# RATIONAL KNOTS AND A THEOREM OF KANENOBU

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Current version: September 13, 2000   First version: August 18, 1999

**Abstract.** We give examples where Kanenobu's rationality criterion fails.

*AMS subject classification:* 57M25 (primary).

## 1. Introduction

The present note is devoted to the following theorem of Kanenobu [K]:

**Theorem 1.1** (Kanenobu [K]) If $K$ is a rational (2-bridge) knot (or link), then

$$\left(-u-u^{-1}\right)\,\left[Q_K(-u-u^{-1})-1\right] = 2[V_K(u)V_K(u^{-1})-1]\,, \tag{1}$$

where $V$ is the polynomial of [J] and $Q$ this of [BLM, Ho].

Apart from its elegance, the formula (1) attracted my attention in particular because it provides a simple criterion to decide about the non-rationality of a knot (apart from considering Schubert's classification [S] or knot group arguments).

In the following, we will construct some infinite series of knots, which are not rational, in fact even non-alternating, but for which (1) is satisfied. These examples have been suggested by empirical calculations (explained subsequently), which nevertheless reveal (1) to be a surprisingly powerful test.

## 2. A systematic collection of examples

The first series of examples we construct suggested by the empirical calculations are basically due to Joan Birman. (We denote by $\sigma_i$ the Artin braid group generators and by $\Delta=\sigma_2\sigma_1\sigma_2$ the square root of the generator of the center of the 3 strand braid group $B_3$, as well as by $[\alpha]$ the exponent sum of $\alpha$, and by $w$, $g$ and $c$ the writhe, genus and crossing number.)

**Proposition 2.1** Let $\alpha\in B_3$ be a 3-braid of the form (i) $\sigma_1^{6k\mp 1}\sigma_2^{\pm 1}$ or (ii) $\sigma_1^{6k+1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ for $k\in\mathbb{N}$, $k>0$ sufficiently large. Then the knot $K=\left(\Delta^{4k}\alpha^{-1}\right)\hat{}$ satisfies (1), but is not rational, even non-alternating. Moreover, infinitely many knots arise this way.

*Supported by a DFG postdoc grant.

The proof we give here is of knot-theoretical flavour, but a more generally applicable argument will be given later.

**Proof.** By [Bi, proposition 2] $K$ and $\hat{\alpha}$ have the same $P$ [H] (so in particular $V$ and $\Delta$ [Al]) polynomial (such paris of knots we will subsequently call Birman pairs), and as $[\Delta^{4k}\alpha^{-1}] = [\alpha]$ we have by [K, theorem 2] that $Q(\hat{\alpha}) = Q(K)$. Thus, as $\hat{\alpha}$ is evidently 2-bridge, (1) holds for $K$. We show now that $K$ is non-alternating. Assume that $K$ were alternating. As $\hat{\alpha}$ is a closed alternating braid, we have that $\Delta(K) = \Delta(\hat{\alpha})$ is a monic polynomial. But by Murasugi [Mu3] (see also [Cr, corollary 5.3]) such a link is fibered, and therefore, by [Mu2, theorem A] the inequality of Morton-Williams-Franks [Mo, FW] is sharp on $K$.

In case $\alpha$ is of type (i), we would then have $K$ to be a $(2,n)$ torus link, and because of $V(K) = V(\hat{\alpha})$ we would have $K = \hat{\alpha}$. But (basically as observed by Birman) Murasugi's formulas [Mu, §9-11] show that for $k$ sufficiently large $\sigma(K) \neq \sigma(\hat{\alpha})$, a contradiction.

In case $\alpha$ is of type (ii), the (closed braid) diagram $\hat{\alpha}$ is reduced, and the Morton-Williams-Franks bound for both $K$ and $\hat{\alpha}$ is sharp and it is 3. Then by [Mu2, corollary 2] a reduced alternating diagram $D$ of $K$ must have the same crossing number as the diagram $\hat{\alpha}$. Therefore, as

$$g(D) = g(K) = \max\deg\Delta(K) = \max\deg\Delta(\hat{\alpha}) = g(\hat{\alpha}),$$

the number of Seifert circles of $D$ is the same as this of $\hat{\alpha}$, namely 3. But by Morton's inequalities [Mo] the $P$ polynomial determines the writhe of a diagram of minimal number of Seifert circles, if the inequality of Morton-Williams-Franks is sharp. Therefore $[\alpha] = w(\hat{\alpha}) = w(D)$. As $D$ as a diagram with 3 Seifert circles can be made into a braid diagram by at most one Vogel move [Vo] and $\sigma$ changes at most by 2 under a crossing change, the remark after proposition 11.1 of [Mu] shows $|\sigma(K) - \sigma(\hat{\alpha})| \leq 2$. However, by Murasugi's signature formulas, for $k$ large enough $|\sigma(K) - \sigma(\hat{\alpha})|$ also gets large enough, a contradiction.

Finally, to show that infinitely many of the knots are distinct, let $k \to \infty$ and use again Murasugi's signature formulas showing $\sigma \to \infty$. $\square$

## 3. An empirical approach

A more realistic estimate for the quality of (1) as a rationality test can be obtained by examining the tables of Thistlethwaite [HT].

First, (1) detected all non-rational prime knots of Rolfsen's [Ro] tables (which are easy to identify from the Conway notation recorded there). For $\geq 11$ crossing knots, Thistlethwaite does not specify which knots in his tables are rational, but the number of such knots for given low crossing number can be obtained by computer in a few seconds by enumerating iterated fractions arising from compositions of the crossing number into the entries of the Conway notation, and considering (only) fractions $p/q$ with $p,q \in \mathbb{N}$ mutually prime and $p$ odd up to the equivalence $p/q_1 \sim p/q_2 \iff q_1 q_2^{\pm 1} = \pm 1$ in $\mathbb{Z}_p^*$ (see, e. g., [K2]). The numbers are

| crossing number | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|
| number of rational knots | 91 | 176 | 352 | 693 | 1387 | 2752 |

(A general formula for these numbers has been proved by Ernst ans Sumners in [ES].)

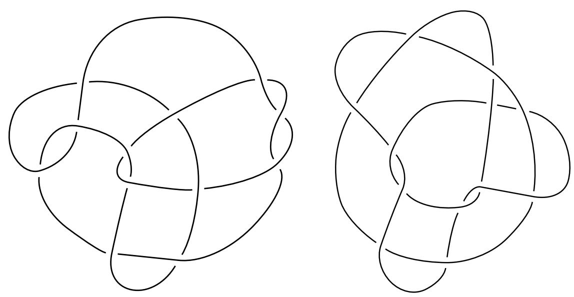
Considering prime alternating knots, I found that the number of knots satisfying (1) coincides up to 16 crossings with the one of the above table, showing that (1) decides about rationality of *any* such knot.

A further check showed (1) to be violated for any composite knot of at most 16 crossings, assuming that so far the crossing number is additive under connected sum and taking from a prime knot and its obverse only one as a factor, as mirroring a composite factor does not change either of the expressions on both hand-sides of (1). Note, that by [Me], [Ki] and [Th], (1) is easily shown to be violated by any *alternating* composite knot, by comparing the edge coefficients.

It is clear that among the non-alternating knots examples should occur, and the simplest ones are two knots of 12 crossings, $12_{2037}$, building a (rather famous, see [LM, example 17]) Birman pair[1] with $7_1$, and $12_{1879}$. The complete list of exceptions (recorded in Dowker-Thistlethwaite [DT] notation) up to 15 crossings is given below:

[1]For this particular pair the coincidence of the $Q$ polynomials was observed, without further explanation, already in [BLM].

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 1879 | 4 | 12 | -18 | 14 | -20 | 2 | 8 | -22 | -24 | -10 | -6 | -16 | | | |
| 12 | 2037 | 6 | 10 | 14 | -18 | 2 | -20 | 4 | 22 | 24 | -8 | -12 | 16 | | | |
| 13 | 7750 | 4 | 12 | 14 | -16 | 18 | -20 | 2 | -22 | 24 | -26 | -6 | 8 | -10 | | |
| 13 | 7960 | 4 | 12 | 16 | -22 | 14 | -20 | 2 | 8 | 24 | 26 | -10 | -6 | 18 | | |
| 14 | 33787 | 4 | 12 | 16 | -14 | -22 | 2 | -20 | 24 | -26 | -28 | -10 | -8 | 6 | -18 | |
| 14 | 43535 | 6 | 10 | 24 | -18 | 2 | -20 | -22 | 26 | 28 | -8 | -12 | 4 | 14 | 16 | |
| 14 | 44370 | 6 | 12 | 16 | -22 | -18 | 4 | -24 | 2 | -26 | -8 | -28 | -14 | -10 | -20 | |
| 14 | 46672 | 8 | -10 | 12 | -18 | 20 | -22 | 24 | 26 | -6 | 28 | -2 | 4 | 16 | 14 | |
| 14 | 46862 | 8 | -12 | -16 | -20 | 22 | -2 | 24 | -4 | 26 | -6 | 28 | 10 | 14 | 18 | |
| 15 | 157719 | 4 | 12 | 14 | -16 | 18 | -28 | 2 | -22 | 24 | -26 | -30 | -6 | 8 | -10 | -20 |
| 15 | 168643 | 4 | 12 | 18 | 14 | 24 | 22 | 2 | -26 | 6 | -28 | -30 | 10 | 8 | -16 | -20 |
| 15 | 233158 | 6 | 12 | 16 | -24 | -20 | -26 | 4 | -28 | 2 | -10 | -8 | -30 | -18 | -14 | -22 |
| 15 | 247180 | 6 | 16 | 14 | 20 | -26 | 18 | -24 | 4 | 2 | 10 | 28 | 30 | -12 | -8 | 22 |



$12_{1879}$ $12_{2037}$

**Figure 1:** The two simplest knots, for which (1) fails as rationality test

Beside $12_{2037}$ proposition 2.1 explains two more of these examples, $14_{43535}$ (associated to $10_2$) and $14_{46862}$ (associated to the (2,11)-torus knot $11_{367}$).

For all 13 knots listed above non-alternation can be proved by the Kauffman [Ka] $F$ polynomial. In fact except for the Birman (pair) knots, and the 2 further examples $13_{7960}$, having the same $V$ and $Q$ (but not $\Delta$) polynomial as $5_2$, and $15_{168643}$, already the criteria for $V$ of [Ka2] and [Th, theorem 1] work.

The small number of exceptions compared to the total number of knots (given in [HTW]) testifies the quality of (1) as rationality test.

## 4. Some more series of examples

It may appear that the 10 knots of the above list outside of the scope of proposition 2.1 satisfy (1) by accidental coincidence. However, there are patterns underlying some of these remaining examples. Drawing the pictures, one reveals striking similarities between some of them, which can be extended to infinite series.

**Example 4.1** For example, the diagrams of $13_{7960}$ and $15_{247180}$ differ just by a $\bar{t}'_2$ move at the encircled crossing (see figure 2). Applying further $\bar{t}'_2$ moves we find that the next 8 diagrams still satisfy (1). Thus we are lead to conjecture that this will hold for the whole series of diagrams.

This can be shown by some messy calculation, or by the following analytic argument.

We need to show that

$$z[Q(z)-1] = 2\left[V\left(\frac{z+\sqrt{z^2-4}}{2}\right)\cdot V\left(\frac{z-\sqrt{z^2-4}}{2}\right)-1\right]$$

for the polynomials $Q_i$ and $V_i$ of the diagrams $D_i$ with $i$ twists. Considering the generating functions

$$g(x,z) := \sum_{i=0}^{\infty} Q_i(z)x^i$$

and

$$f(x,y,z) := \sum_{i,j=0}^{\infty} V_i\left(\frac{z+\sqrt{z^2-4}}{2}\right)\cdot V_j\left(\frac{z-\sqrt{z^2-4}}{2}\right)x^iy^j$$

(both series converge absolutely in a neighborhood of $(x,y,z)=(0,0,0)$ resp. $(x,z)=(0,0)$ because of the exponential growth of the polynomial coefficients in the crossing number) we find by the relations of the $Q$ and $V$ polynomial, that these are rational functions in $x$ and $y$ (with coefficients in the fraction field $\mathcal{F} = \mathcal{F}\,\mathbb{Z}[z,\sqrt{z^2-4}]$ of $\mathbb{Z}[z,\sqrt{z^2-4}]$) whose denominators have the form

$$g(x,z) = \frac{g_1(x,z)}{g_2(x,z)},$$

with $\deg_x g_2 \le 3$ and $(1-x)|g_2$ for $g$ and

$$f(x,y,z) = \frac{P(x,y,z)}{(1-x)(1-y)\big((z+\sqrt{z^2-4})^2x-4\big)\big((z-\sqrt{z^2-4})^2y-4\big)}$$

with $P\in\mathcal{F}[x,y]$, $\deg_x P$, $\deg_y P\le 2$ for $f$. To extract the diagonal part $i=j$ of $f$ we are interested in (we call this 'contracting'), we apply a usual trick from harmonic analysis, obtaining

$$\tilde f(t,z) = \int_0^1 f(\sqrt{t}e^{2\pi i u},\sqrt{t}e^{-2\pi i u},z)\,du = \frac{1}{2\pi i}\oint_{|u|=\sqrt{t}}\frac{1}{u}f\big(u,\frac{t}{u},z\big)\,du\,.$$

The curve integral can be calculated by evaluating the relevant residues for small $t$, namely

$$u=0,\quad u=t,\quad u=t\cdot\left(\frac{z-\sqrt{z^2-4}}{2}\right)^2,$$

obtaining rational expressions in $t$ with denominators composed by the following 5 factors:

$$t^2,\quad t-1,\quad \left(\frac{z+\sqrt{z^2-4}}{2}\right)^2t-1,\quad \left(\frac{z-\sqrt{z^2-4}}{2}\right)^2t-1\quad\text{and}\quad \left(\frac{z+\sqrt{z^2-4}}{z-\sqrt{z^2-4}}\right)^2t-1,$$

and with the numerators of at most the same degree in $t$ as this of the denominators.

Thus the identity we wish to show

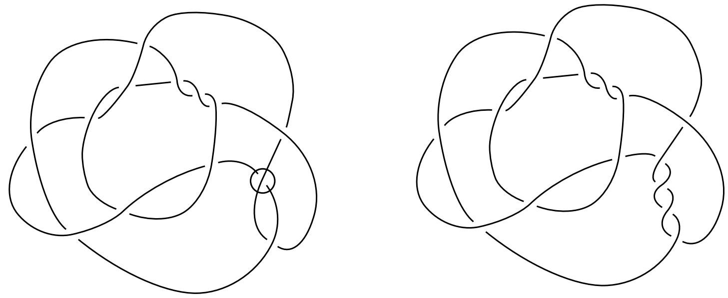
$$\frac{1}{1-t}+\frac{z}{2}\left[g(t,z)-\frac{1}{1-t}\right] = \tilde f(t,z)$$

turns, by multiplying by the LCM of the denominators (which is of degree 8 in $t$), into a polynomial identity of degree at most 8 in $t$. To show it, it suffices to show equality for the first 9 coefficients in the Taylor expansion, which correspond to the first 9 diagrams, the ones we checked above.

**Example 4.2** The knots $13_{7750}$ and $15_{157719}$ on figure 3 differ just by two local replacements of a crossing by a parallel clasp. If we repeat this procedure, adding the same number of crossings at both places, we obtain 6 further diagrams satisfying (1).

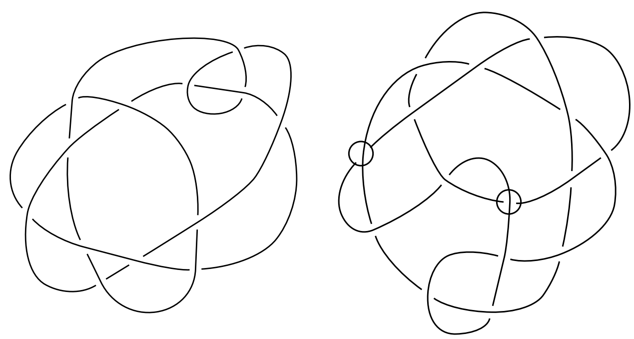
Again, considering this series as the diagonal part of the 2-parameter series $D_{i,i'}$ (with respectively $i$ and $i'$ half-twists inserted: note that for $i+j$ odd these are 2 component link diagrams), one can build

$$f(x,x_1,y,y_1,z) := \sum_{i,j,i',j'\ge 0} V_{i,i'}\left(\frac{z\pm\sqrt{z^2-4}}{2}\right)V_{j,j'}\left(\frac{z\pm\sqrt{z^2-4}}{2}\right)x^ix_1^{i'}y^jy_1^{j'}$$



$13_{7960}$ $15_{247180}$

**Figure 2:** Two examples of non-alternating knots satisfying (1), differing just by a $\vec{t}_2'$ move.



$13_{7750}$ $15_{157719}$

**Figure 3:** Two further similar examples: smoothing out the encircled crossings on the right gives the knot with the left diagram.

and contract 3 times, obtaining a polynomial in $t$ with coefficients lying in some higher-degree algebraic extension of $\mathcal{F}$ ; likewise one builds the corresponding series for $Q$,

$$g(x,x_1,z) := \sum_{i,i'\geq 0} Q_{i,i'}(z)x^i x_1^{i'},$$

and contracts once. One can then show the general case by some finite number of checks (or by some even messier direct calculation).

Instead we show that the knots $D_{i,j}$ are non-alternating (something we would need to show also in the previous example, but which is then a special case of the argument given in the following lines).

We consider the maximal $z$-degree of the Kauffman polynomial and check that for $p+q=6,7$, $p,q>0$ it is $c(D_{p,q})-4$, and that the maximal coefficient of $z$ is of the form $\pm a^k \mp a^{k+4}$ for some $k\in\mathbb{Z}$, which exhibits non-alternation (see [Ka, p. 426-427]). For $p+q>7$, $p,q>0$, the same property follows by induction on $p+q$ by applying the Kauffman relation near a crossing $p$ in the box with $q$ twists with $q\geq p$ and using the general inequality $\max\deg_z F(K)\leq c(K)-1$ for any non-trivial link $K$, applying it on the diagram on which the crossings in the twist box have become nugatory.

There is a further similarity of diagrams, between $14_{44370}$ and $15_{233158}$ (this time involving local changes at 3 crossings), but in this case I could not extend it to an infinite series.

## 5. Questions

We conclude by summarizing the problems suggested by empirical evidence.

**Question 5.1** Is there a composite knot satisfying (1)?

**Question 5.2** Is there a non-rational alternating knot which satisfies (1) (it would need to be prime)?

**Question 5.3** Is there a non-rational knot with the $F$ polynomial of a rational knot?

It should be pointed out, that among rational (and also non-rational, see [L]) knots duplications of $F$ are well-known and have been tabulated by Kanenobu.

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