MINIMAL GRID DIAGRAMS OF THE ALTERNATING LINKS

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Abstract. We prove that every alternating link has a minimal alternating grid diagram, and examine the number of possible alternating grid diagrams of a given alternating link.

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1 Introduction

The arc index was introduced by Cromwell [Cr] as a variant of studying open book decompositions of knots and links. It was later found to have important connections to contact topology (e.g. [Sa]). By the work of Dynnikov and Prasolov, [DP], this is also connected to determining the maximal Thurston-Bennequin invariant of Legendrian knots.

The arc index is not generally algorithmically determinable. The knot-spoke method to obtain an arc presentation of a knot from a given diagram was developed in [BP] and [JP]. For alternating links the Morton-Beltrami bound [MB] shows that this grid diagram is minimal. However, these grid diagrams are unpleasant and difficult to obtain (see [JLS]). We present a much simpler construction, which directly deforms an alternating (minimal crossing) diagram $\mathcal{D}$ into a grid diagram D without adding any crossings. (Brackets denote the forgetful functor from grid to planar diagrams.)

Theorem 1.1 If $\mathcal{D}$ is a twist-reduced alternating link diagram, then there is a grid diagram D of $\mu(D) = c(\mathcal{D}) + 2$ so that $[D] = \mathcal{D}$.

Of course, the proof of existence of grid diagrams means that for every $\mathcal{D}$ (not necessarily alternating), there is a D with $[D] = \mathcal{D}$. This can be accomplished, though, in general only by adding an excessive number of edges, so that $\mu(D)$ is large.

Because of the work of Dynnikov-Prasolov, one can see Theorem 1.1 also as an extension of Tanaka's result [Ta] (see §2.5).

We also give a construction to obtain such grid diagrams from the location of their crossings (Theorem 4.1), which shows that the number of realizations D of a given $\mathcal{D}$ is hyperexponential in the crossing number in average.

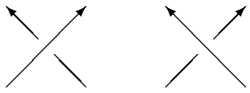
The work on Theorem 1.1 was suggested to me by G.T.Jin. After presenting the proof, I understood that he has now an independent one.

2 Preliminaries

2.1 Link diagrams

All link diagrams and links are assumed oriented. (Orientation is only really needed for §2.5, but it is better to assume it while ignoring it elsewhere.)

Crossings in an oriented diagram $\mathcal{D}$ of a knot K are called *positive* or *negative*.



positive negative (2.1)

Let $c_\pm(\mathcal{D})$ be the number of positive, respectively negative crossings of a link diagram $\mathcal{D}$, so that the *crossing number* of $\mathcal{D}$ is $c(D) = c_+(\mathcal{D}) + c_-(\mathcal{D})$ and its *writhe* is $w(\mathcal{D}) = c_+(\mathcal{D}) - c_-(\mathcal{D})$.

The piece of a solid ine in $\mathcal{D}$ between two crossings is an *edge*.

If we forget crossing information of $\mathcal{D}$, we obtain the resulting *plane curve* $\widehat{\mathcal{D}}$ of $\mathcal{D}$.

The mirror image of K will be written $!K$, and the mirror image of diagram $\mathcal{D}$ (in the form obtained by switching all crossings of $\mathcal{D}$) will be $!\mathcal{D}$.

Before we come to the proof of our results, we will briefly review the necessary, and previously initiated, framework to incorporate them into.

Definition 2.1 The diagram on the right of figure 1 is called *connected sum* $A\#B$ of the diagrams A and B. If a diagram $\mathcal{D}$ can be represented as the connected sum of diagrams A and B, such that both A and B have at least one crossing, then $\mathcal{D}$ is called *composite*; otherwise it is called *prime*.

Equivalently, a diagram is composite if there is a closed curve γ, called *decomposing curve*, intersecting $\mathcal{D}$ in exactly two points (outside of crossings), so that both interior and exterior of γ contain a crossing of $\mathcal{D}$. This curve γ is considered up to isotopy that maintains transversality with $\mathcal{D}$ and does not go through crossings of $\mathcal{D}$, as well as up to moving through ∞ (i.e. exchanging interior and exterior).

Any diagram $\mathcal{D}$ can be written as $\mathcal{D}_1\#\mathcal{D}_2\#\cdots\#\mathcal{D}_n$, so that all $\mathcal{D}_i$ are prime and have at least one crossing. Then $\mathcal{D}_i$ are called the *prime factors* of $\mathcal{D}$. A link L is prime if for any composite diagram $\mathcal{D}_1\#\mathcal{D}_2$ of L, one of $\mathcal{D}_{1,2}$ is an unknot diagram.

Definition 2.2 A diagram $\mathcal{D}$ is *split*, or *disconnected*, if there is a closed curve which does not intersect $\mathcal{D}$, but which contains parts of $\mathcal{D}$ in both its interior and exterior. Otherwise $\mathcal{D}$ is called *connected*. A link is split if it has a split diagram.

Thus, a diagram $\mathcal{D}$ is connected iff its plane curve $\widehat{\mathcal{D}}$ is a connected set.

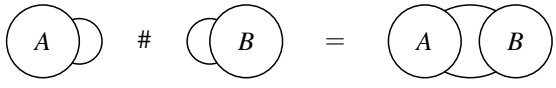


Figure 1

Definition 2.3 A crossing q in a link diagram $\mathcal{D}$ is called *nugatory* if there is a closed (smooth) plane curve intersecting $\mathcal{D}$ transversely in q and nowhere else. A diagram is called *reduced* if it has no nugatory crossings.

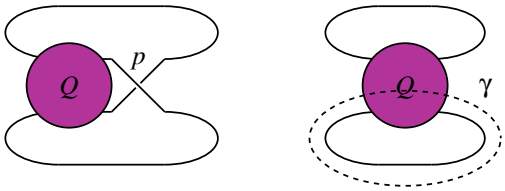
Definition 2.4 A *kink* is a crossing like  or .

A kink is thus a nugatory crossing, but not vice versa. Also, a link diagram with a nugatory crossing is composite, unless it is the 1-crossing (unknot) diagram. (Thus for more than 1 crossing, a diagram being prime implies it being reduced.) In particular, a kink (in a diagram of more than 1 crossing) has its *kink (decomposing) curve* specified by containing only the kink crossing in its interior. Keep in mind these relations.

Theorem 2.5 ([Me]) An alternating reduced diagram of a link L is prime iff L is prime.

The following lemma is rather standard (and easy to prove).

Lemma 2.6 Assume the the diagram in (a) is non-split (connected) and composite with a decomposing curve γ. Then either (1) p is a kink and γ is its kink curve, or (2) γ can be isotoped inside Q as a decomposing curve of (b). In particular, if p is spliced as in (b), then γ is *not* of the form as indicated there (where γ is assumed to intersect exactly two edges inside Q).

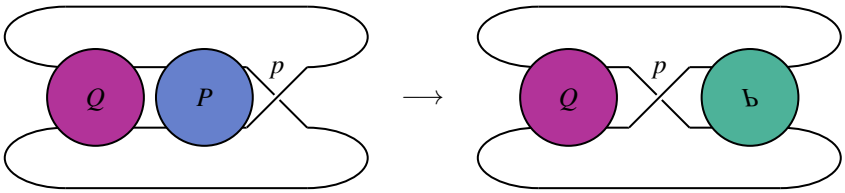


(a) (b) (2.2) □

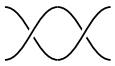
Definition 2.7 Let $c(\mathcal{D})$ be the *crossing number* of a link diagram $\mathcal{D}$. Let $c(L)$ be the crossing number of a link L, which is the minimal crossing number of all diagrams $\mathcal{D}$ of L. A link L is *trivial* if $c(L) = 0$.

Theorem 2.8 (Kauffman–Murasugi–Thistlethwaite) Each alternating reduced diagram is of minimal crossing number (for the link it represents).

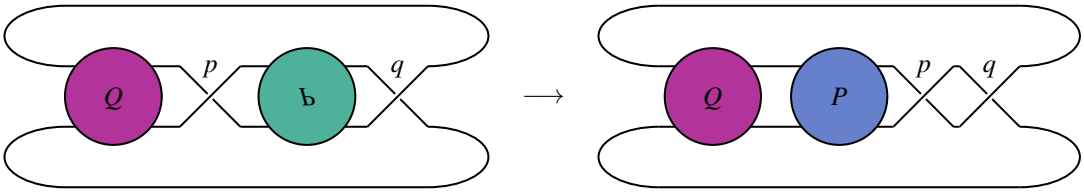
We must introduce the notion of twist equivalence of crossings. The version of this relation we present here is that of [La]. First we introduce a diagram move called *flype*, shown on the following picture:



(2.3)

A *clasp*  is a tangle that consists of two crossings, i.e. a pair of crossings that bound a bigonal (2-corner) region in the plane complement of a diagram.

Definition 2.9 Let $\mathcal{D}$ be a reduced link diagram. We say two crossings p and q of $\mathcal{D}$ to be *twist equivalent* if there exists a flype making p and q into a clasp.



(2.4)

(Note that this formulation assumes a certain correspondence between the crossings in the diagrams on both hand sides of (2.3), but this correspondence is quite obvious.) We call this relation *twist equivalence*.

A diagram is *twist reduced* if for each twist equivalent crossings p,q, there is a sequence $p = p_0, p_1, \ldots, p_k = q$ so that p_i, p_{i+1} form a clasp (like p,q do on the right of (2.4)).

In [MT, MT2] it was proved that alternating diagrams of the same link are related by flypes. And one can use flypes to deduce that every alternating link has a twist reduced alternating diagram.

2.2 General preliminaries on graphs

Below it will be helpful for us to translate the statements we like to prove about link diagrams into statements about planar graphs. As graphs occur throughout the rest of the paper, we first need to set up some related terminology, most of which is standard, and also explained in [MS]. For clarification, we repeat a part of these definitions, and collect several simple facts that will be used later. Some new, more specific to our proofs, notions will be introduced later.

Definition 2.10 A *loop edge* is an edge connecting a vertex with itself. A *multiple edge* e is the set of all n edges connecting the same pair of (distinct) vertices; we assume $n \geq 2$. The number n is called *multiplicity* of e. Simple edges that belong to the same multiple edge (that is, connect the same 2 vertices) are called *parallel*. A graph is called *simple* if it has no multiple edges, and *loop-free* if it has no loops. Below an 'edge' is always understood to be simple, unless evident otherwise from the context (for example, like in 'an edge of multiplicity n').

Definition 2.11 For a graph G, let $E(G)$ resp. $e(G)$ be the set resp. number of edges. An edge of multiplicity n is understood to contribute n to $e(G)$, We call G *trivial* if $e(G) = 0$. An edge between vertices v and w, which is not given a separate name, will be written as vw.

We will assume that graphs are connected and have no loop edges and vertices of valence 1, unless the opposite is clear.

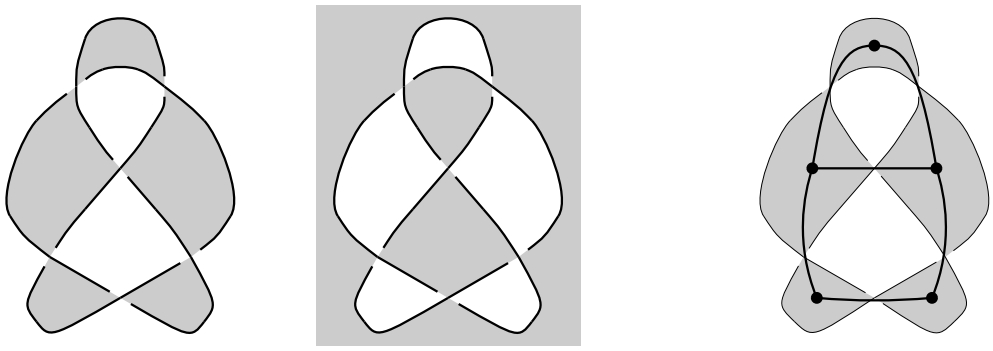
Definition 2.12 A cycle-free graph is a *forest*. A connected forest is a *tree*. If G is a connected graph of n vertices, a *spanning tree* of G is a tree contained in the edges of G, which is maximal, that is, has $n-1$ edges. Let for a graph G by $s(G)$ be denoted the number of its spanning trees (or 0 if G is not connected).

Definition 2.13 If G is a graph and $e_1, \ldots, e_n$ edges in G, whose deletion disconnects G, then we call $\{e_1, \ldots, e_n\}$ an *n-cut*, or, ignoring its size, simply *cut* of G. An edge forming a 1-cut is called an *isthmus*. We say G is *n-connected* if it has no k-cut for any $k < n$. When we delete an edge, we understand that a vertex it is incident to is *not* to be deleted too. In that sense, the set I_v of edges incident to a given vertex v always forms a cut – if we delete these edges, v gets an isolated component, so the graph is not connected anymore. We call v a *cut vertex*, if G gets disconnected when deleting all edges in I_v and *additionally* v itself.

Definition 2.14 The *join* $G = G_1 * G_2$ of two graphs G_1 and G_2 is a graph G obtained by identifying a vertex in G_1 and a vertex in G_2, called *join vertices*. (The identified vertex becomes a cut vertex in G. The choice of join vertices has influence on G, but this choice will turn out irrelevant for us in the following.) Every non-trivial graph G can be written as a join $G_1 * \cdots * G_n$ for some non-trivial graphs G_i, such that no G_i has a cut vertex. We call G_i the *join factors* of the graph G. If the decomposition is non-trivial (that is, $n > 1$), we call G *decomposable*.

By definition, a graph is 2-connected iff it has no isthmus. The following is an easy observation:

Lemma 2.15 $G_1 * G_2$ is 2-connected iff G_1 and G_2 are so. $G_1 * G_2$ has an isthmus or loop iff one of G_1 or G_2 has. □



two checkerboard colorings

the checkerboard graph
of the first coloring

Figure 2: Checkerboard coloring and checkerboard graph

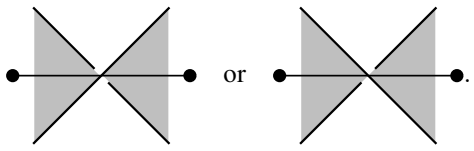
Definition 2.16 A graph G is *planar* if it is embeddable in the plane *and equipped* with a fixed such planar embedding.

Definition 2.17 Assume G is a planar graph. We call a *region* E of G a connected component of the complement of G. Its *boundary* is denoted ∂E and understood as a set of edges of G. If an edge e of G lies in the boundary of regions A and B, then we say that A is *opposite* to B at e, or that A and B are opposite.

Thus G is 2-connected if it is connected and no region is opposite to itself at some edge. Also, G is 3-connected, if it is 2-connected and for each pair of regions there is at most one edge at which they are opposite.

2.3 Checkerboard graphs

To any alternating link diagram $\mathcal{D}$ one can associate its *checkerboard (or Tait) graph* $G = \Gamma(\mathcal{D})$ in the following manner (see figure 2, or also [Ka, DH, St]). Consider the plane curve $\widehat{\mathcal{D}}$ of D as a 4-valent planar graph. There is a black-white coloring of the regions of $\widehat{\mathcal{D}}$, the *checkerboard coloring*, which assigns different colors to each pair of regions opposite at some edge of $\widehat{\mathcal{D}}$. Then $\Gamma(\mathcal{D})$ is defined to have a vertex for each black region of $\widehat{\mathcal{D}}$ and an edge for each crossing of D as follows:



(2.5)

Assuming D is connected, there are only two choices of checkerboard colorings (equivalent up to the interchange of colors) and so $\Gamma(\mathcal{D})$ is uniquely defined *up to duality* (which corresponds to interchanging the colors).

This way the checkerboard graph is a planar graph, in general possibly with multiple edges. Conversely, any such graph is the checkerboard graph of some link diagram. A diagram D is alternating if for all crossings the same one of the two local pictures in (2.5) occurs. Thus the checkerboard graph determines an alternating diagram up to mirroring. The following is also easy to see:

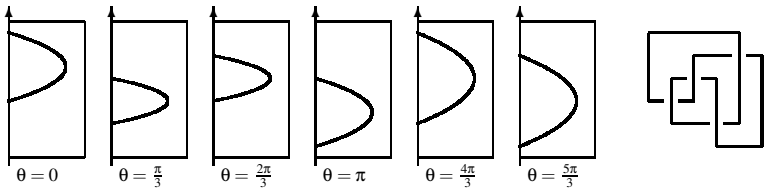
Lemma 2.18 The checkerboard graph $G = \Gamma(\mathcal{D})$ of a diagram D has no cut vertex if and only if D is prime. More precisely, the connected sum of diagrams corresponds to the join of checkerboard graphs: $\Gamma(\mathcal{D}_1 \# \mathcal{D}_2) = \Gamma(\mathcal{D}_1) * \Gamma(\mathcal{D}_2)$. □

In accordance with the ambiguity of $\Gamma(\mathcal{D})$ up to duality, most of the graph properties we will deal with below will be symmetric with respect to taking the dual graph. We can use this symmetry to simplify our arguments at some point. Note furthermore

Lemma 2.19 The property $G = \Gamma(\mathcal{D})$ to be 2-connected and (dually) without loop edges is equivalent to $\mathcal{D}$ having no nugatory crossings. □

2.4 Arc index and grid diagrams

An *arc presentation* of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc.



(2.6)

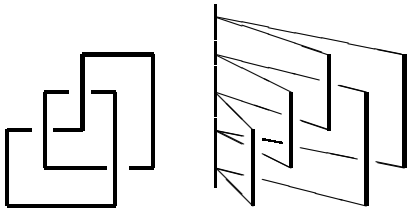
The minimal number of pages among all arc presentations of a link L is called the *arc index* of L [BP, J+] and is denoted by $a(L)$.

A *grid diagram* (for brevity, often below just called a *grid*) is a knot or link diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges. It is essentially a *non-simple* rectilinear polygon [KLA] (and equivalent to it, since the shape of the crossings is canonically fixed). It is not hard to see (and proved by Cromwell [Cr]) that every knot or link admits a grid diagram.

We denote by $\mathcal{D} = [D]$ the *link diagram underlying* the grid diagram D. (We try to use caligraphic letters for link diagrams and Latin ones for grid diagrams.) We say that D is prime if $[D]$ is so.

We will usually denote the *size* of a grid diagram D, which is the number of vertical (or equivalently, horizontal) edges, by $\mu(D)$. To abbreviate, we will often say *grid* for a grid diagram, and *μ-grid* for a size-μ grid (diagram).

The figure below shows that every knot admits an arc presentation.



(2.7)

It also explains that there is a one-to-one correspondence between arc presentations and grid diagrams. In this sense, the grid diagram given on the right of (2.6) corresponds to the open book decomposition whose pages are given on the left. This correspondence also shows that the arc index $a(L)$ is the minimal size of a grid diagram of L. We will henceforth always work with the grid diagram form.

Definition 2.20 A grid diagram D of a link L is *minimal* if $\mu(D) = a(L)$.

One can use a code form of the grid diagram by indicating left-right position of the horizontal segment ends,

read from top to bottom. For example, the grid diagram in (2.6) has the form

$$15 \quad 36 \quad 24 \quad 13 \quad 25 \quad 46. \qquad \begin{matrix} 1&0&0&0&1&0\\ 0&0&1&0&0&1\\ 0&1&0&1&0&0\\ 1&0&1&0&0&0\\ 0&1&0&0&1&0\\ 0&0&0&1&0&1 \end{matrix} \tag{2.8}$$

This is a more memory-efficient variant of what is now called a *Cromwell matrix* (see [Cr, pp. 44-45]), which for the encoding on the left of (2.8) would look like what is given on the right.

Keep in mind that in other places there are several conventions of how to build (2.8). But the important point here is to always understand a grid diagram as a sequence of such integers. That is, we encode the mutual left-right position of vertical segments, and top-down of horizontal ones. Thus, a planar translation does not change a grid diagram for us, but a rotation usually does.

There are several types of *Cromwell moves* that preserve the knot type of a grid diagram (see [AHT]). We do not need these moves a lot here, but note that they can produce grid diagrams which are planar isotopic but still distinct in terms of their code (2.8). It is essential to keep in mind that thus we regard such grid diagrams as distinct.

Also, there are two operations we consider which allow to construct a grid diagram of the mirror image $!K$ from a grid diagram D of K.

First, there is an operation of *h-flip*. It is the planar mirroring/reflection along the horizontal axis (and no switch of crossings). This will yield a grid diagram $\phi(D)$ of $!K$. It reverses the code (2.8) (as a sequence of pairs) of D. The other operation is the *v-flip*, which mirrors along the vertical axis. We denote this operation as $\rho(D)$. It replaces each entry i in (2.8) by $\mu(D)+1-i$.

In general, we may regard $!D$ as any of $\phi(D)$ and $\rho(D)$.

The following type of grid diagram first came to the attention of G.T.Jin, and will be the main focus below.

Definition 2.21 We say a grid diagram D is *proper* if

(a) all horizontal segments except left- and rightmost have exactly one crossing on them

(b) all vertical segments except top- and bottom-most have exactly one crossing on them

(c) the top (horizontal) segment is connected to exactly one of the left- and rightmost segments

(d) the bottom (horizontal) segment is connected to exactly one of the left- and rightmost segments

We can stipulate that the *trivial grid* D with $\mu(D)=2$ is also a proper grid. For example, the grid in (2.7) is proper.

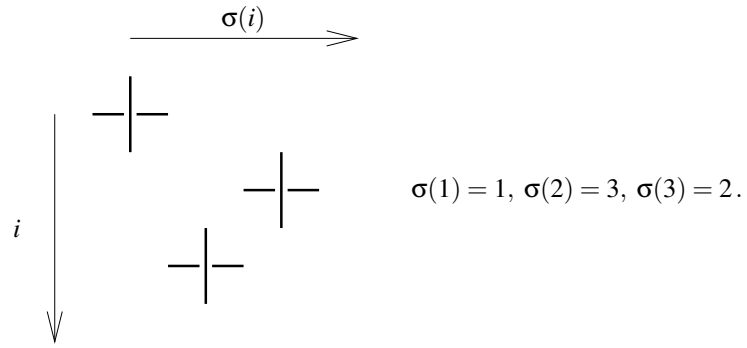
The following is straightforward to observe.

Lemma 2.22 If D is proper, then $[D]$ is an alternating diagram, and D (and $[D]$) is connected.

However, note that $[D]$ may be composite, and have nugatory crossings. This will be the matter of some concern later.

Definition 2.23 (Jin) The permutation $\sigma=\sigma(D)\in S_n$ for a proper grid D where $n=\mu(D)-2$, is defined as follows.

Place D in the coordinate plane so that its top (horizontal) segment lies on the (positive) x-axis and its leftmost (vertical) segment on the (negative) y-axis.



The the crossing points of D are of the form $(j,-i)$ for $1 \le i,j \le n$, with each i and j occurring exactly once. Thus $i \mapsto j$ defines a permutation in $\sigma \in S_n$, which we denote as $\sigma(D)$.

For example, for the grid D in (2.6) we have $\sigma(D)$ is the cycle $1 \mapsto 4 \mapsto 3 \mapsto 1$.

2.5 Thurston-Bennequin invariant

This detour is only briefly included here to make clear the connection to Tanaka's result [Ta].

There is a further correspondence: when a grid diagram of a link L is rotated by $\pi/4$, then it can be seen as a *Legendrian front diagram* of a Legendrian embedding of L. See [LN, Ta].

There is an invariant of Legendrian isotopy, called *Thurston-Bennequin invariant/number*. We define it here equivalently in terms of a grid diagram.

Definition 2.24 The Thurston-Bennequin invariant of a grid diagram D, written $TB(D)$, is defined as

$$TB(D) = -Z(D) + w([D]), \tag{2.9}$$

where $w([D])$ is the writhe of D (taken as a planar diagram of L) and $Z(D)$ the number of NW- or SE-corners of D.

Definition 2.25 The *maximal Thurston-Bennequin invariant* of a link L, written $TB(L)$ is

$$TB(L) = \max\{\, TB(D) : D \text{ is a grid diagram of } L \,\}.$$

Note that $TB(\rho(D)) = TB(\phi(D)) = -\mu(D) - TB(D)$. We writhe this quantity as $TB(!D)$.

Theorem 2.26 (Dynnikov and Prasolov [DP]) A grid diagram D of L is minimal iff $TB(D) = TB(L)$ and $TB(!D) = TB(!L)$.

Modulo the equivalence of front and grid diagrams, the main result of Tanaka [Ta] (with some details from its proof) can be written as follows.

Theorem 2.27 (Tanaka [Ta]) If $\mathcal{D}$ is a twist reduced alternating diagram of L, then there exists a grid diagram D with $[D] = \mathcal{D}$ and $TB(D) = TB(L)$.

Unfortunately, the deformation $\mathcal{D}$ to D Tanaka uses, which bases on a lemma of Yokota, does not apply simultaneously for $!D$. It works for one checkerboard coloring but does not simultaneously respect the other. This is why our result Theorem 1.1 implies Tanaka's but not vice versa. We will also use checkerboard colorings, but mostly via their checkerboard graph, and need a far more delicate deformation process.

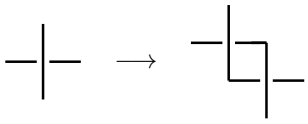
3 Realization: main result

3.1 Polyhedral Tait graphs

This section contains the proof of Theorem 1.1. And we will mainly look at the Tait (checkerboard) graph $G = G(\mathcal{D})$ of an alternating reduced diagram $\mathcal{D}$. (Fix one of the two checkerboard colorings.)

First, to simplify matters, let us exclude composite diagrams $\mathcal{D}$. The below construction would not make problems if G has cut vertices, but it is better not to complicate matters unnecessarily.

Next, we assume as stated, up to flypes, that $\mathcal{D}$ is twist reduced. Because of the transformation



it is evident that we can reduce the proof to alternating diagrams $\mathcal{D}$ with no clasps, which are the *Conway basic polyhedral* diagrams. (A good explanation of the construction is given in [ST].) In particular, since the claim is trivial for the 1-crossing diagram $\mathcal{D}$, we can immediately finish off (and henceforth exclude) the case that $\mathcal{D}$ is an *arborescent* alternating diagram.

Definition 3.1 We say that a planar graph G is *quasi-polyhedral* if it is a loop-free 3-connected simple planar graph without a cut vertex.

In particular this means that G has no vertices of valence 2 (or less).

We thus need to treat only alternating diagrams $\mathcal{D}$ with $G = G(\mathcal{D})$ being quasi-polyhedral.

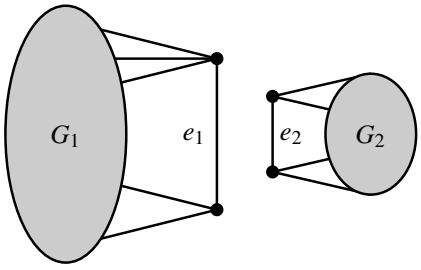
We need the following further definition.

Definition 3.2 We say that a planar graph G is *polyhedral* if G is quasi-polyhedral, and whenever two faces A and B meet at vertices $v_1, v_2 \in V(G)$ (i.e. $v_1, v_2 \in \partial A \cap \partial B$), there is an edge $e \in E(G)$ between v_1, v_2, and A, B meet at e either ($e \subset \partial A \cap \partial B$).

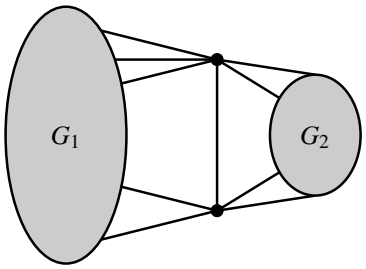
(Obviously, because of both simpleness and 3-connectedness, e must be unique.) These are the graphs G that occur as 1-sceleta of convex Euclidean polyhedra. We will not exploit this fact here, but it is important to keep in mind the difference between "polyhedral" in this and in Conway's sense. It pertains to the fact that Conway basic polyhedra admit non-trivial tangle decompositions.

Quasi-polyhedral graphs can be built of polyhedral ones using a construction of *edge-attachment*, which we explain next.

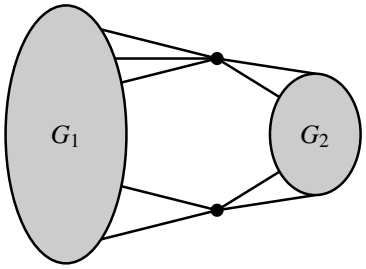
We consider planar graphs G_i with edges e_i in the boundary of their infinite faces for $i > 1$.



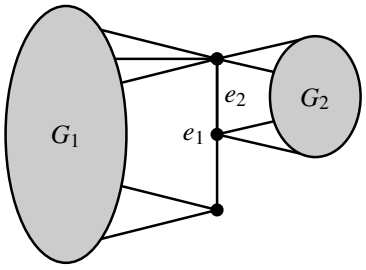
Then one can build a new graph $G_1 \times G_2$ by identifying e_i in G_i. There are several possibilities for this construction, and we try to visualize all. First, one can identify e_i.

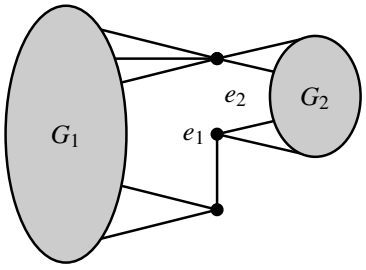


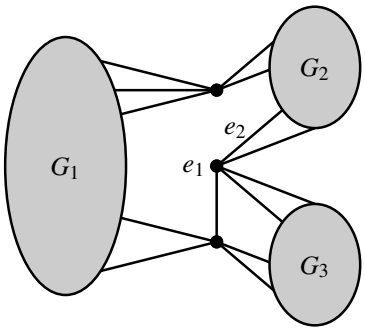
One can also identify e_i but delete the resulting edge.



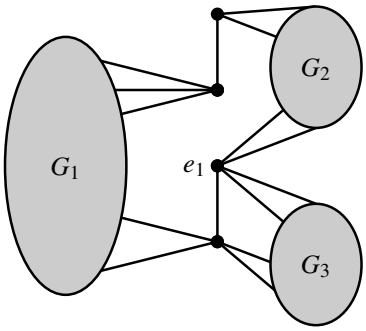
Next, it is possible to bisect e into k edges $e_1^{[i]}$, $i = 1,\dots,k$ and attach graphs G_i, $i = 2,\dots,k$ or G_i, $i = 2,\dots k+1$. That is, one has the option to leave at most one of the edges $e_1^{[i]}$ in G_1 unattached at. Also, as before, after each attachment at $e_1^{[i]}$, this edge can be retained or removed. Typical pictures for $k = 2$ are:







It is also possible that e_i, $i > 1$ is bisected once before identification (which is done with one of the two new edges). In comparison to the above picture, this is done with e_2. (And still the option used to delete one of the two new edges after identification.)



We also keep to attention that only w.r.t. G_i, $i > 1$, does the operation have to involve the unbounded face; see (3.2).

Lemma 3.3 The identification $G_1 \times G_2$ has a cut vertex if and only if G_1 or G_2 have such. □

A clarification is needed on the meaning of these operations on the level of link diagrams. The edge identification is the replacement of a crossing by a tangle. The crossing usually disappears (which corresponds to e being deleted after identification.) However, keep in mind that a crossing in a Conway polyhedron stands for an arborescent tangle to be substituted for it. And when one of the crossings in a clasp is replaced, the clasp disappears. This is why the scenario must also be considered that e is retained after identification. A similar explanation applies to the option of bisecting edges before identifying.

Lemma 3.4 Every quasi-polyhedral graph G can be decomposed $G = G_1 \times \cdots \times G_n$ for polyhedral graphs G_i. □

3.2 Monotonous linear embeddings

In our case $G = G(\mathcal{D})$ and in order to gain the grid D with $[D] = \mathcal{D}$, we need to find a very specific type of planar embedding of G. For this we make a series of definitions. (This paper very much constantly depends on seeing what are the important properties to look for.)

Definition 3.5 A planar embedding of G is *linear* if all edges are straight lines.

It must immediately occur to attention that for linear embeddings multiple edges are not possible (to be drawn as a collection of single edges). Although we have assumed that G has no multiple edges, we will not be able to release ourselves entirely of that assumption in the following arguments. At some places caution is needed (see e.g. (3.1)).

We next stipulate that we regard $\mathbb{R}^2 \supset \widehat{\mathcal{D}}$ as the coordinate plane, and fix the *height function* $h(x,y) = y$.

Definition 3.6 A linear planar embedding of G is *generic* if all vertices have different height, and no 3 vertices are collinear.

Definition 3.7 We say a linear embedding of G is *monotonous* if it is generic, and the intersection of every horizontal line with a bounded face of G is connected (i.e., a single segment, or a point), and if the same is true for the union of all bounded faces, i.e., the complement of the unbounded face of G.

This condition is the same as saying that for every region R (incl, the unbounded one), ∂R is the union of two (and not more) h-increasing paths.

The key lemma we prove in this subsection is the following.

Lemma 3.8 (Monotonicity Lemma) Every quasi-polyhedral graph G has a monotonous embedding.

For the proof, we need to work with a more specific monotonous embedding.

Definition 3.9 Let R be the unbounded face of G, and $e \subset \partial R$ an edge. We say that the monotonous embedding of G is *e-good* if the two vertices v_1 and v_2 of e

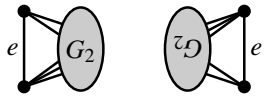
1. Have the same x-coordinate.

2. v_1 has the lowest and v_2 the highest height among all vertices of G.

We also say that for such an embedding the edge e is *good*. We say that G is *good-embeddable* if G has an e-good embedding for every $e \subset \partial R$.

Note: in the following diagrams, we always draw e as if it left of the rest of G. This is not mandatory; it can be easily followed that the same arguments apply also when e is on the right side of G.

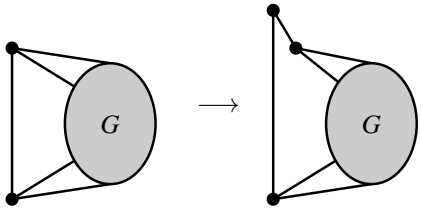
Keep in mind that $G_1 \times G_2$ has an extra twofold ambiguity depending on whether G_2 is attached "from the left" or "from the right" of e in G_1. That is, for fixed $e \in E(G_i)$, there are two possible $G_1 \times G_2$ if the planar isotopy changes the heights of the vertices of e in G_2.



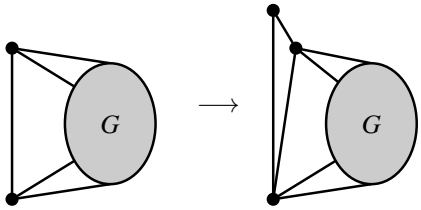
Lemma 3.10 Assume G' is obtained from G by bisecting an edge g. If G has an *e-good* embedding for every edge $e \subset \partial R$, then so has G'.

Proof. Let $e' \subset \partial R$. If e' was not part of e before bisecting, then $e' \in E(G)$. Take an e'-good embedding of G bisect g, and move the new vertex into a generic position (without spoiling monotonicity; since no 3 vertices in G are collinear).

If e' was part of g in G, then choose a g-good embedding of G and use a picture like



Note that even the case that an edge in G becomes multiple is not a problem:



(3.1)

However, this is the situation that a multiple edge forms a bigon. For the rest situation, see the proof of Lemma 3.8. □

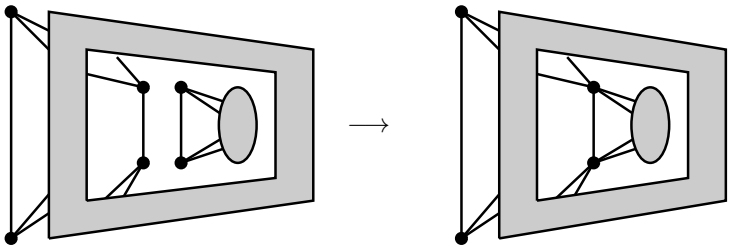
Note that when G is in an e-good embedding with $x(v_i) = x_0$, one has the option of *h-shrinking* this embedding by changing a vertex $v = (x,y)$ to $(x_0 + h(x - x_0), y)$ for some $0 < h < 1$. This still gives an e-good embedding of G.

Similarly, when G has a monotonous (e-good) embedding, then *v-stretching* $(x,y) \mapsto (x,Hy)$ will also give such an embedding for any $H > 1$.

We use these deformations to assume that the slope of every edge (as a linear segment, and regarded without sign) is large enough (or ∞ for a vertical edge). Here we use that in genericity no two vertices have the same height.

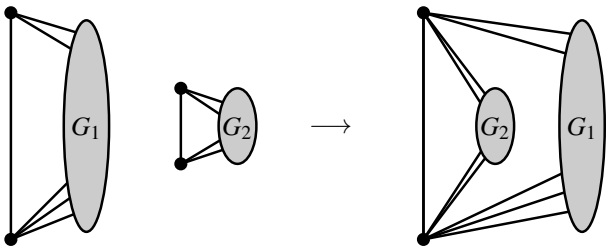
Lemma 3.11 If G_1 and G_2 are good-embeddable, then so is $G_1 \times G_2$.

Proof. Bisecting is handled with Lemma 3.10, and for the identification, use pictures like



(3.2)

and



(3.3)

with slight rotation of G_2 and v-stretching G_1 and h-shrinking G_2.

Keep also in mind that in (3.2) we only need to consider an edge to be good if it is in the boundary of the unbounded region of $G_1 \times G_2$. □

Proof of Lemma 3.8. We prove that every loop-free 2-connected (*not* 3-connected) simple planar graph without a cut vertex is good-embeddable, by induction over $e(G)$.

The cases of low $e(G)$ can be realized directly.

If G is not 3-connected, then G admits a decomposition $G_1 \times G_2$ (Lemmas 3.11 and 3.3), or an unbisection (Lemma 3.10). Note that if, when unbisecting G, one obtains a multiple edge, which does not form a bigon (i.e., one cannot use (3.1)), then G admits a decomposition $G_1 \times G_2$ also. Thus Lemma 3.10 is sufficient.

Therefore, we assume G is 3-connected, and admits no decomposition $G_1 \times G_2$, so is polyhedral.

Now assume $e \subset \partial R$ is fixed for the unbounded face R. If G is a wheel graph [St2, Figure 8], then again construct an e-good embedding directly. Otherwise there is always an edge $g \in E(G)$ disjoint from ∂R.

Let now $G' = G/_g$ be G with g contracted. To restore G later, equip G' with a *memorization marker* $\star$ on two of the corners near the contracted edge $g_* \in V(G')$, which should be the regions in G bounded by g. We like to use induction on G'.

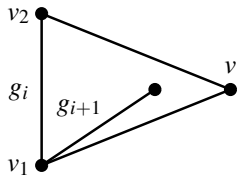
It is now important to observe that when G is polyhedral, then G' has no cut vertex for whatever $g \in E(G)$. Loop-freeness, and 2-connectedness of G' are straightforward. However, G' may not be simple. In fact, if G is a triangulation, then G' will not be simple for whatever $g \in E(G)$. This requires a more careful choice of g.

Definition 3.12 Define $g \in E(G)$ between $v_1, v_2 \in V(G)$ to be *contract-proper*, if the following holds: whenever $v \in V(G)$ is adjacent to both $v_1, v_2 \in V(G)$, then v, v_1, v_2 form the corners of a triangular face of G.

In particular, since such faces are bounded by g, there are at most two such v.

Sublemma 3.13 If G is not a wheel graph, there exists a contract-proper edge g disjoint from ∂R.

Proof. Start with $g_0 = g$ disjoint (with both endpoints) from the infinite face. Assume that it is not contract-proper. Thus g bounds a length-3 cycle with an edge g_1 in its interior (except at most one of its boundary points).

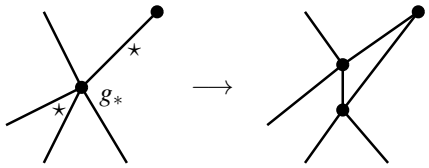


By induction if g_i is not contract-proper, we can find g_{i+1}. But the number of distinct length-3 cycles of G containing in their interior g_i (except at most one of its boundary points) is at least i. This number cannot group infinitely, thus at some stage some g_i must be contract-proper. □

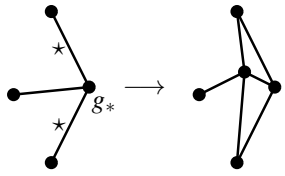
Note also that in case there is a v for g, the above memorization marker $\star$ should mark the edge between $g_* \in V(G')$ and v, while making this double edge simple, and not a corner of g_*.

This making the edge simple may, though, spoil 3-connectedness: G is 3-connected (since polyhedral), but G' may only be 2-connected. This is the reason why the induction process was chosen to run over 2-connected graphs. Since G' has no cut vertex, G' will always be 2-connected (and not only 1-connected), though.

With Sublemma 3.13 we are essentially done. We build $G' = G/_g G$, use induction assumption on G', and decontract g_* (using and preserving genericity) into a short vertical edge g to obtain an e-good embedding of G.



In case there are two $\star$-marked edges the direction of decontracting g must be chosen properly. (But it always can since the embedding of G' is generic and both $\star$-marked edges do not lie over each other.)



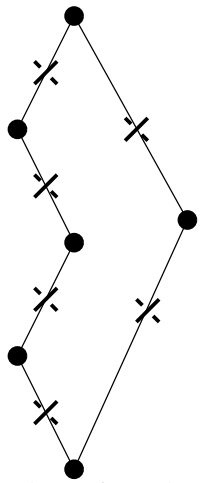
□

3.3 Constructing the grid

Proof of Theorem 1.1. By the Monotonicity Lemma 3.8, we assume an alternating diagram $\mathcal{D}$ whose Tait graph G has a monotonous linear embedding. We also assume it is v-stretched so that the slope of every edge is bigger than 1 (absolutely).

We put small crossing markers somewhere on each edge of G. Then a typical bounded face of G would look

like



Note that the edges of G intersect the markers always from above-below.

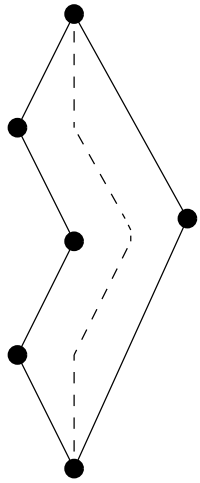
Now we explain how to connect the endpoints of these markers to obtain the desired grid diagram D.

Case 1. Consider marker ends within a bounded region R of G.

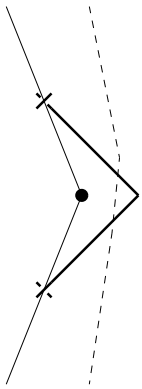
Let v_0, v_1 be the lowest and highest vertex of R. By assumption we can split ∂R into a *left* and *right side*. Define the *median* m_R of R as

$$m_R = \{(x_1+x_2)/2, y) : x_1 < x_2, (x_1,y), (x_2,y) \in \partial R\} \cup \{v_0, v_1\}.$$

Note that the median of R is an increasing PL-line of absolute slope > 1 and lies inside R.

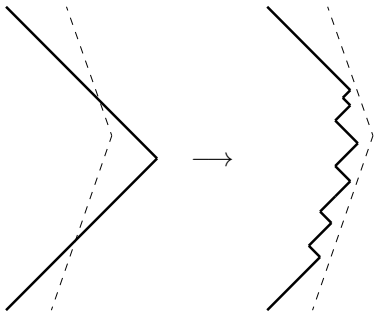


Case 1.1. Two neighbored edges on the left side of ∂R. Extend the upper marker end of the lower crossing and the lower marker end of the upper crossing to the right until they meet.



By assumption these segments do not touch the left side of ∂R. However, they may even go over the right side of R.

If they go over the median, then replace a piece of them by a zig-zag curve near (but left of) the median by segments of slope ± 1. Note that this zig-zag curve can be chosen to have no local maxima.



(3.4)

How to deal with two crossings of neighboring edges on the right side of ∂R should be clear.

Case 1.2. Crossings of the two highest edges in ∂R. Extend the upper right segment of the crossing on the left edge, and the upper left segment on the right edge until they meet. By assumption they run entirely inside R. (The median can be ignored here, and no zig-zag correction is needed.)

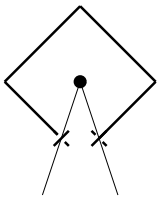
How to deal with the crossing markers of the two lowermost edges in ∂R should be clear.

Case 2. Consider the unbounded face R.

Case 2.1. Two crossings on the right (analogously, left) side of ∂R. Extend the upper right end of the lower crossing marker and the lower right end of the upper crossing marker until they meet (no median considered here).

Case 2.2. Consider now the markers of the two uppermost (analogously, lowermost) edges in ∂R.

Extend the upper left end of the left marker and the upper right end of the right marker long enough, so that adding two crossing-less segments goes "over the top" of ∂R (and G).



(3.5)

In summary, by rotating by $\pm\pi/4$, we have now obtained a grid diagram D with $[D] = \mathcal{D}$. (It realizes the planar embedding of G as its Tait graph.)

After we remove the crossing-less zig-zag segments of (3.4) by Cromwell moves, we see that every segment has exactly one crossing, except the two pairs added in (3.5) and its analogue at the bottom of G. (It it also very easy to infer that these four segments are top/bottom and left/rightmost.) Thus we obtain the sought proper grid diagram D. □

4 Ambiguity: permutation grids

The realization result in the previous section is of course non-unique. For certain links, like $(2,n)$-torus links, it may be possible to describe all grid diagrams directly. However, we will see that the ambiguity is exceedingly large even in average.

Theorem 4.1 For every $\sigma \in S_n$ with $n = \mu(D) - 2$, there is a prime proper grid diagram D_σ with $\sigma(D_\sigma) = \sigma$.

Originally I wrote a program for finding proper grids D without length-1-segments with $\sigma(D) = \sigma$ for a given $\sigma \in S_n$. It is easy to see that for some σ there is more than one D, and that for some σ none is a knot grid. Nevertheless, G.T.Jin used this program to generate proper grids of size 16 from 14-element permutations, and to locate prime alternating 14 crossing knots. This process for finding grids is laborious, as plagued by extensive duplications. We await an implementation of the algorithm of §3.

In order to prove theorem 4.1, we need to look at grid diagrams with specific monotonocity properties, of their top/bottom when looking from left to right (increasing x-coordinate), and of their left/right end, when looking top-down (decreasing y-coordinate).

Definition 4.2 We say a grid diagram D is *top-decreasing* if

$$x \mapsto \max\{y : (x,y) \in \widehat{\mathcal{D}}\}$$

is (non-strictly) decreasing. We say that D is *bottom-decreasing* if

$$x \mapsto \min\{y : (x,y) \in \widehat{\mathcal{D}}\}$$

is decreasing. We say that D is *right-righting* if

$$y \mapsto \max\{x : (x,y) \in \widehat{\mathcal{D}}\}$$

is decreasing, and *left-righting* if

$$y \mapsto \min\{x : (x,y) \in \widehat{\mathcal{D}}\}$$

is decreasing. Similarly we define for every bounded region R of $\mathbb{R}^2 \setminus \widehat{\mathcal{D}}$ by considering the polygon ∂R.

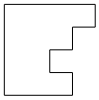
The other 4 terms like *bottom-increasing* and *right-lefting* should be definable analogously. Thus, right-lefting means that the right end goes to the left, when one follows it top-down.

For example, the grid in (2.7) is left-lefting, right-lefting, top- and bottom-increasing, but does not have any of the other 4 properties. (Obviously, there are 4 pairs of properties which for a non-trivial proper grid mutually exclude each other within each pair.)

We now introduce a specific construction of D_σ. For this we need to craft a condition that can get us through an inductive process.

Definition 4.3 Let $B(D)$ for a grid D be the region below the top horizontal segment. We call a grid D *good* if it is proper, top-decreasing, $B(D)$ is bottom-increasing, and the intersection of $B(D)$ with every vertical line is connected.

The below shows a region R which is bottom-increasing but its intersection with some vertical line is disconnected. (It can be observed that when R is bottom-increasing, then the intersection with no vertical line being disconnected means that R is right-lefting.)



The below lemma presents the construction, although we need later some further properties of the thus obtained D_σ (Lemma 4.5).

Lemma 4.4 For every $\sigma \in S_n$, there is a good grid D_σ with $\sigma(D_\sigma) = \sigma$.

Proof. We use induction of $n = \mu(D_\sigma) - 2$. For $n = 0$ we take the trivial grid.

Now assume $n > 0$ and $\sigma \in S_n$.

We consider a permutation $\sigma' \in S_{n-1}$ by "removing" $1 \mapsto \sigma(1)$ in σ. That is, for $2 \le i \le n$, we set

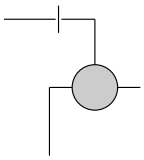
$$\sigma'(i-1) := \begin{cases} \sigma(i) & \text{if } \sigma(i) < \sigma(1) \\ \sigma(i)-1 & \text{if } \sigma(i) > \sigma(1) \end{cases}$$

Let $m = \sigma(1)$. In order not to complicate things more than necessary, let us first deal with $m = 1$. In this case, we just add a kink at the top-left corner of $D_{\sigma'}$ to obtain D_σ. This does not spoil goodness. ($B(D_\sigma)$ is just a square of size 1.)

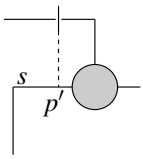
We assume now $m > 1$ and in particular $n > 1$. Now we consider $l(D_{\sigma'})$, the length of the top horizontal segment of $D_{\sigma'}$. Note that it is followed to the right by the vertical segment that has the crossing of $(\sigma')^{-1}(l) \mapsto l$.

There are two cases how to obtain D_σ.

Case 1. $m \le l$. We put the prospective new crossing of $1 \mapsto \sigma(1)$ onto the top segment of $D_{\sigma'}$ so that the vertical strand has the $m+1$-st left-right position counting the leftmost crossing-less segment of $D_{\sigma'}$.

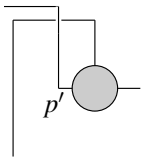


Since the leftmost segment of $D_{\sigma'}$ does not go until the bottom, and $B(D_{\sigma'})$ is bottom-increasing, the segment s is not the bottom-most horizontal segment of $D_{\sigma'}$. Thus it has a unique crossing on it, and again because $B(D_{\sigma'})$ is bottom-increasing, this crossing is right of p'.

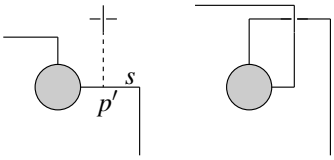


We continue the down segment of p until it hits $\partial B(D_{\sigma'})$ in a segment s in a point p'.

We reconnect as follows to obtain D_σ (and observe that it is good).



Case 2. $m > l$. In that case, the crossing p cannot be placed above the top segment of $D_{\sigma'}$. Proceed thus.



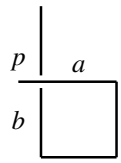
In this diagram the left horizontal segment is the top-most one, and since $D_{\sigma'}$ is top-decreasing, the crossing of the segment s hit by the dotted line (which not the bottom-most of $D_{\sigma'}$!) is left of p'. □

Lemma 4.5 D_σ is proper, it is top-decreasing, bottom-decreasing, left-righting, and right-righting.

Proof. Now, D_σ is proper. In particular, by induction one sees that the leftmost segment does not go to the bottom and rightmost segment does not go to the top. (when $n = \mu(D_\sigma) - 2 > 0$).

Lemma 4.6 Assume a grid D_σ for $\sigma \in S_n$ has a kink crossing. Then it corresponds to $\sigma(1) = 1$ or $\sigma(n) = n$.

Proof. A kink crossing p in a grid looks like



(4.1)

plus rotations. In particular, a kink exhibits two crossing-less segments. Now, D_σ is proper. In particular, one sees that the leftmost segment does not go to the bottom and rightmost segment does not go to the top. So the corner in the kink (4.1) is the top-left or bottom-right corner.

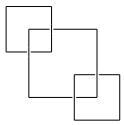
We consider the bottom right corner as in (4.1). Let a be the piece of the horizontal segment between p and the corner. If a has length > 1, then since the second rightmost vertical segment of D_σ cannot be inside the kink, we have contradiction to D_σ being right-righting. Similarly, if b has length > 1, then the second bottom-most segment will give a contradiction to D_σ being bottom-decreasing. Thus p corresponds to $\sigma(n) = n$.

For the π-rotation of (4.1) argue with D_σ being top-decreasing and left-righting to see $\sigma(1) = 1$. □

Definition 4.7 A *cut* of $\sigma \in S_n$ is defined as (C_1, C_2) for $\{1,\dots,n\} = C_1 \cup C_2$ with $C_1 = \{1,\dots,k\}$, $C_2 = \{k+1,\dots,n\}$ for $1 \le k < n$, so that $\sigma(C_i) = C_i$.

Lemma 4.8 Let D_σ be the grid constructed in Lemma 4.4. A decomposing curve γ of $[D_\sigma]$ corresponds to a cut (C_1, C_2) of σ. That is, the interior and exterior of γ contain the crossings of D that correspond to $i \mapsto \sigma(i)$ for $i = C_j$ for fixed $j = 1, 2$.

In the below example, we have $D_{(1\leftrightarrow 2, 3\leftrightarrow 4)}$, and there is an obvious decomposing curve for $k = |C_1| = 2$.



Proof. By induction on n. Start is obvious. If $\sigma(1) = 1$, a new decomposing curve is added for the kink. Otherwise use Lemma 2.6. Notice that one of the two splicings of p in $[D_\sigma]$ corresponding to $1 \mapsto \sigma(1)$ gives $[D_{\sigma'}]$. For the kink case use Lemma 4.6. Otherwise the decomposing curves in $[D_{\sigma'}]$ correspond to cuts (C_1, C_2) of σ'. Let $k = |C_1|$ as in Definition 4.7. For $\sigma(1) > k+1$, the curve γ does not survive in D_σ as a decomposing curve because γ is of the form (b) in Lemma 2.6. □

Remark 4.9 In a similar way it can be seen that a nugatory crossing in $[D_\sigma]$ corresponds to a fixelement $\sigma(k) = k$ where (C_1, C_2) with $k = |C_1|$ is a cut of σ, or $k = n$.

Proof of Theorem 4.1. The diagram D_σ of Lemma 4.4 is of the sought type by Lemma 4.8 unless σ has a cut.

Now assume σ has a cut. One can mirror the construction in Lemma 4.4 by exchanging left and right. Formally, consider $\sigma^*(i) = n + 1 - \sigma(i)$. And build $\rho(D_{\sigma^*})$. It is very easy to see that when σ has a cut, σ^* has none. Thus D_{σ^*}, and hence $\rho(D_{\sigma^*})$ is prime. □

It can be easily estimated that the number of size $\mu = n+2$ grids is, up to polynomial factors in n, asymptotically bounded above by $(n!)^2$. Thus at least (about) the square root of the number of grid diagrams of fixed size are proper (minimal alternating) grid diagrams.

Corollary 4.10 The average number of proper grid diagrams realizing an alternating prime link diagram of n crossings is hyperexponential in n. More precisely, let

$$\mathcal{A}_n := \{\text{ prime alternating link diagrams of } n \text{ crossings }\}$$

and

$$\Gamma_n := \{\text{proper grid diagrams of size } n+2\}.$$

Then

$$\frac{\left|\{D \in \Gamma_n : [D] \in \mathcal{A}_n\}\right|}{|\mathcal{A}_n|} \geq \left(\frac{n}{C}\right)^n$$

for some $C > 0$ and (say) n large enough.

Proof. Obviously D_σ are distinct when $\sigma \in S_n$ are distinct. Use Stirling's formula. It is known that the number of prime alternating link diagrams of n crossings grows exponentially with n (see [ST]). □

We note the following relation to the work in [AHT].

Lemma 4.11 ([AHT]) Assume D_1 and D_2 are grid diagrams with $[D_1] = [D_2]$, and let $n = \max(\mu(D_1), \mu(D_2))$. Then there is a sequence of $O(n^2)$ Cromwell moves transforming D_1 into D_2.

As the paper explains, justifying that such a sequence exists was a gap left out in Cromwell's original "proof" [Cr] that his moves interconvert all grid diagrams of the same link. Their proof is somewhat subtle, and the bound on the number of moves requires to go though the different steps (incl. the Bigon lemma); this bound is not explicitly stated.

What can be inferred from Corollary 4.10 is that even for minimal grid diagrams (i.e., $\mu(D_1) = \mu(D_2) = a([D_i])$) there exists *no sublinear* upper bound of the form $n^{1-\varepsilon}$, $\varepsilon > 0$, on the number of Cromwell moves needed. The number of grid diagrams that can be obtained from D with $\mu(D) = n$ by $n^{1-\varepsilon}$ Cromwell moves is at most

$$(3n)^{2 \cdot n^{1-\varepsilon}} < n!$$

(asymptotically).

Remark 4.12 A final remark is made on practical computations. While the algorithm of §3 awaits implementation, it was much easier to write a program that generates all minimal grids D with $\sigma(D) = \sigma$ for a given σ. Using this program for $\sigma \in S_{14}$ and $\mu(D) = 16$, G.T. Jin was able to obtain minimal alternating grids for all prime 14 crossing knots. He will report his work at a separate place.

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References

[AHT] T. Ando, C. Hayashi and M. Taguchi, *Cromwell moves on grid diagrams for knots and links*, preprint.

[BP] Y. Bae and C.-Y. Park, *An upper bound of arc index of links*, Math. Proc. Cambridge Philos. Soc. **129** (2000), 491-500.

[Cr] P. R. Cromwell, *Embedding knots and links in an open book. I. Basic properties*, Topology Appl. **64** (1995), 37–58.

[DH] O. T. Dasbach and S. Hougardy, *A conjecture of Kauffman on amphicheiral alternating knots*, J. Knot Theory Ramifications **5(5)** (1996), 629–635.

[DP] I. A. Dynnikov and M. V. Prasolov, *Bypasses for rectangular diagrams. A proof of the Jones conjecture and related questions*, Trans. Moscow Math. Soc. **2013**, 97–144, doi.org/10.1090/S0077-1554-2014-00210-7.

[J+] G. T. Jin, H. Kim, G.-S. Lee, J. H. Gong, H. Kim, H. Kim and S. A. Oh, *Prime knots with arc index up to 10*, in "Intelligence of Low Dimensional Topology 2006", Series Knots and Everything, Vol. **40** (World Scientific, Hackensack, NJ, 2007), 65–74.

[JLS] G. T. Jin, H. J. Lee, and A. Stoimenov, *Minimal Grid Diagrams of the Prime Alternating Knots with 13 Crossings*, Bull. Korean Math. Soc. **62(3)** (2025), 799–807, https://doi.org/10.4134/BKMS.b240256.

[JP] G. T. Jin and W. K. Park, *Prime knots with arc index up to 11 and an upper bound of arc index for non-alternating knots*, J. Knot Theory Ramifications **19** (2010), 1655–1672.

[Ka] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[KLA] H. Kim, J. Lee and H.-K. Ahn, *Rectangular partitions of a rectilinear polygon*, Computational Geometry: Theory and Applications **110** (2023), 101965.

[La] M. Lackenby, *The volume of hyperbolic alternating link complements*, with an appendix by I. Agol and D. Thurston, Proc. London Math. Soc. **88(1)** (2004), 204–224.

[LN] C. Livingston and S. Naik, *Ozsváth-Szabó and Rasmussen invariants of doubled knots*, Algebr. Geom. Topol. **6** (2006), 651–657.

[Me] W. W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23(1)** (1986), 37–44.

[MT] W. W. Menasco and M. B. Thistlethwaite, *The Tait flyping conjecture*, Bull. Amer. Math. Soc. **25 (2)** (1991), 403–412.

[MT2] W. W. Menasco and M. B. Thistlethwaite, *The classification of alternating links*, Ann. of Math. **138(1)** (1993), 113–171.

[MB] H. R. Morton and E. Beltrami, *Arc index and the Kauffman polynomial*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 41–48.

[MS] K. Murasugi and A. Stoimenov, *The Alexander polynomial of planar even valence graphs*, Adv. Appl. Math. **31(2)** (2003), 440–462.

[Sa] S. Sarkar, *Grid Diagrams and the Ozsvath-Szabo Tau-Invariant*, Math. Res. Lett. **18(6)** (2011), 1239–1257.

[St] A. Stoimenov, *Square numbers, spanning trees and invariants of achiral knots*, `math.GT/0003172`, Comm. Anal. Geom. **13(3)** (2005), 591–631.

[St2] A. Stoimenov, *Everywhere Equivalent and Everywhere Different Knot Diagrams*, Asian J. Math. **17(1)** (2013), 95–137.

[ST] C. Sundberg and M. B. Thistlethwaite, *The rate of growth of the number of prime alternating links and tangles*, Pacific Journal of Math. **182 (2)** (1998), 329–358.

[Ta] T. Tanaka, *Maximal Thurston-Bennequin numbers of alternating links*, Topology and its Applications **153(14)** (2006), 2476–2483.