HOSTE'S CONJECTURE AND ROOTS OF LINK POLYNOMIALS

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Abstract. We prove that any root z of the Alexander polynomial of a 2-bridge (rational) knot or link satisfies $|z^{1/2}-z^{-1/2}|<2$. This relates to a conjecture of Hoste on the roots of the Alexander polynomial of alternating knots. We extend our result to properties of zeros for Montesinos knots, and to an analogous statement about the skein polynomial. A similar estimate is derived for alternating 3-braid links.

Keywords: alternating knot, Alexander polynomial, skein polynomial, rational link, 3-braid link, Montesinos knot, polynomial root, genus

AMS subject classification: 57M25 (primary), 11C08, 30C15 (secondary)

1 Introduction

The Alexander polynomial Δ remains one of the most fundamental invariants of knots and links in 3-space. Due to its profound importance, many features of the polynomial have been studied over the years in a variety of contexts. Roots of the polynomial are related, among others, to the monodromy and dynamics of surface homeomorphisms [Ro, SW2], divisibility [Mu3] and orderability [PR] of knot groups, and statistical mechanical models of the Alexander polynomial [LW]. They are also studied in connection to Lehmer's question on the existence of a Mahler measure minimizing polynomial [GH, Hi, SW].

The topological understanding of the Alexander polynomial has led a long time ago to the insight what (Laurent) polynomials occur for an arbitrary knot. Ironically, the question to characterize the Alexander polynomials of *alternating* knots turns out to be far more difficult, even although in general alternating knots are much better understood. Hoste, based on computer verification, made the following conjecture about 10 years ago, which was later popularized by Murasugi (see, e.g., [LMu]).

Conjecture 1.1 (Hoste's conjecture) If $z\in\mathbb{C}$ is a root of the Alexander polynomial Δ of an alternating knot, then $\Re e\, z>-1$.

This conjecture is true for knots up to genus 4 [St]. It is true also for special alternating knots (knots which are simultaneously positive and alternating). For such knots all roots of Δ lie on the complex unit circle (and -1 is not a zero of the Alexander polynomial of any knot; for a clarification see [Ga, St2]). The same is true also for special (non-split) alternating links, as explained in [St2, St8]. (This property will play a role in our work here, so it is worth paying some attention to it.)

See §8 below for a more detailed discussion on known and conjectured properties of the Alexander polynomial of an alternating knot.

Recently, Lyubich and Murasugi [LMu] studied the roots of the Alexander polynomial of a 2-bridge (rational) knot or link, motivated by Hoste's conjecture. Although they could not settle the conjecture completely, they proved many special cases, and several results going beyond the statement of the conjecture. One of their results is:

Theorem 1.1 (Lyubich-Murasugi, [LMu, theorem 1]) Let L be a 2-bridge (rational) knot or link and z be a root of the Alexander polynomial $\Delta(L)$. Then $-3 < \Re e\, z < 6$.

Our first result in this paper, proved in §2, is the following.

Theorem 1.2 Let L be a 2-bridge (rational) knot or link and z be a root of the Alexander polynomial $\Delta(L)$. Then

$$\left|z^{1/2} - z^{-1/2}\right| < 2. \tag{1}$$

The condition (1) will play a fundamental role throughout the paper[1]. Let us thus say that a complex number $z \neq 0$ is *internal* if it satifies (1), and *external* otherwise. Let $\mathcal{D}$ be the *domain of internal complex numbers*.

It can be verified (see below lemma 2.1) that if (1) holds, then

$$-\frac{3}{2} < \Re e\, z\,, \tag{2}$$

and

$$|z| < 3 + 2\sqrt{2} \approx 5.8284\,, \tag{3}$$

which improves either estimates in theorem 1.1. Despite that (2) is insufficient for Hoste's conjecture in its strict form, we can confirm the conjecture in certain cases (proposition 2.1). We will see, too, that outside $\mathcal{D}$, one can address it in a larger setting (*cf.* theorem 1.4).

We subsequently found that theorem 1.2 was independently obtained by Koseleff and Pecker [KP]. For other properties of the Alexander polynomial of 2-bridge knots see [Bu, Ha].

A closer look revealed that the proof of theorem 1.2 can be adapted at some cost to 3-braid alternating links. These are considered in §4, where we obtain the following result. Let us stipulate in general below that all constants given in decimal expansion are rounded.

Theorem 1.3 Let L be a (non-split) 3-braid alternating knot or link and z be a root of the Alexander polynomial $\Delta(L)$. Then

$$\left|z^{1/2} - z^{-1/2}\right| < 2.45317\,. \tag{4}$$

Next, we will study Montesinos links. There is a dichotomy between parallel (45) and reverse Montesinos links (26). With regard to Hoste's conjecture we will, in particular, prove:

Theorem 1.4 An *external* zero z of the Alexander polynomial of an alternating Montesinos knot has $\Re e\, z > -1$. In particular, all zeros of the Alexander polynomial satisfy (2). Moreover, for a parallel Montesinos knot z satisfies (3).

There is also a more precise statement possible for reverse knots. See theorem 3.1 in §3. It applies also to alternating Montesinos links (for various component orientations). We do not only restrict the external zeros stronger than predicted by the conjecture, but address also a large family of non-alternating Montesinos links along the way. In opposition to the 2-bridge and 3-braid case, however, the domains we obtain are in general unbounded. (Murasugi has established that zeros cannot be contained in a bounded domain; see example 3.1.)

Parallel Montesinos links are studied in §5, where we obtain the above stated bounds in theorem 5.1. The proof is much more elaborate than in the reverse case and requires a very complex (in either sense) calculation. This is moved out into a separate section §6.

[1] Of course, $z^{-1/2}$ is supposed to be the inverse of the same between the two roots of z chosen to be $z^{1/2}$. Which one is chosen then is no longer relevant, for the norm.

In §7 we address extensions of the zero location to the skein polynomial.

In §8 we discuss the relation of Hoste's conjecture to two other conjectures on the Alexander polynomial of an alternating knot: the well-known Fox trapezoidality conjecture and its extension, the log-concavity conjecture.

We conclude with a few problems in §9.

The practical examples and computations of polynomials and roots were assisted by MATHEMATICA™ [Wo] and KnotScape [HT].

The following abbreviations will be used throughout: 'resp.' will mean 'respectively', 'w.l.o.g.' will stand for 'without loss of generality', and 'r.h.s.' resp. 'l.h.s.' for 'right-hand side' resp. 'left-hand side'.

2 Rational links

Before we turn to knot theory, it will be important to gain a basic description of the domain $\mathcal{D}$ that occurs in theorem 1.2. We will need the below lemma and a part of the calculation for it with a more serious reason later in §6. (We thus conform in the proof to the designations we will use there, except that z will be changed to t.)

Lemma 2.1 The domain $\mathcal{D}$ is bounded by the graphs of the four functions

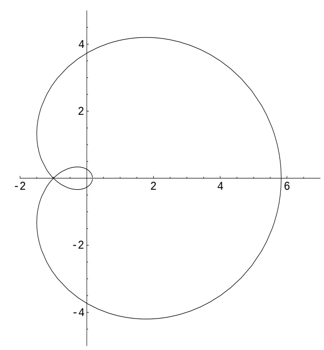
$$\pm f_{\pm}(x) = \pm\sqrt{-x^2+2x+7\pm 4\sqrt{2x+3}}\,. \tag{5}$$

These expressions will be needed below, and thus we will designate them using two functions f_+ and f_-, where the subscript refers to the *inner* '$\pm$' in (5), i.e., the one in front of $4\sqrt{2x+3}$.

We notice that $f_{\pm}$ is defined on $\left[-\frac{3}{2}, 3\pm 2\sqrt{2}\right]$ (for the same among '$\pm$' as the subscript of f). A few special values are

$$f_{\pm}\left(-\frac{3}{2}\right) = \frac{\sqrt{7}}{2}\,, \qquad \begin{array}{l} f_-(-1) = \ 0\,, \\ f_+(-1) = \sqrt{8}\,, \end{array} \quad \text{and} \quad f_{\pm}\left(3\pm 2\sqrt{2}\right) = 0\,. \tag{6}$$

(The two '$\pm$' in the last formula are to be chosen equal.) A MATHEMATICA™ plot on the right shows the functions.



The below calculation is a simple instance of type of arguments which will be needed later. They are not really sophisticated, but create an enormous scope for errors (which have painfully plagued the author during this work). Thus it is compelling to include enough details.

Proof. The condition on the boundary of $\mathcal{D}$ is

$$|z^{1/2}-z^{-1/2}| = 2\,. \tag{7}$$

It can be written as

$$\left|z-2+\frac{1}{z}\right|^2 = 16\,.$$

For

$$z = b+l\sqrt{-1} \tag{8}$$

this becomes

$$\left(b-2+\frac{b}{b^2+l^2}\right)^2+\left(l-\frac{l}{b^2+l^2}\right)^2 = 16\,.$$

Clearing the denominator, using $d=|z|$ with

$$d = \sqrt{b^2+l^2}\,,$$

and setting

$$D = d^2 = |z|^2 ,$$

we have

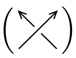
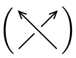
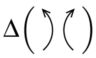
$$((b-2)D+b)^2 + (D-b^2)(D-1)^2 = 16D^2 .$$

This rearranges to a cubic equation in D, but the constant term vanishes, and so, dividing by $D \neq 0$, we have

$$D^2 + D(-4b-14) + (2b-1)^2 = 0 . \tag{9}$$

Solving for D and using $l = \pm\sqrt{D-b^2}$ gives the result (with $x = b$). □

The proof of theorem 1.2, as well as all following knot-theoretic proofs, relies on a recursive calculation based on the skein relation for the Alexander polynomial,

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) = \left(z^{1/2} - z^{-1/2}\right) \Delta\Big($  $\Big)$. (10)

We call the diagram fragments above a *positive*, a *negative*, and a *smoothed out* crossing. The dichotomy between positive and negative crossings will be called *(skein) sign* of the crossing.

In fact, it is more natural, in our case, to set

$$w = z^{1/2} - z^{-1/2} , \tag{11}$$

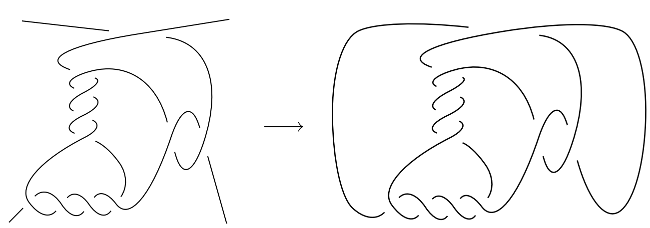
and regard Δ as a (genuine) polynomial in w, which is the *Conway polynomial* ∇. Thus in fact theorem 1.2 is more naturally written in terms of ∇. The variable w will keep its meaning (11) in the following. (The sign ambiguity resulting from the choice of complex root will not create problems.)

A rational (2-bridge) knot or link is represented in Schubert's form [Sh] as $L = S(q,p)$, where p and q are coprime integers with $0 < p < q$. A diagram of $S(q,p)$ can be obtained from any continued fraction expansion of the rational number p/q:

$$\frac{p}{q} = (b_1,\dots,b_n) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \dots \cfrac{1}{b_n}}} \tag{12}$$

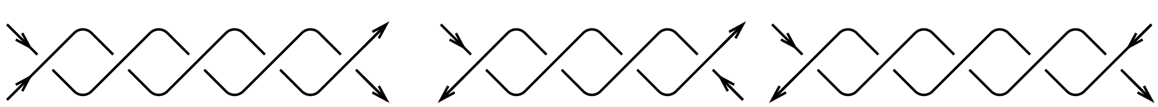
(The numbers b_i are non-zero integers.) In this diagram each b_i corresponds to a group of $|b_i|$ crossings, called in the following a *twist*. $S(q,p)$ a knot for odd q and a 2-component link for even q.

The below diagrams show how to join twists into a rational tangle and how to close it up. (Note that the twists are composed in a non-alternating way when the sign of b_i changes.) The displayed sequence of the type (12) is $(1,2,4,-4) = \dfrac{34}{49}$, and thus the knot depicted is $S(49,34)$.



rational tangle $\longrightarrow$ rational link

It will be important, throughout the paper, to distinguish how twists (of more than one crossing) look like when strands are oriented. We call a twist *reverse* if smoothing out a crossing renders the other crossings nugatory. Otherwise it is *parallel*.



parallel twist reverse twists

There are several standard ways to choose an expansion (12). One is to take all b_i to be positive. This gives an alternating diagram of L (thereby explaining that rational links are alternating); we will use this form in §5. Here we use the *even* continued fraction, determined by demanding that $b_i = 2a_i$ be even (and non-zero). Such a representation gives rise to a (in general non-alternating) diagram of the 2-bridge link $L = S(q, p)$. (Note that an even expansion requires one of p and q to be even; this can be achieved, though, since for odd q, there is an identification $S(q, p) = S(q, q-p)$.)

For a 2-bridge link $L = S(q, p)$, in the diagram coming from the even continued fraction expansion

$$\frac{p}{q} = (2a_1, \dots, 2a_n), \tag{13}$$

each a_i corresponds to a group of $2|a_i|$ crossings in a reverse twist.

Proof of theorem 1.2. In this proof, as well as in §3, we will deal only with reverse twists, before handling the parallel ones in §4 and §5.

We fix now $z \in \mathbb{C}$ for which (1) does *not* hold:

$$\left| z^{1/2} - z^{-1/2} \right| \geq 2. \tag{14}$$

Since the quantitity will be continuously needed, let for the scope of this section,

$$y = |w| = \left| z^{1/2} - z^{-1/2} \right|.$$

Let us meet the convention that when $n = 0$ we have the unknot, and for $n = -1$ the two component unlink. This lets (16) below hold also in these exceptional cases. Let for $L = S(q, p)$ with (13),

$$[2a_1, \dots, 2a_n] := \left| \Delta(L)(z) \right|.$$

The proof of theorem 1.2 is essentially accomplished by the following lemma.

Lemma 2.2

$$[2a_1, \dots, 2a_n] \geq [2a_1, \dots, 2a_{n-1}]. \tag{15}$$

Proof. Use induction on n. For $n = 0$ the claim is trivial: the r.h.s. vanishes and the l.h.s. is 1.

For the induction step we use the skein relation for the Alexander polynomial (10), which implies

$$\Delta(L(2a_1, \dots, 2a_n))(z) = \pm a_n \cdot w \cdot \Delta(L(2a_1, \dots, 2a_{n-1}))(z) + \Delta(L(2a_1, \dots, 2a_{n-2}))(z). \tag{16}$$

The sign before a_n depends on the skein sign (as explained below (10)) of the crossings in the twist corresponding to $2a_n$. This sign changes between a_{n-1} and a_n precisely if the signs of a_{n-1} and a_n are equal. We may thus set the '$\pm$' in (16) to be $(-1)^{n-1}$, when we fix that $a_1 > 0$ and the twist corresponding to a_1 is (skein) positive. Let us take this convention here to omit the '$\pm$'. (The other choice results in mirroring the entire diagram, which has no effect on the zeros of Δ.)

Taking norms, we have

$$[2a_1, \dots, 2a_n] \geq |a_n| \cdot y \cdot [2a_1, \dots, 2a_{n-1}] - [2a_1, \dots, 2a_{n-2}].$$

Now using (14) and the induction assumption $[2a_1, \dots, 2a_{n-1}] \geq [2a_1, \dots, 2a_{n-2}]$, we are done. □

This concludes the proof of theorem 1.2. □

Remark 2.1 It is clear from the proof that in fact for z with (14) we have $|\Delta(L)(z)| \geq 1$. This observation applies also to several of the situations below. On the other hand, the $(2, n)$-torus knots show that the bound 2 in (1) can be arbitrarily approximated, and cannot be improved.

When $y < 2$, the recursive estimate gets gradually ruined, however, it can be salvaged in certain cases. We show the below proposition as an improvement of [LMu, theorem 3]. Similar considerations will be possible later also for (at least reverse) Montesinos links. These, however, so far do little about the remaining situations, and since they do not lead to a complete statement, we will for space reasons no longer dwell upon them.

Proposition 2.1 If in (13) no three consecutive $a_i=\pm1$, then Hoste's conjecture holds. If no a_i is ±1, then $\left|z^{1/2}-z^{-1/2}\right|<1$, and in particular

$$\frac{3}{8}<\Re e\, z \quad \text{and} \quad |z|<\frac{3+\sqrt{5}}{2}\,. \tag{17}$$

Proof. If no a_i is ±1, then $y\ge1$ is enough to make the recursive estimate work to exclude zeros. The condition $y<1$ implies (17) by a calculation similar to the one for lemma 2.1.

If $a_i=\pm1$ occur, one can see that exactly n such consecutive ones (followed, if any, by an a_i with $|a_i|\ge2$), would make the estimate work if

$$\frac{y^n-y^{n-1}-\ldots-y-1}{y^{n-1}-y^{n-2}-\ldots-y-1}\ge\frac{1}{2y-1}\,,$$

and if the numerator on the left is positive. This can be seen (for $1<y<2$) to be equivalent to

$$n\le 1+\log_y\frac{2}{(2-y)(2y+1)}\,. \tag{18}$$

For $\Re e\, z\le-1$ the minimal value of y can be found to be

$$y=\sqrt[4]{12} \tag{19}$$

(attained at $z=1\pm\sqrt{-2}$). One can also check, using the derivative, that the r.h.s. of (18) is increasing in y for $1<y<2$. Setting (19), this r.h.s. evaluates to ≈2.8, so that $n\le2$ (i.e., no 3 consecutive $a_i=\pm1$) is enough for Hoste's conjecture. (The z with $y\le1$ create no problem, as we see from the non-strict version of the first inequality in (17).) $\square$

3 Montesinos links I: reverse links

There was some hope to extend the proof of theorem 1.2. In this realm our aim is to prove Hoste's conjecture for every alternating Montesinos knot. (We will see its assertion fulfilled also for many non-alternating ones.) We also cover alternating (and many non-alternating) Montesinos links. The treatment (including all component orientations for links) is completed only in §5 (see remark 3.3).

A Montesinos link has the presentation

$$L=M(e,p_1/q_1,\ldots,p_n/q_n) \tag{20}$$

with integers e and p_i,q_i satisfying $(p_i,q_i)=1$ and $|q_i|>1$. The notation (20) is well-known, but conventions differ throughout the literature. The convention we use here is as in [St6, St7]. We will repeat only the relevant properties; see *op.cit.* for further explanation (and for the difference to the form used at some other places).

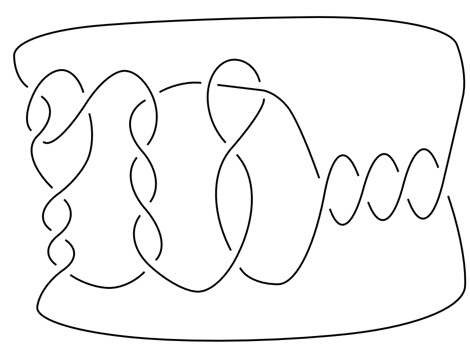


Figure 1: The Montesinos knot with notation $M(4,3/11,-1/4,2/5)$.

The integer e is called *integer part*, and p_i/q_i are the *fractional parts*. The number n is called *length* of the Montesinos link. For $n \le 2$, the Montesinos link becomes a rational link. When all $p_i = \pm 1$, we stipulate that we sign q_i so that $p_i = 1$, and write

$$L = M(1/q_1, \dots, 1/q_n) = P(q_1, \dots, q_n)\,. \tag{21}$$

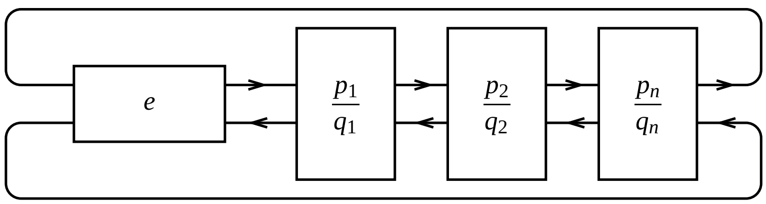
In this case the Montesinos link L is called a *pretzel link*.

The knot in figure 1 should clarify matters. The twist of 4 crossings on the right gives the integer part. The three factional parts are read off from the other twists by

$$(3,1,2) = \frac{3}{11}\,, \quad (-4) = -\frac{1}{4}\,, \quad \text{and} \quad (2,2) = \frac{2}{5}\,.$$

Let us here for clarity introduce two degenerate instances of (20), which we will occasionally use below. If $q_i = 0$, then $p_i/q_i = \pm\infty$, and this means that the Montesinos link becomes the connected sum of rational links $S(q_j, p_j)$ for $j \ne i$. If $p_i = 0$, then the rational tangle p_i/q_i can be omitted in (20). These two cases will be used also in the pretzel notation (21) by setting $q_i = 0$ resp. $q_i = \pm\infty$ (with the understanding $1/\pm\infty = 0$).

Theorem 3.1 Consider the Montesinos link (20), where $|p_i| < q_i$, and with $p_i q_i$ and e even and non-zero. Let L be oriented so that it conforms to the pattern



. (22)

Let z be a root of the Alexander polynomial of such a link and w be as in (11).

1. Then
$$|w| < 2 \quad \text{or} \quad |\Im m\,(w^2)| < \frac{2|w|}{|e|}\,. \tag{23}$$

2. If L is alternating, then
$$|w| < 2 \quad \text{or} \quad \Re e\,(w^2) > -2\,. \tag{24}$$

The form (20) determines an *unoriented* Montesinos link, and is unique up to reversal and cyclic permutations of the vector $(p_1/q_1, \dots, p_n/q_n)$, and up to the identification

$$M(e, \dots, p_i/q_i, \dots) = M(e \pm 1, \dots, (p_i \mp q_i)/q_i, \dots)\,. \tag{25}$$

This shows the existence of one of the standard presentations (20), where $|p_i| < q_i$, and $e \ne 0$ only if all p_i have the same sign as e.

Remark 3.1 It was shown in [LT] that L is alternating if and only if in this standard presentation all p_i have the same sign (but not necessarily $e \ne 0$). We call this below the *alternating form* of L. This criterion also means that in any presentation (20) with $0 < |p_i| < q_i$, the link L is alternating if and only if $|e|$ is not smaller than the number of p_i of opposite sign to e.

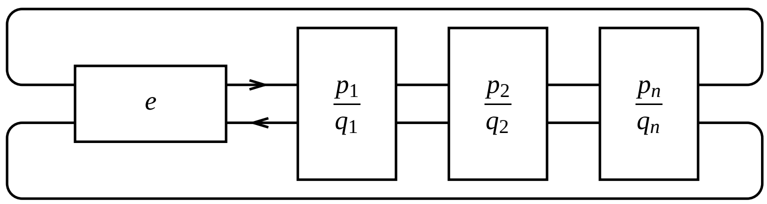
The property (25) shows that choosing $|p_i| < q_i$ and $p_i q_i$ even is no restriction. The restriction in theorem 3.1 comes from the non-vanishing of e and the orientation condition (22). The other (parallel) orientation will be treated in §5. However, the assumption $e \ne 0$ is essential here. For $e = 0$ there are (non-alternating pretzel) links with $\Delta = 0$.

Remark 3.2 It will be relevant to notice that adjusting the parity of $p_i q_i$ when $|p_i| < q_i$ is connected with adding $\mathrm{sgn}(p_i)$ to e and switching the sign of p_i.

Keep in mind that the Alexander polynomial is extremely sensitive to the reversal of orientation of individual (but not all) components of links. We thus next spend some words on orientation.

Remark 3.3 With e and p_iq_i even, there is a choice of component orientation which conforms to (22). For a knot, this is the only choice. For a link, there may be further choices when individual components are reversed. Then one can still achieve (22), but this requires to abandon the evenness of p_iq_i. However, in this section we stick to the evenness condition. In §5 we will see how to handle parallel twists, which will enable us to work (with alternating links) also when p_iq_i is odd. Lemma 5.1 can be used to deal with the remaining cases of orientation for (alternating) links.

When p_iq_i are even, the orientation condition (22) can be satisfied (up to orientation for links) if the twists corresponding to e are reverse:



(26)

(and strands drawn unoriented may have either orientation).

Definition 3.1 We call a Montesinos link as in (26) a *reverse Montesinos link*.

To see how the orientation condition (26) leads to (22), it is enough to see, by remark 3.3, that one can always choose e even. To do so, note first that the connectivity of the tangles p_i/q_i is determined by the parities of p_i and q_i, and we assume p_i or q_i is even. Assume e is odd. When all p_i are even, (26) cannot be satisfied. Thus necessarily some q_i is even. Then one can apply (25) changing the parity of e without changing the oddness of p_i. (Note also that the condition (26) is automatically fulfilled if some q_i is even.)

Note also that when the orientation (22) is given and all p_iq_i are even, then all twists in the diagram are reverse. Our aim is here to see how to adapt the proof of theorem 1.2 by first confining ourselves to reverse twists. As indicated in that proof, the postponement of parallel twists is the main reason for the division of the treatment of Montesinos links into two sections.

Remark 3.4 Note that the second part of condition (24) is equivalent to $\Re e\, z > 0$, so (24) again implies Hoste's conjecture. However, the second part of (23) gives a further restriction also for alternating links. This condition visually means for z that when $|z|$ is large, z is close to the (for alternating links positive) real axis. (Such zeros indeed occur; see the below example.)

Example 3.1 The condition (1) is too strong to assert in theorem 3.1 even for an alternating link. A computer verification in the tables of [HT] exhibited the Montesinos knot $15_{7711} = M(2,1/2,2/5,2/5,2/3)$ and its mutant $15_{10057} = M(2,1/2,2/5,2/3,2/5)$, where the Alexander polynomial has a zero z with $z^{1/2} - z^{-1/2} \approx \pm 2.09826$. Later Murasugi showed me the knots $M(2,2/5,\dots,2/5,5/12)$, for which he proved that they have arbitrarily large real positive Δ zeros.

Proof of theorem 3.1. The existence of the even continued fraction expansion (13) for p_i/q_i is ensured when $|p_i| < q_i$ and p_iq_i is even. These links have, with the orientation we chose, a star-like plumbing Seifert surface. (For rational links the star was just a line.) We use, in analogy to the rational link case, induction over the length of the arms of the star. The problem is to start the induction properly. For the induction step the same argument (and the condition (1)) is sufficient.

The induction start consists in having $p_i = \pm 1$ and q_i even, allowing $q_i = 0$ (with $\pm 1/0 = \pm\infty$), and showing that

$$[M(e,\dots,p_{i-1}/q_{i-1},p_{i+1}/q_{i+1},\dots)] \geq [M(e,\dots,p_{i-1}/q_{i-1},\pm\infty,p_{i+1}/q_{i+1},\dots)]\,. \tag{27}$$

This inequality needs some explanation. Here and below for convenience we switch signs so that $p_i = 1$ but q_i may be negative. (The l.h.s. corresponds for rational links to the role of the unknot, and the r.h.s. to this of the 2-component unlink.)

In the formula (27) again $[L]$ stands for $|\Delta(L)(z)|$ for an *a priori* fixed $z \in \mathbb{C}$ violating (23) resp. (24). The link $M(e,\dots,\pm\infty,\dots)$ means that the twists of $1/q_i$ are removed in such a way that the Montesinos link becomes the connected sum of reverse $(2,q_j)$-torus links for $j \neq i$ (and splits if at least two q_i become zero).

The formula (27) can be established by direct calculation. The case that some $q_j = 0$ for $j \neq i$ is trivial (the r.h.s. is zero). Otherwise, scaling by a power of $w = z^{1/2} - z^{-1/2}$ and $1/2^{n-2}$, we are left to prove

$$\left| -\frac{e}{4} \cdot q_1 \cdot \ldots \cdot q_n \cdot w^2 + \sigma_{n-2,n-1}(q_1,\dots,q_n) \right| \geq \left| \frac{1}{2} q_1 \cdot \ldots \cdot q_n \cdot w \right| . \tag{28}$$

Here we stipulate that q_i is left out everywhere and $\sigma_{k,l}$ is the elementary symmetric polynomial of degree k in l variables. (Note that the twists counted by e have opposite skein sign to those counted by q_i.)

We deal with (28) in two different ways, according to the two assertions of the theorem.

1. The inequality (28) now follows by looking only at the imaginary part of the l.h.s. and using the (logical) negation of (23). With this inequality, we obtain the containment of zeros using the recursive norm estimate

   $$[L] \geq [L'] . \tag{29}$$

   Here again a link L stands for $\Delta(L)(z)$, and w is as in (11). The Montesinos link L differs from L' either by the creation of a new fractional part $\pm 1/q_i$, or by replacing a fractional part with even fraction expansion $(\dots,a_{l-1})$ by $(\dots,a_{l-1},a_l)$, where (a_l is even and non-zero and) the sign of a_l is chosen in (29).

   What (29) accomplishes now is to contain the zeros in the desired domain under successively removing the last entry in the integer sequence (geometrically, deplumbing bands). We can iterate this until we arrive at the reverse $(2,e)$-torus link.

2. If L is alternating, then by remark 3.1 (up to mirroring)

   $$e \geq s := \#\{\, i : q_i < 0 \,\} .$$

   First assume $e > s$. The inequality (28) can be restated

   $$\left| -\frac{e}{4} w^2 + \sum_{j \neq i} \frac{1}{q_j} \right| \geq \frac{|w|}{2} . \tag{30}$$

   Let $s_i := \#\{\, j \neq i : q_j < 0 \,\}$. Thus we have $e > s_i$. Now assume, in opposition to (24), that $\Re e\,(w^2) \leq -2$. We rewrite the above inequality (30) as

   $$\left| \left[\left(-\frac{s_i}{4} \Re e\,(w^2) + \sum_{\substack{j \neq i \\ q_j < 0}} \frac{1}{q_j} \right) + \left(\frac{e - s_i}{4} (-\Re e\,(w^2)) \right) + \left(\sum_{\substack{j \neq i \\ q_j > 0}} \frac{1}{q_j} \right) \right] - \frac{e}{4} \Im m\,(w^2) \cdot \sqrt{-1} \right| \geq \frac{|w|}{2} . \tag{31}$$

   The parenthesized summands in the real part are all positive (since $e > s_i$ and $|q_j| \geq 2$). Looking only at the second term, and using $e > s_i \geq 0$, we see thus that the l.h.s. is at least

   $$\left| -\frac{1}{4} \Re e\,(w^2) - \frac{1}{4} \Im m\,(w^2) \cdot \sqrt{-1} \right| = \frac{|w|^2}{4} \geq \frac{|w|}{2} ,$$

   because $|w| \geq 2$, and this establishes (30).

   Now let us analyze what the consequence (29) means. When $e > 0$, it means that we can keep the containment of zeros when we create new or extend existing fractional parts. This again we reduce the problem to the reverse $(2,e)$-torus link.

There remains the case $e=s$. If in the alternating form all p_iq_i are even, then $e=s=0$ (see remark 3.2), which is out of the scope of theorem 3.1. (It follows from theorem 3.2.) If some p_iq_i are odd, then we need a little more explanation, and postpone the treatment of this situation to lemma 5.1.

The case $e<0$ is analogous (and equivalent under mirroring). $\square$

Remark 3.5 When in the assumption of theorem 3.1, the link L is alternating and $e,p_i>0$, then the first sum in (31) disappears, and we see that (31) can fail only if $\Re e\,(w^2)>0$, which may then replace the second alternative in (24):

$$|w|<2 \quad \text{or} \quad \Re e\,(w^2)>0\,. \tag{32}$$

It should be kept in mind that even if we write the condition on z at the end, the reasoning is reverse: we assume this condition violated for z from the beginning and see that this assumption is sufficient to propagate the estimates inductively.

It remains to treat the links with $e=0$. Note that when $e=0$, the link (20) is alternating iff all p_iq_i have the same sign (see remark 3.1 and [LT]). Thus in the context of theorem 3.1, for L to be alternating, all $p_i>0$. This situation is dealt with in theorem 3.2 below. Unlike theorem 3.1, the alternation condition is essential to have the method working. Keep in mind that the orientation (22) will be completed only with the case $e=0$ of lemma 5.1.

Theorem 3.2 Consider the Montesinos link L in (20), where $0<p_i<q_i$, with p_iq_i even, and $e=0$. Let L be oriented so that it conforms to the pattern (22). Then any root z of the Alexander polynomial $\Delta(L)$ satisfies (1).

Proof. Again we only remark how to start the induction. We have to prove instead of (27),

$$[M(1/q_1,\dots,1/q_n)]\geq[M(1/q_2,\dots,1/q_n)] \tag{33}$$

(for q_i even and possibly 0, with the treatment of $1/0=\pm\infty$ as explained below (27)).

The Alexander polynomials of these pretzel links are just monomials in w, and both inequalities are easily established.

With inequality (33), we obtain the containment of zeros using the premise (29). Now the Montesinos link L differs from L' by replacing a fractional part with even fraction expansion $(\dots,a_{l-1})$ by $(\dots,a_{l-1},a_l)$, where the sign of a_l is chosen in (29).

In opposition to the previous proof, we have not justified with (33) in (29) the creation of a new fractional part when modifying L' to L. This is, however, not necessary. By induction, we can reduce the zero location to the pretzel links $P(q_1,\dots,q_n)$. These links are special alternating, and so all their Δ zeros lie on the unit circle. $\square$

4 Closed 3-braid links

Proof of theorem 1.3. If L is a non-split 3-braid alternating link, then by [St4], it is either the closure of alternating 3-braid, or among a tiny family of pretzel links. Latter links are special alternating. For them (see §1 and [St2]) all roots of $\Delta(L)$ have unit norm, and hence satisfy (1).

We assume now that L is closure $\hat\beta$ of an alternating 3-braid

$$\beta=(p_1,q_1,\dots,p_n,q_n)=\sigma_1^{p_1}\sigma_2^{-q_1}\cdot\ldots\cdot\sigma_1^{p_n}\sigma_2^{-q_n}\,, \tag{34}$$

where n,p_i,q_i are positive integers and σ_i are Artin's standard generators. Let call $\sigma_1^{p_i}$ and $\sigma_2^{-q_i}$ *syllables*, and n the *length* of the alternating braid β. We will keep n with this meaning throughout the proof.

Let again z satisfy (14). We will later see that this assumption must be improved, which will lead to the auxiliary constant on the right of (4).

To do the induction step, let us call an alternating braid β' a *reduction* of an alternating braid β, if β' is obtained by reducing one p_i or q_i by 1 in (34), and possibly changing $(\dots,p_i,0,p_{i+1},\dots)\to(\dots,p_i+p_{i+1},\dots)$ (resp. for q_i).

Oppositely, call β' an *extension* of β if β' is obtained by augmenting one p_i or q_i by 1 in (34). (This is slightly more restrictive than the inverse of a reduction, since n is not changed.)

We are led again to prove an analogue of (15). Here we need to work more carefully, and some modifications are necessary. Let us reintroduce the number w from (11), and write

$$y = |w| - 1\,.$$

The analogue of (15) we will prove is

$$[\beta] \geq y \cdot [\beta'] \tag{35}$$

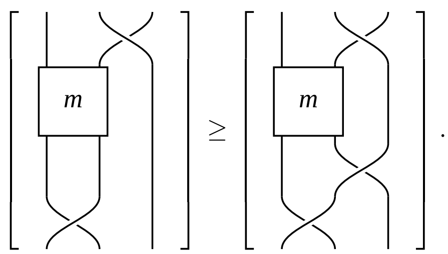
for $[\beta] = |\Delta(\hat{\beta})(z)|$, where β' is a reduction of β and again hat denotes the usual braid closure.

The proof of (35) goes again by induction on the number of crossings (or word length) $\sum_{i=1}^n p_i + q_i$ in (34). For this we use again (14) and the skein relation (10) at a crossing in the syllable reduced in β to obtain β'. This relation now reads as (16), with $a_n = \pm 1$, and the sign chosen as before.

Let us also assume that $n \geq 2$, since the case $n = 1$ is easy.

If the reduction of β does not change n, then induction goes as before. (We previously forgot the extra factor y occurring, since we just needed $y > 1$.) If the reduction of β changes n, i.e., $p_i = 1$ or $q_i = 1$, then we need an extra argument which will lead to the auxiliary constant on the right of (4).

Let us focus on the case $q_i = 1$; the case $p_i = 1$ is analogous. Again by applying the skein relation (10) at the crossing of $q_i = 1$, we see that it is enough to prove



(36)

Here β and β' are depicted only in the differing spot, and a box with label m inside means σ_1^m for $m > 0$. (Note that for $n = 1$ the inequality is false, which forces the assumption $n \geq 2$.)

We use crossing number induction on the combination of (35) and (36).

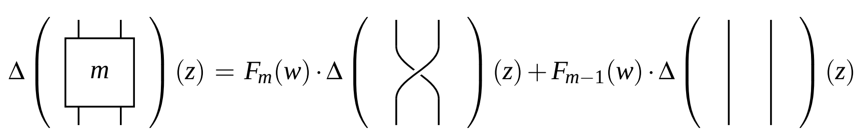
Now we use the Fibonacci polynomials F_n for $n \geq 0$, defined by

$$F_0(z) = 0,\ \ F_1(z) = 1,\ \ \text{and}\ \ F_n(z) = zF_{n-1}(z) + F_{n-2}(z)$$

for $n \geq 2$. It is directly verified by induction that for a complex number w with $|w| > 1$,

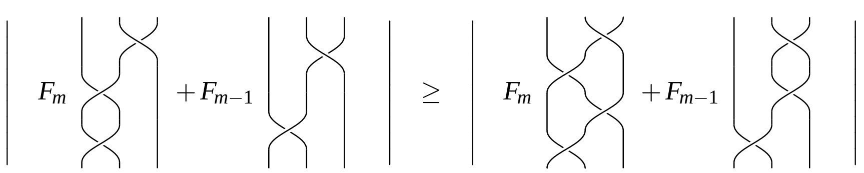
$$\left|\frac{F_{m-1}(w)}{F_m(w)}\right| \leq \frac{1}{|w|-1} = \frac{1}{y}\,. \tag{37}$$

It is an easy consequence of the skein relation (10) for Δ that for $m > 0$,



(38)

Now we expand both hand sides of (36) using (38) at the boxes with m. In the below inequality a braid β stands for $\Delta(\hat{\beta})(z)$, and only the differing parts of the braids are depicted.



(39)

The reason we have done this is that now the second braid on the right simplifies to an alternating braid β', which is an iterated reduction of the first braid β on the left. Similarly is (without simplification) the second braid on the left.

The first braid on the right simplifies by two crossings (at least), but remains non-alternating after that. For it we use (36) as induction assumption. This is justified except if $n=2$ and $q_1=q_2=1$ (since then the assumption $n\geq 2$ no longer holds after the simplification), which we will treat extra later.

We thus use induction assumption to estimate the contributions of the 3 last braids in (39) against the first. For the values $\Delta(z)$ we count by how may crossings the right braids simplify and apply inductively and iteratedly (35). Then we compare the coefficients F_i using (37). We see that then (39) will follow from

$$1-\frac{1}{y^2}\,\geq\,\frac{1}{y^2}+\frac{1}{y^4}\,. \tag{40}$$

This is satisfied if $1/y=1/(|w|-1)$ is not larger than the smallest (real) positive zero t of $-1+2t^2+t^4$, which would lead to a worse bound for y than we stated.

There are some ways to improve (40) by observing where we are too generous in passing from (39). One such place is (37). By iteratedly substituting this estimate into the recursion for F_n, we see that the the r.h.s. can be changed to the continued fractions (defined in (13))

$$(|w|,|w|,\dots,|w|,1)\,\to\,x:=\frac{|w|}{2}-\sqrt{\left(\frac{|w|}{2}\right)^2-1}=\frac{y+1}{2}-\sqrt{\left(\frac{y+1}{2}\right)^2-1}\,. \tag{41}$$

Then (40) modifies to

$$1-\frac{x}{y}\,\geq\,\frac{1}{y^2}+\frac{x}{y^3}\,. \tag{42}$$

This is again regarded, via (41), as an inequality for real y. It is easy to see that x decreases when y increases. Thus the difference of the two hand sides has again a unique positive zero. It is found to be $y\approx 1.45317$, which reflects in the bound given in (4).

In the extra case $n=2$, $q_1=q_2=1$, the first braid in (36) gives a Hopf link, the last one splits (and has zero Alexander polynomial), and the other two give unknots. The polynomial of a Hopf link is $\Delta=\pm w$, and so we see that the inequality (36) holds from (14). $\square$

This proof of theorem 1.3 can be further extended to a positive 3-braid.

Theorem 4.1 Let L be a (non-split) positive braid knot or link of braid index 3 and z be a root of the Alexander polynomial $\Delta(L)$. Then

$$\left|z^{1/2}-z^{-1/2}\right|\,<\,3.274601\,. \tag{43}$$

Proof. By [St5], again a positive braid link of braid index 3 is the closure of a positive 3-braid. In this case '$-q_i$' become 'q_i' in (34).

The sole difficulty is to properly adapt (39). Here we consider the *two* syllables σ_1^m and σ_2^l following the σ_2^{-1}:

$$[\dots\sigma_1^m\sigma_2^l\dots]\,\geq\,[\dots\sigma_2^{-1}\sigma_1^m\sigma_2^l\dots]\,. \tag{44}$$

Again we use (38) to expand this inequality into this time 8 terms for $m,l=0,1$.

In passing to the analogue of (40), the 4 braids on the left-hand side remain positive. The analogue of first braid on the right of (39) simplifies just by 2 crossings, which leads to $1/y$. In two terms on the right the σ_2^{-1} does not cancel. Then we delete the σ_2^{-1} in them using (44) applied as induction assumption (over the crossing number). This leads to the inequality

$$1-\frac{2x}{y}-\frac{x^2}{y^2}\,\geq\,\frac{1}{y}+\frac{x}{y}+\frac{x}{y^2}+\frac{x^2}{y^2}\,,$$

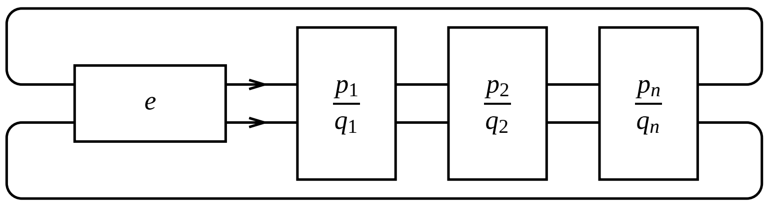
and to (43). (Again, the case $n=1$ in (44) must be excluded, and handled *ad hoc* in (35). The extra care for $n=2$ is no longer needed.) $\square$

Remark 4.1 It is not clear what improvements are possible. However, Hirasawa was quoted in [LMu] considering the example 10_{152}. From his observation it is clear that (1) (and Hoste's conjecture) is not true for positive 3-braid links. Also, there exist (non-split) 3-braid links whose Alexander polynomial vanishes (see [St5]). This hints to why such recursive skein calculations are difficult to work for general 3-braids.

5 Montesinos links II: the parallel case

Definition 5.1 A Montesinos link as in (45) will be called a *parallel Montesinos link*.

Theorem 5.1 Consider the Montesinos link L in (20), where $0 < p_i < q_i$ (with p_iq_i possibly odd), and $e \geq 0$. Let L be oriented (and admit an orientation) so that it conforms to the pattern (45).



(45)

Then any external root z of the Alexander polynomial $\Delta(L)$ satisfies (3). Moreover, $\Re e\, z > -1$.

Proof. In this case we work with the positive but not necessarily even continued fraction expansion of p_i/q_i

$$p_i/q_i = (a_1, \dots, a_n)\,.$$

(Here $a_j = a_{i,j} > 0$ and $a_n > 1$.) The induction step must be refined as follows.

If the twists corresponding to a_n are parallel, we use the skein relation as for (36) to reduce this to

$$[(\dots, a_{n-1}+1)] = [(\dots, a_{n-1}, 1)] \geq [(\dots, a_{n-1}, 0)] = [(\dots)]\,, \tag{46}$$

which works by induction. Here it is understood that we modify only one factional part of the Montesinos link L, and the end of the continued fraction expansion of this factional part is shown only. (The ellipsis stands for the same sequence of positive integers.) Again $[L]$ stands for $|\Delta(L)(z)|$.

If the twists corresponding to a_n are reverse and $a_n = 2p$ is even, we use (16) (replacing a_n by $p = a_n/2$).

If the twists corresponding to a_n are reverse and $a_n = 2p-1$ is odd (with $p > 1$ since $a_n \geq 2$), we use (16) in the form

$$(\dots, a_{n-1}, a_n) = p\cdot w\cdot(\dots, a_{n-1}) + (\dots, a_{n-1}, -1) \,=\, p\cdot w\cdot(\dots, a_{n-1}) + (\dots, a_{n-1}-1)\,, \tag{47}$$

where the links are denoted as in (46), and again L stands for $\Delta(L)(z)$. If $a_{n-1} = 1$, both sequences on the right of (47) simplify by one entry, but the induction assumption still applies.

For all three types of induction step argument, the assumption (14) suffices.

The main difficulty lies again in the induction start. It will be helpful to remember that the Alexander polynomial of the pretzel link $P(q_1, \dots, q_n)$ from (21) is invariant under permutation of the q_i. This can be seen from the explicit formula (51) below, and is more generally owed to the mutation invariance of Δ. With this observation our task lies in showing

$$[P(q_1, \dots, q_n)] \geq [P(q_2, \dots, q_n)]\,, \tag{48}$$

where q_i are odd and positive. Once we have (48), the proof is completed as for theorem 3.2, by reducing the zero location to the special alternating pretzel links $P(q_1, \dots, q_n)$.

We move the proof that the negation of (3) is sufficient for (48) to a separate lemma 6.1, which we work on in the next section. Indeed, some skillful estimation is needed, and our condition (3) is cruder than (1), but it is the optimal bound for $|z|$ that fits with (1).

The second stated estimate was obtained with particular focus on Hoste's conjecture and is given in lemma 6.3. Its proof is considerably longer, and clearly displays the difficulties in seeking further improvement using this method (and moderately manageable calculations). □

We conclude with the remaining case of (reverse) orientation (22) for alternating knots, completing the proof of theorem 3.1 (see the remarks following the theorem, in particular remark 3.2).

Lemma 5.1 Consider the Montesinos link L in (20), where $e \geq 0$ and $0 < p_i < q_i$, with a non-zero number m of p_iq_i odd. Let L be oriented so that it conforms to the pattern (26). Then any root z of the Alexander polynomial $\Delta(L)$ satisfies (32).

For theorem 3.1 we need only the special case $e = 0$, but the other cases are necessary to justify the statement about arbitrarily oriented (alternating Montesinos) links made in remark 3.3. Note also that (23) still holds for the links in the lemma, by (the already established part of) theorem 3.1.

Proof. Observe that the pattern (26) can be achieved up to component orientation if and only if $l := e+m$ is even. Thus the parity of l is implied by (26).

The argument goes as for theorem 5.1. We reduce, again by inductively using (14), the problem to the pretzel links

$$P(\underbrace{1,1,\dots,1}_{l\text{ copies}},q_1,\dots,q_{n-m})\,, \tag{49}$$

with $q_i > 0$ even. In the analogue of (27) we have to prove, either some q_i in (49) is replaced by $\pm\infty$ on the left and by 0 on the right, or an entry 1 in (49) is retained on the left and replaced by 0 on the right. (With this we can reduce the zero location to the reverse $(2,l)$-torus link.) Both situations were previously studied in (28), when $e = l > 0$ and all $q_i > 0$. We finished this case off with the conclusion (32). □

6 The pretzel link estimates

Lemma 6.1 For $t = z$ outside the bound (3) and $q_i > 0$ odd, we have

$$\big|\Delta(P(q_1,\dots,q_{n-1}))(t)\big| \leq \big|\Delta(P(q_1,\dots,q_n))(t)\big|\,. \tag{50}$$

The change of variable from (3) was done with regard to the proof of the lemma. The proof depends heavily on the following explicit formula for the Alexander polynomial of a pretzel link.

Lemma 6.2 Let $u_i = \dfrac{q_i-1}{2}$. Then

$$\Delta(P(q_1,\dots,q_n))(t) = \frac{1}{\sqrt{t}^{\,n-1}(t+1)}\left[\prod_{i=1}^{n}\left(u_i(t-1)+t\right)-\prod_{i=1}^{n}\left(u_i(t-1)-1\right)\right]. \tag{51}$$

Proof. Let us stipulate that a link L stands for $\Delta(L)(t)$. Use iteratedly the identity

$$P(q_1,\dots,q_n) = u_i(t^{1/2}-t^{-1/2})P(q_1,\dots,q_{i-1},q_{i+1},\dots,q_n)+P(q_1,\dots,q_{i-1},1,q_{i+1},\dots,q_n)\,,$$

to get

$$P(q_1,\dots,q_n) = \sum_S (t^{1/2}-t^{-1/2})^{|S|}\left(\prod_{i\in S}u_i\right)P(\underbrace{1,1,\dots,1}_{n-|S|\text{ copies}})\,,$$

where the sum runs over subsets S of $\{1,\dots,n\}$. Now the pretzel link occurring in the sum is just the $(2,k)$ torus link for $k = n-|S|$, whose polynomial is

$$\frac{t^k-(-1)^k}{\sqrt{t}^{\,k-1}(t+1)}\,.$$

The rest follows by grouping terms properly. □

The below term and its notation will be used extensively in the following, so we highlight its definition.

Definition 6.1 Let for $v \in \mathbb{C} \setminus \{0\}$ the *argument* $\arg v \in \mathbb{R}/2\pi$ be the number satisfying $v = e^{\arg v \sqrt{-1}} \cdot |v|$. Unless otherwise noted, the convention we use is that $\arg v \in [0, 2\pi)$.

Proof of lemma 6.1. With lemma 6.2, we have to prove

$$(u_n(t-1)-1)\left|\prod_{i=1}^{n} \frac{u_i(t-1)+t}{u_i(t-1)-1} - 1\right| \geq \sqrt{|t|}\left|\prod_{i=1}^{n-1} \frac{u_i(t-1)+t}{u_i(t-1)-1} - 1\right|.$$

This is equivalent to stating that with $u_i \geq 0$,

$$s = u_n + 1 \geq 1\,, \tag{52}$$

and

$$a_i = \frac{u_i}{u_i+1}\,, \tag{53}$$

the numbers

$$z = \eta_{n-1} = \prod_{i=1}^{n-1} \frac{u_i(t-1)+t}{u_i(t-1)-1} = \prod_{i=1}^{n-1} \frac{t-a_i}{a_i t - 1}$$

satisfy

$$\sqrt{|t|}\,|z-1| \leq \big|z(t-a_n) - (a_n t - 1)\big| \cdot s\,. \tag{54}$$

Our attitude throughout this section will be that we transform (54) into a series of inequalities, each following one of which is at least as strong as the previous one. At some point we will see that under our assumtions some inequality holds, and hence so does (54).

To examine the condition (54) here, we first observe that for $|t| \geq 1$ and $0 \leq a_i < 1$ from (53), we have

$$|t - a_i| \geq |a_i t - 1|\,. \tag{55}$$

Thus

$$|z| = |\eta_{n-1}| \geq 1\,.$$

For (54), it is enough to have

$$\sqrt{|t|}|z-1| \leq \big(|z||t-a| - |at-1|\big) \cdot s$$

and

$$\sqrt{|t|} \leq \frac{|z||t-a| - |at-1|}{|z|+1} \cdot s$$

The minimal value of the r.h.s. over $|z| \geq 1$ is attained for $|z| = 1$, and thus it is sufficient to have

$$\sqrt{|t|} \leq \frac{|t-a| - |at-1|}{2} \cdot s\,. \tag{56}$$

We will next estimate the numerator. Consider figure 2. We assume that $B = t - a$ and $A = at - 1$ for a natural number $s \geq 1$ (with the meaning in (52)) and

$$a = \frac{s-1}{s}\,. \tag{57}$$

The Cosine law in $\triangle OAX$ gives

$$\overline{OA}^2 = \left(\frac{2s-1}{2s}\right)^2 |t-1|^2 + \left(\frac{|t+1|}{2s}\right)^2 - 2\cos(\pi - \alpha)\,\frac{2s-1}{(2s)^2}|t-1||t+1|\,,$$

and in $\triangle OBX$

$$\overline{OB}^2 = \left(\frac{2s-1}{2s}\right)^2 |t-1|^2 + \left(\frac{|t+1|}{2s}\right)^2 - 2\cos(\alpha)\,\frac{2s-1}{(2s)^2}|t-1||t+1|\,.$$

Thus

$$\overline{OB}^2 - \overline{OA}^2 = \frac{2s-1}{2s^2}\big(\cos(\pi-\alpha) - \cos(\alpha)\big)|t-1||t+1|\,. \tag{58}$$

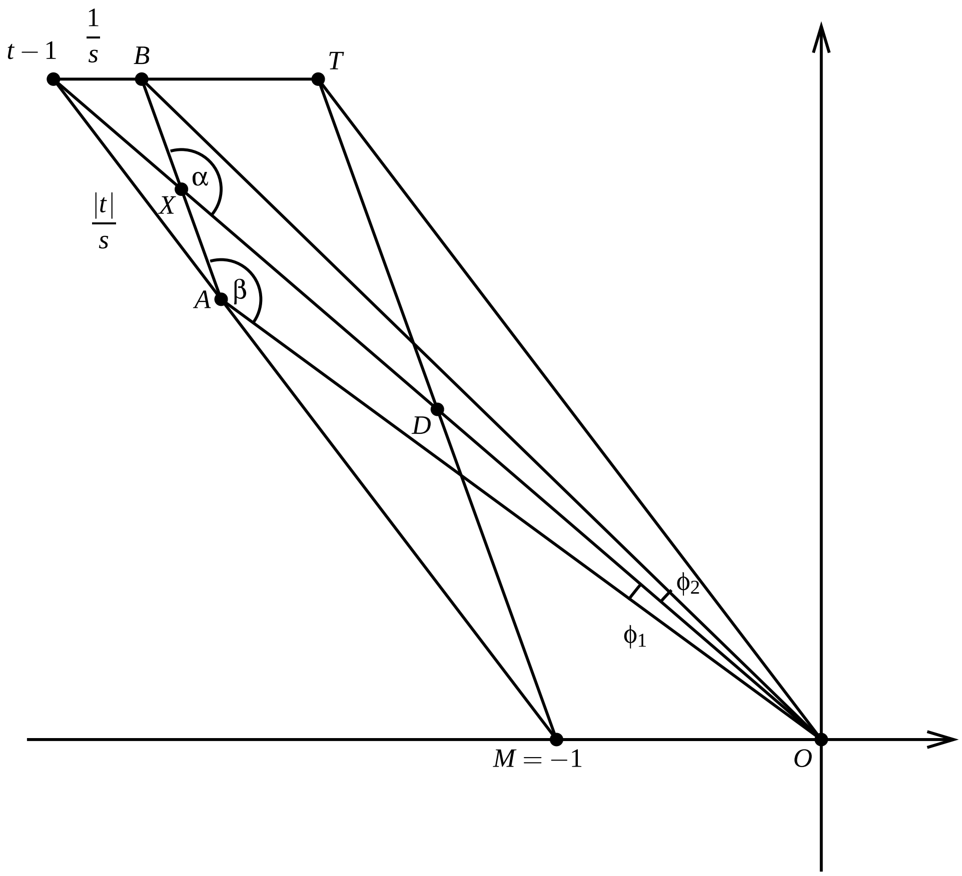


Figure 2: Various notations used in the proof of lemmas 6.1 and 6.3.

Now consider the above picture for $s=1$. The Cosine law in $\triangle ODT$ gives

$$|t|^2 = \left(\frac{|t+1|}{2}\right)^2 + \left(\frac{|t-1|}{2}\right)^2 - 2\cos(\alpha)\,\frac{|t-1||t+1|}{4},$$

and in $\triangle ODM$,

$$1 = \left(\frac{|t+1|}{2}\right)^2 + \left(\frac{|t-1|}{2}\right)^2 - 2\cos(\pi-\alpha)\,\frac{|t-1||t+1|}{4}.$$

Thus

$$-2\left(-|t|^2 + \frac{|t-1|^2}{4} + \frac{|t+1|^2}{4}\right) = -\cos(\alpha)\cdot|t-1||t+1|$$

and

$$2\left(-1 + \frac{|t-1|^2}{4} + \frac{|t+1|^2}{4}\right) = \cos(\pi-\alpha)\cdot|t-1||t+1|.$$

Adding the last two identities and substituting into (58) gives

$$\overline{OB}^2 - \overline{OA}^2 = \frac{2s-1}{s^2}\left(|t|^2-1\right), \tag{59}$$

which is rewritten as

$$|t-a|^2 - |at-1|^2 = \frac{2s-1}{s^2}\left(|t|^2-1\right). \tag{60}$$

Now

$$|t-a| + |at-1| \le (a+1)(|t|+1) = \frac{2s-1}{s}(|t|+1),$$

and so from (60),

$$|t-a|-|at-1| \geq \frac{1}{s}(|t|-1)\,. \tag{61}$$

Combining this with (56) gives the sufficient stronger condition

$$\frac{|t|}{2}-\frac{1}{2} \geq \sqrt{|t|}\,,$$

which is true for $|t| \geq 3+2\sqrt{2}$. □

Remark 6.1 The formula in lemma 6.2 is valid also for $q_i < 0$. One can draw from this also some conclusions on the Δ zeros of certain non-alternating pretzel links. Note that for an alternating pretzel link, the formula combined with (the strict inequality part of) (55) readily proves that a t with $|t| > 1$ (and then also for $|t| < 1$) cannot be a zero, which we knew from the links being special alternating.

This, relatively smooth, estimate is too generous to work for small $|t|$. Under the condition $\Re e\, t \leq -1$ we have made more (laborious) attempts to approach Hoste's conjecture following this strategy. Here is what we could prove:

Lemma 6.3 For an external t with $\Re e\, t \leq -1$, and $q_i > 0$ odd, we have (50).

This lemma requires the most substantial work of the paper. The property that t is external enters again in an essential manner: the serious difficulty we face around $t = -1$ cannot be overcome also in (50). (Compare to the 3-braid calculation, and the comment on it in §9.) An important insight that can be gained from the below proof is that (50) *fails* for certain values of t (with $\Re e\, t \leq -1$) close to -1. For such t a rather different approach might be needed – and it is further motivated by the the length of the following calculation.

We need some preparations. We first have to return to lemma 2.1. According to the notation in its proof, but changing z to t, we will write

$$b = \Re e\, t\,, \quad l = \Im m\, t\,, \text{ and } \quad d = |t| = \sqrt{l^2+b^2}\,. \tag{62}$$

Lemma 6.4 Let $t \in \mathbb{C}$ with $\Re e\, t \leq -1$. Then t is external if and only if b and d in (62) satisfy

$$b \leq \frac{1+d^2}{2}-2d \tag{63}$$

for $|l| \leq \sqrt{8}$ (i.e., $d \leq 3$), and $b \leq -1$ otherwise.

Proof. First we ascertain that on $\left[-\frac{3}{2},-1\right]$ the function f_+ of (5) is increasing, while f_- is decreasing. (We can do this using the derivatives of the squares.) According to (6), this explains that no restriction enters for $|l| \leq f_+(1) = \sqrt{8}$. For the other l, it is enough for (63) to show that equality

$$b = \frac{1+d^2}{2}-2d \tag{64}$$

occurs on the boundary of $\mathcal{D}$. The condition (7) was rewritten as (9). Now, rearranging for b gives

$$b^2-b(1+D)+\frac{1-14D+D^2}{4} = 0\,.$$

Solving this gives two values for b, but since only $b < 0$ is relevant, we obtain the solution (64) (with $d = \sqrt{D}$). □

Lemma 6.5 When $0 \leq x \leq \frac{1}{3}$, we have

$$\arcsin x+\arctan x \leq 2x\,.$$

Proof. Since the statement is clear (as an equality) for $x = 0$, it is enough to check that the inequality of the derivatives is true when $x^2 \le \frac{\sqrt{17}-1}{8} > \frac{1}{9}$. □

Proof of lemma 6.3. We fix some external $t \in \mathbb{C}$ with $\Re e\, t \le -1$ for the entire proof. Assume, to reduce technicalities, w.l.o.g. that

$$\Im m\, t < 0. \tag{65}$$

Then the numbers

$$r_i = \frac{t-a_i}{a_i t - 1} \tag{66}$$

have positive imaginary parts, and

$$0 \le \arg r_i < \frac{\pi}{2}. \tag{67}$$

We will try to understand precisely for which $z \in \mathbb{C}$ the conditions (54) are satisfied. We will see that the products

$$z = \eta_{n-1} = \prod_{i=1}^{n-1} r_i \tag{68}$$

may violate (54) when $a_n = 0$, $\arg \eta_{n-1} \in [0, 2\pi)$ is above π, and $|\eta_{n-1}|$ is close to 1.

To partly remedy this, we first prove a bound on the norm of η_{n-1} in terms of its argument. It turns out that, although r_i can get arbitrary close to 1, there is some lower bound on the ratio $\frac{\log |r_i|}{\arg r_i}$. This means that (for fixed t) products of r_i with argument close to or above π will be bounded in norm above 1. We will prove such a bound in (69).

Consider[2] figure 2, with $B = t - a$ and $A = at - 1$ for a from (57) and a natural number $s \ge 1$.

Lemma 6.6 The angles α and β are not acute.

Proof. Consider first β. It is obvious (and easy to argue exactly) that β increases with s, and so it is sufficient to look at $s = 1$. The claim then follows from $\Re e\, t \le -1$. Since, by looking at $\triangle OAX$ we see that $\alpha = \beta + \phi_1$, it is clear that α is not acute either. □

Lemma 6.7 We have for $z = \eta_{n-1}$ from (68) the inequality

$$|z| \ge 1 + \arg(z) \cdot \tau \tag{69}$$

for

$$\tau := \frac{(|t|-1)|t-1|}{(|t|+1)|t+1|}. \tag{70}$$

Proof. Let $z = r_i$ with $u = u_i$ and $s = u + 1$. We will prove (69) first when $u > 0$; for $u = 0$ we will need later an extra calculation. When we complete the case $u = 0$, we will be done, since when (69) holds for $z = r_i$, it holds for their products, too.

Case 1. Thus assume $u > 0$. Consider r_i in (66). Let $\phi = \arg r_i$. Look again at figure 2. We have

$$\sin \phi_1 \le \frac{\overline{AX}}{\overline{OX}} = \frac{|t+1|}{(2u+1)|t-1|},$$

and because α is not acute by lemma 6.6, also

$$\tan \phi_2 \le \frac{\overline{BX}}{\overline{OX}} = \frac{|t+1|}{(2u+1)|t-1|}.$$

[2] For better visibility, we drew figure 2 when $\Im m\, t > 0$; if necessary, imagine the conjugate (vertical reflection).

Therefore,

$$\phi = \phi_1 + \phi_2 \le \arcsin\left(\frac{|t+1|}{(2u+1)|t-1|}\right) + \arctan\left(\frac{|t+1|}{(2u+1)|t-1|}\right).$$

Now $|t+1| < |t-1|$ when $\Re e\, t \le -1$, and $2u+1 = 2s-1 \ge 3$ (for $s = u+1 > 1$), and thus by lemma 6.5,

$$\arg r_i = \phi \le \frac{2|t+1|}{(2s-1)|t-1|}. \tag{71}$$

Next, we return to (61). From this inequality, we have (for $a = a_i$, since the calculation applies for every index i)

$$|r_i| - 1 = \frac{|t-a| - |at-1|}{|at-1|} \ge \frac{|t|-1}{s|at-1|}, \tag{72}$$

and combining this with (71) gives

$$\frac{|r_i|-1}{\arg r_i} \ge \frac{(|t|-1)|t-1|}{2|t+1|} \cdot \frac{2s-1}{s|at-1|}. \tag{73}$$

Now

$$\frac{2s-1}{s|at-1|} = \frac{2s-1}{|(s-1)t-s|} \ge \frac{2s-1}{(s-1)|t|+s} \ge \frac{2}{|t|+1}. \tag{74}$$

By using (74) in (73), the statement (69) follows for $z = r_i$ when $u > 0$.

Case 2. In the case $u = 0$, we have $r_i = -t$ from (53) and (66), and so the inequality to prove is

$$|t| \ge 1 + \arg(-t)\,\tau.$$

Simplifying, and using $\arg(-t) \le \tan(\arg -t) = \left|\frac{\Im m\, t}{\Re e\, t}\right|$, we see that it is enough to prove

$$\frac{(|t|+1)|t+1|}{|t-1|} \ge \left|\frac{\Im m\, t}{\Re e\, t}\right|.$$

With the designations in (62), this can be written as

$$\frac{(d+1)\sqrt{d^2+2b+1}}{\sqrt{d^2-2b+1}} \ge \frac{\sqrt{d^2-b^2}}{|b|}.$$

Squaring and rearranging terms, and changing b to $-b$, with $1 \le b \le d$, gives

$$(d+1)^2 b^2 \ge (d^2-b^2)\left(1 + \frac{4b}{d^2-2b+1}\right),$$

or

$$d^2b^2 + 2db^2 + 2b^2 - d^2 \ge \frac{4b(d^2-b^2)}{d^2-2b+1}.$$

Use $d^2b^2 - d^2 \ge 0$ on the left, multiply by the denominator on the right, and use therein $d^2+1 \ge 2d$. We obtain the sufficient simpler condition

$$(2db^2+2b^2)(2d-2b) \ge 4bd^2 - 4b^3,$$

or

$$4(1+d)b^2(d-b) \ge 4(d-b)b(d+b),$$

which becomes clear after reducing by common factors. This concludes the proof of lemma 6.7. □

We will in the sequel often write τ for the expression in (70). We will use a part of the following calculation later.

Lemma 6.8 When

$$t = -1 - l\sqrt{-1}, \tag{75}$$

for $l > 0$, then $f(l) = \tau$ from (70) is a strictly increasing function in l.

Proof. The expression in (70) can with (75) be rewritten as

$$\tau = \frac{\sqrt{4+l^2}\left(\sqrt{1+l^2}-1\right)^2}{l^3}. \tag{76}$$

Using this presentation, it is straightforward, but somewhat painful, to verify that the derivative $f'(l)$ is positive for $l > 0$. Another possibly simpler way is to use the substitution

$$d = |t| = \sqrt{l^2+1} \tag{77}$$

in (76) to rewrite (and simplify) τ and τ^2, and obtain

$$\tau = \frac{\sqrt{3+d^2}\left(d-1\right)^2}{\sqrt{d^2-1}^3}, \qquad \text{and} \qquad \tau^2 = \frac{\left(3+d^2\right)\left(d-1\right)}{\left(d+1\right)^3}.$$

The positivity of the derivative (in d for $d > 1$) of τ^2 is likely easier to check. □

The following lemma is an exercise in Euclidean geometry.

Lemma 6.9 For $p_1, p_2 \in \mathbb{C}$ and $m > 1$, the set

$$\Big\{ x \in \mathbb{C} : |x-p_1| = m|x-p_2| \Big\}$$

is a circle with

$$\text{center } \frac{m^2}{m^2-1}p_2 - \frac{1}{m^2-1}p_1 \quad \text{and radius } \frac{m}{m^2-1}|p_2-p_1|. \tag{78}$$ □

Now, with $z \in \mathbb{C}$ satisfying (69), we return to (54).

Case 1. $s = 1$. We have to examine

$$\sqrt{|t|} \le \frac{|-tz-1|}{|z-1|}. \tag{79}$$

Keep in mind that with (65) we have $\arg(-t) \in \left(0, \frac{\pi}{2}\right)$.

Case 1.1. $\arg z < 2\pi - \arg(-tz)$. Let $\gamma = \arg z$ and $k = |z|$. By squaring (79), using Cosine law and the assumption of the case, we see that it is sufficient to have

$$\frac{1+k^2|t|^2 - 2\cos(\gamma)\cdot k|t|}{1+k^2-2\cos(\gamma)\cdot k} \ge |t|.$$

This is equivalent to

$$|t|^2 + 2\cos(\gamma)k\frac{|t|^2-|t|}{1+k^2-2\cos(\gamma)k} - \frac{|t|^2-1}{1+k^2-2\cos(\gamma)k} \ge |t|,$$

and further

$$|t|(|t|-1)\left(1+\frac{2\cos(\gamma)k}{1+k^2-2\cos(\gamma)k}\right) \ge \frac{|t|^2-1}{1+k^2-2\cos(\gamma)k}.$$

This in turn simplifies to

$$1+k^2 \ge \frac{|t|+1}{|t|},$$

which is certainly true, since with $k = |z|$ and $|t|$ both at least 1, the left-hand side is at least 2, while the right-hand side is at most 2.

Case 1.2.
$$\pi \ge \arg z \ge 2\pi - \arg(-tz). \tag{80}$$

In particular, since $\arg(-t) < \dfrac{\pi}{2}$, we must have $\arg z > \dfrac{3\pi}{4}$. We consider again (79). It is easy to see that for z satisfying (80) and having a fixed norm, $|z-1|$ is maximal and $|-tz-1|$ is minimal when z is real negative. Thus assume for $k=|z|\geq 1$ from (69) that

$$z=-k \leq -1\,, \tag{81}$$

so that the condition (79) to test becomes

$$|kt-1|^2 \geq |t|(1+k)^2\,,$$

or with (62),

$$(kb-1)^2+k^2l^2 = k^2d^2-2kb+1 \geq d(1+k)^2\,. \tag{82}$$

When a z as in (81) is given, one sees next that for t of fixed norm $d=|t|$ (*cf.* (62)), $|-tz-1|$ decreases when $b=\Re e\, t<0$ increases.

An upper bound for b is given by the condition (63) for external t. For $d\leq 3$, we can thus work just with those t for which (63) is an equality, as in (64). Using that with (64), we have

$$2(b+d)=(d-1)^2\,, \tag{83}$$

the test (82) can then be rewritten as

$$k^2(d^2-d)-k(d-1)^2+(1-d) \geq 0\,.$$

We have $d\geq 1$, and the case $d=1$ gives $t=-1$, which is not interesting. Thus, dividing by $d-1$, and using (81), we see easily the necessary inequality.

We have to test (82) also for $b=-1$ and $l\leq -\sqrt{8}$, i.e., $d\geq 3$. Substituting $b=-1$ in (82), dividing by $d-1>0$ and simplifying, we obtain

$$k^2d-2k-1 \geq 0\,,$$

which is true for $d\geq 3$ and $k\geq 1$. Thus we are done in this case.

Case 1.3. $\pi<\arg z$. We apply the formula (78) to the circle

$$C := \left\{ x\in\mathbb{C} : \sqrt{|t|}\,|z-x| \leq |y-x| \right\}$$

for $y=-tz$. We are interested whether $1\in C$. The center of C is

$$\frac{|t|}{|t|-1}z-\frac{1}{|t|-1}y\,,$$

and its radius is

$$\frac{\sqrt{|t|}}{|t|-1}\cdot|y-z| = \frac{\sqrt{|t|}}{|t|-1}\cdot|t+1||z|\,.$$

We have $1\in C$ when $|\text{center}|\leq \text{radius}-1$, which can be simplified to

$$\big||t|z-y\big| \leq \sqrt{|t|}\,|t+1||z|-(|t|-1)\,,$$

or to

$$\big|t+|t|\big| \leq \sqrt{|t|}\,|t+1|-\frac{|t|-1}{|z|}\,. \tag{84}$$

Here we notice that for given $|t|$, the value

$$\big||t|+t\big|-\sqrt{|t|}\,|t+1| \tag{85}$$

is increasing with $\Re e\, t\leq -1$. To see this, express (85) using b, l and d as in (62):

$$\begin{aligned}\big||t|+t\big|-\sqrt{|t|}|t+1| &= \sqrt{(b+d)^2+l^2}-\sqrt{d}\cdot\sqrt{(b+1)^2+l^2}\\ &= \sqrt{2d^2+2bd}-\sqrt{d}\cdot\sqrt{d^2+2b+1}\,.\end{aligned} \tag{86}$$

Regard this for fixed d as a function in $b \in [-d, -1]$. The non-negativity of its derivative becomes, after multiplying by the denominators,

$$d\sqrt{d^2+2b+1} - \sqrt{d} \cdot \sqrt{2d^2+2bd} \geq 0\,,$$

or after squaring

$$d^2\left(d^2+2b+1\right) \geq d^2\,(2d+2b)\,,$$

which is seen to hold by simplifying.

With (63) in mind, for $d \leq 3$ we find it thus again enough to evaluate (85) for $b = \Re e\, t$ as in (64). Then we have in particular

$$|t+1| = \sqrt{2}(d-1) \quad \text{and} \quad |t-1| = 2\sqrt{d}\,. \tag{87}$$

Using the first equality for the second term in (86), and (83) for the first, we see that the condition (84) becomes

$$\left(1-\sqrt{2}\right)\sqrt{d}(d-1) \leq -\frac{d-1}{|z|}\,.$$

Dividing by $d-1$ for $d > 1$ (since $d = 1$ is again trivial) and rearranging translates this into

$$|z| \geq \frac{1}{\left(\sqrt{2}-1\right)\sqrt{d}}\,. \tag{88}$$

With (69) and the assumption of the case, the condition (88) will be satisfied when

$$1+\pi\tau \geq \frac{1}{\left(\sqrt{2}-1\right)\sqrt{d}}\,. \tag{89}$$

Now with (70) and (87) we have

$$\tau = \frac{d-1}{d+1}\cdot\frac{|t-1|}{|t+1|} = \frac{\sqrt{2d}}{d+1}\,.$$

Substituting this into (89) and multiplying by all denominators, we obtain

$$\left(\sqrt{2}-1\right)\sqrt{d}(d+1)+\pi\sqrt{2}\left(\sqrt{2}-1\right)d-d-1 \geq 0\,. \tag{90}$$

The first summand is increasing in d for $d > 1$. So is the rest of the l.h.s., which one sees from the linear coefficient $\pi\sqrt{2}\left(\sqrt{2}-1\right)-1 > 0$. Thus it is enough to test (90) for $d = 1$, which is easy.

Next we have to test (84) for $b = -1$ and $d \geq 3$. Substituting $b = -1$ in (86), we see that (84) becomes

$$\sqrt{2d}\sqrt{d-1} - \sqrt{d}\cdot\sqrt{d^2-1} \leq -\frac{d-1}{|z|}\,. \tag{91}$$

Now by lemma 6.8, we have (with f being the function therein) $\tau \geq f\left(\sqrt{8}\right) = \sqrt{\frac{3}{8}}$, so that by (69)

$$|z| \geq 1+\sqrt{\frac{3}{8}}\pi > \frac{5}{2}\,.$$

Thus after dividing by $\sqrt{d-1}$ and rearranging, we see that (91) would be implied by

$$\sqrt{2d}+\frac{2\sqrt{d-1}}{5} \leq \sqrt{d}\sqrt{d+1}\,,$$

which is seen to hold for $d \geq 3$ from $\sqrt{d-1} \leq \sqrt{d}$ and $\sqrt{2}+2/5 \leq 2 \leq \sqrt{d+1}$. Herewith we can conclude the case.

Case 2. $s \geq 2$. Here it turns out unnecessary to assume t is external, and thus (by similar monotonicity arguments) we will substitute only $b = -1$ (for all $d \geq 1$) into certain inequalities we want to show, instead of bothering with (64).

We distinguish three subcases similar to case 1. However, in the conditions there the role of $-t$ is taken by the more general

$$r = \frac{t-a}{at-1} \tag{92}$$

with a as in (57), now satisfying $a \geq \frac{1}{2}$.

Case 2.1. $2\pi - \arg rz \geq \arg z$. Then it is easy to see that (for $\Re e\, t \leq -1$)

$$|at-1| \geq \left|\frac{1}{2}t-1\right| > \sqrt{|t|}. \tag{93}$$

For the right inequality,

$$\left|\frac{1}{2}t-1\right| > \sqrt{|t|}, \tag{94}$$

note that among t with $\Re e\, t \leq -1$ and given norm, $\left|\frac{1}{2}t-1\right|$ is smallest when $\Re e\, t = -1$, so it is enough to check (94) only for such t. This is easily done by squaring twice both hand sides.

Now (54) can be written

$$\frac{\sqrt{|t|}}{s|at-1|} \leq \frac{|rz-1|}{|z-1|}. \tag{95}$$

We know from (94) that the l.h.s. of (95) is smaller than 1, while from (72) we have $|r| \geq 1$. Under the assumption of this case the conclusion follows easily. (Formally, one can repeat the calculation in case 1.1, replacing $|t|$ by $|r|$.)

Case 2.2. $2\pi - \arg rz \leq \arg r$ and $\arg r \leq \pi$. Looking at (95) and using (93), we see that it is enough to prove

$$|rz-1| \geq |z-1|. \tag{96}$$

Now, for $\arg z$ as chosen, multiplying the complex numbers z and rz by $e^{\sqrt{-1}(\pi-\arg z)}$ will increase the right-hand side, and decrease the left-hand side. Thus it is enough to test (96) when z is real negative.

Then we see that (96) will in particular follow if $\Re e\, r \geq 1$. Looking at the diagram in figure 2, we see that this is equivalent to the angle β not being acute, a property proved in lemma 6.6. This argument finishes case 2.2.

Case 2.3. $\arg r \geq \pi$. Thus we have (69). Now consider

$$C := \left\{ x \in \mathbb{C} : \frac{s|at-1|}{\sqrt{|t|}}|y-x| \geq |z-x| \right\}$$

with $y = r \cdot z$ and a from (57). By (78) and (93), this set C is the the exterior of a circle. The center of the circle is

$$\frac{s^2|at-1|^2}{s^2|at-1|^2-|t|}y - \frac{|t|}{s^2|at-1|^2-|t|}z. \tag{97}$$

Using

$$\left|\frac{t-a}{at-1}-1\right| = \left|\frac{t-a-at+1}{at-1}\right| = \left|\frac{(t+1)(1-a)}{at-1}\right| = \frac{|t+1|}{s|at-1|},$$

the radius evaluates to

$$\frac{s|at-1|\sqrt{|t|}}{s^2|at-1|^2-|t|}|y-z| = \frac{s|at-1|\sqrt{|t|}}{s^2|at-1|^2-|t|}|z|\left|\frac{t-a}{at-1}-1\right| = \frac{|t+1||z|\sqrt{|t|}}{s^2|at-1|^2-|t|}. \tag{98}$$

We need $1 \in C$. This is implied by $|\text{center}| \geq \text{radius}+1$. This inequality can be rewritten and strengthened by clearing denominators in (97) and (98), using (92) and decreasing the left-hand side by the triangular inequality:

$$\left(s^2|at-1||t-a|-|t|\right)|z| \geq |t+1||z|\sqrt{|t|}+s^2|at-1|^2-|t|.$$

This can be further reorganized as

$$|t|(|z|-1) \le s^2|at-1|\Big(|z||t-a|-|at-1|\Big)-\sqrt{|t|}|t+1||z|\,.$$

Since, by (55) or (72), $|r|\ge 1$ in (92), it is sufficient that

$$|t|(|z|-1) \le s^2|at-1|(|z|-1)|t-a|-\sqrt{|t|}|t+1||z|\,. \tag{99}$$

We simplify (99) to

$$|t| \le s^2|at-1||t-a|-\frac{\sqrt{|t|}\,|z||t+1|}{|z|-1}\,.$$

Then we use (69). So it is enough to have

$$|t| \le s^2|at-1||t-a|-\left(1+\frac{1}{\pi\tau}\right)\sqrt{|t|}\,|t+1|\,.$$

Furthermore, we need to look only at $s=2$ (and $a=1/2$), which minimizes $|at-1|$ and $|t-a|$:

$$|t| \le |t-2||2t-1|-\left(1+\frac{1}{\pi\tau}\right)\sqrt{|t|}\,|t+1|\,. \tag{100}$$

Here we see again that it is enough to test t in (75), since among t with $\Re e\, t\le -1$ and given $|t|$, the one with $\Re e\ t=-1$ makes $|2t-1|$ and $|t-2|$ smallest. It also makes $|t-1|/|t+1|$ smallest, and thus minimizes τ. Setting t as in (75) and encountering the expression in (76), let us simplify the calculation by using that

$$\tau = \frac{\sqrt{4+l^2}\big(\sqrt{1+l^2}-1\big)^2}{l^3} \ge \frac{l}{2\sqrt{4+l^2}}\,.$$

Then we have

$$\frac{|t+1|}{\tau} \le 2\sqrt{l^2+4}\,. \tag{101}$$

By expanding the parenthesis in (100), applying (101) on the last term, and then making the substitution (77), we have

$$d \le \sqrt{d^2+8}\cdot\sqrt{4d^2+5}-\sqrt{d^2-1}\cdot\sqrt{d}-\frac{2}{\pi}\sqrt{d}\cdot\sqrt{d^2+3}\,. \tag{102}$$

It is enough to prove that (102) holds for all $d\ge 1$. By using $d\le d\sqrt{d}$ on the left and $\sqrt{d^2-1}\le d$ and $\sqrt{4d^2+5}\ge 2d\ge\sqrt{d^2+3}$ on the right, this can be simplified (and strengthened) to

$$\left(1+\frac{2}{\pi}\right)\sqrt{d} \le \sqrt{d^2+8}\,,$$

which now can be easily checked (after squaring) for $d\ge 1$.

With this the case distinction, and the proof of lemma 6.3, is complete. $\square$

7 Extensions to the skein polynomial

An advantage of using the skein relation (10), in contrast to the Seifert matrix, is that, to a limited extent, it offers some information beyond the Alexander polynomial. We have obtained the following about the skein (HOMFLY-PT) polynomial P [LMi], in analogy to theorem 1.2. We use the convention of P with the variables v,w, the unknot having polynomial $P=1$, and the skein relation

$$v^{-1}P\Big(\quad\Big)-vP\Big(\quad\Big) = wP\Big(\quad\Big)\,.$$

(This is the form used in [Mo], except that we changed z to w to avoid confusion with the variable z in (1). This way, w correctly reflects the role it plays in (11).)

Theorem 7.1 Let L be a 2-bridge link, and $(v,w)\in\mathbb{C}^2$, $v,w\neq 0$ be a root of $P_L(v,w)$ with $|v|\neq 1$. Then

$$|w| < \max_{k>0} \frac{\big(|v|^{2k}+1\big)\,|v-v^{-1}|}{|1-v^{2k}|}\,. \tag{103}$$

(We leave it to the reader to see why we can write 'max' above, whereas in general we would have to write 'sup', and that this maximum is finite.)

Proof. This is mainly an adaptation of the proof of theorem 1.2. We fix $v\in\mathbb{C}$ with $v\neq 0$ and $|v|\neq 1$, and study $P_L(v,w)$ as a polynomial in w using a recursive norm estimate.

Assume again w violates (103). For $k=\pm 1$ this condition simplifies to

$$|w| < |v| + \frac{1}{|v|}\,. \tag{104}$$

When (104) fails, we have for the polynomial of the two component unlink U,

$$\big|\,P_U(v,z)\,\big| = \left|\,\frac{v-v^{-1}}{w}\,\right| \leq 1\,.$$

This replaces the comparison between the unknot and two component unlink polynomial in the proof of theorem 1.2.

For the induction step, we have to modify (16). It depends now slightly on the sign of a_n. Let $P_n := P(L(2a_1,\dots,2a_n))$. For $a_n>0$ the analogue of (16) reads

$$P_n(v,w) = v\cdot w\cdot \frac{1-v^{2a_n}}{1-v^2}P_{n-1}(v,w) + v^{2a_n}P_{n-2}(v,w)\,. \tag{105}$$

For $a_n<0$, replace in the above formula a_n by $-a_n$ and v by $-v^{-1}$.

We want to conclude from (105) inductively over n that

$$|P_n| \geq |P_{n-1}|\,. \tag{106}$$

Again, taking norms, using $|P_{n-1}| \geq |P_{n-2}|$, and dividing by $|P_{n-1}|$, we see that (106) will follow from

$$\left|\,v\cdot w\cdot\frac{1-v^{2a_n}}{1-v^2}\,\right| - \big|\,v^{2a_n}\,\big| \geq 1\,. \tag{107}$$

The condition (103) is then made so that its violation to ascertain this inequality. Therein k takes the role of a_n in (107). Since the fractional expression on the right of (103) is the same for $\pm k$, maximizing only over $k>0$ is enough. $\square$

Remark 7.1 One can restrict the maximum in (103) further to those k being a divisor of the leading coefficient $\mu = \max\operatorname{cf}\nabla$ of the Alexander-Conway polynomial $\nabla(w) = P(1,w)$ (*cf.* below (11)). This is because in the presentation (13), the coefficient expresses as $\mu = \max\operatorname{cf}\nabla = \pm\prod_{i=1}^{n} a_i$. In such a way, we can make sense of (103) also when $|v|=1$, except for the 2μ-th roots of unity. (Note that when $v=\pm 1$, we have the Alexander-Conway polynomial, for which we saw before the recursion working in a slightly different way.) In particular, for a fibered 2-bridge link ($\max\operatorname{cf}\nabla = \pm 1$) one needs to take only $k=\pm 1$, and obtains (104) instead of (103) (which is then valid except if $v=\pm 1$).

Used as a practical test, the various Alexander polynomial conditions we obtained would apply only to more complicated examples. In contrast, our skein polynomial criterion for a 2-bridge link has some significance also among relatively simple knots, as we show in the following.

Example 7.1 We tested the condition (103) for several values of v on the alternating Rolfsen [Ro, appendix] knots. It can identify as non-rational at least the following ones: 8_{15}, 9_{35}, 9_{38}, 9_{39}, 9_{41}, 10_{49}, 10_{53}, 10_{63}, 10_{69}, 10_{78}, 10_{96}, 10_{97}, 10_{101}, and 10_{120}. The improvement explained in remark 7.1 rules further out from being rational 9_{25}, 10_{55}, 10_{58}, 10_{66}, and 10_{80}.

The complexity of (103), compared to (1), already suggests that statements about the skein polynomial become increasingly technical. Neither turn they out very practically useful, with the Morton-Williams-Franks (MWF) inequality at hand [Mo]. We thus omit the discussion for space reasons. Suffice it to say that experiments with knots in the tables of [HT] have not turned up examples where the new evaluation estimates we obtained outperform the MWF 3-braid test.

For similar (and even more compelling) reasons, I have not attempted either a (skein polynomial) refinement of the Montesinos link calculation.

Moreover, our recursive skein aproach is diffcult to use for another important special case of P, the Jones polynomial V. This expresses as $V(z)=P(z,w)$, with w related to z as in (11). (Note, e.g., that under this relation, the restriction (103) is always satisfied, and so theorem 7.1 is useless.) There is indeed a denseness result in [JZDT] about Jones polynomial roots of alternating pretzel links. Thus caution is needed among what classes of links the question about location of roots makes sense.

8 Log-concavity and zeros of the Alexander polynomial

In this (mainly expository) section we put Hoste's conjecture in relation to other properties of the Alexander polynomial.

Let here the Alexander polynomial $\Delta(t)\in\mathbb{Z}[t,t^{-1}]$ of knots be regarded in the formal variable t (as opposed to z, which we use to indicate a concrete complex number). Below $\Delta(t)$ is normalized so that

$$\Delta(1/t)=\Delta(t) \quad \text{and} \quad \Delta(1)=1\,. \tag{108}$$

It is very well known what Alexander polynomials occur for an arbitrary knot: the conditions (108) precisely characterize such polynomials. (We will call the first property *reciprocity*.) Regarding the (much harder) question about those polynomials occurring for an alternating knot, several results and conjectures have been treated over a period of time.

Let $\max\deg\Delta$ be the *maximal degree* of Δ (maximal power of t with non-zero coefficient). Because of reciprocity the minimal degree of Δ, defined analogously, is $\min\deg\Delta=-\max\deg\Delta$. Crowell and Murasugi [Cw, Mu] proved that when K is alternating, the polynomial $\Delta_K(t)$ is *alternating*, i.e. all coefficients of $\Delta_K(-t)$ are positive or all are negative. We call such polynomials also *positive* resp. *negative*. This is to mean in particular that all coefficients between $\min\deg\Delta$ and $\max\deg\Delta$ are non-zero. To avoid ambiguities, let us assume that all Alexander polynomials we treat from now on have this property (which is not automatic).

The work of Crowell-Murasugi shows also that for K alternating, the maximal degree of Δ gives the *genus* $g(K)$ (and canonical genus $g_c(K)$) of K:

$$g_c(K)=g(K)=\max\deg\Delta_K\,. \tag{109}$$

We are here in particular motivated by the log-concavity conjecture made in [St2]. Let $[X]_k$ for $k\in\mathbb{Z}$ be the coefficient of t^k in a Laurent polynomial $X\in\mathbb{Z}[t^{\pm1}]$. Call a polynomial X to be *log-concave*, if $[X]_k$ are log-concave, i.e.

$$[X]_k^2\geq[X]_{k+1}[X]_{k-1}\geq 0 \tag{110}$$

for all $k\in\mathbb{Z}$. (We assume the non-negativity of these expressions for technical reasons: we want to regard only positive and alternating polynomials as log-concave.)

Conjecture 8.1 (log-concavity conjecture) If K is an alternating knot, then $\Delta_K(t)$ is log-concave.

The log-concavity conjecture is a natural strengthening of a much older conjecture formulated by Fox, which is now referred to also as 'Trapezoidal' conjecture.

Conjecture 8.2 (Fox's Trapezoidal conjecture) If K is an alternating knot, then there is a number $0\leq n\leq g(K)$ such that for $\Delta_{[k]}:=\big|[\Delta_K]_k\big|$ we have

$$\begin{array}{ll}\Delta_{[k]}=\Delta_{[k-1]} & \text{for } 0<|k|\leq n,\\ \Delta_{[k]}<\Delta_{[k-1]} & \text{for } n<|k|\leq g(K)\,.\end{array} \tag{111}$$

We call polynomials of this form *trapezoidal*. Since log-concave polynomials are trapezoidal, the log-concavity conjecture implies Fox's conjecture.

The Trapezoidal conjecture has received some attention in the literature. It was verified for rational (2-bridge) knots [Ha] (see also [Bu]) and later for a larger class of alternating algebraic knots [Mu2]. More recently, some linear inequalities on the coefficients of Δ coming from Ozsváth-Szabó's knot Floer homology [OS] have been seen to imply the conjecture for $g = 2$, and also to settle (for general g) in (111) the case $|k| = g(K)$. Jong [Jn, Jn2] has proved independently the Trapezoidal conjecture up to genus 2 using the generator description in [St9], and observed that for genus 2 the log-concavity of Δ easily follows from trapezoidality. Then, in [St] the proof of log-concavity was extended to genus at most 4 using similar techniques. (We use our generator classification and an appropriate calculation to prove there also Hoste's conjecture up to genus 4.)

One can find individual examples showing that trapezoidality (or log-concavity) of an Alexander polynomial does not imply Hoste's condition, however, there is a more meaningful generalization of this intuition. It turns out that log-concavity poses (essentially) no constraints on the location of zeros of Alexander polynomials, in the following sense:

Theorem 8.1 The zeros of log-concave alternating Alexander knot polynomials are dense in $\mathbb{C}$.

This implies that Hoste's conjecture is (almost) independent from Fox's or the log-concavity conjecture. There are minor relations, e.g., an alternating polynomial cannot have a real negative zero. There are also conditions arising when restricting the degree of the polynomial. For example, when $\max\deg\Delta = 2$, Murasugi has verified that every alternating Alexander knot polynomial satisfies Hoste's condition. (More precisely, every root z has $\Re e\, z \in (0,3)$, or $|z| = 1$, or $z \in \mathbb{R}_+$ is real.)

Compare theorem 8.1 also to the result in [JZDT] regarding the Jones polynomial roots.

The proof of theorem 8.1 is elementary, and (with length concerns in mind) we decided to move it out, trying to focus on the preceding more substatial details.

9 Problems

It would be interesting to what extent our skein method can be used with regard to Hoste's conjecture, and link polynomial properties more broadly. Some open problems seem somehow related.

- For Montesinos links, a more general problem, formulated in [St3], is whether there exists (and if so, to find) a condition on the Alexander polynomial of an arbitrary Montesinos link. Our method leads to such a condition in many cases in §3 and §5, but the non-alternating links excluded in theorems 3.2 and 5.1 still remain difficult to treat. This is not surprising, since, as we cautioned, that there are (non-alternating) Montesinos links with vanishing polynomial. Thus whatever zero location technique is used, it must naturally bypass such examples.

- Another related problem is to prove that there exist knots without matched diagrams [Ki, problem 1.60, p.42]. Our proof of theorem 1.2 reflects somehow the existence of such diagrams for rational knots (and links), but they have very special properties which make a recursive calculation convenient. (The knots in example 3.1 have very similar matched diagrams.)

- Finally, Hoste's conjecture remains open also for 3-braids. A computer test of alternating 3-braids of even length up to 18 determined the maximum of the left hand-side of (1) to be ≈ 1.94, which suggests that (1) may still be true. However, (36) is not always true in (and close to) $z = -1$. More generally, no refinement of the argument is visible to overcome the barrier when approaching $|w| = 2$. See also remark 4.1.

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