize markings, the situation is completely different. Here we state for knots, to make clear the difference from (4.1), but the same conclusion holds for links and their maximum euler characteristics.

Proposition 4.3. *For every knot K, there exists a diagram D of K and a geometric marking C of D such that $g(K) = g(S_{\mathbb{D}(D,C)})$. Namely,*

$$\min\{g(S_{\mathbb{D}(D,C)}) \mid (D,C) \textit{ is a diagram of } K \textit{ with geometric marking } C\} = g(K)\,.$$

Proof. It is known that every knot K has a closed braid representative β (over the band generator $\{a_{i,j}\}$) such that its braided surface S_β attains the minimum genus [Rud1, Section 3] (see also [BF, Theorem 4.2]). Let D be the closed braid diagram D_β. For each band generator

$$a_{i,j}^{\pm 1} = \underline{(\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j-1})^{-1}}\ \sigma_i^{\pm 1}\ \underline{(\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j-1})}$$

of β, we regard the underlined crossings as a marking C. Then C is a geometric marking and $S_{\mathbb{D}(D,C)} = S_\beta$. □

To investigate how to use markings, we consider the following properties.

Definition 4.4. Let D be a diagram of a link L. We say that a geometric marking C of D is

- *maximum* if $\#C$ is maximum among all the geometric markings of D.
- *locally maximum* if there is no geometric marking C' such that $C \subsetneq C'$.
- *full* if C contains all the negative crossings of D.
- *tight* if $\chi(S_{\mathbb{D}(D,C)}) = \chi(L)$.
- *quasipositive* if $S_{\mathbb{D}(D,C)}$ is quasipositive.

By definition, full implies maximum and maximum implies locally maximum. A quasipositive marking is always tight. None of the converse of these three implications are true.

Example 4.5. For the closed 4-braid diagram $\sigma_3^{-1}\underline{\sigma_2^{-1}}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3\underline{\sigma_2}\sigma_1$, the marking $C = \{\sigma_2^{-1}, \sigma_2\}$ indicated by underline is locally maximum. However, C is not maximum since $C' = \{\sigma_1^{-1}, \sigma_3^{-1}, \sigma_1, \sigma_3\}$ is a marking. Furthermore, C' is maximum and tight. However, C' is neither full nor quasipositive.

To explain the reason why we would like to care about a (locally) maximum marking, we observe the following.

Lemma 4.6. *Let D be a diagram of a link L, and let C and C' be geometric markings of D such that $C \subsetneq C'$. Then*

$$\chi(S_{\mathbb{D}(D,C)}) < \chi(S_{\mathbb{D}(D,C')}).$$

Thus a tight marking is locally maximum.

Proof. Let U_C and $U_{C'}$ be the unlink diagram consisting of the Seifert circles of $\mathbb{D}(D,C)$ and $\mathbb{D}(D,C')$, respectively. By definition, U_C is obtained from $U_{C'}$ by resolving the crossings of $C' \setminus C$. Since each crossing of $U_{C'}$ connects distinct components, the first crossing change reduces the number of components. Furthermore, since each crossing change increases or decreases the number of components by one,

$$\begin{aligned} s(\mathbb{D}(D,C)) &= \#\text{components of the unlink } U_C \\ &\leq \#\text{components of the unlink } U_{C'} + \#C' - \#C - 2 \\ &= s(\mathbb{D}(D,C')) + \#C' - \#C - 2. \end{aligned}$$

Thus

$$\begin{aligned} \chi(S_{\mathbb{D}(D,C)}) &= s(\mathbb{D}(D,C)) - (c(D) - \#C) \\ &\leq s(\mathbb{D}(D,C')) - (c(D) - \#C') - 2 \\ &= \chi(S_{\mathbb{D}(D,C')}) - 2\,. \end{aligned}$$

□

On the other hand, a maximum marking may not be tight. In particular, there is a diagram without tight marking.

Example 4.7. Let D be the closed braid diagram of the 4-braid (word)

$$\sigma_3^{-1}\sigma_3^{-1}\underline{\sigma_2}\sigma_3^{-1}\sigma_2\sigma_1\sigma_1\sigma_1\underline{\sigma_2^{-1}}\sigma_1\sigma_2^{-1}$$

which represents the unknot U. (This is Morton's example of a closed irreducible, i.e. non-destabilizable, 4-braid whose closure is the unknot [Mo].) The marking C indicated by underline is a maximum (hence locally maximum) marking but $\chi(S_{\mathbb{D}(D,C)}) = 4 - 9 = -5 < 1 = \chi(U)$, thus C is not tight. Furthermore, D has no tight marking.

Here we give one sufficient condition for the tightness of markings.

Proposition 4.8. *A geometric marking C is tight if C is full and $s(\mathbb{D}(D,C)) = s(D)$.*

Proof. Since the marking C contains all the negative crossings of D, $\#C = 2c_-(D)$, where $c_-(D)$ is the number of negative crossings of D. From the assumption $s(\mathbb{D}(D,C)) = s(D)$,

$$\chi(S_{\mathbb{D}(D,C)}) = s(\mathbb{D}(D,C)) - c(D) + 2c_-(D) = s(D) - w(D)\,.$$

Thus by Bennequin's inequality we conclude

$$s(D) - w(D) = \chi(S_{\mathbb{D}(D,C)}) \leq \chi(L) \leq -\overline{sl}(L) \leq -sl(D) = s(D) - w(D)\,.$$

Thus C is tight. □

Proposition 4.8 often allows us to find a minimum genus Seifert surface from a diagram, even though the canonical Seifert surface does not attain the minimum.

Example 4.9. From a diagram D of Figure 8, the marking C indicated by shaded crossings is geometric. It contains all the negative crossings and $s(\mathbb{D}(D,C)) = s(D) = 4$, so by Proposition 4.8 it is a tight marking.

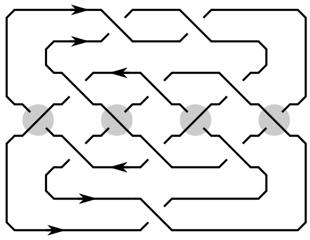


Figure 8. Example of tight marking

5. From generalized banded diagram to ordinary banded diagram

In this section we discuss an extension of Theorem 2.11 for generalized banded diagrams. We develop a technique to convert a geometric generalized banded into an ordinary banded diagram, preserving its canonical Seifert surface.

5.1. **Regions.** Let $\mathbb{D}$ be an abstract generalized banded diagram. For a Seifert circle s, let D_s be the disk bounded by s. Some of the other Seifert circles of $\mathbb{D}$ not contained in D_s intersect with D_s as arcs. They cut D_s into several connected components. We call these connected components of D_s the *regions* of s.

We view the regions as N-gons by regarding the intersections of Seifert circles as their corners.

Definition 5.1. Let R be a region of a Seifert circle s.

- R is *trivial* if it is a 0-gon, namely, $R = D_s$.
- R is *simple* if every corner of R is an intersection of s and other Seifert circles.
- R is *innermost* if R contains no non-trivial regions of other Seifert circles.

Example 5.2. Let us consider the abstract banded diagram in Figure 9 and its Seifert circle s. The shaded disk represents D_s. In the definitions concerning regions, bands are irrelevant, thus we omit bands not to make the diagram unnecessarily complicated.

The Seifert circle s has six regions (a)–(f). To emphasize the polygon structure of regions, we add vertices at each corner, the intersection point of Seifert circles. A white vertex is an intersection of s and other Seifert circles, and a black vertex is an intersection of Seifert circles other than s. The condition that R is simple says that ∂R contains no black vertices.

The regions (a),(b) are simple. The regions (d),(e),(f) are innermost.

5.2. **B-move.** Our elementary but critical observation is that a certain type of bigon region, which we call *admissible* as defined below, can be removed by isotopy. (By definition, a bigon region is always simple.)

Definition 5.3 (Admissible bigon region)**.** Let $\mathbb{D}$ be a geometric banded diagram. Let R be a bigon region for a Seifert circle s, cut from a subarc of another Seifert circle s'. We put $s_R = \partial R \cap s$ and $s'_R = \partial R \cap s'$. We say that a bigon region R is *admissible* if it satisfies the following two conditions.

(a) Both s_R and s'_R are oriented subarcs from one corner p to the other corner q.

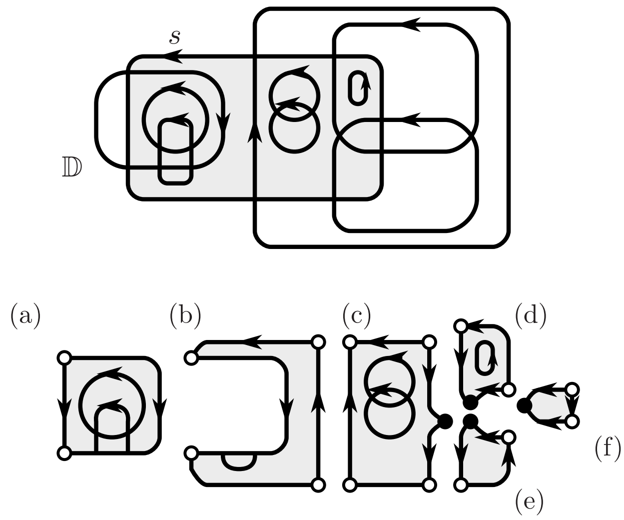


FIGURE 9. Regions of a Seifert circle s for an abstract banded diagram $\mathbb{D}$. (Since the bands are irrelevant, we omit displaying them.)

(b) $R \subset D_{s'}$.

Definition 5.4 (B-move)**.** Let $\mathbb{D}$ be a geometric banded diagram and let R be an admissible bigon region. Assume that the interior $\mathrm{Int}(R)$ of R contains no Seifert circles.

Arcs of $\mathbb{D}$ having non-empty intersections with R are classified into the eight types as depicted in Figure 10 because otherwise they contradict the assumption that $\mathbb{D}$ is geometric.

We push the disk D_s along the bigon R to get a geometric generalized banded diagram $\mathbb{D}'$. The canonical Seifert surface $S_{\mathbb{D}'}$ is isotopic to $S_{\mathbb{D}}$ (see Figure 10). We call the operation the *B-move* along the admissible bigon region R.

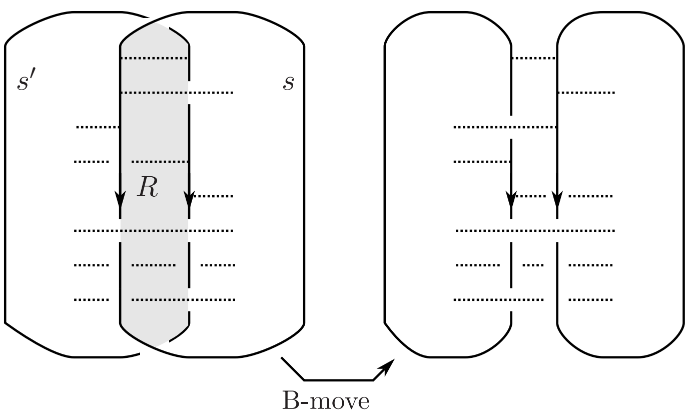


FIGURE 10. B-move (the case $h(s) > h(s')$). There are only 8 types of bands near an admissible bigon region R without other Seifert circles inside it.

In a similar vein, for later use, we introduce the following basic operation similar to the B-move, which we call A-move (annulus move).

Definition 5.5 (A-move)**.** Let $\mathbb{D}$ be a geometric generalized banded diagram. Assume that two Seifert circles s and s' satisfy the following properties:

(a) $D_{s'} \subset D_s$, s and s' are coherent, and $s \cup s'$ cuts an annulus A from $\mathbb{R}^2$.
(b) s and s' are disjoint from other Seifert circles.
(c) The interior $\mathrm{Int}(A)$ of A contains no Seifert circles.
(d) $h(s) > h(s')$. Furthermore, $h(s'') > h(s')$ holds for every Seifert circle s'' contained in s'.

Then, as in the B-move case, arcs of $\mathbb{D}$ having non-empty intersection with A are classified into the seven types as depicted in Figure 11. The property (d) means that for each intersection point x of an arc and s', we have $h'(x) = o$. Namely, arcs always lie above of s'. This implies that the one can put the canonical Seifert surface $S_{\mathbb{D}}$ so that the interior of the annulus $A \times \{h(s')\}$ is disjoint from $S_{\mathbb{D}}$. Therefore, by enlarging the Seifert circle s' along $A \times \{h(s')\}$, we get a new generalized banded diagram $\mathbb{D}'$ whose canonical Seifert surface $S_{\mathbb{D}'}$ is isotopic to $S_{\mathbb{D}}$. We call this operation A-*move*.

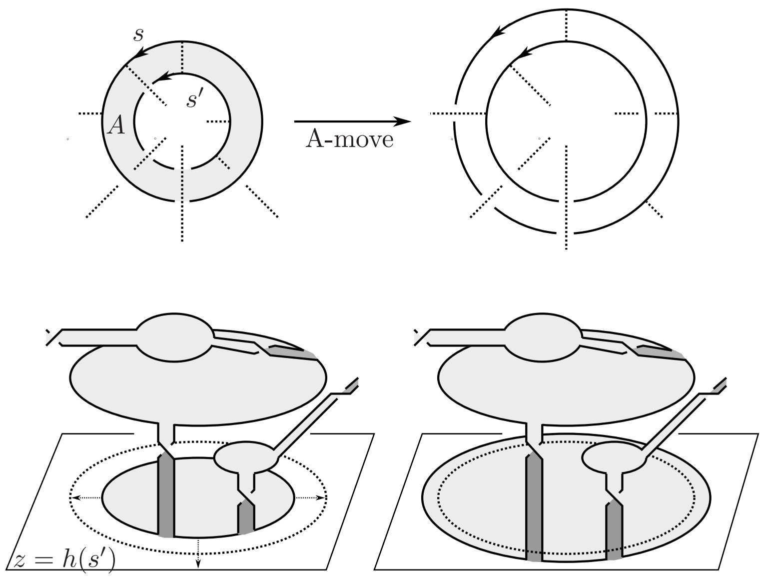


FIGURE 11. A-move

5.3. **Drilling Y-move.** In this section we give a certain extension of the Y-move for a generalized banded diagram.

Although it is interesting and important to develop Y-moves in full generality, here we do this for the simplest case which is sufficient in our purpose.

Let $\mathbb{D}$ be a generalized banded diagram with height assignment $H = (h, h')$. Let s and s' be incoherent Seifert circles of $\mathbb{D}$. Here we straightforwardly extend Definition 2.2 of (in)coherence, for two disjoint Seifert circles. This means that, whenever we say that Seifert circles s and s' of a generalized banded diagram are (in)coherent, it implies that s and s' are disjoint.

Assume that there exists an oriented simple arc γ from a point on s to a point on s', such that the interior of γ is disjoint from $\mathbb{D}$. Recall that as we have seen in Figure 3, there are there cases according to the positions of s and s'.

(i) s and s' are non-nested.
(ii) s' is contained in s.
(iii) s is contained in s'.

By reversing the orientation of γ, the case (iii) can be changed to the case (ii). Here we present a generalization of Y-move for the case (ii), which we call *drilling Y-move*.

Definition 5.6 (Drilling Y-move for generalized banded diagram)**.** In the setting above, we replace a Seifert circle s with a new Seifert circle s_γ, which is the band sum of s and s' along γ, to get an abstract generalized banded diagram $\mathbb{D}_\gamma$. Thus the set of Seifert circles $\mathcal{S}(\mathbb{D}_\gamma)$ of $\mathbb{D}_\gamma$ is $\mathcal{S}(\mathbb{D}_\gamma) = (\mathbb{D} \setminus \{s\}) \cup \{s_\gamma\}$.

The height assignment $H_\gamma = (h_\gamma, h'_\gamma)$ is defined as follows. As for the height $h_\gamma : \mathcal{S}(\mathbb{D}_\gamma) \to \mathbb{R}$ of Seifert circles, we take a natural choice.

$$h_\gamma(u) = \begin{cases} h(u) & u \neq s_\gamma \\ h(s) & u = s_\gamma \end{cases}$$

Let X_γ be the crossings of bands and the Seifert circle s_γ created by the operation. Then $X(\mathbb{D}_\gamma) = X(\mathbb{D}) \cup X_\gamma$. Assume that $x \in X_\gamma$ is a crossing of a band a and s_γ. Since the band a cannot intersect with s' more than once, one of the endpoints of x lies inside s'. Let s_x (possibly s') be the Seifert circle of $\mathbb{D}_\gamma$ that contains such an endpoint.

Then we define

$$h'_\gamma(x) = \begin{cases} h'(x) & x \notin X_\gamma \\ o & x \in X_\gamma, \text{ if } h(s_x) > h(s') \\ o & x \in X_\gamma, \text{ if } h(s_x) = h(s') \text{ and } h(s) < h(s') \\ u & x \in X_\gamma, \text{ if } h(s') > h(s_x) \\ u & x \in X_\gamma, \text{ if } h(s_x) = h(s') \text{ and } h(s) > h(s') \end{cases}$$

Here we remark that $h(s_x) = h(s')$ implies $s_x = s'$ because h is injective.

We call the generalized banded diagram $\mathbb{D}_\gamma$ *the generalized banded diagram* by applying *Y-move* along γ (see Figure 12).

Although the definition looks a bit complicated, the effect of a Y-move is easily understood. The canonical surface $S(\mathbb{D}_\gamma)$ is obtained from $S_{\mathbb{D}}$ by drilling the disk D_s. Thus, if $\mathbb{D}$ is geometric, so is $\mathbb{D}_\gamma$, and their canonical Seifert surfaces are isotopic. This is why we call the Y-move 'drilling'.

5.4. **Bigon removal lemma.** We are ready to state and prove the most useful and fundamental lemma for geometric generalized banded diagram.

Theorem 5.7 (Bigon removal lemma)**.** *Let $\mathbb{D}$ be a geometric generalized banded diagram. Let R be an innermost admissible bigon region R (see Definition 5.1 for the definition of innermost in our sense). By drilling Y-moves inside R and by a B-move, we remove the bigon R to be able to get a new generalized banded diagram $\mathbb{D}'$ such that $sc(\mathbb{D}') = sc(\mathbb{D}) - 2$.*

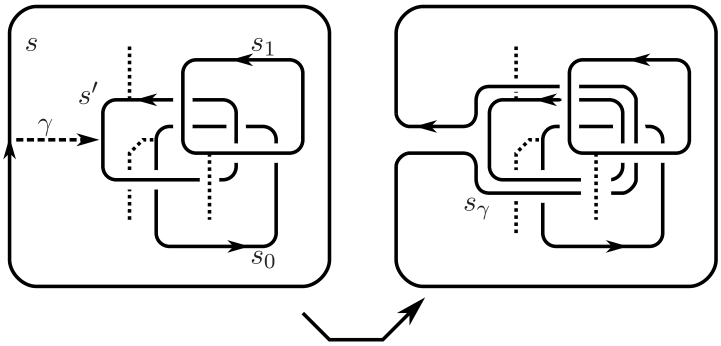


FIGURE 12. Y-move for a generalized banded diagram for the non-nested case, case (ii). Here the height assignment is $h(s_0) < h(s') < h(s_1) < h(s)$.

Proof. Let s' be the Seifert circle that cuts a region R of s. Since the condition (b) of the admissibility says that R can be also viewed as a region of s', we may freely interchange the role of s and s' if needed.

Assume that $\mathrm{Int}(R)$ contains Seifert circles $s_1, \dots, s_m$. Since we are assuming R is innermost, $s_1, \dots, s_m$ are mutually disjoint. Let p and q be the corner of the bigon R, taken so that $s_R := \partial R \cap s$ and $s'_R := \partial R \cap s'$ are oriented from p to q.

Let D_i be the disk bounded by s_i and $X = R \setminus \bigcup_{i=1}^m D_i$. We view a band $a \in \mathcal{A}$ of the banded diagram $\mathbb{D}$ as a properly embedded arc of R by taking $R \cap a$. Then these arcs cut X into several connected components. Let a be an arc that connects a point x on s_R or s'_R and a point on s_i. By interchanging the role of s and s' if needed, we assume that there is an arc a connecting s_R and s_i.

Among such arcs, we take one which is the closest to p. Namely, all arcs whose endpoint sits on a subarc of s_R between p and x connects s_R and s'_R. Then we find an arc γ in X connecting s'_R and s_i whose interior is disjoint from $\mathbb{D}$. Thus by applying a drilling Y-move along γ, we can remove the Seifert circle s_i from the inside of bigon R.

By repeating the same procedure, we eventually remove all Seifert circles inside R. Then we apply B-move to remove R.

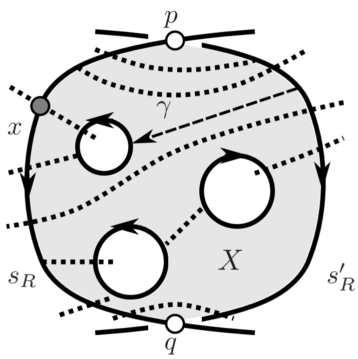


FIGURE 13. When an innermost admissible bigon R contains Seifert circles, then it admits a drilling Y-move.

□

5.5. **Simple and admissible generalized banded diagram.** We introduce the following class of generalized banded diagrams that can be effectively simplified by bigon removals.

Definition 5.8 (Simple generalized banded diagram)**.** A generalized banded diagram $\mathbb{D}$ is *simple* if for each Seifert circle s, all the regions of s are simple.

Definition 5.9 (Admissible generalized banded diagram)**.** A generalized banded diagram $\mathbb{D}$ is *admissible* if for each pair of Seifert circles (s,s'), $s \cup s'$ forms an abstract generalized banded diagram $\mathbb{D}_k$ $(k \geq 0)$ in Figure 14.

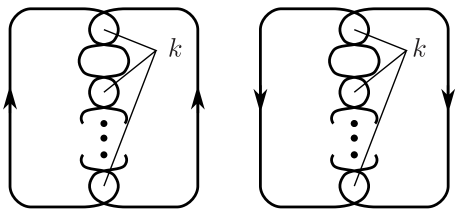


FIGURE 14. Diagram $\mathbb{D}_k$ $(k \geq 0)$. $\mathbb{D}_k$ has exactly k admissible bigons. $\mathbb{D}_0$ is just a disjoint union of non-nested coherent circles.

The diagram $\mathbb{D}_k$ can be made disjoint by removing admissible bigon regions.

Proposition 5.10. *Let $\mathbb{D}$ be a geometric generalized banded diagram. If $\mathbb{D}$ is admissible and simple, then by Y-moves and B-moves one can change $\mathbb{D}$ into a new generalized banded diagram $\mathbb{D}'$ such that $sc(\mathbb{D}') = 0$.*

Proof. Take an innermost bigon region R of $\mathbb{D}$. By assumption R is admissible. Since bigon removal in Theorem 5.7 preserves the property that $\mathbb{D}$ is admissible and simple, applying Theorem 5.7 finitely many times we eventually remove all the intersections of Seifert circles. □

Although a geometric generalized banded diagram $\mathbb{D}$ with $sc(\mathbb{D}) = 0$ obtained in Proposition 5.10 is close to an ordinary banded diagram, it has still significant differences – the height assignment, the height of disks and bands may be complicated.

However, as the next theorem shows, by Y-moves and A-moves one can covert such a generalized banded diagram into an ordinary banded diagram.

Theorem 5.11. *Let $\mathbb{D}$ be a geometric generalized banded diagram. If $sc(\mathbb{D}) = 0$ (i.e. all the Seifert circles are disjoint as circles in $\mathbb{R}^2$), then $\mathbb{D}$ can be converted to an ordinary banded diagram by Y-moves and A-moves.*

Proof. By Proposition 3.8 it is sufficient to show that one can achieve the property

$$h(s) > h(s') \text{ whenever } s \text{ is contained in } s'. \tag{5.1}$$

Let $n = s(\mathbb{D}) = \#\mathcal{S}(\mathbb{D})$ be the number of Seifert circles. With no loss of generality, we assume that $h(\mathcal{S}) = \{1, 2, \ldots, n\}$.

Let s be a Seifert circle such that the height condition (5.1) fails. Namely, there is a Seifert circle s' contained in s but $h(s) > h(s')$. Among such Seifert circles s, we take one so that its height $h(s)$ is minimum. Let $\{s_1, \ldots, s_m\}$ be the Seifert circles contained in s. We assume that $h(s_1) < h(s_2) < \cdots < h(s_m)$.

Let $X = D_s \setminus \bigcup_{i=1}^{m} \mathrm{Int} D_{s_i}$. We denote by $s_{a_1}, \ldots, s_{a_\ell}, s$ $(a_1 < a_2 < \cdots < a_\ell)$ the Seifert circles that appear as the boundary of X.

If s_{a_i} and s are incoherent for some i, then we can find an arc in X connecting s and s_{a_i} whose interior is disjoint from $\mathbb{D}$. In this case, by applying a drilling Y-move for s, we reduce the number m.

Thus we assume that the Seifert circles s_{a_i} and s are coherent for all i. Since s_{a_i} and s_{a_j} are non-nested, this means that s_{a_i} and s_{a_j} are incoherent whenever $i \neq j$.

If X is an annulus (i.e., $\ell = 1$), the assumptions (a)–(c) of A-move in Definition 5.5 are satisfied (with $s' = s_{a_1}$). Since we are assuming that the Seifert circle s is taken so that its height is minimum among all Seifert circles violating (5.1), the assumption (d) of A-move is also satisfied. Therefore we can apply the A-move to reduce the number m.

When $\ell > 1$, we take an arc γ in X connecting s_{a_1} and s_{a_2} so that its interior is disjoint from $\mathbb{D}$. The height s_{a_1} is the minimum of $\{h(s_1), \ldots, h(s_m)\}$, the height of Seifert circles contained in s, because by our choice of the Seifert circle s, all the Seifert circles s' contained in s_{a_i} satisfy $h(s') > h(s_{a_i}) \geq h(s_{a_1})$. In particular, $h(s_{a_1}) < h(s)$.

Thus we may assume that the interior of bands of $\mathbb{D}$ are disjoint from $D_s \times \{h(s_{a_1})\}$. Thus we can apply an (ordinary) Y-move to change the Seifert circle s_{a_1} to shallow s_{a_2} (see Figure 15). This reduces the number ℓ of the boundary components of X.

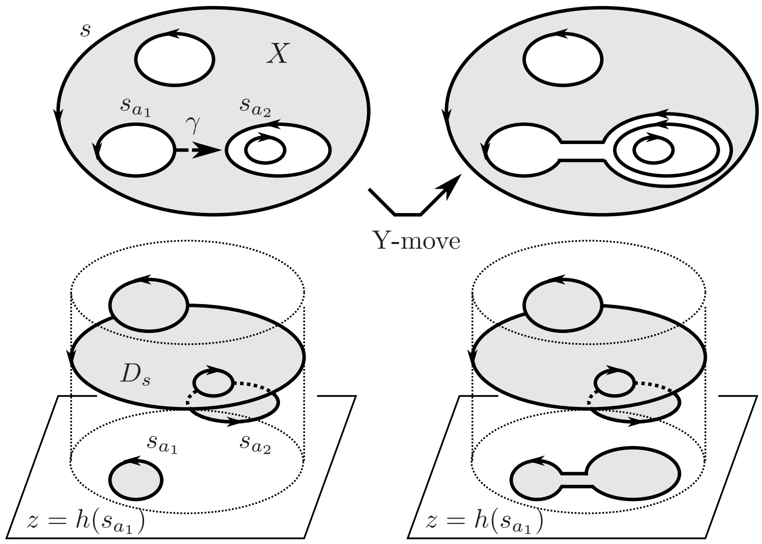


FIGURE 15. (Ordinary) Y-move that decreases the number ℓ of the boundary components of X.

All these operations preserve the height of Seifert circles. Therefore by iterating these procedures we eventually convert $\mathbb{D}$ into an ordinary banded diagram.

$\square$

Theorem 2.11, Proposition 5.10 and Theorem 5.11, prove our following main theorem.

Theorem 5.12. *Let $\mathbb{D}$ be a geometric generalized banded diagram of a link L. If $\mathbb{D}$ is simple and admissible, then by finitely many Y-moves, B-moves, and A-moves, we can make $\mathbb{D}$ into an ordinary banded diagram $\mathbb{D}'$ such that all the Seifert circles of $\mathbb{D}'$ are coherent (i.e. $\mathbb{D}'$ is a closed braid diagram).*

We remark that our proof of Theorem 5.12 is algorithmic. Since a Y-move, B-move and A-move preserve the self-linking number and the number of Seifert circles, Theorem 5.12 extends Corollary 2.12 and Corollary 2.16 for geometric, simple and admissible generalized banded diagrams.

Corollary 5.13. *Let $\mathbb{D}(L)$ be the set of geometric, simple and admissible generalized banded diagrams of a link L. Then*

$$b(L) = \min\{s(\mathbb{D}) \mid \mathbb{D} \in \mathbb{D}(L)\}, \ \overline{sl}(L) = \max\{sl(\mathbb{D}) \mid \mathbb{D} \in \mathbb{D}(L)\}.$$

Furthermore, we have the following criteria and construction of strongly quasipositive links and its quasipositive Seifert surface.

Corollary 5.14. *If a link L is represented by a geometric, simple and admissible generalized banded diagram $\mathbb{D}$ which has no negative bands, then L is strongly quasipositive and $S_{\mathbb{D}}$ is quasipositive. In particular, $\chi(S_{\mathbb{D}}) = \chi(L) = -\overline{sl}(L)$, and L admits a strongly quasipositive braid representative of $s(\mathbb{D})$ strands.*

We say that a marking C of a diagram D *simple* (resp. *admissible*) if $\mathbb{D}(D, C)$ is simple (resp. admissible). By Corollary 5.14 we have the following useful sufficient condition for strong quasipositivity.

Corollary 5.15. *If a geometric marking C of D is full, simple and admissible, then C is quasipositive.*

For the convenience to readers, we summarize the implications among the properties of geometric markings.

$$\begin{tikzcd}
\text{quasipositive} \arrow[rr] & & \text{tight} \arrow[d] \\
\text{full} \arrow[u, "(*)"] \arrow[urr, "(**)"'] \arrow[r] & \text{maximum} \arrow[r] & \text{locally maximum}
\end{tikzcd}$$

($*$) If simple and admissible (Corollary 5.15)
($**$) If $s(\mathbb{D}(D, C)) = s(D)$ (Proposition 4.8)

6. Applications

In this section, as an application of the generalized banded diagram technique, we prove strong quasipositivity for various links.

Definition 6.1 ((Weakly) successively almost positive diagram)**.** A diagram D is *successively almost positive* if all the negative crossings of D appear successively along a single overarc of D. Similarly, a diagram D is *weakly successively almost positive* if all the negative crossings lie on a single overarc of D. Such an overarc is called the *negative overarc* (see Figure 16).

We say that a link L is *successively almost positive* (resp. *weakly successively almost positive*) if L can be represented by a successively almost positive diagram (resp. weakly successively almost positive diagram).

Successively almost positive and weakly successively almost positive diagrams/links are introduced in [It] and [IS] respectively, as an 'appropriate' generalization of positive links. Indeed, as we have seen in [IS], weakly successively positive links share

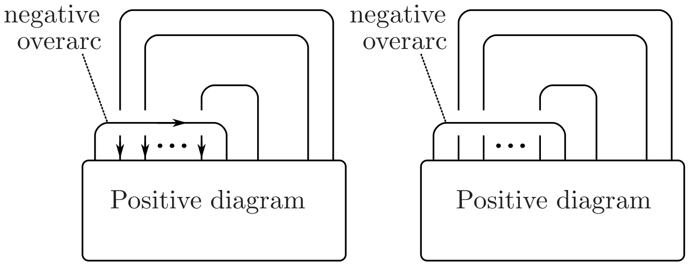


FIGURE 16. Successively almost positive diagram (left) and weakly successively almost positive diagram (right).

many properties as positive links. They provide a unified framework to study positive links – unlike the class of positive diagrams, weakly successively almost positive diagrams are closed under a suitable skein relation that allows us to use induction arguments. In fact, some results on weakly successively almost positive links proven in [IS] are new, even for positive links that have been studied for a long time.

The most important open problem for weakly successively almost positive links is the following.

Question 6.2. [IS, Question 12.1 (a)] *Is every weakly successively almost positive link strongly quasipositive ?*

Our examples and results provide supporting evidence for the affirmative answer, and are interesting in their own right.

Indeed, the generalized banded diagram technique allows us to show that many weakly successively almost positive links are strongly quasipositive and makes it possible to visualize its quasipositive Seifert surface.

6.1. **Braid diagrams and successively almost positive braids.** A *braid word* is a word over the standard generators $\{\sigma_1,\dots,\sigma_{n-1}\}$ and their inverses of the n-braid group B_n. Similarly a *band braid word* is a word over the band generators $\{a_{i,j}\}_{1\leq i<j\leq n}$ and their inverses. (See Example 2.7.) Since $a_{i,i+1}=\sigma_i$, sometimes we view a braid word as a special case of a band braid word.

To avoid cumbersome distinctions, in the following we will often exchange the roles of a braid word w and the braid β represented by w. That is, by *braid* we mean both an element of B_n and its particular braid word representative. In particular, we denote by D_β the link diagram obtained by closing β (which, of course, depends on the word w). As for band braid words, we will again switch the meaning of a band braid word w and the braid β represented by w. We denote by $\mathbb{D}_\beta$ the ordinary banded diagram obtained by closing β.

A braid word w is *positive* (resp. *almost positive*) if it is a product of positive generators $\sigma_1,\dots,\sigma_{n-1}$ (resp. it has at most one negative generator σ_i^{-1}). A link L is a *positive braid link* (resp. *almost positive braid link*) if L is the closure of a positive (resp. almost positive) braid.

Definition 6.3 (Successively almost positive braid)**.** An n-braid word w for $\beta\in B_n$ is *successively almost positive* (s.a.p. in short), if up to cyclic permutation it is of the form

$$w=(\sigma_j^{-1}\sigma_{j-1}^{-1}\cdots\sigma_i^{-1})v\,, \tag{6.1}$$

for some $1 \leq i < j \leq n-1$, where v is a positive braid word. A link L is a *successively almost positive braid link* (s.a.p. braid link, in short) if L is the closure of a successively almost positive braid.

Since

$$\begin{aligned}\beta &= (\sigma_j^{-1}\sigma_{j-1}^{-1}\cdots\sigma_i^{-1})\alpha\\ &= (\sigma_j^{-1}\cdots\sigma_1^{-1})(\sigma_1\cdots\sigma_{i-1})\alpha\\ &\sim (\sigma_{n-1}^{-1}\cdots\sigma_1^{-1})(\sigma_1\cdots\sigma_{i-1})\alpha(\sigma_{j+1}\cdots\sigma_{n-1})\end{aligned}$$

(here $\sim$ denotes the cyclic permutation), when L is a successively almost positive braid link, we are always able to take its successively almost positive braid word so that $i = 1$ and $j = n-1$ in (6.1).

A successively almost positive braid might not be a strongly quasipositive braid. For example, $\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}$ is not strongly quasipositive, but successively almost positive.

However, we show that a successively almost positive braid *link* is strongly quasipositive.

Theorem 6.4. *A successively almost positive braid link is strongly quasipositive.*

To prove this theorem, we give the following algorithm to find a maximum marking C from a successively almost positive braid word, which, in particular, computes the maximum euler characteristic of s.a.p. braid link L, since

$$-\chi(L) = -n + \ell(\beta) - \#C\,. \tag{6.2}$$

In the proof of Theorem 6.4 we will use C just as the marking thus obtained. It will follow from the proof of the theorem that C is in fact a maximal marking.

In a presentation of a word in a group by letters, we use the following notions.

Definition 6.5. Let $v = a_1a_2a_3\cdots a_k$ be a word where a_i are letters.

- We say that v is a *non-consecutive subword* of a word w if for all i, a_i occurs in w right of a_{i+1}, but there may be (arbitrary) letters in between.
- We say v is a *(consecutive) subword* of w if there are fixed letters $\{a_i\}_{i=1}^k$ in w so that a_{i+1} occurs in w immediately following the occurrence of a_i. (Subwords are stipulated by default consecutive.)

Example 6.6. Let $w = \sigma_1\sigma_2\sigma_3^{-1}\sigma_3\sigma_2\sigma_2$.

- $\sigma_1\sigma_3$ is a non-consecutive subword, but $\sigma_3\sigma_1$ is not a non-consecutive subword.
- $\sigma_1\sigma_2\sigma_3$ is a non-consecutive subword of w.
- $\sigma_1\sigma_2\sigma_3^{-1}$ is a (consecutive) subword of w.

When the input braid word α is of length ℓ, then Step (2) of Algorithm 1 is done in time $\ell(\alpha) = \ell$. Since we return to (2) to at most $\frac{n}{2}$ times, the maximum marking C (and hence $\chi(L)$) is computed in time $\mathcal{O}(n\ell)$.

A successively almost positive braid link $L = \widehat{\beta}$ is successively almost positive, hence by [IS], we have $1 - \chi(L) = \max\deg\nabla_L(z)$ for non-split L, where $\nabla_L(z)$ is the Conway polynomial of L. Thus one can also compute the maximum euler characteristic by computing the Conway polynomial. However, the algorithm is much faster than computing the Conway polynomial.

Algorithm 1 Finding a maximum marking of successively almost positive braid word

Input : A successively almost positive braid word $\beta = (\sigma_{n-1}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1})\alpha$
Output: A maximum marking C of D_β

(1) Set $k_1 := 1$, $j := 1$, and $C := \emptyset$.
(2) Let k'_j be the maximal index such that $w_j := \sigma_{k_j}\sigma_{k_j+1}\ldots\sigma_{k'_j}$ (with indices increasing by 1) occurs as a non-consecutive subword of α. For each index $i = k_j, \ldots, k'_j$, we always take the first (i.e., leftmost) letter σ_i in α that allows us to build such a non-consecutive subword w_j.
(3) If such k'_j does not exist (i.e., the braid α contains no letter σ_{k_j}), put $k_j := k_j + 1$. If $k_j \leq n-1$, go back to (2), and otherwise to (6).
(4) If k'_j exists, then put $C := C \cup \{\sigma_{k_j}^{-1}, \ldots, \sigma_{k'_j}^{-1}, \sigma_{k_j}, \ldots, \sigma_{k'_j}\}$, where σ_{k_*} is the crossing that corresponds to a letter in the non-consecutive subword w_j in (2).
(5) If $k'_j < n-2$, put $k_{j+1} := k'_j + 2$ and $j := j+1$, and go back to (2).
(6) Stop.

Example 6.7. Here to make notation simpler, we denote by i and $-i$ the generator σ_i and σ_i^{-1}, respectively. Let $\alpha = 572327161332174124 \in B_8$.

- Step (2) finds a non-consecutive subword $w_1 = 12$ $(572327\underline{1}613321741\underline{2}4)$, so $k_1 = 1$, $k'_1 = 2$.
- Since $k'_1 = 2 < 8-2 = 6$, we put $k_2 = k'_1 + 2 = 4$ and we go back to (2) – this time we find a non-consecutive subword $w_2 = 4$ $(57232716133217\underline{4}124)$ so $k'_2 = 4$.
- Since $k'_2 = 4 < 8-2 = 6$, we put $k_3 = k'_2 + 2 = 6$ and go back to (2) again. Then we find a non-consecutive subword $w_3 = 67$ $(5723271\underline{6}13321\underline{7}4124)$ so $k'_3 = 7$.
- Since $k'_3 = 7 \geq 8-2 = 6$, the algorithm stops and we get a maximum marking

$$\beta = \underline{-7}\,\underline{-6} - 5\underline{-4} - 3\underline{-2}\,\underline{-1}572327\underline{1}\,\underline{6}133\underline{2}1\underline{7}\,\underline{4}124.$$

Thus for the link $L = \widehat{\beta}$, we have $-\chi(L) = -8 + 25 - 10 = 7$.

Proof of Theorem 6.4. Let us put

$$\nu_n = \sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1} \in B_n$$

and $\beta = \nu_n\alpha \in B_n$, where α is a positive braid (word).

We prove the theorem by induction on the string number n.

Let C be the marking of β obtained by Algorithm 1. For notational simplification write

$$S_C := \{\, i \,|\, \sigma_i^{-1} \text{ is in } C \,\}\,.$$

If $\#C = 2(n-1)$, then by Corollary 5.15, C is quasipositive, so we assume $\#C/2 < n-1$, i.e., there is a negative crossing $\sigma_{i_0}^{-1}$ in β which is not involved in the marking C. We take the minimal $1 \leq i_0 < n$ with this property, i.e.,

$$i_0 = \min\Big(\{1, \ldots, n-1\} \setminus S_C\Big)\,.$$

Figure 17 shows the example $n = 8$, $i_0 = 4$.

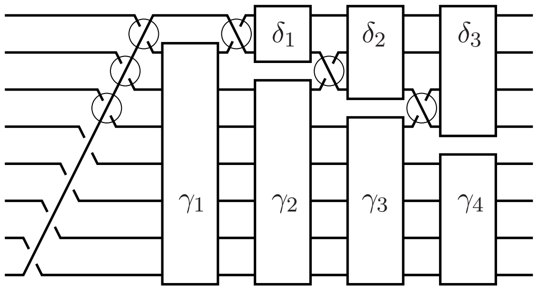


FIGURE 17. Successively positive braid with marking. The circled crossings represent the marking.

Thus we can write

$$\beta = \nu_n\alpha = \nu_n(\gamma_1\sigma_1\delta_1)(\gamma_2\sigma_2\delta_2)\cdots(\gamma_{i_0-1}\sigma_{i_0-1}\delta_{i_0-1})\gamma_{i_0}\,,$$

where γ_i contains only $\sigma_{i'}$ for $i' > i$, and δ_i contains only $\sigma_{i'}$ for $i' \leq i$. Thus γ_i commutes with $(\sigma'_i\delta_{i'})$ for all $i' < i$, and we can write

$$\beta = \nu_n\alpha = \nu_n(\gamma_1\gamma_2\cdots\gamma_{i_0})(\sigma_1\delta_1)(\sigma_2\delta_2)\cdots(\sigma_{i_0-1}\delta_{i_0-1}).$$

By using braid relations

$$\nu_n\sigma_j^{\pm 1} = \sigma_{j-1}^{\pm 1}\nu_n \quad (j = 2, \ldots, n-1),$$

one can slide $(\gamma_1\gamma_2\cdots\gamma_{i_0})$ across ν_n to get

$$\beta = \nu_n\alpha = (\gamma'_1\gamma'_2\cdots\gamma'_{i_0})\nu_n(\sigma_1\delta_1)(\sigma_2\delta_2)\cdots(\sigma_{i_0-1}\delta_{i_0-1}),$$

where γ'_i are the words obtained from γ_i by replacing each σ_j by σ_{j-1}. By taking conjugates we get

$$\beta' = \nu_n(\sigma_1\delta_1)(\sigma_2\delta_2)\cdots(\sigma_{i_0-1}\delta_{i_0-1})(\gamma'_1\gamma'_2\cdots\gamma'_{i_0})\,.$$

Since γ'_i and $(\sigma_i\delta_i)$ contain no $\sigma_{n-1}^{\pm 1}$, the braid β', which is conjugate to β, has an isolated σ_{n-1}^{-1}.

Thus we can (negatively) destabilize β' to a braid $\beta'' \in B_{n-1}$, which is still of the s.a.p. form $\beta'' = \nu_{n-1}\alpha'$ with α' positive.

What remains to see is that if we apply Algorithm 1 on β'', we obtain a marking C'' with $\#C'' = \#C$. But this is an easy combinatorial observation which we leave to the reader. More precisely,

$$S_{C''} = \{1, \ldots, i_0 - 1\} \cup sh(S_C \cap \{i_0 + 1, \ldots, n-1\})\,,$$

where sh is the shift map $i \mapsto i - 1$.

□

Corollary 6.8. [HIK, Theorem 3.1] *An almost positive braid link is strongly quasipositive.*

The following result which appeared in the proof deserves to be mentioned.

Corollary 6.9. *Let L be the closure of a successively almost positive n-braid $\beta = (\sigma_1\cdots\sigma_{n-1})^{-1}\alpha$. If D_β does not admit a marking that contains all the negative crossings (in the terminology of Definition 6.5 above, it means that $\sigma_1\sigma_2\cdots\sigma_{n-1}$*

is not a non-consecutive subword of α*), then* β *admits a negative destabilization up to conjugacy. In particular,* $\frac{1}{2}span_v P_L(v,z) + 1 \leq b(L) < n$.

Here $\mathrm{span}_v P_L(v,z)$ is the span of the variable v in the HOMFLY polynomial $P_L(v,z)$ of L. This makes a sharp contrast to the following famous result.

Theorem 6.10. [FW]. *If* L *is the closure of a positive braid of the form* $\beta = (\sigma_1 \cdots \sigma_{n-1})^n \alpha$*, where* α *is a positive braid, then* $b(L) = \frac{1}{2}span_v P_L(v,z) + 1 = n$.

6.2. **Almost positive links are strongly quasipositive.** An almost positive diagram D is a diagram that has exactly one negative crossing c, which can be seen as a special case of (weakly) successively almost positive diagram. In [FLL], it is proven that almost positive links are strongly quasipositive. Their proof is based on a characterization of quasipositive Seifert surface given in [Rud2].

Here we give an alternative proof that also provides information about the braid index of strongly quasipositive representatives.

The almost positive diagrams are classified into the following two types [St1]:

- **Type I:** There are no positive crossings that connect the same pair of Seifert circles as c.
- **Type II:** There is a positive crossing that connects the same pair of Seifert circles as c.

Theorem 6.11. *Almost positive links are strongly quasipositive. More precisely,*

- *If a link* L *has an almost positive diagram* D *of type I, then* L *admits a strongly quasipositive braid representative with* $(s(D)-1)$*-strands.*
- *If a link* L *has an almost positive diagram* D *of type II, then* L *admits a strongly quasipositive braid representative with* $s(D)$*-strands.*

Proof. If D is of type I, we use the following operation which we call the *Chalcraft-Murasugi-Przytycki's move* (CMP move in short). Assume that the negative crossing c connects the two Seifert circles s and s'. Since D is of type I, no other crossings connect s and s'. We move the underarc of the negative crossing c across one of the Seifert circles s', swallowing the Seifert circles and crossings adjacent to s' as closely as possible (see Figure 18).

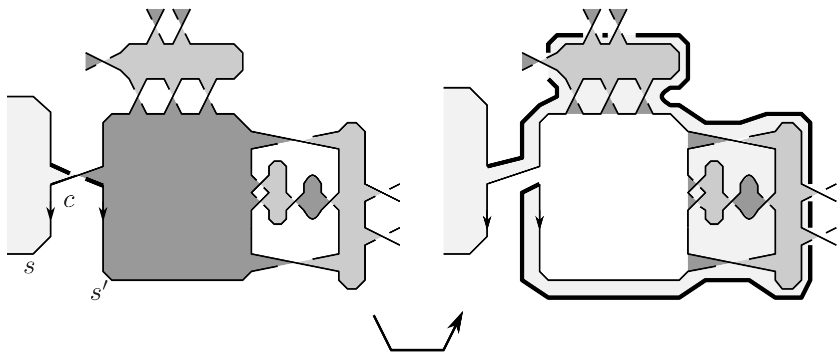


FIGURE 18. Chalcraft-Murasugi-Przytycki's move at a negative crossing c along a Seifert circle s'.

As is clear from Figure 18, the CMP move can be regarded as a move of canonical Seifert surfaces, rather than just as a move of diagrams or links. The move of the underarc of c is achieved by flyping the twisted band at c and the disk $D_{s'}$ bounded by s'. As a consequence, we can regard the diagram D' obtained by the CMP move as an ordinary banded diagram $\mathbb{D}$ in an obvious way. Since this procedure removes the negative crossing (band) and the Seifert circle s',

$$sl(\mathbb{D}) = sl(D) + 2, \quad s(\mathbb{D}) = s(D) - 1 \tag{6.3}$$

holds. $\mathbb{D}$ is a (ordinary) banded diagram without negative bands, so by Corollary 2.13 we may convert it to a strongly quasipositive closed braid diagram of $(s(D)-1)$ strands.

If D is of type II, we take a positive crossing c' connecting the same Seifert circles as the negative crossing c. Then $C = \{c, c'\}$ is a geometric, simple marking[2] of D. Furthermore, by taking an isotopy of diagrams in S^2, we may assume that the marking C is admissible. Then

$$sl(\mathbb{D}(D,C)) = sl(D), \quad s(\mathbb{D}(D,C)) = s(D). \tag{6.4}$$

Thus by Corollary 5.14, we see that $\mathbb{D}(D,C)$ can be converted to a strongly quasipositive closed braid diagram of $s(D)$ strands. $\square$

It is worth pointing out the below corollary, which follows from (6.3) and (6.4).

Corollary 6.12. *An almost positive diagram D of a link L, $sl(D) = \overline{sl}(L)$ if and only if it is of type II.*

Remark 6.13. In [St2] it is explained precisely when the argument for type I can be generalized to multiple negative crossings, i.e., when can CMP moves at all negative crossings be performed to remove them and obtain a strongly quasipositive surface.

As is clear from the proof, our argument can applied for many n-almost positive diagrams D of a link L; for each negative crossing c of D, if there are no other crossings connecting the same pair of Seifert circles as c, we try to apply the CMP move. Similarly, if there is a positive crossing connecting the same pair of Seifert circles as c, using such a positive crossing, we try to view D as a generalized banded diagram so that the negative crossing c appears as a crossing of Seifert circles. If these procedures yield a (geometric, simple and admissible) generalized banded diagram with no negative band, then we conclude that L is strongly quasipositive.

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