New knot and link invariants

Thomas Fiedler

Laboratoire E. Picard, Université P. Sabatier,

118, Route de Narbonne, 31062 Toulouse Cedex, France,

e-mail: `fiedler@picard.ups-tlse.fr`

Alexander Stoimenow

Ludwig-Maximilians University Munich, Mathematics

Institute, Theresienstraße 39, 80333 München, Germany,

e-mail: `stoimeno@informatik.hu-berlin.de`,

WWW: `http://www.informatik.hu-berlin.de/~stoimeno`

Current version: February 1, 2002   First version: December 9, 1996

This is a minorly updated version compared to the printed one.

Abstract

We study the new formulas of the first author for the degree-3-Vassiliev invariants for knots in the 3-sphere and solid torus and present some results obtained by them. We show that a knot with Jones polynomial consisting of exactly two monomials must have at least 20 crossings.

Keywords: Vassiliev invariants, Gauß sums, mutation, positive knots, Jones polynomial

AMS subject classification: 57M25

Contents

1   Introduction

Recently, the first author [Fi] introduced some new knot invariants by generalizing the approach of Polyak and Viro [PV] of Gauß sums. Conversely to the Polyak-Viro invariants, which are known to be of finite (Vassiliev) type [BL, BN, BN2, BS, St, Va, Vo], and therefore, at least up to degree ≤ 12 [K] orientation non-sensitive, the main hope of this generalization was to find some new invariants that *do* distinguish knot and link orientation in S^3.

In this mainly expository note we give a definition of these invariants and examples using the computer calculations of the program [St2]. We announce some results, supported by these examples, about these invariants applied to link mutants in S^3. We present some applications of the Gauß sum formulas for the degree-3-Vassiliev invariant to crossing number inequalities and positive knots.

A mathematically more detailed description of the first author's approach, including invariance proofs, can be found in his forthcoming monography [Fi2], especially from the (more interesting) point of view of these invariants in the solid torus.

2   Gauß diagrams and Gauß sums

Consider a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ (S^1 and $\mathbb{R}^3$ oriented). Decompose $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ so that the projection (henceforth called *knot diagram*) of K into $\mathbb{R}^2$ is generic. To this projection we can assign a *Gauß diagram* (GD), a circle with oriented chords, by connecting points in S^1 mapped to a crossing and orienting the chord from the preimage of the undercrossing to the preimage of the overcrossing. By forgetting the orientation of each chord in a GD we obtain its underlying *chord diagram* (CD) [BN].

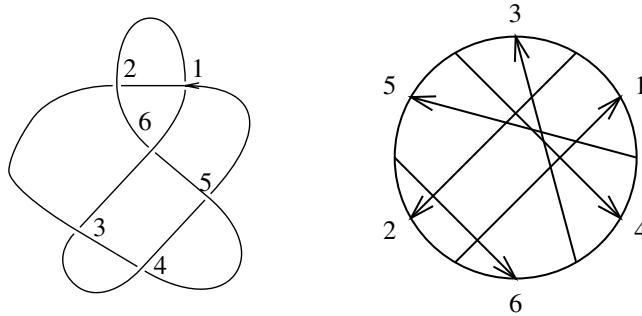Figure 1 shows the knot 6_2 in its standard projection and the corresponding Gauß diagram.

Figure 1: The knot 6_2 and its Gauß diagram.

A Gauß sum of degree k is a term assigned to a knot diagram, which is of the following form

$$\sum_{\substack{\text{ordered choices of } k \text{ crossings of the} \\ \text{knot diagram, satisfying certain condi-} \\ \text{tions}}} \text{function} \big(\text{ data, assigned to the crossings} \big) .$$

Each summand we will call *weight* and the function *weight function* (not to be confused with those in [BN]!).

Sums of this type ("small state sums") have been studied for the 1st time for degree ≤ 2 by the first author [Fi3, Fi4] and later by Polyak-Viro [PV]. As data of the crossings they considered the writhe (which is $+1$ on a positive and -1 on a negative crossing) and as weight functions the elementary symmetric polynomials. The aim was to find linear combinations of such terms, invariant under the 3 Reidemeister moves, and therefore giving formulas for knot invariants. All such invariants turned out to be of finite type. This was stated by the authors and proved by Oestlund.
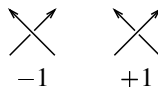
Figure 2: The writhe

Theorem 2.1 (Viro-Polyak-Oestlund) *All invariant Gauß sums of degree k with symmetric polynomials in the writhes of the crossings are Vassiliev invariants (VI) of degree $\leq k$.*

Definition 2.1 *The* winding index *of a plane curve $C \subset \mathbb{R}^2 = \mathbb{C}$ around a point $p \notin C$ is*

$$w(C,p) := \frac{1}{2\pi i} \oint_C \frac{1}{z-p}\, dz.$$

Pictorially it measures how many times the curve "walks" around p, counting reverse walk negatively.

Definition 2.2 *The* Whitney index $n(C)$ *of a plane curve C is the degree of the map*

$$\frac{C'}{||C'||} : S^1 \longrightarrow S^1.$$

The Whitney index of a knot diagram is the Whitney index of its underlying plane curve.

Definition 2.3 *The* writhe $w(D)$ *of a knot diagram D is the sum of the writhes of all crossings (see figure 2).*

Example 2.1 The standard projection of 6_2 has Whitney index 1 and writhe -2.

In the following we will decisively use the following fact.

Lemma 2.1 *Two plane diagrams of the same knot (i.e., ambient isotopic) with the same n and w are regular isotopic.*

In other words, the ambient isotopy classes of knots decompose into regular isotopy classes parametrized by two numbers n and w. The only restriction to n and w is that their sum is odd.
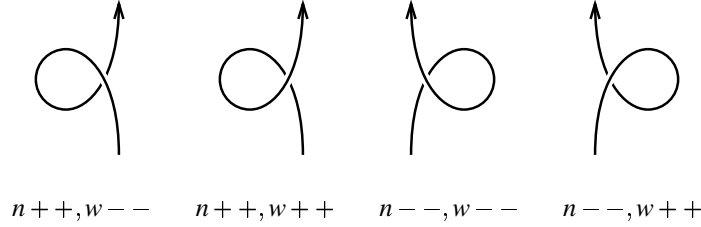
There are four ways to modify n and w by Reidemeister I moves, as depicted on figure 3 (where '$n \pm \pm$' denotes the operation '$n \to n \pm 1$' and the result of the move on a strand without crossings is displayed).

In the following we will look for regular isotopy invariants of the knot diagram.

Definition 2.4 *A* smoothing *of a crossing is the procedure*
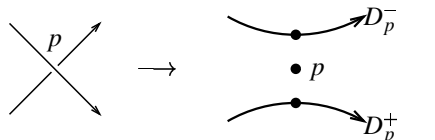
*where D_p^+ denotes the component, where the under- is smoothed to the overcrossing. Note, that apart from the link diagram resulted after this operation, we have the 2 "traces" of p on two of the components of the link diagram, as well as the "trace" of p in its complement. We will sometimes forget about the one or the other and simply omit the corresponding mark in the picture.*

$$n++,w-- \qquad n++,w++ \qquad n--,w-- \qquad n--,w++$$

Figure 3: The four ways to modify n and w by Reidemeister I moves.

Definition 2.5 *Smoothing simultaneously 2 crossings p,q in a knot diagram, the resulting link may have 1 or 3 components. In former case call p,q* linked, *else not linked. In terms of the Gauß diagram the chords corresponding to linked crossings intersect.*

Definition 2.6 *Call in a pair p,q of linked crossings p* distinguished, *if D_p^+ contains the* under*crossing of q, i.e., if the Gauß diagram looks as*
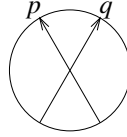
Apart from w, for each crossing p we have four[1] more data:

$$i_p^{\pm} := w(D_p^{\pm},p), \qquad n_p^{\pm} := n(D_p^{\pm}).$$

(Here in the definition of $i_p^{\pm}$ by p we mean the trace of p in the complement, as described above.) Set

$$i_p := i_p^+ + i_p^-, \qquad n_p := n_p^+ + n_p^-, \qquad \delta_p := i_p^+ - i_p^-.$$

3   The degree-3-Vassiliev invariant and positive knots

3.1   A formula for the degree-3-Vassiliev invariant

Smoothing simultaneously 3 crossings and counting the traces of all crossings on the components of the resulting link, we obtain a partition of 6. The following 5 are possible. Each one corresponds to one of the 5 degree 3 CD's underlying the Gauß diagram (which is not true in higher degree), see figure 4. This partition we will call (rough) configuration.

If p and q are not linked, either $p \in D_q^+$ or $p \in D_q^-$. We will write this sign before q, and analogously the corresponding one before p. The configuration $(4,2)0$ is made of triples of crossings (p,q,r) with (p,r) and (q,r) linked and p and q not linked, $p \in D_q^+$ and $q \in D_p^-$, i.e. "$-p \quad +q$" (so the rough configuration of $(4,2)0$ is $(4,2)$).

To introduce some notation, write a configuration as a bracketed sequence of 6 entries of the form '[sign] letter [digit]'. Walking around the circle, the letter indicates the name of the crossing the chord represents (p,q or r), a basepoint of which we pass, the sign whether we pass the over-$(+)$ or undercrossing $(-)$, and the digit after the first occurence of a letter the type[2] of the crossing the chord represents.

---

[1] In fact, these are only 3 data, since for all p we have $n_p^+ + n_p^- = n$.

[2] The type of a crossing will be defined in the next section, so at present simply forget about it and about the option to write the digit.
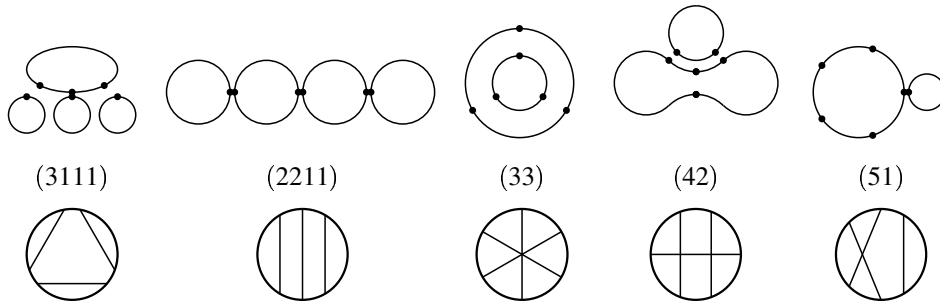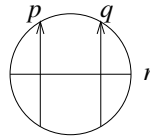
Figure 4: The rough configurations

Figure 5: The Gauß diagram of the configuration $(4,2)0$. Both orientations of r are possible.

The sign and the digit are optional. If we omit one of them we do not fix the orientation of the chord or the type of the crossing.

A Gauß sum we will write as "<config>{<weight>}", e.g.

$$[+q + pr - p - qr]\{w_p + w_q + w_r\} \qquad \text{means} \qquad \sum_{(4,2)0} w_p + w_q + w_r.$$

If we omit the weight, we will mean "by default" the product of the writhes of the involved crossings[3]. Everything up to now works as well for 2 crossings instead of 3 with 4 entries within the brackets. So, interpreted as a Gauß sum, e.g.

$$[+q + p - q - p] := \sum_{\substack{p,q \text{ linked} \\ p \text{ distinguished}}} w_p \cdot w_q.$$

The summation has to be understood so that we sum over distinct unordered p, q, r but if a configuration has a non-trivial automorphism (i.e., e.g. if p, q, r match, so do q, r, p, as in the case of $(3,3)$), only *one* of the matching permutations is taken.

Here is a formula for the degree-3-Vassiliev invariant (in one of its variations modulo degree-2-Vassiliev invariants).

$$vt_3 = [pqrpqr] + [+q + pr - p - qr] + \frac{1}{2}[pqpq]\{w_p + w_q\}. \tag{1}$$

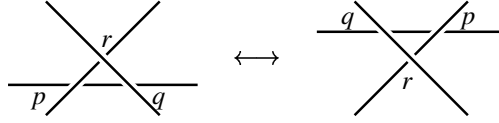The idea of finding it is (roughly) the following (for an exact proof, see [Fi2]).

We have to check how a linear combination of Gauß sums

$$v = \sum_{\text{conf. } \eta} c_\eta \sum_{\substack{\text{crsg. choices} \\ \text{of conf. } \eta}} \text{function}(\dots) \tag{2}$$

---

[3]It will always be clear from the context whether by this notation we mean the Gauß sum or the configuration.

behaves under, say, a type III move.

To measure the difference of each Gauß sum on both sides of such a move, we have to care of the choices of 3 crossings which land in different configurations η, μ in both diagrams. For such configurations at least 2 of the chosen crossings must be among p, q, r. In case exactly 2 of them are among p, q, r, we have to deal with an unknown $b_{\eta\mu}$ – the number of choices of the third one (or rather the sum of "function$(\dots)$" over such choices). The difference of the Gauß sum on both sides is a certain linear combination of all $b_{\eta\mu}$'s. Each $b_{\eta\mu}$ may appear multiple times for different summands in (2).

There exist linear dependencies between the $b_{\mu\eta}$'s which allow to eliminate some in the sum. For the rest, the difference of v to be 0 (i.e., v to be invariant under Reidemeister III), we have to ensure that all the coefficients of the remaining b's vanish. These are in turn linear combinations of various c_μ's. This way we obtain linear equations for the c_μ's. At this point the system could still be solved by hand, due to the limited number of configurations.

Remark 3.1 As for Vassiliev invariants of the same degree there may be many formulas, differing by scaling and lower degree Vassiliev invariants, it appears desirable to introduce some "standard" set of low-degree Vassiliev invariants, and relate everything to it. vt_3 satisfies the relation

$$vt_3 = 4v_3 = -\frac{1}{3}V^{(2)}(1) - \frac{1}{9}V^{(3)}(1),$$
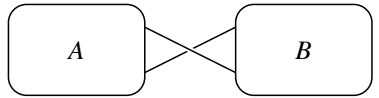
where v_3 is the formula of [PV]

$$v_3 = \frac{1}{2}[-q+p-r-p+q+r] + [+p-q+r-p+q-r], \tag{3}$$

and V is the Jones polynomial [J].

Here is a consequence we announce of formula (1) (see [St4] for a proof and a sharper result).

Definition 3.1 *Call a knot diagram* positive *if all its crossings have writhe 1.*

Definition 3.2 *Call a knot diagram* reduced *if it does not look as*

i.e., its corresponding Gauß diagram has no isolated chord.

Theorem 3.1 *The number of crossings of a reduced positive diagram of a knot K is at most $vt_3(K)$.*

This renders it decidable whether for a given knot such a positive diagram exists.

Example 3.1 We have $vt_3(6_2) = 4$, so for our 6_2 there is no such diagram (it would have to have ≤ 4 crossings).

As another application, it follows from (1), that vt_3 is always integral and even more, the identity $vt_3 = 4v_3$ shows, that $vt_3(K)$ is always even for any K. In fact it is even always divisible by 4. This can be shown either via Vassiliev theory arguments or from the other known conditions to the values of the Jones polynomial on knots (summarized by Jones [J2, §12]).

3.2   A lower bound for the crossing number

Here a final consequence of formula (1). Obviously in a diagram of c crossings, we have

$$|vt_3| \leq \binom{c}{2} + \binom{c}{3} \leq \frac{c^3}{6},$$

whence

$$c \geq \sqrt[3]{6|vt_3|} = \sqrt[3]{\left| 2V^{(2)}(1) + \frac{2}{3}V^{(3)}(1) \right|}. \tag{4}$$

This way the Jones polynomial can be sometimes more powerful to give lower bounds for the crossing number of a knot than by its span [Ka3, Mu, Th]. In all such cases the knot is non-alternating. A similar lower bound for c was given by Polyak and Viro [PV2] in terms of the degree-2 (Casson) invariant: $c \geq \sqrt{8v_2}$.

There are examples, where the inequality (4) does better than either this of [Ka3, Mu, Th] and [PV2].

Example 3.2 The knot 16_{809894} of Thistlethwaite's tables [HT] (see figure 6) has a non-monic Jones polynomial of span 9, from which one finds $v_2 = 12$, and hence the inequalities of [Ka3, Mu, Th] and [PV2] both give 10 as a lower bound for its crossing number. The diagram of figure 6 has 5 Seifert circles, and the Morton-Williams-Franks inequality [Mo, FW] gives 5 as lower bound for its braid index, hence it indeed has braid index 5 by [Ya]. If 16_{809894} had a representation as a closure of a homogeneous 5 braid, its crossing number 16 would show (see, e. g., corollary 4.1 of [Cr]) that $\max \deg \Delta \geq 1/2(16 - 5 + 1) = 6$. However, $\max \deg \Delta = 5$ for this knot, so 16_{809894} is not a closed homogeneous 5 braid, and [Mu2, proposition 7.4] cannot be applied to it. However, $v_3 = 42$, which shows $c \geq 11$, as $24v_3 > 1000$.

In general the combination of the v_2 and $\mathrm{span}\,V$ bounds will perform well, and examples like the one above, at least among not too complicated knots, are difficult to find, but considered separately, one can construct larger classes of knots where (4) gives better results.

For $\mathrm{span}\,V$ an appealing example are torus knots (or a connected sum of such with a knot like 5_2, which leads out of the scope of proposition 7 of [Mu2]) by [J2, proposition 11.9] and the formulas for v_3 on torus knots due to Simon Willerton in [Wi], who used an unpublished (hard) result of Thang Le (see also [St3, §8]).

As for the v_2 bound, one can show from [St8, theorem 3.1] that for all but finitely many positive knots of given genus the v_2 bound for the crossing number will minorate the one of v_3.
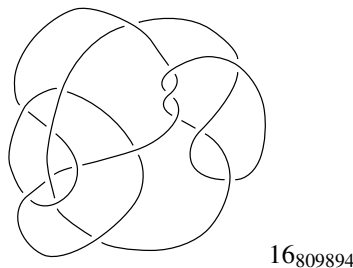

16_{809894}

Figure 6: A knot showing that the inequality (4) is sometimes better than the ones of [Ka3, Mu, Th] and [PV2].

The above mentioned conditions of Jones to the value range of the Jones polynomial imply that no other Jones polynomial than the unit one can have only one monomial and that the only polynomials

of exactly two monomials, which can occur as Jones polynomials of some knot, are of the form $(a+1)t^{12ay} - at^{12(a+1)y}$ with $a, y \in \mathbb{Z}$, $a \neq 0, -1$ and $y \neq 0$ (in particular no Jones polynomial of some knot can have span exactly one). The simplest such example is $2t^{12} - t^{24}$. The existence of a knot with such a polynomial is unclear. In contrast to it there is a remarkable result of Traczyk about an infinite number of 2-component links with Jones polynomial with 2 monomials (reproduced by Przytycki in [P3]). Using (4) we can now show that such a knot must have at least 20 crossings. Namely, for a (hypothetic) knot with $V = 2t^{12} - t^{24}$ the inequality (4) gives $c \geq \left\lceil \sqrt[3]{6912} \right\rceil = 20$, whereas the span of V is 12 and $\left\lceil \sqrt{8v_2} \right\rceil = 10$. By a further bit of argument you can exclude crossing number 20 using that the minimal coefficient is 1 in a positive diagram, a consequence of Kauffman's state model (see [St5]).

As a final application of (1) we mention, that from it it straightforwardly follows, that the $(2, 2n-1)$ torus knot diagram is the unique diagram maximizing v_3 over all diagrams of this (and lower) crossing number. A bit more on argument shows that in fact it maximizes v_3 in crossing number $\leq 2n$.

4   Refined Gauß diagrams

Here we present a generalization of the degree-3-Vassiliev invariant to knots in the solid torus [Go].

Consider a two component link $K \cup T$ in S^3 where T is the trivial knot (unknot). Let K, T, S^3 be oriented. Deform $K \cup T$ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ so that $\infty \in T$. This isotopy is unique up to isotopy. Such a link we can represent choosing an appropriate projection $\mathbb{R}^3 \to \mathbb{R}^2$ as knot with a point in its complement, on which T projects, assuming the orientation of T to be from the sheet of paper to the reader's eye.

Luckily, lemma 2.1 can be generalized to such diagrams.

Lemma 4.1 *If $K \cup T$ and $K' \cup T'$ isotopic, implying that K and K' isotopic within the solid torus $S^3 \backslash T \simeq S^3 \backslash T'$, and in the above described projection K and K' have equal n and w, then K and K' are regular isotopic in $S^3 \backslash T$.*

The new idea of the first author was to *refine* the crossings in a Gauß diagram, i. e. to assign to each one a *type*. In order to bound the number of configurations and the dimension of the equation system, at present this has been only done for types in $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Here for our introductory purpose we consider $\mathbb{Z}_2$ graduation of the crossings.

Definition 4.1 *The* type *of a crossing p in a refined diagram is $w(D_p^+, T)$ mod 2.*

We will distinguish two cases of refined diagrams according to the parity of $w(K, T)$.

In view of lemma 4.1 the receipt to find invariant Gauß sums remains the same, however, we gain with the type a new useful ingredient.

The price we pay is that the combinatorial trouble arising here is serious – the linear equation systems became too large and had to be solved by computer. We shall later say a word on dimensions. The solutions have been computed by K. Mohnke (even linking number) and myself (odd linking number) using MATHEMATICA™[Wo]. Many of them (and the really interesting ones) are very long and our lack of deeper insight into the solutions, which is a constant negative companion of electronical calculations, prevents us from finding nice expressions for them. Therefore, here we will record just some short (but, unfortunately, less interesting) ones.

We can incorporate the study of knots into this scheme by considering T the meridian of the knot[4] K (of course, we could also try it with $K \cup T$ the split link, but this does not give anything new). T carries a natural orientation from K and S^3. K is isotopic to K' exactly if so are $K \cup T$ and $K' \cup T'$. Since for T the meridian of K we have $w(K, T) = 1$, from this point of view the odd linking number invariants are more interesting. However, by applying cabling operations we can also hope to gain some information for knots from the even linking number invariants.

---

[4]In this case we will omit T and identify the link with the knot.
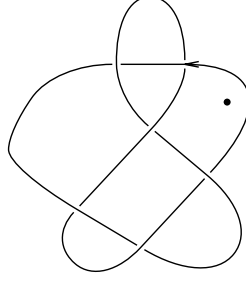
Figure 7: The knot 6_2 with its meridian

As, as we saw, one can treat knots in S^3 as knots with meridian, it was for a brief period tempting to find out what the invariants can do for knots in S^3 this way. In the present note we content ourselves to some failed examples with knots serving to explain the (cabling) idea, which can be more successfully applied to links. We will give more interesting examples in a later note.

Although, after testing the examples given below, finally we found out, that for *knots* this way the solid torus Vassiliev invariants and their cablings don't give more information than the usual degree-3-Vassiliev invariant in S^3, we will give some evidence that (especially at finer crossing type) they may be better than the degree-3-Vassiliev invariants for 2-component links.

4.1   Even linking number case

For the case $w(K,T) \equiv 0 \pmod 2$ the linear equation system was with 50 equations and 48 variables. We obtained 34 invariants. 15 of them are stable and 19 non-stable.

Definition 4.2 *A knot invariant v of regular isotopy is called k-stable for some $k \in \mathbb{Z}$, if there exist numbers $n_1, w_1, \ldots, n_k, w_k \in \mathbb{N}$ and a function $f_v : \mathbb{Z}^{k+2} \to \mathbb{Z}$, such that for all knots K and all $n, w \in \mathbb{Z}$*

$$v(K,n,w) = f_v(v(K,n_1,w_1), \ldots, v(K,n_k,w_k), n, w).$$

Here we use the parametrization of the regular isotopy classes discussed after lemma 2.1 and by "K" we mean the ambient isotopy class of K. "Stable" means "1-stable".

In other words, "stable" means that the change of the invariant under change of n or w (Reidemeister I) does not depend on the concrete knot. Roughly, an unstable invariant contains more information than any ambient isotopy invariant derived from it, together with n and w.

Examples of stable invariants are all Vassiliev invariants (where f depends on $v(K,n_1,w_1)$ only) and the Kauffman bracket [Ka] (where f depends on $v(K,n_1,w_1)$ and w).

Here is an example of such an invariant.

$$
\begin{aligned}
vt_3^{(1)} \quad = \quad & -[+p0+r0-p-r]\{w_r w_p(i_r-i_p)\} + [+p0+q1+r1-q-p-r] \\
& -[+r1+p0+q1-r-q-p] + [+p1-q0-r0+q-p+r] \\
& -[+q0-p1-r0+p-q+r] - [+p1+q0+r0-p-q-r] \\
& -[+r1+p0+q0-r-q-p] - \frac{1}{2}[+p1+r0-p-r]\{w_p\} \\
& +[+p0+q0+r1-p-q-r] + [+p0+q0+r1-q-p-r] \\
& +\frac{1}{2}[+p0+r1-p-r]\{w_p\}.
\end{aligned}
$$

This invariant is stable, asymmetric (changes the sign under mirroring[5] $K \cup T$) and ambient invariant (that is, invariant under Reidemeister I).

$K \cup T$ is invertible (that is, isotopic to $-(K \cup T) := -K \cup -T$; we reverse the orientation of *all* components!) exactly if $-(K \cup T)$ and $flip(K \cup T) := flip(K) \cup -T$ are regular isotopic (as their n, w are equal) and this is exactly if so are $flip(K) \cup T$ and $-K \cup T$.

Figure 8 shows the 3 operations on a link with $K = $ trefoil.

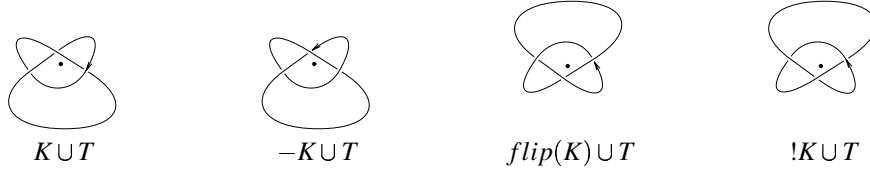$\qquad K \cup T \qquad\qquad\qquad -K \cup T \qquad\qquad\qquad flip(K) \cup T \qquad\qquad\qquad !K \cup T$

Figure 8: The 3 operations on a link

Another invariant is

$$
\begin{aligned}
vt_3^{(2)} \;=\; & <\text{ formula for } vt_3 \text{ only over crossings of type } 0 > \\
& +4[+p1 - q0 + r0 - p + q - r] + [+p1 - q0r0 + q - pr] \\
& + [+p1 - q0r1 + q - pr] + [-p1 + q0r0 - q + pr] \\
& + [-p1 + q0r1 - q + pr] - [+p1 - q1r0 + q - pr] \\
& - [+p1 - q1r1 + q - pr] - [-p1 + q1r0 - q + pr] \\
& - [-p1 + q1r1 - q + pr] + [+p0 - q0r1 + q - pr] \\
& + [-p0 + q0r1 - q + pr] .
\end{aligned}
$$

It has the same properties as $vt_3^{(1)}$.

4.2   Odd linking number case

For the case $w(K, T) \equiv 1 \pmod 2$ we had 120 equations and 111 variables. We obtained 33 invariants, 14 stable and 19 non-stable ones. The most complicated ones have 55 terms. Here is a simple one

$$
\begin{aligned}
vt_3^{(3)} \;=\; & -[+p1 - q1 + r1 - p + q - r] - [+p1 - q0 + r0 - p + q - r] \\
& - [+p1 - q1 - r0 + q - p + r] - [+p1 - q1 - r1 + q - p + r] \\
& + [+p0 - r0 - p + q1 + r - q] + [+p0 - r1 - p + q1 + r - q] \\
& + [+p1 - q1 + r0 + q - r - p] + [-q0 + r1 + p1 - p + q - r] \\
& + [-q1 + r0 + p1 - p + q - r] + [+p1 - q0 + r1 + q - r - p] \\
& + [+p1 - q1 + r1 + q - r - p] + [+p1 - q0 + r0 + q - r - p] \\
& + [-q1 + r1 + p1 - p + q - r] + [-q0 + r0 + p1 - p + q - r] .
\end{aligned}
$$

This invariant is non-stable and asymmetric. One can make the observation that augmenting of n by 2 at fixed w adds one and the same contribution to the value of $vt_3^{(3)}$. We have

$$
vt_3^{(3)}(K, n, w) \;=\; vt_3^{(3,1)}(K) + vt_3^{(3,2)}(K) \cdot n + vt_3^{(3,3)}(K) \cdot w,
$$

where the $vt_3^{(3,i)}$ are ambient isotopy invariants, and therefore $vt_3^{(3)}$ is 3-stable.

---

[5]Note that mirroring on $K \cup T$ means mirroring in $\mathbb{R}^2$ of the diagram of K with the point T and *not* crossing change, since we need T to point to us.

Remark 4.1 In fact, $vt_3^{(3,2)} = vt_3^{(3,3)}$, so $vt_3^{(3)}$ is 2-stable. This can be deduced from the fact, that we have no $(5,1)$-configuration with isolated chord of type 0. Analogously, if we had no $(5,1)$-configuration with isolated chord of type 1 instead of 0, then we had $vt_3^{(3,2)} = -vt_3^{(3,3)}$.

Remark 4.2 Generally, a Gauß sum invariant induces on an ambient isotopy class a polynomial of n and w, henceforth called (n,w)-polynomial, where w and n appear with exponents ≤ 1 resp. $\leq 1 + \max\deg_n(\text{all weights})$, where $\deg_n$ is the degree with respect to n's of the crossings only.

Remark 4.3 In fact, $vt_3^{(3,2)} = vt_3^{(3,3)}$, so $vt_3^{(3)}$ is 2-stable. This can be deduced from the fact, that we have no $(5,1)$-configuration with isolated chord of type 0. Analogously, if we had no $(5,1)$-configuration with isolated chord of type 1 instead of 0, then we had $vt_3^{(3,2)} = -vt_3^{(3,3)}$.

4.3 Chirality

From the ambient invariance and asymmetry of $vt_3^{(2)}$ it follows that each amphicheral link must have zero invariant. This can be used to detect chirality of links. The following example is due to the first author.

Example 4.1 For the link $K \cup T$ on figure 9 with K the figure 8 knot we have $vt_3^{(2)} = 4$, so it is non-amphicheral. The components themselves are amphicheral, and all other invariants of degree ≤ 3 (*lk* and the two component link degree 3 Gauß sum of [PV]) are 0 and fail.
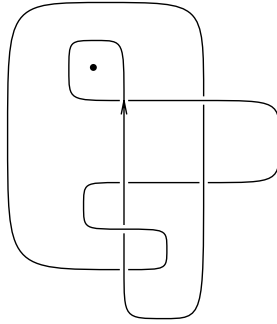

Figure 9: A non-amphicheral link with even linking number.

Two hard examples of chiral knots are [Ka, p.218] 10_{48} (where the HOMFLY polynomial [H] fails to detect chirality) and 9_{42} (where the HOMFLY and Kauffman [Ka2] polynomials fail). After we experimentally observed and the second author subsequently proved more generally in [St3] that our solid torus invariants will not be able to do it either, there is another idea based on Gauß sums, which can be used at least in the case the knot is alternating.

The flyping theorem of Menasco and Thistlethwaite [MT] shows that in alternating diagrams not only vt_3 is an invariant, but all its three single terms in (1) are! The first and third one are clearly invariants of the intersection graph of the associated Gauß diagram.

The intersection graph of a Gauß diagram is a graph with vertices corresponding to arrows in the Gauß diagram and edges connecting intersecting arrows/vertices. It is an exercise to see that the intersection graph is preserved by flypes, and therefore so are the single terms in (1). (A little bit more argument shows invariance also for the second term in the Polyak-Viro formula (3) and hence also for the other one.)

The third of (1) term is easily computed on 10_{48} and its mirror image to be ± 8 (note, that all three terms are negated by obversion), and its non-vanishing renders it easy to decide about 10_{48}'s chirality. The first term deals (even easier) with the other three alternating troublemakers in Rolfsen's tables [Ro, appendix A] – 10_{91}, 10_{104} (HOMFLY) and 10_{71} (HOMFLY and Kauffman). The argument does not work for the non-alternating 9_{42} and 10_{125} (HOMFLY), but they both have signature ± 2, and so a signature trick works there (see [St6]).

4.4   Skein equivalence and mutation

As Gauß sums change unpredictably under crossing changes, we could hope that they manage to detect skein equivalent knots. However, checking one such example – the skein equivalent triple 8_8, $!10_{129}$ and 13_{6714} (see figure 10) discovered by Thistlethwaite [LM] (and later in greater variety by Kanenobu [Kn]) – we observed the same disappointment.

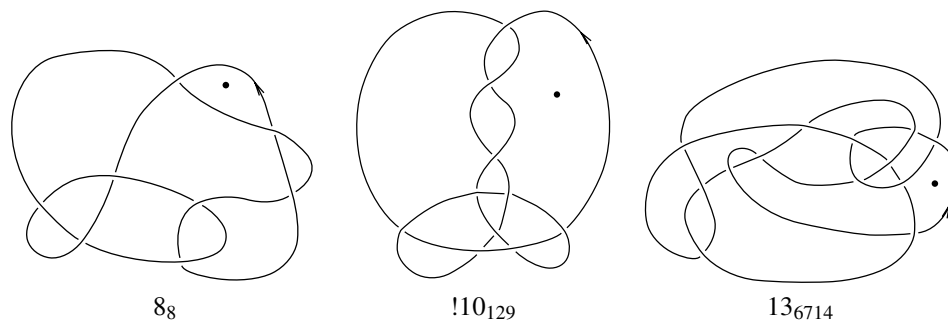8_8 $!10_{129}$ 13_{6714}

Figure 10: The skein equivalent knots of Thistlethwaite. They form the beginning of a larger series of skein equivalent knots constructed by Kanenobu [Kn].

It is known that mutants always make trouble to distinguish. According to Przytycki, one applicable method is to calculate truncations (in the Alexander variable) of the HOMFLY polynomial of a 3-cable of the link. Nothing simpler works: that the knot polynomials themselves do not detect mutation is an easy consequence of the local diagrammatic relations they satisfy, and subsequently it was shown that all cablings of the Jones polynomial [MT], and the 2-cablings of the HOMFLY [P, P2] and Kauffman [LL] polynomial fail. Such truncations have been shown to have subexponential complexity by Przytycka and Przytycki [PP] and later with sharper estimates by Vertigan (see appendix to [PP2] and [Ki, p. 111]) and were used by Morton and Traczyk and independently by J. Murakami to distinguish, *inter alia*, the Kinoshita-Terasaka and Conway knots (figure 11), the most famous example of mutants.

Again, there is no sense in trying to distinguish mutant knots by applying the solid torus invariants on the meridian of the knots or their cables. However, interestingly, some invariants *do* distinguish (and even without cabling) pairs of solid torus knots which are not (necessarily) mutants in the solid torus but give mutated links when considered with T in S^3. The second author gave such examples in [St7].

5   The computer programs

The program `gsinv` [St2] can compute the Fiedler solid torus invariants. It is written in C++ [S] and at present it has ≈ 4000 lines. Here is a brief summary of its features.

- It reads in a knot from a projection file produced by J. Weeks' program SnapPea[6] [W], generates the associated extended Gauß diagram (Gauß diagram with all data assigned to each chord) and computes a given Gauß sum on it.

---

[6]Unfortunately, the link entry program of SnapPea is not yet available on other systems than Apple[TM].
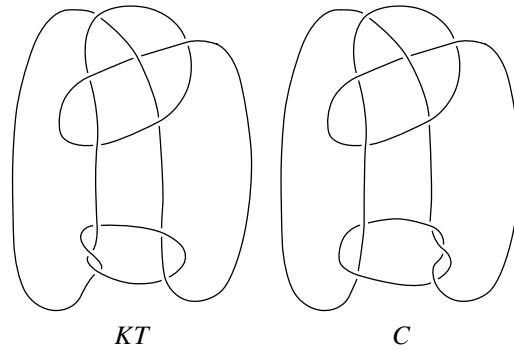
Figure 11: The Kinoshita-Terasaka/Conway mutants

- It can also create a PostScript (PS) [Ad] image of the knot. This feature has been extended to handle arbitrary regular link and tangle projections (i.e., every component has at least three edges, so that a dotted component is no longer allowed; see -l option). The pictures in this paper have been generated this way.

- Another functionality is that it can output the Millett-Ewing format of the link (but not tangle!) projection, making possible by their powerful programs the calculation of the various polynomials of the link.

- It contains detailed comments, so we hope that it will be (re)usable.

Below we provide instructions for use and we are always grateful for comments and suggestions about it.

A modification of this program is possible to handle general diagrams by using the extended Dowker notation (see §5.3), but we will develop one only at request.

An additional small program br2proj is provided to generate input files of a closed braid and its axis (so as to examine the invariants as conjugacy invariants for braids) from a braid word. Therefore, this way it is possible to use the invariants (at least for braids) without SnapPea's graphical interface.

Another program lsd2sp is provided to convert the format of the editor "LinkSmith" in KnotScape to the SnapPea link projection format (at least in so far that it is recognized as such by gsinv).

A final tool is v23 which contains an implementation of the Polyak-Viro formula for v_2 and the Fiedler formula for $v_3/4$ reading in the knot(s) in the notation of Dowker and Thistlethwaite (the convention for obversion being that the sign of the first entry is equal to the writhe of the crossing it represents).

5.1 How to install and use them

The (gzipped) tar-archive [St2] should be expanded on a UNIX machine with a command like

```
tar -xzvf gsinv.tgz
```

It contains the following files:

gsinv.C – the main program. It is to be compiled using the GNU C++ compiler g++, version 2.7.0 ff., with a command like

```
                                 g++ -o gsinv gsinv.C &
```

(Note: due to the large number of complicated expressions for the invariants with $\mathbb{Z}_3$ graduation, compiling will probably last up to an hour, so it is a good idea to do it in the background.)

`comp.h`,   `rat.h`, – files included into `gsinv.C`.
`mydefs.h`, `x.h`, `y.h`, `eveninv.h`, `oddinv.h`, `lk0mod3.h`, `lkZ_0.h` and `lk1mod3.h`

`8-17-2-1` – the SnapPea projection file of the 2-cable of 8_{17} provided to reproduce the below example.

`tang.lsd` – the LinkSmith format file of a tangle provided to reproduce the below example 5.4.

`mypic.psh` – header file for `dvips` for including the graphics output of the program into a TeX or LaTeX document using the `epsf` package. Include the header file using the command

```
            \special{header=mypic.psh}
```

on the first page of the TeX or LaTeX document.

`br2proj.C` – This program should be compiled with

```
            g++ -o br2proj br2proj.C
```

The program reads in a braid (word) and generates an input file with the link consisting of its closure (which must be a knot) and axis (the point *T*). The format of this file slightly differs from this of SnapPea, so to make `gsinv` recognize it, invoke it with the additional parameters '`-f 1`'. The braid has to be sent to the program `br2proj` from `stdin`, indicating first the length of the braid (i. e., the number of generators and their inverses counted with multiplicities if appearing in powers) and then the braid word starting with a '`{`', followed by a comma separated list of non-zero integers and terminated by a '`}`'. An integer *i* represents σ_i or σ_{-i}^{-1} according to its sign. The number of strands is computed automatically. The braid must close to a knot. The program writes output on `stdout`, so redirect it into a file. See example 5.3.

`lsd2sp.C` – This program should be compiled with

```
            g++ -o lsd2sp lsd2sp.C
```

The program reads in from the file specified as argument a LinkSmith format of a link or tangle diagram and outputs on `stdout` the corresponding SnapPea projection file. See example 5.4.

`v23.C` – This program should be compiled with

```
            g++ -o v23 v23.C
```

The program reads in from `stdin` the Dowker-Thistlethwaite notation of a (prime) knot diagram in the format: "crossing number, knot identifier, Dowker-Thistlethwaite notation" and outputs on `stdout` the sequence "crossing number, knot identifier, v_2, $v_3/4$". The input may contain multiple lines of the given format, which are processed separately, until all input is exhausted. See example 5.5.

A call of the program `gsinv` from a UNIX shell should look like

```
gsinv [ -ps ] [ -i <inv#s> ] [ -w <PSwgt> ] [ -m <mode> ] [ -q ]
   [ -K ] [ -mod [ 1 | 2 | 3 ] ] [ -h <homcl> ]   [ -a <arsz> ]
   [ -aw <arwid> ] [ -T ] [ -af ] [ -ken ]
   [ -c <crwid> ] [ -J ] [  [ -f [ 0 | 1 ] ] ] -l  <input file> |
   <input file> [ <Re(T)> <Im(T)> ] |  [ -f [ 0 | 1 ] ]  -p
   [ -l ]  { <input file>  } ]
```

Here "`[ thing_1 | thing_2 ]`" means *either* `thing_1` *or* `thing_2`, "`[ thing ]`" means at *most* one occurence of `thing` and "`{ thing }`" means at *least* one occurence of `thing`.

Unlike many other programs, if an option of this program requires a parameter, it has to be supplied as the next command line argument, not as the rest of the argument indicating the option. E.g., "`-i6`" is not correct, you have to write "`-i␣6`". Options are processed in order of appearance from left to right on the command line; a later option may override a previous one.

When an option of this program requires a parameter, latter *must* be specified; no defaults exist. However, all option parameters have default settings, which are taken when the option is not specified on the command line.

The program has two modi. In invariant mode (which is the default), each input file is processed by writing the results on `stdout` (unless the `-q` option is specified) and into a log-file whose name is obtained by adding "`.log`" to the name of the input file. In PS mode (activated by the `-ps` option) no output is generated on `stdout` (so the `-q` option is obsolete); instead PS images of the input files are written into corresponding `.eps` files (see example 5.2). An exception from these modi is the `-ken` option (see below).

The meaning of the options is the following:

`-ps` sets the PS mode – write PS images instead of computing invariants. The `-ps` option overrides the `-mod` option.

`-m` The next argument should be a number between 1 and 31. Its lowest 5 bits indicate whether an action should be done for $K, -K, flip(K), !K$ and $flip(!K)$. Default setting is 1 for PS (only K) and 15 for invariant computation (first 4).

`-K` This option indicates in PS mode that the dot for T should not be drawn.

`-h` The next argument should be a non-zero integer indicating the homology class a to be used in $\mathbb{Z}$-graduation. In $\mathbb{Z}$-graduation, the Gauß sums are computed like in $\mathbb{Z}_3$-graduation with the only difference that a chord labeled by '1' matches (only) a crossing of type a and a chord labeled by '2' matches a crossing of type $-a$, and the linking number must coincide with the one the invariants are designed for in $\mathbb{Z}$ (and not only $\mathbb{Z}_3$). Presently only invariants for zero linking number are available. Default setting for a is 1. This option is obsolete when not specifying '`-mod 1`' (see `-mod` option).

`-f` Next argument should be either `0` or `1`. 0 (default setting) makes the program expect SnapPea format in the input file(s), 1 sets the format produced by the program `br2proj`. Both inputs formats simultaneously are not allowed. If 1 is set, the possibility to indicate coordinates of T on the command line (see `-p` option) is not allowed.

-w   Sets the PS weight to the next argument. This should be a number with $1/2 < \# \leq 1$ indicating the degree of smoothing the vertices. 1 (default setting) draws sharp vertices, $1/2 + \varepsilon$ smooth vertices. A good value is 0.8. A higher value may be useful to avoid intersections of the smoothed curves which can occur at long edges in the diagram polygon. This option automatically sets `-ps`.

-i   Only some special invariants with $\mathbb{Z}_2$ graduation given in the next argument should be computed. This option was included at some evolutionary stage and is not particularly recommended. The next argument should be something like "`2,5,7-10,4`" (meaning that we are interested in invariants 2,4,5 and 7 to 10 only). Default (without the `-i` option) is all invariants. However, only special numbers in the list should be used. For even linking number they are 1 for $vt_3^{(1)}$, 16 for $vt_3^{(4)}$ and 5 for $vt_3^{(2)}$, for odd linking number – 15 for $vt_3^{(3)}$, 16 for $vt_3^{(4)}$ and 17 for $vt_3^{(5)}$. Additionally for both linking numbers the usual Gauß sum formulas are available (evaluated on K by forgetting T) – invariant number 2 for vt_3 and 3 for v_3. The invariants provided with this option are *not* included in the invariant list computed without the `-i` option, but are linear combinations of invariants in the list. A list of $\mathbb{Z}_2$-invariants available for computation is given in [St7]. The `-i` option overrides the `-mod` option.

-p   Indicates that all arguments remaining after parsing in all options should be taken as file names of input files to process (options are all leading arguments starting with a '−' and the arguments required to follow some of them). In this case each input file is assumed to be a SnapPea projection file of a knot with a point for T therein (resp. a file of the format rendered by `br2proj`, when '`-f 1`' is specified). The point should be added *after* drawing the knot by a double mouse click. (Note that such a projection is useless for SnapPea, as it is not a "regular" link projection.) If without this option there is only 1 argument remaining after parsing in all options, the program assumes a SnapPea projection file of a knot with a point therein. If (without this option) after parsing in all options there are 3 arguments remaining, the program assumes in the file only a knot and reads in the coordinates of the point T from the last 2 arguments. (If the `-l` option is used, the possibility to give the second and third argument is not allowed.) Another number of remaining arguments than 1 or 3 is incorrect without this option.

-q   Quiet mode. Do not print anything on `stdout`, just into the log-files. Obsolete in PS mode.

-J   Compute the series of (formally infinite sum) invariants J of degree 2, depending on a parameter a. The program automatically computes the highest absolute value of a type of a crossing and gives the series only for a between its negated and itself. For each non-zero $a \in \mathbb{Z}$ the invariant J is defined by

$$J(a) := [+pa - q0 - p + q] + \sum_{\substack{m > 0, n < 0, \\ \gcd(m, -n) = a}} [+pm - qn - p + q],$$

where crossing (type) graduation is taken in $\mathbb{Z}$.

-mod   The next argument should be either '2' (default setting), '1' or '3' indicating the invariants using which graduation should be computed. '1' means $\mathbb{Z}$-graduation (see the `-h` option).

-c   The next argument should be a number. It explains which part of the diagonal length of the PS image should be taken as crossing size. When passing an undercrossing the PS pen is taken off the sheet at this distance from the crossing and put back at the same distance on the other side.

-a   The next argument should be a number. It explains which part of the diagonal length of the PS image should be taken as arrow length. A high value (e.g., 1000) causes the arrow to become invisibly small, and should be specified if an arrow is not desired.

-l   This option forces the program to read from each input file a *regular* link or tangle projection for SnapPea and to generate a PS image file of the link. It is therefore similar to the `-ps -K` options, with the difference that no point T is allowed in the input files and the link may

have multiple (not only one or two) components. This option is hence unrelated to the main functionality of the program. The −l option sets automatically the −ps option and overrides the −mod option.

−aw This option specifies the width of the arrow. The next argument should be a number c indicating the ratio between length and width of the arrow, so that if $(0,0)$ is the top of the arrow, coming along the positive real line, its two ends are (proportionally) at $(c, \pm 1)$. Default value for c is 5. This value may be lowered, if the line segment just before the arrow is strongly curly, so that it leaves the angle spanned by the two arrow ends.

−ken This option indicates that the Millett-Ewing notation of the read in link projection(s) should be printed. (This enables to use the Millett-Ewing programs for calculating the classical link polynomials.) Neither any invariants are calculated, nor PostScript is rendered.

−T This option indicates that, when reading in a projection in the solid torus link format, in the PS image the dot indicating the trivial component should be omitted.

−af If this option is invoked, it has the effect of suppressing the drawing of an arrow in orientation direction after the first segment of non-closed components, and at the closed components. Without this option, on a tangle component, an arrow (in orientation direction) is drawn also after the first segment (and not only at the end). This is useful when drawing tangle diagrams with many wired strands to facilitate the recognition of their orientation. If many strands, however, are short, then this is not desirable, and the −af is thought for such cases. Note, that the arrow at the end of non-closed components is drawn even if −af is specified; to avoid this, use the −a option (see there).

Example 5.1 If we write

```
gsinv -m 6 -i 15 8-17-2-1
```

assuming in the file 8−17−2−1 the knot $(8_{17})_{2,1}$ with its own meridian, we obtain the following output

```
file: 8-17-2-1
lk(K,T)%2=1
w=1
n=6

inv #15:
vt33(-k)= 10
vt33(flip(k))= 10
```

The program computes the parity of $lk(K,T)$ and depending on it chooses the right set of invariants. Each invariant is indicated by its name (e.g., vt33 for $vt_3^{(3)}$). So we see that, unfortunately, $vt_3^{(3)}$ does not detect the (possible) non-invertibility of $(8_{17})_{2,1}$ (implying this of 8_{17}).

Example 5.2 If we want to see how $K = (8_{17})_{2,1}$ with its own meridian looks like, type

```
gsinv -ps -m 15 8-17-2-1
```

You obtain 4 PS files: 8−17−2−1.eps with K, 8−17−2−1−.eps with $−K$ and 8−17−2−1_fl.eps and 8−17−2−1_mi.eps with $flip(K)$ and $!K$ respectively. You can include them into a LATEX file as described above.

Example 5.3 To write the image of the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-3}$ and its axis into the file 6_2.out, type

```
echo "6 {1,-2,1,-2,-2,-2}" | br2proj > 6_2.out
```

and to create an image of it, type

```
gsinv -ps -w 0.8 -f 1 6_2.out
```

Example 5.4 To write the SnapPea format of the LinkSmith file `tang.lsd` into the file `tang.sp`, type

```
lsd2sp tang.lsd > tang.sp
```

To make a picture of the result, type for example

```
link -ps -w 0.8 -c 70 -aw 3.0 -af -l -a 1000 tang.sp
```

Example 5.5 To see the invariants v_2 and $v_3/4$ of the positive (right-hand) trefoil, given in its Dowker-Thistlethwaite notation "`4 6 2`", type

```
echo "3 1 4 6 2" | v23
```

(Remember to always specify crossing number and identifier before the Dowker-Thistlethwaite notation.) The result is as expected

```
     3      1      1      1
```

Remark 5.1 The polynomial complexity of the program `gsinv` allows it to handle examples of about 80 crossings in a few hours. However, at least at about 100 crossings it exhausts the limit of virtual memory available on an average-sized computer.

5.2   Transforming diagrams

Given a diagram with a point the invariants can be effectively computed. However, usually a link may be given by a regular (oriented) diagram consisting of components K and T (with T unknotted), so it is worth saying a word on how to transform such a regular diagram into a diagram of the kind we need.

   If we do not pose any condition on the (regular) diagram we start with, such a procedure would in particular mean to transform the diagram of T therein into the zero crossing diagram, which is a very subtle question and easily fills several dozens of pages for itself (see [Bi] for an exposition on the problem and some recent results). Hence, we assume that the diagram of T has no self-crossings. E. g., the (closed) mixed braid diagrams of [La] are of this kind.
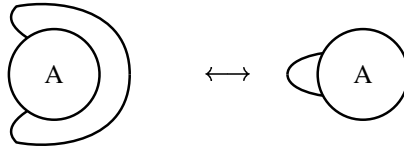
   Given such a diagram of $K \cup T$,

1) use moves of the kind (called in [A, fig. 5.58] "tongue moves")
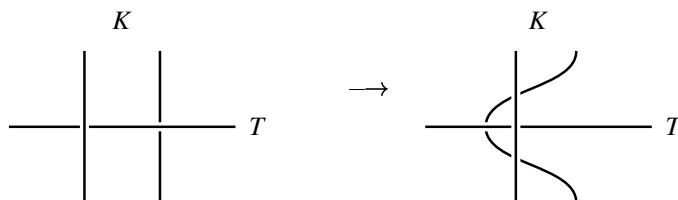
   to permute the over- and undercrossings of T with K (at the cost of adding self-cossings of K) so that, running along (the orientation of) T, we have just one sequence of overcrossings and one sequence of undercrossings, i. e., only one *bridge* of T;

2) use moves in the 2-sphere

to get the component of the complement of the diagram of K, where the bridge of T ends (that is, the component entered by T by an overcrossing and exited by an undercrossing)
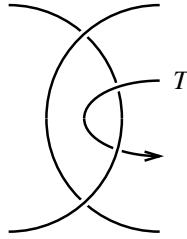
to be the unbounded component in the plane, and finally

3) delete T from the diagram, placing a point into the component of the complement of the diagram of K, where the bridge of T begins (that is, the component entered by T by an undercrossing and exited by an overcrossing).
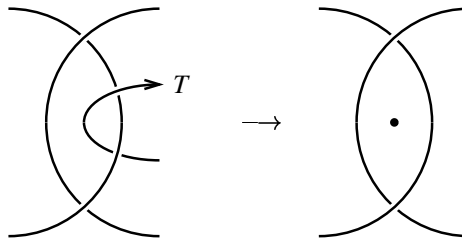
To reverse the procedure (to come from our type of diagram to a regular one) you just need to reverse the last step.

5.3   A notation for diagrams

It is often convenient to represent link diagrams in an alphanumeric format, different from braids. The first systematic way of describing link diagrams in such a way was introduced by Conway [Co]. His notation was developed rather to reflect algebraic properties of the knot (or link) and is hence not particularly convenient for a computer. In this regard much better does the notation of Dowker and Thistlethwaite [DT] for knot diagrams. It has been used to compile by computer prime knot tables up to 16 crossings [HT].

A Dowker notation of an n crossing knot diagram is a permutation (vector) $d = (d_1, \dots, d_n)$ of the numbers $2, 4, \dots, 2n$, with possibly some of them negated. When walking along the line in a knot diagram and numbering the over/under-crossings in this order from 1 to $2n$, the n over- and n under-crossings pair up in n pairs of an over- and under-crossing, such that any pair consists of an even and odd entry. If the i-th entry d_i in a Dowker notation is positive, it indicates that d_i should be the over-crossing in a crossing, where $2i - 1$ is the under-crossing, else d_i the under-crossing of the over-crossing $2i - 1$. (See, e. g., [A] for more details.) This determines an (oriented) knot diagram up to mirroring of each composite component and up to moves in S^2. There was no convention made by the authors how to fix the mirroring ambiguity, but one way to do so is to take the sign of the left-most entry in the vector d corresponding to a crossing in a given composite component to be equal to the writhe of the

crossing it denotes. (For connected diagrams, the program KnotScape [HT] apparently uses exactly this convention, taking the sign of the first entry to be the writhe of the corresponding crossing).

Here we propose how to extend this notation for our purpose (and call it extended Dowker notation).

For a link diagram of our type we must additionally to the Dowker notation of the diagram of K specify (first) the connected component of the complement of the knot diagram of K, where the point T lies (the one, where the bridge of T begins), and (second) the unbounded one (where the bridge ends). We use for this the convention, that for $1 \leq i \leq 2n$ the component described by i (resp. $-i$) is the one to the left (resp. right) of the segment of the line between over/under-crossing i and $i \mod (2n) + 1$ in orientation direction.
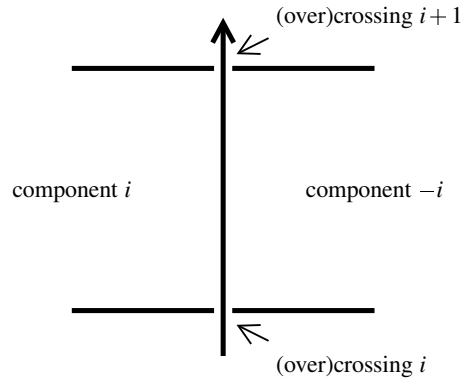
Example 5.6 The diagram of the knot 8_8 and its meridian on figure 10, with starting counting the over/under-crossings at the position of the arrow, could be denoted by $10\ 8\ 2\ 14\ 4\ 16\ 6\ 2 \quad 16 - 16$ or by $10\ 8\ 2\ 14\ 4\ 16\ 6\ 2 \quad 10\ 3$ (note, that the crossing $(1,10)$ is positive).

References

Here are the email addresses of some of the authors of the cited papers. We hope email correspondence could be helpful. Note that email addresses often change, so some may be already out of date. We are grateful for any hints and corrections. A more detailed list of email addresses is maintained by Bar-Natan [BN2].

Dror Bar-Natan   drorbn@math.huji.ac.il
Joan S. Birman   jb@math.columbia.edu
Louis H. Kauffman   kauffman@uic.edu
Xiao-Song Lin   xl@math.ucr.edu
Hugh Morton   su14@liv.ac.uk

Jozef Przytycki	`przytyck@math.gwu.edu`
Vladimir Turaev	`turaev@math.u-strasbg.fr`
Oleg Viro	`viro@math.uu.se`
Pierre Vogel	`vogel@mathp7.jussieu.fr`
J. Weeks	`weeks@geom.umn.edu`

[A]   C. C. Adams, *Das Knotenbuch*, Spektrum Akademischer Verlag, Berlin, 1995 (*The knot book*, W. H. Freeman & Co., New York, 1994).

[Ad]   Adobe Systems Inc., *PostScript Language, Tutorial and Cookbook*, Addison-Wesley, 1993.

[BN]   D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423–472.

[BN2]   —— " ——, *Bibliography of Vassiliev invariants*, available via anonymus file transfer from `ftp.ma.huji.ac.il`, subdirectory `drorbn`.

[BS]   —— " —— and A. Stoimenow, *The Fundamental Theorem of Vassiliev invariants*, "Geometry and Physics", Lecture Notes in Pure & Appl. Math. **184**, M. Dekker, New York, 1996, 101–134.

[Bi]   J. S. Birman, *Recognizing the unknot*, preprint.

[BL]   —— " —— and X-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993) 225–270.

[Co]   J. H. Conway, *On enumeration of knots and links*, in "Computational Problems in abstract algebra" (J. Leech, ed.), 329-358. Pergamon Press, 1969.

[Cr]   P. R. Cromwell, *Homogeneous links*, J. London Math. Soc. (series 2) **39** (1989), 535–552.

[DT]   C. H. Dowker and M. B. Thistlethwaite, *Classification of knot projections*, Topol. Appl. **16** (1983), 19–31.

[Fi]   Th. Fiedler, *Talks given in Toulouse & Berlin*, summer 1996.

[Fi2]   —— " ——, *Gauss sum invariants for knots and links*, monography, to appear.

[Fi3]   —— " ——, *A small state sum for knots*, Topology **32 (2)** (1993), 281–294.

[Fi4]   —— " ——, *Die Casson Invariante eines positiven Knotens ist nicht kleiner als sein Geschlecht*, talk given at the knot theory workshop in Siegen, Germany, 1993.

[FW]   J. Franks and R. F. Williams, *Braids and the Jones-Conway polynomial*, Trans. Amer. Math. Soc. **303** (1987), 97–108.

[Go]   V. Goryunov, *Finite Order Invariants of Framed Knots in a solid torus and in Arnold's J^+-Theory of plane curves*, "Geometry and Physics", Lecture Notes in Pure & Appl. Math. **184**, M. Dekker, New York, 1996, 549–556.

[H]   P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–246.

[HT]   J. Hoste and M. Thistlethwaite, *KnotScape*, a knot polynomial calculation program, available at `http://www.math.utk.edu/~morwen`.

[J]   V. F. R. Jones, *A polynomial invariant of knots and links via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[J2]   —— " ——, *Hecke algebra representations of of braid groups and link polynomials*, Ann. of Math. **126** (1987) 335–388.

[Kn]   T. Kanenobu, *Examples of polynomial invariants of knots and links I*, Math. Ann. **275** (1986), 555–572.

[Ka]   L. H. Kauffman, *Knots and physics* (second edition), World Scientific, Singapore 1993.

[Ka2]   —— " ——, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[Ka3]   —— " ——, *New invariants in the theory of knots*, Amer. Math. Mon. **3** (1988), 195–242.

[Ki]   R. Kirby (ed.), *Problems of low-dimensional topology*, book available on `http://math.berkeley.edu/~kirby`.

[K]   J. Kneissler, *The number of primitive Vassiliev invariants up to degree twelve*, University of Bonn preprint, June 1997. See also `q-alg/9706022`.

[La]   S. Lambropoulou, *Solid torus links and Hecke algebras of B-type*, Proceedings of the Conference on Quantum Topology (D. N. Yetter, ed.), World Scientific, 1994, 225–245.

[LL]   W. B. R. Lickorish and A. S. Lipson, *Polynomials of 2-cable-like links*, Proc. Amer. Math. Soc. **100** (1987), 355–361.

[LM]   —— ” —— and K. C. Millett, *A polynomial invariant for oriented links*, Topology **26 (1)** (1987), 107–141.

[MT]   W. W. Menasco and M. B. Thistlethwaite, *The Tait flyping conjecture*, Bull. Amer. Math. Soc. **25 (2)** (1991), 403–412.

[Mo]   H. R. Morton, *Seifert circles and knot polynomials*, Proc. Camb. Phil. Soc. **99** (1986), 107–109.

[MT]   —— ” —— and P. Traczyk, *The Jones polynomial of satellite links around mutants*, In 'Braids', (Joan S. Birman and Anatoly Libgober, eds.), Contemporary Mathematics **78**, Amer. Math. Soc. (1988), 587–592.

[Mu]   K. Murasugi, *Jones polynomial and classical conjectures in knot theory*, Topology **26** (1987), 187–194.

[Mu2]   —— ” ——, *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326 (1)** (1991), 237–260.

[PV]   M. Polyak and O. Viro, *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Notes **11** (1994), 445–454.

[PV2]   —— ” —— and —— ” ——, *On the Casson knot invariant*, preprint `math.GT/9903158`.

[PP]   T. Przytycka and J. Przytycki, *Subexponentially computable truncations of Jones-type polynomials*, in "Graph Structure Theory", Contemporary Mathematics **147** (1993), 63–108.

[PP2]   —— ” —— and —— ” ——, *Subexponentially computable truncations of Jones-type polynomials*, with appendix on Vertigan's algorithm. Preprint **22**, Odense Univ., 1992.

[P]   J. H. Przytycki, *Search for different links with the same Jones' type polynomials: Ideas from graph theory and statistical mechanics*, Panoramas of Mathematics, Banach Center Publications, Vol. **34**, Warszawa 1995.

[P2]   —— ” ——, *Equivalence of cables of mutants of knots*, Canad. J. Math., **41 3(2)**, 1989, 250–273.

[P3]   —— ” ——, *Symmetric knots and billiard knots*, in "Ideal Knots", Series on Knots and Everything, Vol. **19** (A. Stasiak, V. Katrich, L. Kauffman, eds.), World Scientific, 1999, 374–414.

[Ro]   D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[S]   Bjarne Stroustrup, *The C++ Programming Language*, Addison-Wesley, 1995.

[St]   A. Stoimenow, *On Harrison cohomology and a conjecture by Drinfel'd*, diploma thesis, Humboldt University Berlin, 1996.

[St2]   —— ” ——, *Program for computing Gauß sums*, C++ files available as `gsinv.tgz` on my web page.

[St3]   —— ” ——, *Gauß sum invariants, Vassiliev invariants and braiding sequences*, Jour. of Knot Theory and its Ramifications **9(2)** (2000), 221–269.

[St4]   —— ” ——, *Positive knots, closed braids and the Jones polynomial*, preprint `math/9805078`.

[St5]   —— ” ——, *On some restrictions to the values of the Jones polynomial*, Humboldt University Berlin preprint, October 1997.

[St6]   —— ” ——, *Some minimal degree Vassiliev invariants not realizable by the HOMFLY and Kauffman polynomial*, C. R. Acad. Bulgare Sci. **54(4)** (2001), 9–14.

[St7]   —— ” ——, *Mutant links distinguished my degree-3-Gauß sums*, Proceedings of the International Conference on Knot Theory "Knots in Hellas, 98", Series on Knots and Everything **24**, World Scientific, 2000.

[St8]   —— ” ——, *Knots of genus one*, Proc. Amer. Math. Soc. **129(7)** (2001), 2141–2156.

[Th]   M. B. Thistlethwaite, *A spanning tree expansion for the Jones polynomial*, Topology **26** (1987), 297–309.

[Va]   V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Vo]   P. Vogel, *Algebraic structures on modules of diagrams*, Université Paris VII preprint, June 1996, to appear in Inv. Math.

[W]   J. Weeks, *Documentation for SnapPea Vers. 2.0 Projection file format*, available from the author of the program.

[Wi]   Simon Willerton, *Cabling the Vassiliev invariants*, preprint.

[Wo]   S. Wolfram, *Mathematica — a system for doing mathematics by computer*, Addison-Wesley, 1989.

[Ya]   S. Yamada, *The minimal number of Seifert circles equals the braid index*, Invent. Math. **88** (1987), 347–356.