# APPLICATIONS OF JONES SEMIADEQUACY TESTS

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## 1   Introduction

This paper contains two applications of the semiadequacy formulas for the edge Jones polynomial coefficients [St3], found independently by Dasbach-Lin [DL, DL2].

We will deal with the construction of infinitely many positive braid links which admit no minimal string positive braid representative in §4. These examples simultaneously serve to exhibit a non-monotonous (in increasing strand number) minimal braid length function. In §5 we treat a construction of minimal crossing almost alternating diagrams. (More background and motivation will be provided at the beginning of the respective sections.)

Other uses of these formulas are for instance [St2] (for non-triviality of the Q polynomial of 3-braid links) or [St6] (for the classification of 3-braid links with partial inversion symmetries).

## 2   Basic preliminaries

### 2.1   Kauffman bracket, Jones polynomial and semiadequacy

The Jones polynomial is useful to define here via Kauffman's state model; we follow the setting of [St3], which is the main reference throughout. Recall, that the Kauffman bracket $\langle D \rangle$ of a link diagram $D$ is a Laurent polynomial in a variable $A$, obtained by summing over all states $S$ the terms

$$A^{\#A(S)-\#B(S)} \left( -A^2 - A^{-2} \right)^{|S|-1}, \tag{1}$$

where a *state* is a choice of *splicings* (or *splittings*) of type $A$ or $B$ for any single crossing (see figure 1), $\#A(S)$ and $\#B(S)$ denote the number of type A (resp. type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in $S$.

We call the *A-state* the state in which all crossings are $A$-spliced, and *B-state* is defined analogously. We call a trace $a$ in the $A$-state *dual* to a trace $b$ in the $B$-state, if $a$ and $b$ correspond to the same crossing as in figure 1.

The Jones polynomial of a link $L$ can be specified from the Kauffman bracket of some diagram $D$ of $L$ by

$$V_L(t) = \left( -t^{-3/4} \right)^{-w(D)} \langle D \rangle \Big|_{A=t^{-1/4}}, \tag{2}$$

with $w(D)$ being the *writhe* of $D$.
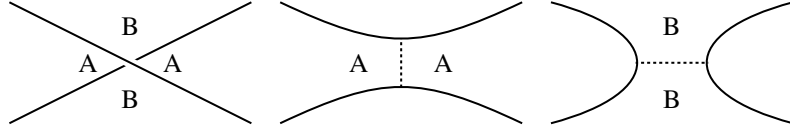
**Figure 1:** The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing. It is useful to put a "trace" of each splitted crossing as an arc connecting the loops at the splitted spot.

Let $S$ be the $A$-state of a diagram $D$ and $S'$ a state of $D$ with exactly one $B$-splicing. If $|S| > |S'|$ for all such $S'$, we say that $D$ is $A$-*adequate*. Similarly one defines a $B$-adequate diagram $D$. See [LT, Th]. Then we set a diagram to be

$$
\begin{aligned}
\text{adequate} &= \text{A-semiadequate } and \text{ B-semiadequate}, \\
\text{semiadequate} &= \text{A-semiadequate } or \text{ B-semiadequate}, \\
\text{inadequate} &= neither \text{ A-semiadequate } nor \text{ B-semiadequate}.
\end{aligned}
$$

(Note that inadequate is a stronger condition than not to be adequate.)

A link is called ($A$ or $B$-)adequate, if it has an ($A$ or $B$-)adequate diagram. A link is *semiadequate* if it is $A$- or $B$-adequate. A link is *inadequate*, if it is neither $A$- nor $B$-adequate.

As noted, semiadequate links are a much wider extension of the class of alternating links than adequate links. For example, only 3 non-alternating knots in Rolfsen's tables [Ro, appendix] are adequate, while all 55 are semiadequate.

**Definition 2.1** When

$$
V_K = a_0 t^k + V_1 t^{k+1} + \ldots + a_d t^{k+d} \tag{3}
$$

with $a_0 \neq 0 \neq a_d$ is the Jones polynomial of a knot or link $K$, we will write for $d$ the *span* $\operatorname{span} V_K$ of $V$, for $k$ the *minimal degree* $\operatorname{min deg} V_K$ and for $k+d$ the *maximal degree* $\operatorname{max deg} V_K$. We will use throughout the paper the notation $V_i = V_i(K) = a_i$ and $\bar{V}_i = \bar{V}_i(K) = a_{k-i}$ for the the $i+1$-st or $(i+1)$-last coefficient of $V$ (since these terms will occur often, and to abbreviate the clumsier alternative $[V]_{\operatorname{min deg} V + i}$ resp. $[V]_{\operatorname{max deg} V - i}$).

## 2.2 HOMFLY and Kauffman polynomials

We include a description of the other polynomials that will make appearance in the paper, at least to clarify conventions.

The *skein (HOMFLY) polynomial* $P$ is a Laurent polynomial in two variables $l$ and $m$ of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$
l^{-1} P\big(\!\times\!\big) + l P\big(\!\times\!\big) = -m P\big)\big(\;. \tag{4}
$$

This convention uses the variables of [LMi], but differs from theirs by the interchange of $l$ and $l^{-1}$. We call the three diagram fragments in (4) from left to right a *positive* crossing, a *negative* crossing and a *smoothed out* crossing (in the skein sense).

A diagram is called *positive*, if all its crossings are positive. A(n oriented) link is positive, if it admits a positive diagram (see for example[1] [Cr2, O]).

The *Kauffman polynomial* [Ka2] $F$ is usually defined via a regular isotopy invariant $\Lambda(a, z)$ of unoriented links.

We use here a slightly different convention for the variables in $F$, differing from [Ka2, Th] by the interchange of $a$ and $a^{-1}$. Thus in particular we have the relation $F(D)(a, z) = a^{w(D)} \Lambda(D)(a, z)$, where $w(D)$ is the writhe of a

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[1]Occasionally, the name is assigned to positive *braid* links, but we frown upon this obsolete (and confusing) convention here.

link diagram $D$, and $\Lambda(D)$ is the writhe-unnormalized version of the polynomial. $\Lambda$ is given in our convention by the properties

$$\Lambda(\asymp) + \Lambda(\asymp) = z\left(\Lambda(\smile\frown) + \Lambda()\,()\right),$$
$$\Lambda(\bigcirc) = a^{-1}\,\Lambda(\,|\,); \quad \Lambda(\bigcirc) = a\,\Lambda(\,|\,),$$
$$\Lambda(\bigcirc) = 1.$$

Note that for $P$ and $F$ there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity.

The Jones polynomial $V$ can be obtained from $P$ and $F$ (in our conventions) by the substitutions (with $i = \sqrt{-1}$; see [LMi] or [Ka2, §III])

$$V(t) \quad = \quad P(-it, i(t^{-1/2} - t^{1/2})) \quad = \quad F(-t^{3/4}, t^{1/4} + t^{-1/4}). \tag{5}$$

By $[P]_M$ we denote the coefficient of the monomial $M = l^x m^y$ in the polynomial in $P$. If $P$ has a single variable, then we use the exponent rather than the whole monomial for $M$. (For example, $[V]_3 = [V]_{t^3}$ for $V \in \mathbb{Z}[t^{\pm 1}]$.)

## 2.3 Crossing numbers and critical line polynomials

Let $c(D)$ be the crossing number of a diagram $D$, and $c(K)$ the crossing number of a knot or link $K$,

$$c(K) := \min\{c(D) : D \text{ is a diagram of } K\}.$$

We call $D$ *minimal* if $c(D) = c(K)$.

For reasons that will become clear immediately, let us also define $c_\pm(D)$ to be the number of positive resp. negative crossings of a diagram $D$, and set

$$c_\pm(K) := \min\{c_\pm(D) : D \text{ is a diagram of } K\}.$$

Obviously, $c(K) \geq c_+(K) + c_-(K)$, but knots like Perko's with minimal diagrams of different writhe show that the inequality can be strict. Nevertheless, in general one can gain good (and often sharp) estimates of $c(K)$ by estimating $c_+(K)$ and $c_-(K)$.

Thistlethwaite proved in [Th] that (with our convention for $\Lambda$) for a link diagram $D$ of $c(D)$ crossings we have $[\Lambda(D)]_{z^l a^m} \neq 0$ only if $l + |m| \leq c(D)$, and that $D$ is $A$ resp. $B$-adequate iff such a coefficient does not vanish for some $l$ and $m$ with $l - m = c(D)$ resp. $l + m = c(D)$. These properties imply most of his results, incl. the main one, the crossing number minimality of adequate diagrams.

The coefficients of $\Lambda(D)$ for which $l \pm m = c(D)$ form the "critical line" polynomials $\phi_\mp(D)$. Thistlethwaite expresses these polynomials in terms of some graph invariants, so they clearly encode combinatorial information of the diagram. Unfortunately, we do not know how to interpret most of this information, i.e., to say what tangible features of the diagram it measures.

Still the minimal and maximal degree of $\phi_\pm(D)$ do have a "visual" meaning. In §3 of [St5] we translated this meaning to our present context. One degree can be expressed also in terms of the writhe and Jones polynomial, thus giving a new obstruction to semiadequacy (proposition 4.3 below). However, for semiadequate links this obstruction (consequently vanishes and) gives no new information. In contrast, the other degree (see proposition 3.2 below) gives a new invariant for semiadequacy. This will be used crucially in the last sections, together with our work of "decoding" in such a visual way the Jones polynomial coefficients.

Let us, though, better specify $\phi_\pm$ directly in terms of $F(L)$. (For the opposite signs, mirror the link.) Let

$$a_-(L) = \max\{m - l : [F(L)]_{a^l z^m} \neq 0\}/2, \tag{6}$$

and

$$\phi_+(z) = \sum_{i=0}^{\max\deg_z F(L)} z^i \cdot [F(L)]_{z^i a^{i - 2a_-(L)}}.$$

(Note the sign switch between $\phi_+$ and $a_-$; this is not a typo, but our tribute to the incommensurability of all mnemonic and related historical notation.) Then Thistlethwaite proves that $c_-(D) \geq a_-(D)$ for every diagram $D$ of $L$, and equality holds if and only if $D$ is an $A$-adequate diagram. In that case $\phi_+(z)$ is a non-negative polynomial.

## 2.4 Knot numbering

Knots of $\leq 10$ crossings will be denoted according to Rolfsen's tables [Ro, appendix], and for $\geq 11$ crossings according to Hoste and Thistlethwaite's program KnotScape [HT]. However, I reorganize the KnotScape tables so that non-alternating knots are appended after alternating ones (of the same crossing number), instead of using "a" and "n" super/subscripts. For example, there are 367 prime alternating knots of 11 crossings. Thus a knot commonly named $11_{a111}$ in the table will be written as $11_{111}$, while $11_{n111}$ as $11_{478}$ (for $111 + 367 = 478$).

Knot diagrams of 17 and 18 crossings were generated by the algorithm (and software, temporarily, but unfortunately no longer available) of [RFS], and are applied for some computational checks, but a numbering is never used.

The obverse (mirror image) of $K$ is denoted by $!K$. Fixing a mirroring convention of the table knots will not be very relevant.

To save space, knot diagrams are displayed in terms of their DT notation [DT], in the way used in [HT]. (The second entry, knot identifier, is an artifact of the way examples were tracked down and has no real significance.)

# 3 Jones polynomial of (semi)adequate links

The following is a brief summary of the formulas and methods developed in [St3] that will be needed later. (See there for full proofs.)

## 3.1 The second coefficient

We consider the bracket [Ka] (rather than Tutte) polynomial. The $A$-state of $D$, the state with all splicings $A$, is denoted by $A(D)$. (Occasionally, we omit the argument $D$ in this notation, if no ambiguity arises.) For us a state is always understood as a planar picture of loops (solid lines) and traces connecting these loops (dashed lines). Then it is clear that and how to reconstruct $D$ from $A(D)$.

**Definition 3.1** One fundamental object exploited in this paper is the *A-graph* $G(A) = G(A(D))$ of $D$. It is defined as the planar graph with vertices given by loops in the $A$-state of $D$, and edges given by crossings of $D$. (The trace of each crossing connects two loops.) The analogous terminology is set up also for the $B$-state.

Clearly the $A$-state determines the $A$-graph, but not conversely. Their distinction is relevant in some situations. However, $G(A(D))$ (including its planar embedding) determines $A(D)$ if $D$ is alternating; then sometimes $G(A(D))$ is called the Tait graph of $D$. Note also that, for alternating $D$, the duality of crossing traces between $A(D)$ and $B(D)$ corresponds to the duality (in the usual graph-theoretic sense) of edges in the planar graphs $G(A(D))$ and $G(B(D))$.

If $D$ is a connected diagram, then $G(A(D))$ is also connected. If $D$ is positive, then $G(A(D))$ is the *Seifert graph*, and its vertices correspond to the Seifert circles of $D$.

Let $v(G)$ and $e(G)$ be the number of vertices and edges of a graph $G$. Let $G'$ be $G$ with multiple edges removed (so that a simple edge remains). We call $G'$ the *reduction* of $G$.

We will write sometimes

$$s_+(D) = v(G(A(D))) = v(G(A(D))'), \qquad s_-(D) = v(G(B(D))) = v(G(B(D))').$$

The definition of $A$-adequate can be restated saying that $G(A(D))$ has no edges connecting the same vertex. For $B$-splicings the graph $G(B(D))$ and the property $B$-adequate are similarly defined (and what is stated below proved).

In the following, we shall explain the second and third coefficient of the Jones polynomial in semiadequate diagrams.

**Proposition 3.1** If $D$ is $A$-adequate connected diagram, then in the representation (3) of $V_D$ we have $V_0 = \pm 1$, $V_1 V_0 \leq 0$, and

$$|V_1| = e(G(A(D))') - v(G(A(D))') + 1 = b_1(G(A(D))') \tag{7}$$

is the first Betti number of the reduced $A$-graph.

The formula (7) was also explained in [DL2].

**Lemma 3.1** If $G$ is a planar simple graph (no multiple edges), then $b_1(G) \leq \left\lfloor \frac{2}{3}e(G) \right\rfloor - 1$. This inequality is sharp for proper $G$ when $e(G) > 2$.

(Here and below $\lfloor x \rfloor$ is the largest integer not greater than $x$.)

## 3.2 Conditions for positivity

Even although alternating knots are not generally positive, in the following positivity arguments will be essential. It is well-known that if a diagram $D$ is positive, then it is $A$-adequate: the $A$-state of $D$ is just the Seifert picture of $D$ (and the $A$-state loops are the Seifert circles). This point should be kept in mind throughout.

Since $A$-adequacy is an unoriented condition, $D$ would remain $A$-adequate even if we alter orientation of some components. We say that an unoriented diagram $D$ *admits a positive orientation*, or is *positively orientable*, if it arises from such a diagram by forgetting orientation. We observed in [St3] the following lemma, which specifies which $A$-adequate diagrams are positively orientable. Since the lemma will be of considerable importance, its short argument is reproduced as well.

**Lemma 3.2** Let $D$ be $A$-adequate. Then $D$ is positively orientable iff $G(A(D))$ is a bipartite graph.

**Proof.** If $D$ is positively orientable, its graph $G(A(D))$ is the Seifert graph of a (positive) diagram, and hence bipartite. Conversely, if $G(A(D))$ is bipartite, it is possible to orient the loops in $A(D)$ so that each trace looks locally like $\uparrow\!\cdots\!\uparrow$. Then it is clear that it is possible to extend this loop orientation to an orientation of $D$, and with that orientation $D$ becomes positive. $\square$

**Corollary 3.1** Let $L$ be an $A$-adequate link, with an $A$-adequate connected diagram $D$, then

$$1 - \left\lfloor \frac{2}{3}c(D) \right\rfloor \leq V_0 V_1 \leq 0.$$

If $V_1 = 0$, then $D$ admits a positive orientation, and (with this orientation) $L$ is fibered.

This gives a new semiadequacy test. Beside that Thistlethwaite's condition on the positive critical line coefficients [Th] involves the Kauffman polynomial, which is considerably slower to calculate, the Jones polynomial features sometimes prove essential, as we showed by some examples.

The minimal number $e(n) = e_n$ of edges needed for a planar simple graph to have given $b_1 = n$ is by lemma 3.1

$$e_n = \begin{cases} \dfrac{3}{2}(n+1) & n \text{ odd} \\[2mm] \dfrac{3}{2}n + 2 & n \text{ even} \end{cases}.$$

For later discussion, it is useful to define some properties of graphs.

**Definition 3.2** The *join* (or *block sum*, *cf.* [Cr2]) '$*$' of two graphs is defined by
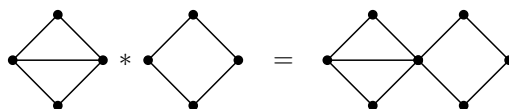


This operation depends on the choice of a vertex in each one of the graphs.

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We call $v$ a *cut vertex* of a graph $G$, if $G$ gets disconnected when deleting all edges incident to $v$ and *additionally* $v$ itself. (When we delete an edge, we understand that a vertex it is incident to is *not* to be deleted, too.)

Every connected non-trivial (i.e., with at least one edge) graph $G$ can be written as a join $G_1 * \ldots * G_n$ for some non-trivial connected graphs $G_i$, such that no $G_i$ has a cut vertex. We call $G_i$ the *join factors* of the graph $G$. The number $a(G) = n$ of join factors of $G$ is called *atom number* of $G$. (This should not be confused with the critical line quantities $a_\pm(L)$ in (6).)

We set $a(A(D)) = a(G(A(D)))$.

**Proposition 3.2** ([St5]) We have $\max \deg_z \phi_+(D) = c(D) - a(A(D))$ if $D$ is connected and $A$-adequate.

Keep in mind that if $D$ is positive, then the atoms are the *Cromwell blocks* [Cr2] (or Murasugi atoms [QW]) $D_i$ of $D$.

**Corollary 3.2** If $L$ is non-split and $e(|V_1|) > \mathrm{span} V(L)$, then $L$ is non-alternating.

Below we will modify this reasoning for a *bipartite* planar simple graph (see lemma 5.1).

Note that in corollary 3.1 the case $V_1 = 0$ poses strong additional restrictions to $A$-semiadequacy. The change of component orientation alters $V$ only by a positive unit [LMi], so that several positivity criteria still apply. Among others, with $n(L)$ the number of components of $L$, we have $V_0 = (-1)^{n(L)-1}$ (*cf.* proposition 4.3 below). For knots positively orientable is the same as positive, and we have a series of further properties. For example $\min \deg V_L = g(L)$, the genus of $L$, which is in particular always positive.

The next corollary determines asymptotically the minimal value of $V_0 V_1$ for positive link diagrams.

**Corollary 3.3** Let $L$ be a positive $n(L)$-component link, with a positive connected diagram $D$. Then

$$-\frac{1}{2} c(D) \leq (-1)^{n(L)-1} V_1 \leq 0,$$

and the left inequality is asymptotically sharp (for large $c(D)$).

## 3.3 The third coefficient

The third coefficient in semiadequate diagrams can still be identified in a relatively self-contained form. It depends, however, on more than just $G(A(D))'$ or $G(A(D))$.

**Definition 3.3** We call two edges $e_{1,2}$ in $G(A(D))'$ *intertwined*, if the following 3 conditions hold:

1. $e_{1,2}$ have a common vertex $v$.

2. The loop $l$ of $v$ in the $A$-state of $D$ separates the loops $l_{1,2}$ of the other vertices $v_{1,2}$ of $e_{1,2}$.

3. $e_{1,2}$ correspond to at least a double edge in $A(D)$, and there are traces of four crossings along $l$ that are connected in cyclic order to $l_1, l_2, l_1, l_2$.
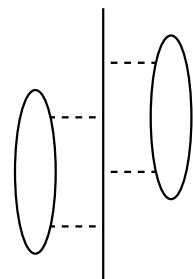
It should be made clear that intertwinedness of edges in $G(A(D))'$ still depends on the *state $A(D)$*, rather than just the graph $G(A(D))'$ or $G(A(D))$: in the graphs the intertwining information becomes lost.

**Definition 3.4** A *connection* in $A(D)$ is the set of traces between the same two loops, i.e., an edge in $G(A(D))'$. A connection $e$ in $A(D)$ is said *multiple*, if it consists of at least two crossing traces. More generally we can define the *multiplicity* of a connection as the number of its traces.

Then we can speak also of intertwined (multiple) connections. We will later occasionally relax terminology even more and speak just of intertwined loops $M$ and $N$, when their connections to a third loop $L$ are intertwined. This is legitimate, because one can determine $L$ from $(M,N)$ uniquely.

Call below a loop in $A(D)$ *separating* if it is connected by crossing traces from either side. So if connections $(M,L)$ and $(N,L)$ are intertwined, then $L$ would be separating, and connected by $M$ and $N$ from opposite sides. Clearly there can be only one such $L$ for given $M$ and $N$. (Moreover, almost throughout where we will apply this terminology below, there will be in fact only one separating loop in $A(D)$.)

**Definition 3.5** Define the *intertwining graph $IG(A(D))$* to consist of vertices given by multiple connections in $A(D)$ (or multiple edges in $G(A(D))$), and edges connecting pairs of intertwined connections.

This is the second graph associated to $D$, which plays a fundamental role in the whole paper. Note that this is a simple graph (no multiple edges), but not necessarily planar or connected. It may also be empty. For example, for an alternating diagram this graph has no edges, so is a (possibly empty) set of isolated vertices. With the preceding remark, $IG(A(D))$ is determined by the state $A(D)$, but not (in general) by the graph $G(A(D))$.

**Proposition 3.3** If $D$ is $A$-adequate, then

$$V_0 V_2 = \binom{|V_1|+1}{2} - \triangle G(A(D))' + \chi(IG(A(D))) = \binom{|V_1|+1}{2} + e_{++}(A(D)) - \triangle G(A(D))' - \delta A(D)',$$

where $\triangle G(A(D))'$ is the number of triangles (cycles of length 3) in $G(A(D))'$, $\delta A(D)' = e(IG(A(D)))$ the number of intertwined edge pairs in $G(A(D))'$, and $e_{++}(A(D)) = v(IG(A(D)))$ the number of multiple connections in $A(D)$.

This formula was obtained independently and simultaneously by Dasbach-Lin [DL2]. Their proof is longer, but it unravels the underlying combinatorics completely, while in [St3] we helped ourselves with some skein theoretic arguments about positive braid links.

**Remark 3.1** Note that in alternating diagrams $\delta A(D)' = 0$, since there are no separating loops, while in positive diagrams $\triangle G(A(D))' = 0$ because $G(A(D))'$ is bipartite. Note also that a pair of intertwined edges does not occur in a triangle.

## 3.4 Legs and traces

Let the *intertwining index* of an edge pair in $G(A(D))$ be half the number of interchanged connections from either side of $l$. For example the intertwining index of
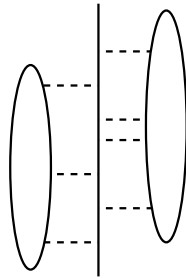


is 3, and edges are intertwined iff their intertwining index is $\geq 2$. (We will from now on, to save space, draw in diagrams only a part of $l$ that contains its basepoints. So the straight line, that represents $l$, is understood to be closed up.)

We assume in the following that an *edge* in $A(D)$ stands for a possible (but non-empty) collection of parallel traces:



$$\hspace{1em} \tag{8}$$

(The term "edge" thus assumes a clear separation between the state $A(D)$ and the graph $G(A(D))$.) Traces between loops $a$ and $b$ are *parallel* if between their basepoints on both $a$ and $b$ no traces connecting $a$ or $b$ to other loops occur. With this convention we identify, and do not display several, parallel traces in diagrams. A connection in $A(D)$, which is the set of all traces or edges that connect the same two loops, in general decomposes into several edges.

An edge obviously has a *multiplicity* by the number of faces it contains. Accordingly it is *even* or *odd*. In the case of the $A$-state of a positive diagram $D$ of a (positive) fibered link, one can obtain $A(D)$ by starting with some single loop, and then attaching new loops with all their traces.



$$\tag{9}$$
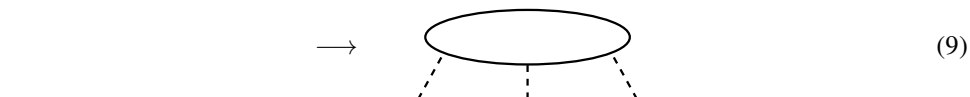
Since $G(A(D))'$ is a tree, we can assume that we attach the traces of the new loop to fragments of the *same* previous loop. This condition of 'attachment' will be assumed consistently below.

Every loop $l$ which is not in the outer cycle is attached to a single previous loop $m$. This means that all traces of crossings, that connect $l$ at one end connect to $m$ on the other. The parallel equivalence classes of such traces are called below *legs*. So legs are edges in $A(D)$ connecting an attached loop. As edges, they can also be labeled even or odd.

The number of legs of a loop $l$ is called the *valence* of $l$. As in (8), and unless stated clearly otherwise, we group parallel traces into a single one (with multiplicity indicated, or explained from the context) when drawing loop diagrams. So a dashed line in a diagram starting from a loop attached by (9) (usually) stands for a leg.

It is clear that the legs of each attached loop form a multiple connection of traces. So the existence of attachments forces the intertwining graph $IG(A)$ to be non-empty (i.e., have at least one vertex). Then the condition $\chi(IG) = 0$ means that we must have a cycle in $IG(A)$. The existence of this cycle will be helpful at several places below.

In our case we will attach a loop to another atom, but we can maintain the above terminology of (even/odd) legs.

# 4 The minimal braid length and minimal minimal braid length of a knot

## 4.1 The examples

Braids are now inseparably intertwined with links, and are used for a variety of applications of or to link theory. In some laborious project, a set of useful braid representatives of prime knots up to 13 crossings were compiled in [St8].

A braid is minimal if it realizes the braid index, and minimal length if it has shortest length word (in the standard Artin generators) among all braid representatives of its closure.

There is an a priori *upper* bound on the length of a braid representative for a given knot in terms of its crossing number: it is quadratic, proved by Vogel [Vo]. It is kind of conjecture that quadratic is the best possible, and examples are twist knots, but the problem is we have no good *lower* bounds (except the crossing number itself).

For most applications minimal width matters more than length, and that is what was focused on in [St8]. It is clear that minimal words may not necessarily be realized on a minimal number of braid strands.

We write for $\mathrm{mmbl}(K)$ the minimal minimal braid length and $\mathrm{mbl}(K)$ the minimal braid length of $K$.

As I wrote on my page [St8], I had known about $10_{136}$, as example where the minimal width and minimal length braids are not the same. Perhaps this feature was first known to Gittings [Gi], and was further studied by van Cott [vC]. One main question there was how to systematically exhibit such examples. The problem was mildly interesting, so I did not compile more.

But recently Boden and Shimoda [BS] raised this issue with more emphasis, in relation to simple walks used in calculation of the colored Jones polynomial. (Perhaps one can gain braid length bounds from simple walks. That would be really something to study.) So I looked a little further into it.

I noticed that I had published examples where the minimal width minimal braid length and minimal braid length are not the same in Figure 7 of [St4], in the context of positive braid representatives.

In this remark we observe two related facts.

**Proposition 4.1** There exist infinitely many knots with a positive braid representative but not minimal such one.

**Proposition 4.2** There exist infinitely many knots where minimal minimal braid length and minimal braid length are not the same.

One can extend the second example in Figure 7 of [St4], $16_{1223549}$, to an infinite family $K_k$, given by the 4-braids

$$-3\,2\,3^{2k}\,2\,2\,1\,1\,2\,2\,1\,3\,2\,2\,1\,1\,2.$$

(Here $i$ stands for $\sigma_i$ if $i > 0$ and for $\sigma_{-i}^{-1}$ otherwise.)

We call, as in [St3], a braid $\beta$ to be $A$-adequate, if it admits a word so that the diagram $\hat{\beta}$ is $A$-adequate. (Similarly can be done with $B$-adequate and adequate.) By Thistlethwaite's work [Th], this must a minimal length word in Artin's generators (even up to conjugacy), and if one minimal word is $A$-adequate, all other such will be. One reason for introducing this concept was to prove and use that all 3-braids are semiadequate, which in particular gives a solution of the minimal conjugacy word problem in the 3-braid group $B_3$.

Without constantly repeating this below, there are straightforward analogues of every concept/statement under mirror image.

The proofs of proposition 4.1 is just a minor shortening of the one of proposition 4.2. We will use the language of the latter one for contextual reasons.

**Proof.** Setting $k = 1$ gives $16_{1223549}$. More generally all $15 + 2k$ crossing 4-braids give a knot diagram which simplifies to a $14 + 2k$-crossing positive 5-braid diagram.

$$1\,2\,3\,3\,4\,2\,2\,3\,1\,2\,4\,2\,3\,3\,2\,4^{2k-1}$$

The only point to essentially prove is the following.

**Lemma 4.1** The crossing number of $K_k$ is $14 + 2k$.

Assume the lemma is proved.

First, the braid index of $K_k$ is 4. This can be seen by direct calculation of the skein polynomial and MWF, but another argument is thus. If $b(K_k) \leq 3$, and as $K_k$ is a closed positive braid, then by [St2], $K_k$ is a closed positive 3-braid, and by looking at the genus, we see that this 3-braid must have $12 + 2k$ crossings, a contradiction to lemma 4.1.

Given the lemma 4.1, we have $\mathrm{mmbl}(K_k) = 14 + 2k$, but a 4-braid with knot closure must have odd crossing number, so $\mathrm{mbl}(K_k) = 15 + 2k$. $\qquad\qquad\square$

**Remark 4.1** By [Cr, O], the knots are prime.

**Proof of lemma 4.1.** For lemma 4.1, we use the following tool. We write $c_-(D)$ for the number of negative crossings of a diagram $D$, and $c_-(K)$ for the the minimal number of negative crossings of a diagram of $K$.

Following Thistlethwaite, we defined in [St5] a quantity $a(K) = a_-(K)$ from the Kauffman polynomial, with the property that $c_-(K) \geq a_-(K)$ and equality holds if and only if $K$ is $A$-adequate. Similarly let $a_-(K) = a_+(!K)$ for the mirror image $!K$ of $K$. We also considered the critical line polynomial $\phi_\pm(D) = \phi_\pm(K)$. (Note that with our signing $a_-(K) = c_-(D)$ if and only if $\phi_+(D) \neq 0$.) And $V_L$ is the Jones polynomial.

**Proposition 4.3** ([St5, Corollary 3.10]) If a link $L$ is non-split and $A$-adequate, then we have

$$2 \min \deg V_L = \min \deg_z \phi_+(L) - 3a(L)$$

and the trailing coefficient of $V_L$ is

$$\min \mathrm{cf} V_L = (-1)^{\min \deg_z \phi_+(L)}.$$

This gives a fairly handy semiadequacy test. For instance, it rules out one of the two adequacies for 33 of the 38 non-alternating (prime) 10 crossing knots. (3 of the 42 knots are adequate, and the Perko knot has both semiadequacies without being adequate.) Unfortunately, $10_{136}$ is among the remaining 5.

The test does work for (or rather against) $B$-adequacy of $16_{1223549}$, which is the starting point of the argument. Since $K_k$ are positive, obviously $a_-(K) = 0$. We calculate $a_+(K) = 13 + 2k$ from some easy recursive calculation of the Kauffman polynomial. Thus $c(K_k) = 13 + 2k$ and if equality holds, $K_k$ would be $B$-adequate.

By direct calculation, we find proposition 4.3 obstructing $B$-adequacy, so the proof is complete. □

Note that if proposition 4.3 fails, the work in [St3] gives enough insight, in principle, to rule out $B$-adequacy for general $k$. But we would have to go through some fairly laborious process, as outlined in [St7] for the odd crossing number amphicheiral knots. (Such complexity will hardly be warranted by the importance of our examples.) On the opposite end, it is obvious that many similar constructions are possible with the introduced tool.

## 4.2 The conjecture

Originally we thought that the following is a construction of knots where the difference of minimal minimal braid length and minimal braid length can get arbitrarily large.

We write $w(\beta)$ for the writhe (exponent sum) of a braid $\beta$.

**Proposition 4.4** Let $K$ admit a $B$-adequate braid $\beta$ and a minimal braid $\beta'$ with $w(\beta') < w(\beta)$. Then the iterated connected sum $K_k = \#^k K$ satisfies $\mathrm{mmbl}(K_k) - \mathrm{mbl}(K_k) \geq k$.

**Proof.** Let us write $c(\beta)$ for the word length of $\beta$ in Artin's generators (and inverses). Also write $c_-(\beta)$ for the minimal number of inverse Artin's generators in a word for $\beta$.

By $B$-adequacy $a_+(K) = c_+(\beta) \leq c_+(\beta')$ by Thistlethwaite for whatever braid word $\beta'$ with $\hat{\beta}' = K$. Since $w(\beta')$ is unique by the proof of the Jones conjecture [LM, DP] for any word of a minimal braid $\beta'$, we have $c_-(\beta') > c_-(\beta)$. Thus $c(\beta') = c_+(\beta') + c_-(\beta') > c_+(\beta) + c_-(\beta) = c(\beta)$. By choosing $\beta'$ with $\mathrm{mmbl}(K) = c(\beta')$, we have $\mathrm{mmbl}(K) \geq c(\beta) + 1 \geq \mathrm{mbl}(K) + 1$.

If we take $K_k$, then $w(\beta'_k) = kw(\beta')$ for a minimal braid $\beta'_k$ of $K_k$ using [BM], and $a_+(K_k) = ka_+(K)$ by $B$-adequacy, so $\mathrm{mmbl}(K_k) - \mathrm{mbl}(K_k) \geq k$. □

**Example 4.1** $10_{136}$ (properly mirrored) admits a $B$-adequate braid $\beta$ of 5 strands and writhe $-2$. It also admits a (minimal) 4-braid, but of writhe $-1 > -2$, not $-3 < -2$.

The below problem, which we like to highlight, is the product of our continuous failure to find examples subject to proposition 4.4.

A braid is smaller if it has fewer strands.

**Conjecture 4.1**    1. If $\beta$ is B-adequate, then $\beta$ is not Markov equivalent to a smaller braid of smaller writhe.

2. If $\beta$ is A-adequate, then $\beta$ is not Markov equivalent to a smaller braid of larger writhe.

3. If $\beta$ is adequate, then $\beta$ is minimal.

Obviously parts 1 and 2 are equivalent and (either of) both imply part 3. (Unlike for knots, A- and B-adequate together imply adequate for braids, for the writhe is a braid word invariant.)

In fact 'smaller braid' can be replaced by 'minimal braid'. (Then 1 and/or 2 would not quite imply 3.) But this begs additional insight into the braid index, which is a separate difficulty.

The easiest way to test the claim is by sharpness of the right MWF inequality [Mo, FW]:

$$\max\deg_l P(\hat{\beta}) = w(\beta) + n - 1. \tag{10}$$

(The skein polynomial $P$ is used with the variables of [LMi] but exchanging $l$ and $l^{-1}$.) This implies that an $n'$-braid Markov equivalent to $\beta$ for $n' < n$ has $w(\beta') \geq w(\beta) + n - n' > w(\beta)$.

We compiled the list of B-adequate (prime) braids with knot closure up to 18 crossings and tested (10). It failed only in a handful of cases. Most of these $B$-adequate braids could be proved minimal by 2cMWF (see e.g. [St]). Only for 18 crossings some 5-braids can reduce to 4-braids, but these are minimal and of the "wrong" writhe (as occurs for $10_{136}$).

A few known results fit into the conjecture.

Positive braids are A-adequate and (by Bennequin's or Morton's inequality e.g.) do not admit Markov equivalent smaller braids of larger writhe. Part 3 of the conjecture is true for alternating braids ([Mu]) and positive braids without trivial syllables (by an argument essentially due to Nakamura [Na]; see also [St2, §5]).

There is also a possible weaker version:

**Conjecture 4.2**    1. If $\beta$ is B-adequate, then $\beta$ is not conjugate to a positive stabilization.

2. If $\beta$ is A-adequate, then $\beta$ is not conjugate to a negative stabilization.

3. If $\beta$ is adequate, then $\beta$ is irreducible.

The problem of detecting stabilization up to conjugacy is a fundamental problem in braid-link theory. This will be discussed in a separate paper. But in either form, the conjecture remains more a speculation at this point. Still it highlights a distinct possibility of a connection between the combinatorics of adequacy and braid-theoretic features, which is quite elusive at present.

# 5   Minimal crossing almost alternating diagrams

Almost alternating diagrams were introduced by Adams et al. [A+]. While many links have such diagrams, their construction usually requires one to go up in crossing number. Thus the question was raised about minimal crossing almost alternating diagrams [Bu]. By inspection, such diagrams do not occur for knots up to 12 crossings, but there are a few of 13 crossings. One is

$$13 \ 235 \ 4 \ 8 \ 14 \ 2 \ -16 \ 18 \ 20 \ 6 \ 22 \ 24 \ 26 \ 10 \ 12 \tag{11}$$

of $13_{5491}$. However, the verification of minimality essentially depends on enumerating lower crossing diagrams, which again leads to the problem of finding diagrams of this type of higher (and arbitrarily high) crossing number.

This turns out not so easy. Here we introduce a test that, although at first sight somewhat involved, is successfully applicable. Kauffman, HOMFLY, and Jones, all are needed. From some (sufficiently many, but by far not all) of the examples we give, infinite families can be constructed by iterated double (positive) parallel clasping (12). This yields:

**Proposition 5.1** For every crossing number $n \geq 13$, there exists a minimal crossing almost alternating diagram of a knot.

For the main idea behind the description of diagrams, one applies moves, which replace a positive crossing by a parallel positive clasp:

$$\times \quad \longrightarrow \quad \bowtie \mathrel{\cdot} \tag{12}$$

We call this procedure below a *(parallel) clasping*. (It is enough to restrict ourselves to positive crossings here.) A diagram $D$ of $n$ components can be transformed into a knot diagram $D'$, with $1 - \chi(D') = n - \chi(D)$, by repeatedly applying clasping on crossings of two different components. Note that in $A(D)$ a clasping doubles a trace (replaces it with two parallel ones).

Again, we write $b_1 = e - v + 1$ for the Betti number of a connected graph of $e$ edges and $v$ vertices. The following is a matter of simple counting (and using that all cycles have length at least 4).

**Lemma 5.1** When $G$ is a planar bipartite connected graph (no multiple edges), then $b_1 \leq v - 3$ and $e \leq 2v - 4$. $\qquad\square$

We present now the following test (with the caveat of remark 5.1); nothing less involved succeeded in collecting practical examples.

**Theorem 5.1** Assume $D$ is a diagram of a knot $K$, satisfying the following conditions:

1. $a_-(D) = 0$ and $a_+(D) = c(D) - 2$ and proposition 4.3 obstructs $B$-adequacy.

2. $\max \deg_z \phi_+(D) = c(D) - 3$ (keep in mind our signing of $\phi_\pm$).

3. $V_1(D) \neq 0$.

Then

1. If $c(D) - 2\min\deg_t V(D) - |V_1(D)| < 4$ then $D$ is minimal

2. If
$$c(D) - 2\min\deg_t V(D) - |V_1(D)| = 4 \tag{13}$$
   then

   (a) If $|V_1| \geq 2\min\deg_t V$ or $|V_2| \leq \binom{|V_1| + 1}{2}$, then $D$ is minimal.

   (b) Assume $D$ is not minimal. Then there exists a special alternating prime knot diagram $D'$ with the following properties:

      i. (one can choose which of the two crossing numbers)
      $$c(D') = c(D) - 2 \tag{14}$$

      or
      $$c(D') = 2c(D) - 4\min\deg_t V(D) - 6 + |V_1(D)| = 3|V_1(D)| + 2, \tag{15}$$

      ii. $\min\deg_t V(D') = \min\deg_t V(D) + \mu$ with
      $$\mu = \frac{c(D') - c(D)}{2} + 1$$

      iii. $|V_1(D')| = |V_1(D)|$

      iv. $|V_2(D')| > \binom{|V_1(D')| + 1}{2}$ if (14), or $1 \leq |V_2(D')| - \binom{|V_1(D')| + 1}{2} \leq |V_1(D)|$ if (15)

      v. $\max\mathrm{cf}_m P(D') = \max\mathrm{cf}_m P(D) \cdot l^{2\mu}$

The alternative (14) works better for smaller $c(D)$, while for increasing $c(D)$, the option (15) becomes (far) more viable. A crucial reason why this test is often stable under parallel clasping is that (13) remains valid, and that (15) does not increase. Often do the other invariants involved not change either (as long as we do not switch number of components). Proposition 4.3 does require an iterated calculation of the Kauffman polynomial under parallel (double) clasping, but although technical, this can be considered straightforward.

If we had alternating *link* tables available, we could content ourselves with lower-crossing number link diagrams $D'$ instead.

**Example 5.1**

$$D = 15\ 61262\ 6\ 10\ 16\ 30\ 4\ 22\ 26\ 2\ 8\ 24\ 28\ 14\ 20\ 12\ \text{-}18$$

This satisfies conditions 1 and 2. It goes in case 2: we have $\min\deg_t V = 3$, $c(D) = 15$ and $|V_1| = 5$. Then consider test 2b. We have $\operatorname{span} V(D) = 12$, so (15) and parity gives $c(D') = 13$.

We then test 13 crossing prime special alternating knots of genus 3, and $\max\operatorname{cf}_m P(D) = l^6 - 5l^8 + 4l^{10} - l^{12}$ does not occur as leading $P$-term. Thus $D$ is minimal (it belongs to $15_{100431}$).

This argument is stable after claspings, since (15) gives $c(D') = 17$ and that $l^4 \max\operatorname{cf}_m P(D)$ does not occur in genus 5 special alternating knot diagrams of 17 crossings. (See the proof of proposition 5.1.)

**Example 5.2** For
$$D = 16\ 1497\ 6\ 14\ 22\ 26\ 30\ \text{-}16\ 4\ 24\ 28\ 8\ 2\ 12\ 20\ 32\ 10\ 18$$
we have (after doing all previous necessary checks) that $\max\operatorname{cf}_m P(D) = l^6 - 6l^8 + 10l^{10} - 3l^{12}$. Testing $c(D') = c(D) - 2$, we find that this $\max\operatorname{cf}_m P(D)$ occurs among special alternating genus 3 prime 14 crossing knots only for $14_{16368}$. But for that knot $|V_1| = 6$ and $21 = |V_2| = \binom{|V_1|+1}{2}$, so that condition 2(b)iv fails, and $D$ is minimal. In fact, $D$ depicts $16_{894574}$. (Since (15) yields $c(D') = 20$, this special argument will likely fail after claspings on $D$.)

The following is a list of almost alternating diagrams (not claimed all to represent mutually distinct knots), which can be proved minimal.

On the 16 crossing diagrams also test 2a applies. This test will likely remain restricted under claspings for these examples. We do not know if test 1 can ever apply.

```
14 526 6 12 16 22 26 4 20 2 24 -8 14 28 10 18      15 4872 4 8 14 2 26 18 6 -30 24 10 16 28 20 12 22
14 647 6 12 22 14 26 4 -20 28 24 8 2 16 10 18      15 7320 4 8 16 2 26 18 22 6 -30 24 14 28 10 20 12
14 648 6 12 22 14 26 4 20 28 24 -8 2 16 10 18      15 20589 4 10 -18 26 2 20 28 8 24 30 14 6 16 22 12
14 694 6 12 22 28 14 4 18 24 10 26 2 16 -8 20      15 61262 6 10 16 30 4 22 26 2 8 24 28 14 20 12 -18
14 721 6 12 24 14 18 4 -22 28 10 26 8 2 16 20      15 61266 6 10 16 30 4 24 20 2 8 26 12 28 14 22 -18
14 722 6 12 24 14 18 4 22 28 10 26 -8 2 16 20      15 66178 6 12 16 22 28 4 20 2 -10 26 14 30 8 18 24
14 731 6 12 26 -16 22 4 20 24 28 10 14 8 2 18      15 66783 -6 12 18 22 28 4 20 26 2 10 16 30 8 14 24
                                                   15 67274 6 12 20 14 28 4 -18 24 8 2 26 16 30 10 22
16 3514 6 10 16 32 4 26 20 2 8 24 28 14 30 22 12 -18   15 67679 -6 12 20 24 14 4 18 28 10 2 26 8 30 16 22
16 4531 6 12 18 32 26 4 30 22 2 -10 28 14 20 8 24 16   15 68163 6 12 20 30 24 4 28 22 26 2 -10 16 8 14 18
16 1136 6 12 22 14 28 4 20 32 26 -8 2 16 30 10 18 24   15 68457 6 12 22 14 28 4 -20 30 10 26 2 16 8 18 24
16 1806 6 12 22 28 -14 4 20 26 32 10 2 18 30 8 16 24   15 68457 6 12 22 14 28 4 20 30 -10 26 2 16 8 18 24
16 2822 -6 12 24 16 30 4 26 20 32 8 28 2 10 22 14 18   15 68610 -6 12 22 16 28 4 20 26 30 10 2 18 8 14 24
16 88 6 12 26 32 16 4 30 22 8 14 28 18 2 -10 20 24     15 68610 6 12 22 -16 28 4 20 26 30 10 2 18 8 14 24
16 544 -6 12 28 16 22 4 26 20 32 8 14 30 10 2 24 18    15 69290 6 12 22 30 16 4 28 24 8 26 2 -10 18 14 20
16 877 6 12 30 -16 24 4 20 26 32 10 28 14 8 22 2 18    15 69989 6 12 24 30 16 4 20 26 8 14 28 2 -10 18 22
16 59 6 14 20 24 -30 16 4 28 10 2 26 8 32 12 18 22     15 70360 6 12 28 14 18 4 -26 20 10 24 30 16 8 2 22
16 109 6 14 20 26 16 30 4 24 8 2 28 -10 18 32 12 22    15 70360 6 12 28 14 18 4 26 20 10 24 30 16 -8 2 22
16 1497 6 14 22 26 30 -16 4 24 28 8 2 12 20 32 10 18   15 72845 6 14 20 26 16 30 4 24 8 2 -12 28 18 10 22
                                                   15 74824 6 14 24 30 16 20 4 28 10 26 8 2 -12 18 22
15 3689 4 8 14 2 18 26 6 -30 22 12 28 16 20 10 24     15 77643 6 16 24 30 18 26 20 4 28 12 8 2 -14 10 22
```

**Proof of theorem 5.1.**   When we want to study almost alternating diagrams, they are neither $A$- nor $B$-adequate. By [Th], if $D$ is almost special alternating, then $a_-(D) < c_-(D) = 1$, so $a_-(D) = 0$ and $a_+(D) < c_+(D) = c(D) - 1$, so $a_+(D) \leq c(D) - 2$.

When $a_+(D) = c(D) - 2$ and proposition 4.3 obstructs $B$-adequacy, then we know that $c_+(D) > a_+(D) \geq c(D) - 1$, so a diagram $\tilde{D}$ of fewer crossings than $D$ must have $c(D) - 1$ crossings and be positive (and no diagram of $K$ has fewer than $c(D) - 1$ positive crossings). So the question is to find practically working way to exclude such a diagram $\tilde{D}$.

We assume throughout for simplicity that $\tilde{D}$ is a knot diagram. We have $g(\tilde{D}) = \min\deg_t V(\tilde{D})$, so

$$s(\tilde{D}) = c(\tilde{D}) - 2\min\deg_t V + 1 = c(D) - 2\min\deg_t V$$

is the number of Seifert circles of $\tilde{D}$.

Now $G(A(\tilde{D}))$ is the Seifert graph.

Furthermore $c(\tilde{D}) - \max\deg_z \phi_+(\tilde{D}) = 2$, so the Seifert graph has 2 Murasugi atoms. We will assume there is one separating Seifert circle $s_0$. (The case $\tilde{D}$ is a connected sum can be easily ruled out, but it will be enough to remedy it by attaching the connected sum factors on opposite site of the Seifert circle. The arguments that follow are valid for that situation as well.)

We need a graph $G(A(\tilde{D}))'$ of $b_1 = |V_1|$ with $s(\tilde{D})$ vertices and one cut vertex. If $s(\tilde{D}) < |V_1| + 4$, there is no way to create $b_1$ cycles with the number of vertices. This explains the (so far fictitious) test 1.

So assume $s(\tilde{D}) = |V_1| + 4$, in the test 2. Then the only option is there is a single Seifert circle $s_1$ attached (say) outside $s_0$ as like in (9).

To explain condition 2a, note that, since in $\tilde{D}$ at least the connection between $s_0$ and $s_1$ is obviously multiple, we have

$$|V_1(D)| = |V_1(\tilde{D})| = b_1(A(\tilde{D})') < b_1(A(\tilde{D})) = 1 - \chi(\tilde{D}) = 2\min\deg_t V(\tilde{D}) = 2\min\deg_t V(D)$$

Vertices of $IG(A(\tilde{D}))$ are always pairs of loops in $A(\tilde{D})$ (but only pairs of loops with at least 2 traces between them, i.e., multiple connections). Also as the connection between $s_0$ and $s_1$ is multiple, $(s_0, s_1)$ is a vertex of $IG(A(\tilde{D}))$. (We will below label vertices of $IG(A(\tilde{D}))$ by pairs of loops this way.) So $IG(A(\tilde{D}))$ has at least one vertex.

If $\chi(IG(A(\tilde{D}))) = 0$, then it must contain a cycle, but it is easy to see that with only $s_1$ on one side of $s_0$, this is not possible. Note that an edge in $IG(A(\tilde{D}))$ is between intertwined (multiple) connections. Two connections inside $s_0$ can never be intertwined, because they are not on opposite sides of a separating loop. (Because the inside of $s_0$ is an atom, there are no separating loops inside $s_0$.) An edge in $IG(A(\tilde{D}))$ can exist only between $(s_0, s_1)$ and $(s_0, s')$, where $s'$ is a loop inside $s_0$ (because $s_1$ is outside $s_0$, and is the only such loop). This means that $(s_0, s_1)$ is a vertex of every possible edge in IG(A(D~)), so $IG(A(\tilde{D}))$ is a "star" plus possible other isolated vertices, but cannot have a cycle, and $\chi(IG(A(\tilde{D}))) > 0$.

This explains conditions 2a.

Now move to test 2b. The idea now is that one can detach $s_1$ from $s_0$ at the cost of making some edges inside $s$ multiple, thus yielding the claimed diagram $D'$. (Thus the move between $D$ and $\tilde{D}$ is an isotopy of the knot $K$, but in $D'$ the knot type will generally change.) By Murasugi-Przytycki [MP], the process of detaching a leg of $s_1$ (as long as other legs remain) changes $\max \mathrm{cf}_m P$ only up to units. By a trivial skein relation argument, the same holds under changing the multiplicity of a leg of $s_1$ as well as for any edge inside $s_0$ (as long its multiplicity remains non-zero).

A bit care is needed how to make $D'$ a knot (rather than a multi-component link) diagram.

Obviously $s_1$ has an odd (multiplicity) leg, otherwise $\tilde{D}$ is not a knot diagram. Also $s_1$ has at least 2 traces (connected to $s_0$), otherwise $\tilde{D}$ can reduce in number of positive crossings, in contradiction to $c_+(K) = c(\tilde{D}) = c_+(\tilde{D})$.

Let use first treat the alternative (14). To obtain a diagram $D'$ of $c(D') = c(D) - 2$,

1. detach all even legs of $s_1$ and add their traces to any edge inside $s_1$ (this is the effect of positive parallel claspings),

2. every time you detach an odd leg, find a proper edge inside $s_1$ where the leg (i.e., the set of its traces) can be added so that the result is still a knot (this is possible!),

3. detach this way all odd legs except the last,

4. for the last odd leg, detach all traces except one, and add them to some edge inside $s_0$,

5. and finally delete $s_1$ with its last trace.

Since $s_1$ has more than one trace connected to $s_0$, (otherwise $c_-(\tilde{D}) > c_-(K)$; cf. the beginning of the proof), obviously at least one clasping is applied inside $s_1$, so that (the case of (14) in) 2(b)iv holds.

To obtain a diagram $D'$ of fewer crossings, we need $\leq 2v - 4$ crossings for the atom inside $s_0$, where

$$v = v(G(A(D'))) = v(G(A(D'))') = s(D') = s(\tilde{D}) - 1$$

(because $s_1$ is discounted), i.e., at most

$$2(c(D) - 2\min\deg_t V(D) - 1) - 4\,.$$

So make all edges inside $s_0$ of multiplicity 1. Every time a leg of $s_1$ is removed, add no trace to any edge inside $s_0$.

Write below

$$b_1 = b_1(G(A(\tilde{D}))') = |V_1(\tilde{D})| = |V_1(D)|\,.$$

Now the diagram may not a knot diagram. But it has at most $b_1 + 1$ components. By $b_1$ claspings one can get back to a knot diagram. Break up a cycle inside all faces of $G(A(\tilde{D}))'$.

Thus we can choose $D'$ to have at most

$$2c(D) - 4\min\deg_t V(D) - 6 + |V_1| \tag{16}$$

crossings, which leads to (15). Also $s(D') = s(\tilde{D}) - 1$, of the same parity of $s(D)$, so to have $D'$ being a knot diagram, we need $c(D') - c(D)$ even. But this is ascertained because $|V_1|$ has same parity to $c(D)$ by (13).

We applied claspings at at most $b_1$ edges, in $G(A(D'))$ at most $b_1$ edges are multiple. So with $IG(A(D'))$ having no edges, and $\triangle G(A(D'))' = 0$, we have

$$0 \leq V_2(D') - \binom{|V_1(D')| + 1}{2} = e_{++}(G(A(D'))) = \chi(IG(A(D'))) \leq b_1 = b_1(G(A(\tilde{D}))')\,.$$

If

$$c(D') < (16)$$

(the difference is even) add double claspings at edges which are already multiple. If all edges are simple, do claspings at some edge until $c(D') = (16)$. This will not augment the number of multiple edges inside $s_0$ unless all edges are simple. Then it will augment 0 to 1. But we excluded the case $b_1 = |V_1| = 0$ (it is not very practically relevant). Thus the number of multiple edges $= \chi(IG(A(D')))$ is at most $b_1$.

Also since $b_1 > 0$ and $c(D') = (16)$, we have $c(D') > 2v - 4$, not all edges inside $s_0$ can be simple, so $\chi(IG(A(D'))) \geq 1$. Then (the case of (15) in) 2(b)iv holds. $\qquad\square$

**Remark 5.1** The test can be (probably) applied on many almost positive diagrams. It is just our interest in almost alternating diagrams that led us to apply it on almost positive-alternating diagrams (or almost special alternating diagrams). What the test can in general do is to prove that a certain almost positive knot diagram is minimal. It is not claimed that the test rules out the knots in question from being positive. Even another minimal crossing diagram could potentially be positive. It is also known that some positive knots have only non-minimal positive diagrams.

**Remark 5.2** If

$$c(D) - 2\min\deg_t V(D) - |V_1(D)| < 4 \tag{17}$$

and $\max\mathrm{cf}_z P$ does not factor non-trivially into (non-strictly) alternating polynomials, then similar argument works, except the right equality in (15) should be revised. (The alternating property of $\max\mathrm{cf}_z P$ translates under convention change from Cromwell's positivity [Cr2, Corollary 4.3].) I have not experimented with this idea, though.

**Proof of proposition 5.1.** For $n = 13$ we have to serve ourselves upon some auxiliary example like (11).

For $n = 15 + 2k$, one family is obtained by iterated double clasping at (one of the crossings of) the parallel clasp in the diagram of example 5.1.

For $n = 14 + 2k$, do the same in the first diagram given above. This gives the sequence $14_{41015}$, $16_{1177477}$, etc. $\square$

The question how to deal with minimality of non-special almost alternating diagrams remains to be considered.

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