SOME EXAMPLES RELATED TO KNOT SLICENESS

A. Stoimenow*

Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan
e-mail: stoimeno@kurims.kyoto-u.ac.jp
WWW: http://www.kurims.kyoto-u.ac.jp/~stoimeno/

Abstract. It is known that the linking form on the 2-cover of slice knots has a metabolizer. We show that several weaker conditions, or some other conditions related to sliceness, do not imply the existence of a metabolizer. We then show how the Rudolph-Bennequin inequality can be used indirectly to prove that some knots are not slice.

1. Introduction and statement of results

Algebraic topology has attempted the study of topological equivalence problems by means of algebraic invariants. In the case of classical knots (knots in 3-space), much of the theory concerns such invariants derived from abelian coverings of the knot complement. The algebraic information of these coverings is contained in the Alexander module, the homology group of the infinite cyclic cover, on which the deck transformation acts. This module carries a bilinear form, the Blanchfield pairing. Despite being previously well known, the interest of classical abelian invariants is that only they remain easily and generally computable, and therefore practically useful, despite their various recently proposed (non-abelian [6, 16] and quantum [9]) modifications.

In this paper we will study some properties of algebraic invariants for slice knots, i.e. knots trivial in the topological concordance group. Our aim will be to find, partly by computation, knots that demonstrate the failure of possible implications between sliceness obstructions involving various algebraic invariants. We believe that it is useful to have concrete knots in hand to illustrate the occurring phenomena, even though theory may suggest the existence of such examples. The recent expansion of knot tables [13] and computational tools has led to a series of such examples related to other questions. In the present context, we are mainly concerned by the undue lack of computations, even for the (presumably easy to handle) abelian invariants. In the final section, we will turn to smooth concordance, and give some related examples.

1.1. Topological concordance

Classical (topological) knot concordance was introduced by Milnor and Fox [7] in the 1950s. The decision whether two knots are equivalent in that sense is a longstanding problem, related to singularity theory, and the classification of topological four-manifolds. Levine [18] made substantial early progress in the late 1960s, by introducing an algebraic structure called the algebraic knot concordance group, and using it to solve the problem in dimension at least 4. While the algebraic concordance group is of fundamental importance also in dimension 3, Casson and Gordon [5] showed that Levine's homomorphism from the classical to the algebraic concordance group has a non-trivial kernel.

*Supported by 21st Century COE Program.

AMS subject classification: 57M25 (primary), 11E04, 11T30 (secondary),
Key words: quadratic form, finite commutative group, 4-genus, signature, slice knot, Alexander polynomial.

The algebraic knot concordance group C has a description within a Witt group of quadratic forms (see for example [42], mainly §5). Levine's approach is to consider the Witt group of isometric structures of the symmetrized Seifert form over $\mathbb{Q}$. (Working over $\mathbb{Z}$ is much harder, and the arising invariants were given later by Stoltzfus [49].) The Witt group splits along primes p in the ring $\mathbb{Q}[t]$ of polynomials, which are characteristic polynomials of the isometric structure. Embedding $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, we obtain an integer invariant (signature) for polynomials p (regarded now in $\mathbb{C}[t]$) with zeros on the complex unit circle S^1. Factoring out p (i.e. working over $\mathbb{Q}[t]/\langle p \rangle$), we have two $\mathbb{Z}_2$ valued invariants, the discriminant and Hasse-Witt invariant (latter is not a homomorphism). Levine shows that the vanishing of all these invariants implies the vanishing of the form in the Witt group (Theorem 21 in [20]) and that finite order elements in C have order 1, 2, or 4 (Proposition 22 (b) ibid.).

Now Levine defines, in [18], further algebraic knot concordance invariants by Tristram-Levine signatures, and the Alexander polynomial modulo Milnor-Fox factorizations. These must thus find themselves in the above set of complete invariants of C. In [28] the relation is shown between Tristram-Levine signatures and the real Witt group signatures. So by Proposition 22 (a) in [20] invariants detecting the (infinite rank) torsion-free part of C correspond exactly to Tristram-Levine signatures. The Milnor-Fox condition relates to the discriminants (see proposition 5 in [18]), and so if it holds, the algebraic concordance class of the knot is completely determined by the Hasse-Witt invariants. In theory these invariants can be realized non-trivially, but no examples seem to have been elaborated on.

1.2. Linking pairings and metabolizers

A different description of C is in terms of Witt classes of the Blanchfield pairing (valued in $\mathbb{Q}[t]$), modulo pairings with a self-annihilating submodule. There is a correspondence between this and Levine's approach, first proven by Cherry Kearton [15]. (Another source is the appendix A of Litherland's note [23], but beware of diverse typos in the published version.) Under the restriction $\mathbb{Z} \to \mathbb{Z}_n$ for any prime n, the Alexander module turns into the homology group of the n-fold branched cyclic cover, and the Blanchfield pairing determines a (non-singular) linking form on the torsion homology group of this cover. In this paper we will be concerned with the case $n = 2$.

Let K be a knot, D_K be its double branched cover, and λ the linking form on its $\mathbb{Z}$-homology group $H_1 = H_1(D_K)$ [21]. The (finite and odd) order of $H_1(D_K)$ is called the determinant $\det = \det(K)$ of K. The quadratic form λ takes values in $\mathbb{Z}/\det(K)$, which is identified with the subset of $\mathbb{Q}/\mathbb{Z}$ of fractions with denominator (dividing) $\det(K)$.

If K is slice (bounds a topological locally-flat disk in B^4), then it is algebraically slice, i.e. all algebraic concordance invariants of K vanish. This occurs if and only if the Blanchfield pairing has a self-annihilating submodule. So if the knot is (algebraically) slice, then the n-fold cover linking forms are metabolic as well. Specifically for $n = 2$ this means that then λ vanishes on a subgroup M of $H_1(D_K)$ of order $\sqrt{\det}$, equal to its annihilator. M is called a *metabolizer*. That $\det(K)$ ought to be a square is well-known from the condition of Milnor-Fox [7] that the Alexander polynomial is of the form $\Delta_K(t) = f(t)f(1/t)$ for some $f \in \mathbb{Z}[t]$, since $\det(K) = |\Delta_K(-1)|$. A further condition for K being slice is that its (Murasugi) signature σ [33] vanishes, and so do the generalized (or Tristram-Levine) signatures σ_ξ, when ξ is a unit norm complex number and $\Delta(\xi) \neq 0$. (We have $\sigma = \sigma_{-1}$.)

The metabolizer existence condition is useful in some theoretical situations, where the calculation of other invariants is more tedious. See for example [24, 25]. The present work is mainly motivated by the interest in concrete examples showing that this criterion is essential, in particular as opposed to the other conditions for sliceness. We also investigate the size of the isotropic cone Λ_0 of the linking form. We will find, in §3, examples illustrating possible phenomena concerning Λ_0. First we give in §3.2 computational examples, obtained from the tables of [12, 13], that show

Theorem 1.1 For each one of the three conditions below, there exist knots satisfying this condition, which have zero Tristram-Levine signatures and Bennequin numbers, and an Alexander polynomial of the Milnor-Fox form.

(a) Λ_0 is trivial, i.e. $\{0\}$,

(b) $1 < |\Lambda_0| < \sqrt{\det}$,

(c) $|\Lambda_0| \geq \sqrt{\det}$, but Λ_0 contains no subgroup of order $\sqrt{\det}$.

(Here 'det' refers to the above introduced number $\det(K) = |\Delta_K(-1)|$. The meaning of Bennequin numbers is related to smooth sliceness and explained in §4.)

In §3.3 we find, now applying more systematical constructions, examples that refine theorem 1.1. We make decisive use of the realization of any admissible Alexander polynomial by an unknotting number one knot. (This result was proved first by Sakai [41] and Kondo [17], and later by several other authors, with a very recent construction due to Nakamura [34].)

Theorem 1.2 For any of the properties (a)-(c) in theorem 1.1, we can find knots whose H_1 has additionally no $4k+3$ torsion (or whose determinant is not divisible by $4k+3$).

We can in fact classify all trivial cone forms on such groups H_1 (theorem 3.1), although it is not clear which forms are indeed realizable by knots.

The motivation for excluding $4k+3$ torsion lies in the structure of the Witt group of $\mathbb{Z}_p$-forms, and the possibility to rule out concordance order two, when there is such torsion. For $4k+3$ torsion, recent further-going work of Livingston-Naik [26, 27], gives in fact sufficient conditions on infinite concordance order. In §3.4 we will give examples where the isotropic cone can exclude concordance order two, but Livingston-Naik's criterion does not apply.

Theorem 1.3 For the properties (b) and (c) in theorem 1.1, we can find knots $K = \hat{K}\#\hat{K}$ so that the p-Sylow subgroup of $H_1(D_{\hat{K}})$ for any prime $p = 4k+3$ is not cyclic of odd p-power order.

For property (a) such knots do not exist. This follows from an exact description of trivial cone forms on $\hat{K}\#\hat{K}$ given in proposition 3.2 (which is similar to theorem 3.1, although all occurring forms are easily realizable).

1.3. Smooth concordance invariants

Knots are smoothly slice if they bound a smooth disk in four-space. In the 1980s, Andrew Casson, using deep results of Freedman and Donaldson, gave the first example of a topologically slice knot which is not smoothly slice. Such knots known by now remain scarce (they all have trivial Alexander polynomial), despite that some new candidates are recently suspected. The difficulty in exhibiting such examples clearly displays the problems with the methods we apply to study both types of concordance.

In recent years, the separation between topological and smooth concordance grew wider with a vast development of new techniques in the smooth category. One such is the inequality of Rudolph-Bennequin [3, 38, 39], which emerged in the early 1990s, and sometimes proves (smooth) non-sliceness. More recently, Ozsvath and Szabo [35] used Floer homology to define an invariant τ that detects some non-slice knots. This invariant behaves similarly to the (Murasugi) signature and simultaneously improves upon the Rudolph-Bennequin inequality. The Ozsvath-Szabo invariant, in turn, motivated Rasmussen [36] to define a (conjecturedly equivalent) signature-like invariant s from Khovanov homology.

In §4 we show how the inequality of Rudolph-Bennequin can prove that a knot K is non-slice, by applying this inequality on knots K' different from K. We also discuss the relation to the recent knot homological "signatures" of Ozsvath-Szabo-Rasmussen, and their status in the examples of §3.

In §5, we explain concludingly how to construct prime (in fact, hyperbolic and of arbitrarily large volume) knots with the previously chosen properties.

2. Preliminaries and notation

2.1. Knots, linking form and sliceness

In the following knots and links will be assumed oriented, but sometimes orientation will be irrelevant.

For a knot K, its *obverse*, or mirror image $!K$, is obtained by reversing the orientation of the ambient space. The knot K is called *achiral* (or synonymously *amphicheiral*), if it coincides (up to isotopy) with its mirror image, and *chiral* otherwise. When taking the knot orientation into account, we write $-K$ for the knot K with the reversed orientation. We distinguish among achiral knots K between $+$achiral and $-$achiral ones, dependingly on whether K is isotopic to $!K$ or $!-K$.

Prime knots are denoted according to [37, appendix] for up to 10 crossings and according to [12] for ≥ 11 crossings. We number non-alternating knots after alternating ones. So for example $11_{216} = 11_{a216}$ and $11_{484} = 11_{n117}$. We write $K_1 \# K_2$ for the connected sum of K_1 and K_2, and $\#^k K$ for the connected sum of k copies of K.

Let G be a finite group and let p be a prime. A *Sylow p-subgroup* of G is a subgroup H such that p does not divide $|G|/|H|$. If G is abelian, the *p-primary component* of G, written G_p, is the subgroup of all elements whose order is a power of p. It is the unique Sylow p-subgroup of G. By the *p-torsion subgroup* we mean the subgroup of all elements whose order is equal to p.

By D_K we denote the *double branched cover* of S^3 over a knot K. (See [5, 37].) By $H_1 = H_1(D_K) = H_1(D_K,\mathbb{Z})$ we denote its homology group over $\mathbb{Z}$. (The various abbreviated versions will be used at places where no confusion arises; H_1 will be used throughout the paper only in this context, so that, for example, when we talk of H_1 of a knot, always H_1 of its double cover will be meant.) H_1 is a finite commutative group of odd order. This order is called the *determinant* of a knot K, and it will be denoted as $\det = \det(K)$. (It generalizes to links L, by putting $\det(L) = 0$ to stand for infinite $H_1(D_L)$.) By the classification of finite commutative groups, H_1 decomposes into a direct sum of finite (odd order) cyclic groups $\mathbb{Z}_k = \mathbb{Z}/k = \mathbb{Z}/k\mathbb{Z}$; their orders k are called *torsion numbers*.

$H_1(K)$ is also equipped with a bilinear form $\lambda : H_1 \times H_1 \to \mathbb{Q}/\mathbb{Z}$, called the *linking form* (see [21, 32] for example). Since λ in fact takes values of the form $n/\det$ for $n \in \mathbb{Z}$, we can identify them with $\mathbb{Z}_{\det}$.

A knot K is called *slice* if it bounds a disk in B^4. Except for §4, and unless pointed out explicitly otherwise, we work in the topological category. In §4, we will consider smooth sliceness.

It is known that if K is topologically slice, then λ is *metabolic*. This means, there is a subgroup M of $H_1(D_K)$ of order $\sqrt{\det}$, which is equal to its annihilator

$$M^{\perp} = \{\, g \in H_1 : \lambda(g,h) = 0 \text{ for all } h \in M \,\}.$$

M is called a *metabolizer*. Whenever λ is non-degenerate, we have for any subgroup G of H that

$$|G^{\perp}| \cdot |G| = |H|. \tag{1}$$

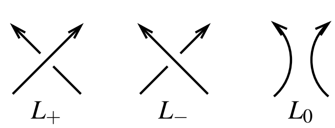
We also recall a few basic facts from number theory we will require in the study of λ. We apply for example Dirichlet's theorem on infinitely many primes contained in arithmetic linear progressions. We use also that every odd number n is the sum of two squares, if any only if every prime $p \equiv 3 \bmod 4$ has even multiplicity $2e$ as factor of n, and then p^e divides a and b in any solution of $a^2 + b^2 = n$. Such facts can be found in standard books on number theory; my personal favorites are [11, 55].

2.2. Knot polynomials and signatures

The *skein polynomial* P (introduced in [8]; here used with the convention of [22], but with l and l^{-1} interchanged) is a Laurent polynomial in two variables l,m of oriented knots and links, and can be defined by being 1 on the unknot and the *(skein) relation*

$$l^{-1}P(L_+) + l\,P(L_-) = -m\,P(L_0). \tag{2}$$

Herein $L_{\pm,0}$ are three links with diagrams differing only near a crossing.



(3)

We call the crossings in the first two fragments respectively *positive* and *negative*, and a crossing replaced by the third fragment *smoothed out*. A triple of links that can be represented as $L_{\pm,0}$ in (3) is called a *skein triple*. The sum of the signs (± 1) of the crossings of a diagram D is called the *writhe* of D and written $w(D)$. The smoothing of all crossings of D yields the *Seifert circles* of D; each crossing in D can be viewed as connecting two Seifert circles. Let $s(D)$ be the number of Seifert circles of D, and $s_-(D)$ be the number of those circles to which only negative crossings are attached. We call such Seifert circles *negative Seifert circles*.

The substitution $\Delta(t) = P(-i, i(t^{1/2} - t^{-1/2}))$ (with $i = \sqrt{-1}$) gives the (one variable) *Alexander polynomial* Δ, see [22]. It allows to express the determinant of K, as $\det(K) = \left|\Delta_K(-1)\right|$. The (possibly negative) minimal and maximal power of l occurring in a monomial of $P(K)$ is denoted $\min\deg_l P(K)$ and $\max\deg_l P(K)$. Alexander polynomials (and factors thereof) will sometimes be denoted by parenthesized list of their coefficients, putting the absolute term in brackets. As an example of such a notation, $1/t - 1 - t^2 = (\ 1\ [-1]\ 0\ -1)$.

The *signature* σ is a $\mathbb{Z}$-valued invariant of knots and links. Originally it was defined in terms of Seifert matrices [37]. We have that $\sigma(L)$ has the opposite parity to the number of components of a link L, whenever the determinant of L is non-zero (i.e. $H_1(D_L)$ is finite). This in particular always happens for L being a knot, so that σ takes only even values on knots.

Most of the early work on the signature was done by Murasugi [33], who showed several properties of this invariant. In particular the following property is known: if $L_{\pm,0}$ form a skein triple, then

$$\sigma(L_+) - \sigma(L_-) \quad \in \quad \{0,1,2\}\,, \tag{4}$$

$$\sigma(L_\pm) - \sigma(L_0) \quad \in \quad \{-1,0,1\}\,. \tag{5}$$

(Note: In (4) one can also have $\{0,-1,-2\}$ instead of $\{0,1,2\}$, since other authors, like Murasugi, take σ to be with opposite sign. Thus (4) not only defines a property, but also specifies our sign convention for σ.) We remark that for knots in (4) only 0 and 2 can occur on the right.

Let M be a Seifert matrix for a knot K, and $\xi \in S^1$ a unit norm complex number (S^1 denoting the set of such complex numbers). The *Tristram-Levine* (or generalized) *signature* $\sigma_\xi(K)$ of K is defined as the signature of the (Hermitian) form $M_\xi = (1-\xi)M + (1-\bar{\xi})M^T$, where bar denotes complex conjugation, and $\cdot^T$ means transposition. We call ξ and σ_ξ *non-singular* if their corresponding form M_ξ is so, that is, $\det(M_\xi) \neq 0$, which is equivalent to $\Delta_K(\xi) \neq 0$. For a fixed knot K we obtain a function $\sigma_*(K) : S^1 \to \mathbb{Z}$ given by $\xi \mapsto \sigma_\xi(K)$. It is called the Tristram-Levine signature function of K. We have $\sigma = \sigma_{-1}$, so Murasugi's signature is a special value of σ_*. If $\sigma_\xi(K)$ is non-singular, then it is even. (Since a knot has non-zero determinant, Murasugi's signature is always non-singular.) Also $\sigma_*(K)$ is locally constant around non-singular ξ, that is, it changes values ("jumps") only in zeros ξ of Δ_K. The properties (4) and (5) hold also for σ_ξ.

Signatures (at least all those we talk about in this paper) change sign under mirroring and are invariant under orientation reversal, and so vanish on amphicheiral knots. They are also additive under connected sum.

Let $g_t(K)$ be the topological *4-ball genus* of a knot K. Then it is known, by Tristram-Murasugi's inequality, that if σ_ξ is non-singular, then

$$|\sigma_\xi(K)| \leq 2g_t(K)\,. \tag{6}$$

So if K is topologically slice (that is, $g_t(K) = 0$), all non-singular σ_ξ vanish. Since a concordance between K_1 and K_2 is equivalent to the sliceness of $K_1\#-!K_2$, this implies that $\sigma_*(K)$ is a topological concordance invariant *outside* the zeros of the Alexander polynomial. That is, if $K_{1,2}$ are concordant, and $\Delta_{K_1}(\xi) \neq 0 \neq \Delta_{K_2}(\xi)$, then $\sigma_\xi(K_1) = \sigma_\xi(K_2)$. (In general one cannot say much about the behaviour of singular σ_ξ under concordance [19].)

Now more sophisticated methods are available to obstruct sliceness in certain cases, like Casson-Gordon invariants [5] and twisted Alexander polynomials [52, 16]. Indeed, the Milnor-Fox and Tristram-Murasugi conditions can be generalized to signatures and twisted Alexander polynomials of certain non-abelian representations of the knot group [16]. The general computability of such invariants is still difficult, though (see [50]). A similar disadvantage reveal, for *smooth* sliceness, the very recent knot homological (signature-like) concordance invariants τ and s of Ozsvath–Szabo and Rasmussen. The determination (or estimation) of these invariants is often easier indirectly, using their properties, rather than their definition. (We will later make a comment on their calculation in relation to the Rudolph-Bennequin inequality.)

We invite the reader to consult [44, 45, 46] for more on the use of notation and (standard) definitions.

3. The metabolizer criterion

3.1. Initial observations and remarks

When looking for a metabolizer M of λ, there are some simple, but important, observations to make.

First, M always exists, whenever H_1 is cyclic. This already restricts the search space for interesting examples, since about 80% of the prime ≤ 16 crossing knots with square determinant have cyclic (or trivial) H_1. It also suggests why composite knots (where H_1 is more often non-cyclic) are likely to be of interest.

The second, and for us more relevant, observation is that clearly any element in $g \in M$ must have $\lambda(g,g) = 0$. That is, each M is contained in the *isotropic cone*

$$\Lambda_0 := \{\, g \in H_1(D_K) : \lambda(g,g) = 0 \in \mathbb{Q}/\mathbb{Z} \,\}$$

of λ. Note that if the isotropic cone contains a subgroup G, then G is always isotropic (the linking form is zero on G); this is a bit more than a tautology, but follows from the identity

$$2\lambda(g,h) = \lambda(g+h, g+h) - \lambda(g,g) - \lambda(h,h)\,, \tag{7}$$

since we have no 2-torsion in $\mathbb{Z}_{\det}$. Thus a natural way to find (or exclude the existence of) M is to determine Λ_0 and seek for subgroups of H_1 of order $\sqrt{\det}$ contained in Λ_0. Any such subgroup is a metabolizer (that is, equal to its annihilator) because of (1).

A third observation is that the linking form λ, restricted to the p-torsion subgroup of H_1 for a prime p, modulo metabolic forms, naturally defines an element in the Witt group of nonsingular $\mathbb{Z}_p$ forms. This group is either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$, depending on whether $p \equiv 1$ or 3 mod 4 (see lemma 1.5, p. 87 of [29]). Thus, if $4k+3$-torsion exists in $H_1(\hat{K})$, one may detect $\hat{K}$ being 4-torsion in the algebraic concordance group. The other invariants (signatures, Alexander polynomial, etc.) can only detect (up to) 2-torsion. This suggests that if $\sigma(\hat{K}) = 0$ and $4k+3 \mid \det(\hat{K})$, then $K = \hat{K}\#\hat{K}$ may be an example whose non-sliceness is only detectable by the linking form. Therefore, in the search for interesting examples $K = \hat{K}\#\hat{K}$, we are led to consider (prime) knots $\hat{K}$ with $\sigma = 0$ and determinant divisible by $4k+3$.

In the case of $4k+3$-torsion, there is, though, Livingston and Naik's result [26], that if a prime $p = 4k+3$ has single multiplicity as divisor in det, then the knot has infinite order in the classical knot concordance group. In [27] Livingston and Naik generalized their result to groups H_1 whose Sylow p-subgroup is cyclic of odd power order. Therefore, for interesting non-slice examples $K = \hat{K}\#\hat{K}$, we should consider in particular knots $\hat{K}$ with determinant divisible by a prime $p = 4k+3$, but whose Sylow p-subgroup of H_1 is not of the stated type for any such p.

For the computational part in the examples in §3, we applied the computer program for calculating the linking form [44]. This program was written in C originally by Thistlethwaite, and later extended by myself. It calculates the torsion numbers of H_1 and the corresponding generators out of a Goeritz matrix [10] of a knot diagram. The further algebraic processing was done with MATHEMATICA™ [54]. Still computation alone often did not suffice to find proper examples, and we were led to argue about their (non-)existence mathematically.

From the explanation in §1.2 is it clear that Witt group invariants of the forms on all n-fold covers could tell something about (algebraic) non-sliceness. But, again, we are unaware of any concrete computations, even for the very restricted case $n = 2$ we consider. The study of higher n, from such computational point of view, is certainly also worthwhile, and may be a future project. (It is the lack of description in terms of Goeritz matrices of higher cyclic cover homology groups that prevented from their study.) It seems unclear (and may not be true) that the Witt class of all finite cover forms recovers the Witt class of the Blanchfield pairing.

3.2. Examples with trivial, small or large cone

The first series of examples shows that the existence of M is essential as opposed to the previously mentioned conditions on Alexander polynomial, Rudolph-Bennequin numbers, and signature.

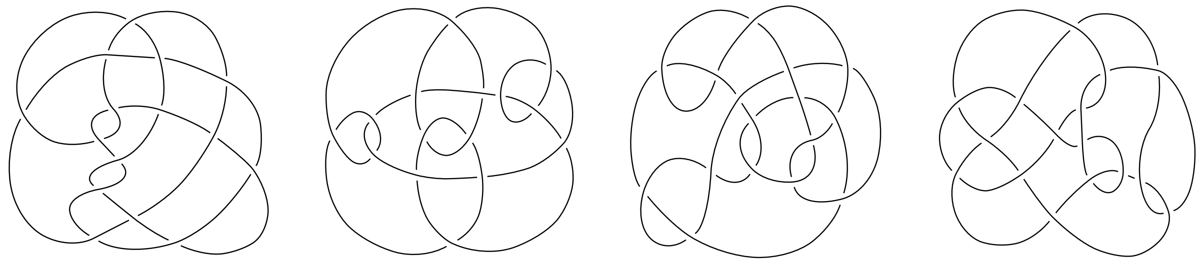
Example 3.1 Consider the knots on figure 1. The first (alternating) knot 15_{77828} has $\sigma = 0$ and Alexander polynomial

$$\Delta = (\ 1\ \ -8\ 32\ \ -82\ 152\ \ -216\ [243]\ \ -216\ 152\ \ -82\ 32\ \ -8\ 1\ ),$$

which is of the Milnor-Fox form $f(t)f(1/t)$ with $f = (\ [1]\ \ -4\ 8\ \ -9\ 8\ \ -4\ 1)$. (We explained in the appendix of [45] that for given Δ only finitely many f come in question, and one can make the search for f very efficient. Thus the Milnor-Fox test is easy to perform.) The Rudolph-Bennequin inequality (in the *smooth* setting; see §4, and in particular remark 4.1) is also trivial on its 15 crossing diagrams. This knot has $H_1 = \mathbb{Z}_{35} \oplus \mathbb{Z}_{35}$. Two particular generators $g_1 = (1,0)$ and $g_2 = (0,1)$ of the cyclic factors have

$$\lambda(g_1,g_1) = 8/35,\ \ \lambda(g_1,g_2) = 18/35,\ \ \lambda(g_2,g_2) = 4/35.$$

This shows that $\Lambda_0 = \{(0,0)\}$. Hence 15_{77828} is not slice. The knots 15_{158192}, 16_{705153}, 16_{747143} and 16_{850678}, three of which also appear in figure 1, are of similar nature. They all have $\Delta = (\ 1\ \ -10\ 43\ \ -100\ [133]\ \ -100\ 43\ \ -10\ 1)$ of Milnor-Fox form, $H_1 = \mathbb{Z}_{21} \oplus \mathbb{Z}_{21}$, and trivial Λ_0. Particularly interesting is 16_{850678}, because it is $-$achiral. Thus for it all (including the singular) Tristram-Levine signatures vanish. (Similarly, as will follow from the explanation in §4, the Rudolph-Bennequin method fails – with certainty; quite likely it also fails for the others.)



15_{77828} 15_{158192} 16_{705153} 16_{850678}

Figure 1: Knots with Milnor-Fox condition on the Alexander polynomial and signature 0, but with trivial isotropic cone of the linking form. The last one, 16_{850678}, is $-$achiral (and fibered of genus 4).

Example 3.2 The fact that in the above series Λ_0 is always trivial originally led to suspect that when Milnor-Fox holds and $\sigma = 0$, then $\Lambda_0 \neq 0$ may already imply $\Lambda_0 \supset M$. There is no reason why this should be true, but the examples found to refute it required a considerable quest. There were a total of 10 prime knots of ≤ 16 crossings (all alternating 16 crossing knots), for which determinant is a square, $\sigma = 0$ and $\Lambda_0 \neq 0$, but $\Lambda_0 \not\supset M$. Seven of them (three of which are shown in figure 2, and are the knots to the left) have $\Delta(t) = f(t)f(1/t)$. For all 10 knots $H_1 = \mathbb{Z}_{135} \oplus \mathbb{Z}_{15}$, with $\Lambda_0 = \{(0,\ 0),\ (45,\ 0),\ (90,\ 0)\}$, which is still too small.

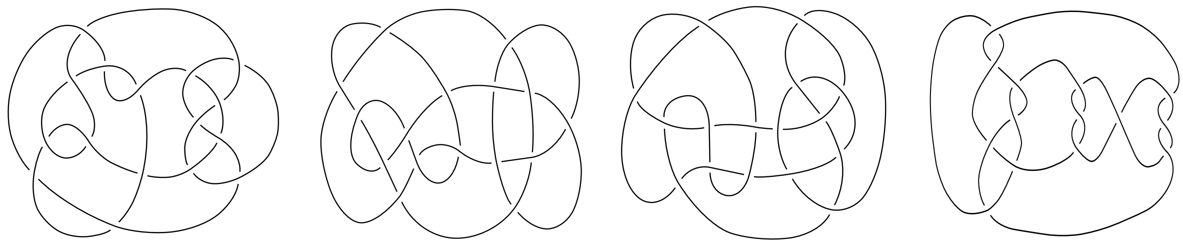
Thus a further-going and more complicated question is what occurs if we do not desire that $\Lambda_0 \supset M$ is excluded already because of cardinality reasons, that is, if $|\Lambda_0| \geq \sqrt{\det}$.

Example 3.3 Consider the knot $K = 3_1\#3_1\#!14_{16777}$. (The knot $!14_{16777}$ is given on the right in figure 2.) We have $\sigma(14_{16777}) = 4$ and $\Delta(14_{16777}) = \Delta(3_1)^2 f(t)f(1/t)$, where $f(t) = -2+2t-2t^2+t^3$. Clearly $\sigma(K) = 0$, but since $f(t)$ has no zeros on the unit circle, we can conclude that even all non-singular (i.e. not corresponding to the zeros of $\Delta(3_1)$) Tristram-Levine signatures of K vanish. We have $\det = 3969$, and $H_1 = \mathbb{Z}_{147} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. It turns out that $|\Lambda_0| = 231 > 63 = \sqrt{\det}$, but Λ_0 contains no subgroup of order 63.

A more complicated knot with simpler factors is $K' = 5_2\#8_{18}\#!9_{12}$. There we have

$$\Delta(t)\ =\ (1-t+t^2)^2 \cdot (1-3t+t^2)^2 \cdot (2-3t+2t^2)^2/t^6\,.$$

Since both $1-t+t^2$ factors come from (the Alexander polynomial of) 8_{18}, which is amphicheiral, one can similarly conclude that all non-singular Tristram-Levine signatures of K' vanish. Now $H_1 = \mathbb{Z}_{105} \oplus \mathbb{Z}_{105}$, and $|\Lambda_0| = 117$, but Λ_0 contains no subgroup of order 105.



16_{79084} 16_{95785} 16_{288707} $!14_{16777}$

Figure 2: Knots providing examples with the Milnor-Fox condition on the Alexander polynomial and signature 0, and with non-trivial isotropic cone of the linking form, which still does not contain a metabolizer. The first three have $|\Lambda_0| = 3$, while the last one's connected sum with $3_1\#3_1$ has $|\Lambda_0| \geq \sqrt{\det}$.

Remark 3.1 The most general condition under which one can seek a large metabolizer, of course, is when $\det(K)$ is a square. (Otherwise, the definition of a metabolizer does not make sense.) It turned out that even in this most general setting, among *prime* knots up to 16 crossings, there was a single example, and it does not fall into any of the further specified categories. This example is 15_{197573}. (This knot has $\sigma = -4$ and $\Delta(t) \neq f(t)f(1/t)$ and 15 crossing diagrams with non-trivial Rudolph-Bennequin numbers.) It has $H_1 = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and $|\Lambda_0| = 21$, but Λ_0 contains no subgroup of order 9.

It is also interesting to ask what values $|\Lambda_0|$ can attain. In particular how large can $|\Lambda_0|$ become for given determinant? Can $|\Lambda_0|$ be relatively prime to det? Obviously it does not need to share all determinant's prime divisors.

3.3. Examples with no $4k+3$-torsion

For some explicitly named knots of 11 or more crossings that occur in the examples of §3.3 and §3.4, see figure 3.

Example 3.4 The search among connected sums allows also to find knots K, where H_1 has no $4k+3$-torsion. They were motivated by the remarks about the Witt group in §3.1. One least crossing number example we found is $K = 6_3\#12_{1152}$. (Here $H_1 = \mathbb{Z}_{65} \oplus \mathbb{Z}_{65}$.) One verifies similarly all Alexander polynomial and signature conditions, but $|\Lambda_0| = 1$.

All other examples we found have trivial Λ_0 as well. However, there is a systematic way of constructing knots with large Λ_0, which we explain.

Proposition 3.1 There are knots K with $4k+3 \nmid \det$, with $\Delta = f(t)f(1/t)$ and zero Tristram-Levine signatures, with $|\Lambda_0| > \sqrt{\det}$ but no metabolizer.

Proof. Let $d \equiv 1 \bmod 4$ have no $4k'+3$ divisor, and let it contain any prime with multiplicity one. Let $\Delta(t) \in \mathbb{Z}[t,t^{-1}]$ be some polynomial with $\Delta(t) = \Delta(1/t)$, $\Delta(1) = 1$, no zero on the unit circle and $\Delta(-1) = d$. That such Δ exists is easy. Consider the base-4-expansion

$$d = \sum_{i=0}^{n} e_i 4^i$$

of d (with $0 \leq e_i \leq 3$ and $e_0 = 1$), and take $\Delta(t) = \nabla(t^{1/2} - t^{-1/2})$ where $\nabla(t) = \sum e_i(-t^2)^i$. By Kondo's result [17] there is an unknotting number one knot K with $\Delta(K) = \Delta$. By [53], $H_1(K)$ is cyclic, and by [21, 44], there is a generator g of H_1 with $\lambda(g,g) = 2/d$ (note that $\sigma(K) = 0$).

The prime condition assures that H_1 has no elements of non-trivial prime power order, and then the metabolicity of λ is equivalent to the metabolicity of its reductions on the p-torsion subgroups, for all primes p dividing d. Then by the Witt group argument, $\lambda \oplus \lambda$ on $H_1 \oplus H_1$ is metabolic.

Since $\lambda \oplus \lambda$ is metabolic, $|\Lambda_0(K^{\#2})| \geq \det(K) = d$. Now, because of (7),

$$\Lambda_0(K^{\#4}) \supset \Lambda_0(K^{\#2}) \oplus \Lambda_0(K^{\#2}),$$

and $|\Lambda_0(K^{\#4})| \geq d^2$. We want to show now that this inequality is strict, and so we must show that the inclusion is proper. Now, when $d = 4k+1$ has no $4k'+3$ divisors, there are a and b relatively prime to d with $a^2+b^2 = d$. Then $(a,0,b,0)$ lies in $\Lambda_0(K^{\#4})$ but not in $\Lambda_0(K^{\#2}) \oplus \Lambda_0(K^{\#2})$.

Then for any knot K_0 with non-metabolic λ, the knot $K_0\#K^{\#2k}$ will also have non-metabolic λ, but for k large enough $|\Lambda_0| > \sqrt{\det}$. □

In the case of trivial cone, we can in fact classify the forms algebraically in a slightly more general situation. This explains the nature of the knots found computationally in example 3.4.

Theorem 3.1 Assume H is a finite commutative group of odd order d, and that d has no $4k+3$ divisors. Let $\lambda : H \times H \to \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form with trivial cone Λ_0. Then (and only then) $H = \mathbb{Z}_{q'} \oplus \mathbb{Z}_q$, where q' is a product of distinct primes $p_i = 4l_i+1$, and $q \mid q'$, and for each prime $p_i \mid q$, there is a basis (g_1, g_2) of the p_i-Sylow subgroup $\mathbb{Z}_{p_i} \oplus \mathbb{Z}_{p_i}$ of H such that with (e_1,e_2) denoting $e_1g_1 + e_2g_2$ we have

$$\lambda\big((e_1,e_2),(e_1,e_2)\big) = e_1^2 + \beta e_2^2, \tag{8}$$

where β is not a square residue mod p_i.

Proof. Of course if $\Lambda_0 = 0$, then λ is non-degenerate. We have Λ_0 if and only if $[\Lambda_0]_{p_i} = 0$, where $[\Lambda_0]_{p_i}$ is the reduction of Λ_0 on the p_i-Sylow subgroup (or p_i-primary component) H_{p_i} of H.

Fix a prime $p = p_i \mid d$. If $\mathbb{Z}_{p^2} \subset H_p$, then one easily finds non-trivial elements in $[\Lambda_0]_p$. Thus $H_p = \mathbb{Z}_p^{\oplus k}$. If $k=1$ then $[\Lambda_0]_p = 0$, since λ does not degenerate on H_p. Consider then $k=2$. Then $H_p = \mathbb{Z}_p \oplus \mathbb{Z}_p$ with a basis (g_1,g_2). We write (e_1,e_2) for $e_1g_1+e_2g_2$. Then for some $a,b,c \in \mathbb{Z}_p$ we have

$$\lambda\big((e_1,e_2),(e_1,e_2)\big) = ae_1^2 + be_1e_2 + ce_2^2. \tag{9}$$

We can assume $a \neq 0 \neq c$, else $\Lambda_0 \neq 0$. Then since 2 is invertible in $\mathbb{Z}_p$, we have

$$\lambda\big((e_1,e_2),(e_1,e_2)\big) = a\left(e_1 + \frac{b}{2a}e_2\right)^2 + \left(c - \frac{b^2}{4a}\right)e_2^2.$$

Now $\left(e_1 \mapsto e_1 + \frac{b}{2a}e_2,\, e_2 \mapsto e_2\right)$ is bijective, and so we can assume w.l.o.g. that in (9) we have $b = 0$.

Now the multiplicative group $\mathbb{Z}_p^*$ of units of $\mathbb{Z}_p$ is cyclic. Thus there are two equivalence classes of non-trivial residue classes modulo p up to multiplication with squares. If a and c lie in the same class then we can assume w.l.o.g. that $a = c$, find $4k+1 = p = e_1^2 + e_2^2$, and have $\Lambda_0 \neq 0$. If a and c are in different classes, then one can make (exactly) one of them equal 1, and λ has the form in (8).

Now let $k \geq 3$. Consider $\mathbb{Z}_p^{\oplus 3} \subset H_p$ with a basis (g_1,g_2,g_3). Then one can again assume that $m_i = \lambda(g_i,g_i) \neq 0$, and by substitutions diagonalize λ. That is, we can assume w.l.o.g. that $\lambda(g_i,g_j) = 0$ when $i \neq j$. Then, since at least two of m_1,m_2,m_3 are in the same equivalence class modulo squares in $\mathbb{Z}_p^*$, by the previous argument we can find a non-trivial element in the cone. □

Remark 3.2 If now det is a square, then for trivial Λ_0 we must have $q' = q$. Among the knots obtained from our calculations we had q being the product of two primes, $p_1 = 5$ and $p_2 \in \{13,17,29,37\}$. (For each of these p_2 examples are $4_1\#6_3\#12_{1722}$, $8_3\#14_{2160}$, $11_{76}\#11_{160}$ and $8_{17}\#14_{15965}$ resp.) No knots occurred where q is only a single prime. More particularly, from 122,624 prime ≤ 16 crossing knots with determinant $d = 4k+1$ prime and $\sigma = 0$, all have $\lambda(g,g) = 2/d$ for some generator g of H_1. (For any d a knot K_1 of such linking form always exists, so if another knot K_2 fails to realize this form, then $K_1\#K_2$ is a potential candidate for trivial Λ_0.) This phenomenon seems related to some property of Minkowski units, but so far I cannot work out an exact explanation.

Now we explain how to construct knots with no $4k+3$-torsion and $1 < |\Lambda_0| < \sqrt{\det}$.

Let K_2 be a knot with determinant d being a product of distinct primes $p = 8k+5$, with a generator of (the necessarily cyclic) H_1 having $\lambda(e_2,e_2) = 1/d$. Let K_1 be a knot with $\det = d^3$ and cyclic H_1 with $\lambda(e_1,e_1) = 2/d^3$.

Then, for each prime $p \mid d$, consider $\Lambda_0(K_1\#K_2)$ on the p-Sylow subgroup $\mathbb{Z}_{p^3}\oplus\mathbb{Z}_p$ of $H_1(K_1\#K_2)$. We calculate $\big|[\Lambda_0]_p\big|$. We have in $\mathbb{Q}/\mathbb{Z}$

$$\lambda\big((e_1,e_2),(e_1,e_2)\big) = 2e_1^2/p^3 + e_2^2/p\,.$$

If $p \nmid e_1$, then the first term on the right has denominator d^3, and so $\lambda \neq 0$. Thus $p \mid e_1$. Let $e_1' \in \mathbb{Z}_p$ be the reduction of $e_1/p \in \mathbb{Z}_{p^2}$ modulo p. Then $e_1', e_2 \in \mathbb{Z}_p$ satisfy $2e_1'^2 + e_2^2 = 0$. Since $p \equiv 5 \bmod 8$, we have that ± 2 is not a quadratic residue, and $e_1' = e_2 = 0$. Thus the vectors $v = (e_1,e_2)$ with $\lambda(v,v) = 0$ are multiples of $(p^2,0)$, and so $\big|[\Lambda_0]_p\big| = p$.

Thus $|\Lambda_0| = d$, while $|H_1| = d^4$. With this idea in mind we can find examples.

Example 3.5 Consider the knot $K_2 = 11_{333}$, which is also the 2-bridge knot $(65,14)$. It has signature $\sigma = 0$, determinant $d = 65$ and a generator g of H_1 with $\lambda(g,g) = 14/65$, which is equivalent to $1/d$ up to squares. (By remark 3.2, this is apparently the smallest d for which we can find K_2.) The Alexander polynomial

$$\Delta = (\ 4\ -16\ 25\ -16\ 4)$$

has no zero on the unit circle. Let K_1 be a knot of unknotting number one, whose Alexander polynomial Δ_{K_1} has no zero on the unit circle, is of the form $\Delta_{K_2} f(t) f(1/t)$ and $\Delta_{K_1}(-1) = 65^3 = 274{,}625$. (For example take $\Delta_{K_1} = \Delta_{K_2}^3$.) Then $K_1\#K_2$ is a knot of the type we sought.

3.4. Excluding concordance order 2

Here we present some examples where our method prohibits the sliceness of the connected sum of a knot with itself, i.e. rules out (topological) concordance order 2. Among $\hat{K}\#\hat{K}$ type examples, we can find or construct the following knots.

Example 3.6 The simplest example arising is $7_7\#7_7$ (with $H_1 = \mathbb{Z}_{21}\oplus\mathbb{Z}_{21}$). It has trivial Λ_0. This can be easily explained, since 7_7 has unknotting number one, and $a^2+b^2 = 0$ has no non-trivial solutions in $\mathbb{Z}_{21}$.

There are, however, no examples of trivial cone, to which the Livingston-Naik result does not apply. This is explained followingly.

Proposition 3.2 $\lambda(\hat{K}\#\hat{K})$ has trivial isotropic cone, if and only if $\det(\hat{K}) = d$ is a product of distinct primes p, all of which congruent 3 mod 4.

Proof. Note that $\Lambda_0(\hat{K}\#\hat{K})$ is trivial, if and only if it is so in every reduction to prime torsion subgroups of H_1.

If the determinant d is of the described exceptional type, then the reduction of Λ_0 to $\mathbb{Z}_p\oplus\mathbb{Z}_p$ is trivial, since p is not of the form a^2+b^2. So Λ_0 is trivial.

Assume now the determinant d is not of the specified type. If a prime $p = 4k+1$ divides the determinant d, then the reduction of $\Lambda_0(\hat{K}\#\hat{K})$ on the p-torsion subgroup is metabolic by the Witt group argument, and so Λ_0 cannot be trivial.

Now let $p = 4k+3$ be a prime with p^2 dividing d. If $\mathbb{Z}_{p^2}$ occurs as subgroup, one immediately finds non-trivial zero-linking elements. Thus the p-Sylow subgroup of $H_1(\hat{K})$ must be a multiple of $\mathbb{Z}_p$, and since we assume $p^2 \mid \det$, we have at least two copies of $\mathbb{Z}_p$. Since $\lambda(\hat{K})$ is non-degenerate, each generator of each $\mathbb{Z}_p$ has non-zero linking. Now the multiplicative group $\mathbb{Z}_p^*$ of units of $\mathbb{Z}_p$ is cyclic, and so any element is a plus or minus a square. Then on $\mathbb{Z}_p\oplus\mathbb{Z}_p \subset H_1(\hat{K})$ we have in $\mathbb{Z}_p$ up to sign

$$\lambda((e_1,e_2),(e_1,e_2)) = e_1^2 \pm e_2^2 + qe_1e_2 \tag{10}$$

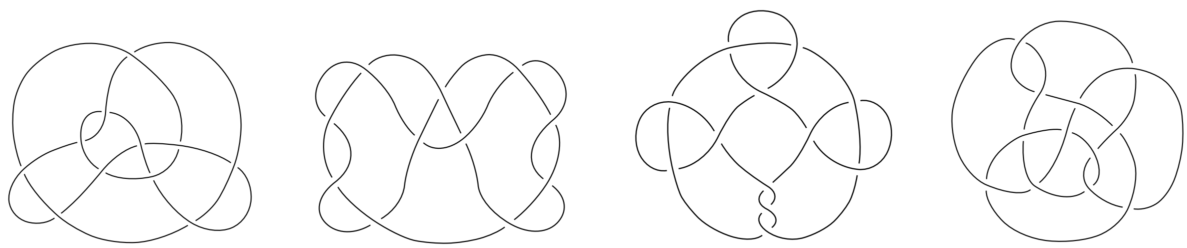
for some $q \in \mathbb{Z}_p$. Since 2 and 4 are invertible in $\mathbb{Z}_p$, we can write in $\mathbb{Z}_p^{\oplus 4} \subset H_1(\hat{K}\#\hat{K})$

$$\lambda\big((x,y,z,w),(x,y,z,w)\big) \,=\, \Big(x+\frac{q}{2}y\Big)^2+\Big(z+\frac{q}{2}w\Big)^2+q'(y^2+w^2)\,, \tag{11}$$

with $q' = \pm 1 - {}^{q^2}\!/_4$.

If we have negative sign in (10), then $(x,x,x,-x)$ for every x is isotropic. So assume we have positive sign. If $q=\pm 2$, then in (10) we have $\lambda = (e_1 \pm e_2)^2$, and we are easily done.

So assume $q \neq \pm 2$. Then $-q'$ is invertible in $\mathbb{Z}_p$. Consider the arithmetic progression $-1/q'+k'p$, and the subprogression in it made of numbers $4k+1$. If $-1/q'$ is even, then k' is odd and vice versa. Thus we have a progression $a'+b'\cdot(2p)$, where $(a',2p)=1$. This progression contains a prime r by Dirichlet's theorem, and since $r=4k+1$, we have $-1/q'+k'p = r = y^2+w^2$ for some y,w (obviously not both divisible by p, since $(p,r)=1$). Then in $\mathbb{Z}_p$ we have $q'(y^2+w^2)=-1$, and for these y and w we can find x and z with $x+{}^q\!/_2 y=0$ and $z+{}^q\!/_2 w=1$, so by (11) we are done. □



11_{274} 11_{333} 12_{554} 12_{1152}

Figure 3

Example 3.7 Among small cone type examples, we found the knot $11_{274}\#11_{274}$. The Alexander polynomial of $\hat{K} = 11_{274}$ is

$$(\ 1\ \ -6\ \ 18\ \ -35\ \ [45]\ \ -35\ \ 18\ \ -6\ \ 1)$$

with determinant 165, and no zeros on the unit circle. In this case $\hat{K}\#\hat{K}$ has $|\Lambda_0| = 9$. A similar example is $\hat{K} = 11_{280}$.

To find examples, to which the Livingston-Naik results do not apply, let $\hat{K} = \hat{K}_1\#\hat{K}_2$, where $\hat{K}_{1,2}$ are unknotting number one knots, whose Alexander polynomial has no zero on the unit circle. We choose the determinants of $\hat{K}_{1,2}$ to be d^2 and d resp., and d to be a product of an even number of different primes $p_i = 4k_i+3$.

To calculate $|\Lambda_0(\hat{K}\#\hat{K})|$ it suffices to consider its restriction on the p-Sylow subgroup for p being any of the p_i. This subgroup is $\mathbb{Z}_{p^2}\oplus\mathbb{Z}_{p^2}\oplus\mathbb{Z}_p\oplus\mathbb{Z}_p$. Now if p divides a^2+b^2, then so does p^2, and p divides both a and b. This means first that the p-adic valuation of the restriction of λ on $\mathbb{Z}_{p^2}\oplus\mathbb{Z}_{p^2}$ is 0 or -2, so that $[\Lambda_0]_p$ splits into a direct sum over its part in $\mathbb{Z}_{p^2}\oplus\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p\oplus\mathbb{Z}_p$. It means second that former summand of $[\Lambda_0]_p$ has size p^2, while latter summand is trivial. Thus $\big|[\Lambda_0]_p\big| = p^2$, and $|\Lambda_0| = d^2$.

Example 3.8 We found among low crossing knots one single example of the large cone type. Here $\hat{K} = 12_{554}$. This knot has $H_1 = \mathbb{Z}_{21}\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$. The Alexander polynomial

$$\Delta = \ (\ -2\ \ 15\ \ -45\ \ [65]\ \ -45\ \ 15\ \ -2)$$

has determinant 189 and no zeros on the unit circle. We have for $K = \hat{K}\#\hat{K}$ that $|\Lambda_0| = 225$, but Λ_0 contains no subgroup of order 189.

For an example, to which the Livingston-Naik result does not apply, take $\hat{K} = 7_7\#12_{554}^{\#2k}$. Since $H_1(12_{554})$ has no elements of non-trivial prime power order, the Witt group argument ensures that $\lambda(\hat{K}\#\hat{K})$ is not metabolic, because it is not so for $k=0$. But since $|\Lambda_0(12_{554}^{\#2})| > \det(12_{554})$, we obtain for large k again large cone.

4. Indirect Rudolph-Bennequin inequality

For the final contribution of the paper, we turn to *smooth* sliceness. In contrast to the topological case in (6), let $g_s(K)$ be the smooth 4-genus of K. So K is smoothly slice if and only if $g_s(K)=0$.

Rudolph [38] showed the "extended slice Bennequin inequality" (later proved also, and slightly more clarified by Kawamura [14]), which gives a lower estimate for $g_s(K)$ from a reduced diagram D of K:

$$g_s(K) \geq \frac{w(D)-s(D)+1}{2} + s_-(D) =: rb(D)\,. \tag{12}$$

As we explained in §2.2, by $w(D)$ we denote the writhe of D, and by $s(D)$ and $s_-(D)$ the number of its Seifert circles resp. negative Seifert circles. (Kawamura remarked that we must exclude diagrams with negative Seifert circles adjacent to nugatory crossings.) We call $rb(D)$ the *Rudolph-Bennequin number* of D. Inequality (12) is an improvement of Bennequin's original inequality [3, theorem 3], which estimates the ordinary genus $g(K)$ of K by the *Bennequin number*

$$b(D) = \frac{w(D)-s(D)+1}{2}\,,$$

in which the $s_-(D)$ term is missing. Rudolph showed prior to the above improvement (12), that $b(D)$ also estimates $g_s(K)$ ("slice Bennequin inequality").

This quantity $b(D)$ has another upper bound, namely the minimal degree $\min\deg_l P(K)$ of the skein polynomial P, as proved by Morton [30]. In particular, if

$$\delta(P(K)) \,=\, \max\bigl(\min\deg_l P(K), -\max\deg_l P(K)\bigr) \leq 0\,,$$

the original Bennequin number $b(D)$ will be useless in showing that K is not slice, whatever diagram D of K (or its mirror image) we apply it to. The $s_-(D)$ term, however, lifts the skein polynomial obstruction, and in [46] we showed that indeed $\min\deg_l P = 0$ can still allow the existence of diagrams with $rb > 0$. (Our example 13_{6374} has also $\Delta = 1$, so that any other previous method to prohibit sliceness fails.)

In theory thus, for many non-slice K, we could have $rb(D) > 0$ for some D. In practice, however, to find such D for most K is a tedious, or even pointless, undertaking. It may well be that D does not exist, and it is not worth checking more than a few diagrams that can be easily obtained. Since the improvement involving $s_-(D)$ is modest, $\delta(P) \ll 0$, even if not a definite obstruction, still remains at least a good heuristic evidence that such D is unlikely to exist.

There is one particular situation, in which one can definitely exclude the existence of D. Namely, note that it would imply that

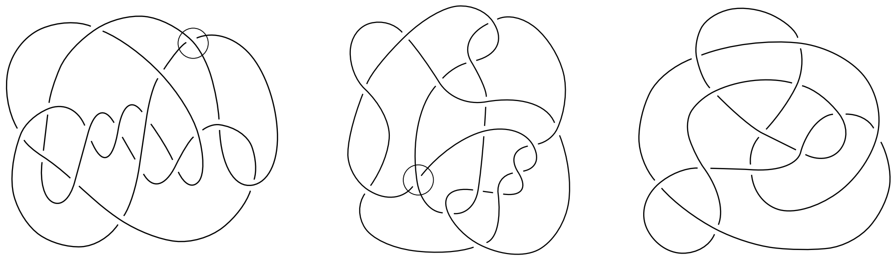
$$\lim_{n\to\infty} g_s(\#^n \pm K) \,=\, \infty\,, \tag{13}$$

as rb is additive under connected sum of diagrams (if it is properly performed, and unless the estimate is trivial), and invariant under reversal of knot orientation. Thus in particular if K is of finite order in (smooth) concordance *as an unoriented knot*, D cannot exist. The finite concordance order amounts, at least in practice, to saying that K is slice or achiral (of either sign), explaining the special role of 16_{850678} in example 3.1.

As a contrast to the intuition described so far, we conclude by showing how the Rudolph-Bennequin inequality can prove indirectly that some knots are not slice, when the quest for a diagram D with $rb(D) > 0$ (among the suggestive candidates) fails. See figure 4.

Example 4.1 Consider the knot $!16_{953447}$. Using Thistlethwaite's tools, one can generate 253 different 16 crossing diagrams of this knot, all of which, however, have $rb \leq 0$. This knot has the Alexander polynomial of the square knot, and $\sigma = 0$. (This again implies that all Tristram-Levine signatures vanish.) Its H_1 is cyclic, and hence λ is metabolic.

Now we pursue the following idea. If K' differs from K by one crossing change (performed in whatever diagram D of K), then $|g_s(K') - g_s(K)| \leq 1$, so that if K is slice, $g_s(K') \leq 1$. However, if we find a diagram D' of K' with $rb(D') \geq 2$, then $g_s(K') \geq 2$, and so K can not be slice. Consider the diagram of $K = !16_{953447}$ on the left of figure 4. Switching the encircled crossing turns K into $K' = 12_{1609}$. When K' is depicted in the diagram D' on the right of figure 4, then $b(D') = rb(D') = 2$, and so we can conclude indirectly that $!16_{953447}$ is not slice.



$!16_{953447}$ $\quad$ $16_{1335658}$ $\quad$ 12_{1609}

Figure 4: The two 16 crossing knots satisfy the Milnor-Fox condition on the Alexander polynomial (the second one has trivial polynomial), have signature 0, and cyclic $H_1(D_K)$. On any of their 16 crossing diagrams that could be found, the Rudolph-Bennequin inequality is also trivial. However, in the diagrams depicted, a crossing switch results in a diagram of 12_{1609}. This knot has the diagram on the right, with (Rudolph-)Bennequin number 2. Thus the 4-genus is at least 2, and so the original 16 crossing knots are not slice.

Example 4.2 The same argument (again with $K' = 12_{1609}$) applies to $16_{1335658}$. One can also handle in a similar way (using another knot $K' = 14_{27071}$) the $(-3,5,7)$–pretzel knot $P(-3,5,7) = 15_{199038}$. In fact, $P(-3,5,7)$ has also a diagram D with $rb(D) > 0$ (one comes from a 32-crossing braid representation; see [45]). This shows, that for some knots we may be lucky to find a diagram that excludes sliceness directly, and working with K' is not necessary. Of course, by Fintushel-Stern's, and later Rudolph's work (see [38]), the non-sliceness of this famous example has been dealt with before. Note contrarily that, since both knots have $\Delta = 1$, they *are topologically* slice by Freedman's theorem.

We found in total 5 prime knots up to 16 crossings (including the 3 so far mentioned), for which the indirect Rudolph-Bennequin inequality proved essential in excluding sliceness. All these knots have $\min\deg_l P = 2$, however, so that on some more complicated diagram the direct inequality may apply – as seen for $P(-3,5,7)$. Note that, when the indirect argument works, still (13) holds, so that achiral knots cannot occur. The attention to our examples was drawn by the problem of [45] to find slice knots with $\min\deg_l P > 0$. Since the described method ruled out all candidates for such a knot we had, the problem remains open.

Remark 4.1 Recently, Ozsvath and Szabo [35] defined a new "signature" invariant τ (for *knots*) using Floer homology, and Rasmussen [36] a conjecturedly equivalent invariant s using Khovanov homology. This invariant lies between the two hand-sides of the slice Bennequin inequality $b(D) \le g_s(K)$. Thus it must confirm, too, the non-sliceness of the examples in this section. However, it is still non-trivial to calculate, and thus Rudolph-Bennequin numbers remain a useful tool – in fact, one can estimate s often easier from them than calculating it directly. (Nonetheless Shumakovitch [43] computed s on a number of knots with $\Delta = 1$. He found some knots of $s \ne 0$, where $\delta(P) = -4$ is relatively small, and so the existence of non-trivial Rudolph-Bennequin numbers seems unlikely.) On the opposite side, the calculation of s is easy for alternating knots by virtue of being equal to the usual signature σ. This fact shows the failure of the new invariant, too, for many of our examples, including the first knot in figure 1, and the knots in figures 2 and 3. It also explains the failure of the Bennequin numbers for such knots.

5. Fibered, prime, arborescent and hyperbolic knots

Motivated by Nakamura's construction [34], in [47], we gave another proof of Sakai-Kondo's result. (We found subsequently that our construction was given, in a different context, also in [31].) The method explained in [47] allows us to choose our examples to have a few special properties. Above, we considered a knot K, found by computation, and a knot L of unknotting number one with suitable Alexander polynomial. Then the knots $K_k :=$

$K\#\#^kL$ can be modified to prime knots K'_k using tangle surgery. See proposition 5.1 below. Also, [47] shows that one can choose L fibered, if its Alexander polynomial is monic. So one can obtain (composite) fibered examples K_k if K is fibered. (How to keep fiberedness in going from K_k to K'_k is not clear at this point, though.)

Proposition 5.1 One can make a composite knot K into a (prime) hyperbolic knot K' of arbitrarily large volume, preserving Δ, H_1, λ, all non-singular Tristram-Levine signatures σ_ξ, and the Rasmussen invariant s. If K is connected sum of arborescent knots, one can also choose K' to be arborescent.

The same argument works for the Ozsvath-Szabo signature τ instead of s. If τ is equivalent to s, then there is anyway nothing to do. If, however, τ is not equivalent to s, and one would like to keep track of τ and s *simultaneously*, an extra argument must be provided.

Proof. In [47] we showed that by tangle surgery one can make K into a hyperbolic knot of arbitrarily large volume. If K is arborescent connected sum, one can also choose K' to be arborescent. (More generally, the tangle surgery can be chosen so as to preserve the Conway polyhedron of the diagram.)

Now, for tangle surgery one chooses the numbers of twists to satisfy congruences modulo the determinant. Then K and K' have Goeritz matrices [10] G and G', such that $d := \det(G) = \det(G')$ and $G \equiv G' \bmod d$. This implies that H_1 and λ are preserved.

The tangle surgeries preserve the Alexander polynomial. It can also be easily observed that they preserve all the non-singular Tristram-Levine signatures σ_ξ. Namely, by (in this or the reverse order) changing a positive crossing (to become negative), applying concordance, and changing a negative crossing, one obtains the original knot. So σ_ξ is changed by at most ± 2. But if $\Delta(\xi) \neq 0$, the sign of $\Delta(\xi)$ determines σ_ξ mod 4; see [48]. So the failure of σ_ξ to exhibit non-sliceness persists under the tangle surgery even if the Alexander polynomial has zeros on the unit circle.

Since s is not connected to Δ, so far tangle surgery may alter it. We will argue how to remedy this problem.

Consider the pretzel knots of the form $P(-p, p+2, q)$, with $p, q > 1$ odd and chosen so that $\Delta = 1$. First note that all such knots have $s = 2$. Namely, by the main Theorem in §1 of [40], these pretzel knots are quasipositive, and by proposition 5.3 of [40] have slice genus 1. By [36] one has then $s = 2$.

Now with a connected sum of a proper number k of $P(-p, p+2, q)$ or their mirror images, we can make s vanish. (We require that k is bounded by the number of surgeries in a way independent on the number of twists in the surgered tangles.) In [47] we showed that the tangle surgery making the connected sum prime can preserve the smooth concordance class (and hence s).

To show hyperbolicity, now note that we can augment p arbitrarily. Similarly we can augment the dealternator twists in the surgered tangles (also within a proper congruence class to keep track of H_1 and λ). Then by Thurston's hyperbolic surgery theorem [51] and the result of Adams on the hyperbolicity of augmented alternating links [2], the knots will be hyperbolic for large number of crossings in the twists. To obtain arbitrarily large volume, we augment the number of twist classes of crossings, and use Adams' lower estimate on the volume of the augmented alternating link in terms of the number of components [1]. (For an explanation, one may consult also [4].) □

Acknowledgement. Large parts of this paper were written during my stay at the Advanced Mathematical Institute, Osaka City University. I would like to thank Professor Akio Kawauchi for his invitation to Osaka and the First KOOK Seminar International for Knot Theory and Related Topics in Awaji-Shima (July 2004), and for all his support. Stefan Friedl, Tomomi Kawamura, Darren Long, Kunio Murasugi and particularly Chuck Livingston contributed with several useful remarks. However, the most substantial improvements (incl. a complete reorganization of the introductory part), were suggested by the referee. I owe to thank him and editor Chuck Weibel for the care in dealing with my paper.

References

[1] C. Adams, *Volumes of N-cusped hyperbolic 3-manifolds*, J. London Math. Soc. **38** (1988), 555–565.

[2] ——— ” ———, *Augmented alternating link complements are hyperbolic*, Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984), London Math. Soc. Lecture Note Ser. **112**, Cambridge Univ. Press, Cambridge, 1986, 115–130.

[3] D. Bennequin, *Entrelacements et équations de Pfaff*, Soc. Math. de France, Astérisque **107-108** (1983), 87–161.

[4] M. Brittenham, *Free genus one knots with large volume*, Pacific J. Math. **201** (2001), 61–82.

[5] A. Casson and C. McA. Gordon, *On slice knots in dimension 3*, "Geometric Topology" (R. J. Milgram, ed.), Proc. Symp. Pure Math. XXXII, AMS, Providence 1978, 39–53.

[6] T. D. Cochran, *Noncommutative knot theory*, Algebr. Geom. Topol. **4** (2004), 347–398.

[7] R. H. Fox and J. W. Milnor, *Singularities of* 2*-spheres in* 4*-space and cobordism of knots*, Osaka J. Math. **3** (1966), 257–267.

[8] P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–246.

[9] S. Garoufalidis and A. Kricker, *Finite type invariants of cyclic branched covers*, Topology **43(6)** (2004), 1247–1283.

[10] C. McA. Gordon and R. A. Litherland, *On the signature of a link*, Invent. Math. **47 (1)** (1978), 53–69.

[11] G. H. Hardy and E. M. Wright, *Einführung in die Zahlentheorie* (German), R. Oldenbourg, Munich, 1958 (third edition).

[12] J. Hoste and M. Thistlethwaite, *KnotScape*, a knot polynomial calculation and table access program, available at `http://www.math.utk.edu/~morwen`.

[13] ——— " ———, ——— " ——— and J. Weeks, *The first 1,701,936 knots*, Math. Intell. **20 (4)** (1998), 33–48.

[14] T. Kawamura, *On unknotting numbers and four-dimensional clasp numbers of links*, Proc. Amer. Math. Soc. **130(1)** (2002), 243–252.

[15] C. Kearton, *Cobordism of knots and Blanchfield duality*, J. London Math. Soc. (2) **10(4)** (1975), 406–408.

[16] P. Kirk and C. Livingston, *Twisted knot polynomials: inversion, mutation and concordance*, Topology **38(3)** (1999), 663–671.

[17] H. Kondo, *Knots of unknotting number* 1 *and their Alexander polynomials*, Osaka J. Math. **16(2)** (1979), 551–559.

[18] J. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

[19] ——— " ———, *Metabolic and hyperbolic forms from knot theory*, J. Pure Appl. Algebra **58(3)** (1989), 251–260.

[20] ——— " ———, *Invariants of knot cobordism*, Invent. Math. **8(2)** (1969), 98–110.

[21] W. B. R. Lickorish, *The unknotting number of a classical knot*, in "Contemporary Mathematics" **44** (1985), 117–119.

[22] ——— " ——— and K. C. Millett, *A polynomial invariant for oriented links*, Topology **26 (1)** (1987), 107–141.

[23] R. A. Litherland, *Cobordism of satellite knots*, Four-manifold theory (Durham, N.H. 1982), Contemp. Math. **35**, Amer. Math. Soc., 1984, 327–362.

[24] C. Livingston, *The slicing number of a knot*, Algebr. Geom. Topol. **2** (2002), 1051–1060.

[25] ——— " ———, *Infinite Order Amphicheiral Knots*, Algebr. Geom. Topol. **1** (2001), 231–241.

[26] ——— " ——— and S. Naik, *Obstructing four-torsion in the classical knot concordance group*, J. Differential Geom. **51(1)** (1999), 1–12.

[27] ——— " ——— and ——— " ——— , *Knot concordance and torsion*, Asian J. Math. **5(1)** (2001), 161–167.

[28] T. Matumoto, *On the signature invariants of a non-singular complex sesquilinear form*, J. Math. Soc. Japan **29(1)** (1977), 67–71.

[29] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **73**, Springer, New York-Heidelberg, 1973.

[30] H. Morton, *Seifert circles and knot polynomials*, Proc. Camb. Phil. Soc. **99** (1986), 107–109.

[31] H. Murakami, *Delta-unknotting number and the Conway polynomial*, Kobe J. Math. **10 (1)** (1993), 17–22.

[32] ——— " ——— and A. Yasuhara, *4-genus and 4-dimensional clasp number of a knot*, Proc. Amer. Math. Soc. **128(12)** (2000), 3693–3699.

[33] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.

[34] T. Nakamura, *Braidzel surfaces and the Alexander polynomial*, Proceedings of the Workshop "Intelligence of Low Dimensional Topology", Osaka City University (2004), 25–34.

[35] P. Ozsváth and Z. Szabó, *Knot Floer homology and the four-ball genus*, Geometry & Topology **7** (2003), 615–639.

[36] J. Rasmussen, *Khovanov homology and the slice genus*, preprint `math.GT/0402131`.

[37] D. Rolfsen, *Knots and links*, Publish or Parish, 1976.

[38] L. Rudolph, *Positive links are strongly quasipositive*, Geometry and Topology Monographs **2** (1999), Proceedings of the Kirbyfest, 555–562. See also `http://www.maths.warwick.ac.uk/gt/GTMon2/paper25.abs.html`.

[39] ——— " ———, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. (N.S.) **29(1)** (1993), 51–59.

[40] ——— " ———, *Quasipositive pretzels*, Topology Appl. **115(1)** (2001), 115–123.

[41] T. Sakai, *A remark on the Alexander polynomials of knots*, Math. Sem. Notes Kobe Univ. **5(3)** (1977), 451–456.

[42] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften **270**, Springer-Verlag, Berlin, 1985.

[43] A. Shumakovitch, *Rasmussen invariant, Slice-Bennequin inequality, and sliceness of knots*, preprint `math.GT/0411643`.

[44] A. Stoimenow, *Polynomial values, the linking form and unknotting numbers*, `math.GT/0405076`, Math. Res. Lett. **11(5-6)** (2004), 755–769.

[45] ——— " ——— , *Some inequalities between knot invariants*, Internat. J. Math. **13(4)** (2002), 373–393.

[46] ——— " ——— , *Some examples related to 4-genera, unknotting numbers, and knot polynomials*, Jour. London Math. Soc. **63(2)** (2001), 487–500.

[47] ——— " ——— , *Alexander polynomials and hyperbolic volume of arborescent links*, preprint.

[48] ——— " ——— , *On some applications of Tristram-Levine signatures and a relation to Vassiliev invariants*, Advances in Math. **194(2)** (2005), 463–484.

[49] N. W. Stoltzfus, *Unraveling the integral knot concordance group*, Mem. Amer. Math. Soc. **12 (192)** (1977), iv+91 pp.

[50] A. Tamulis, *Knots of ten or fewer crossings of algebraic order 2*, J. Knot Theory Ramifications **11(2)** (2002), 211–222.

[51] W. P. Thurston, *Three-dimensional geometry and topology, Vol* **1**, (Silvio Levy, ed.), Princeton Mathematical Series **35**, Princeton University Press, Princeton, NJ, 1997.

[52] M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology **33(2)** (1994), 241–256.

[53] H. Wendt, *Die Gordische Auflösung von Knoten*, Math. Z. **42** (1937), 680–696.

[54] S. Wolfram, *Mathematica — a system for doing mathematics by computer*, Addison-Wesley, 1989.

[55] D. B. Zagier, *Zetafunktionen und quadratische Körper*, Eine Einführung in die höhere Zahlentheorie (German), Hochschultext. Springer-Verlag, Berlin-New York, 1981.