APPLICATIONS OF THE JONES SEMIADEQUACY TESTS

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Abstract. This paper contains three applications of the semiadequacy properties for the edge Jones polynomial coefficients. We will deal with (a) the construction of infinitely many positive braid links which admit no minimal string positive braid representative, (b) a construction of minimal crossing almost alternating diagrams for almost all crossing numbers, and (c) the construction of infinitely many homogeneous links which are not alternative.

1. Introduction

This paper contains some applications of the semiadequacy properties for the edge Jones polynomial coefficients. It had been known, essentially by definition of semiadequacy (see §2 and [LT]), that the leading/trailing terms of the Jones polynomial of a semiadequate link must be monic. But this relationship was further depend in [St3], where formulas for the second and third coefficient were worked out. These formulas were found independently by Dasbach-Lin [DL, DL2] (for a detailed review, see §3).

Here we will use these properties, along with some others, and deal with the construction of infinitely many positive braid links which admit no minimal string positive braid representative in §5. These examples simultaneously serve to exhibit a non-monotonous (in increasing strand number) minimal braid length function.

In §6 we treat a construction of minimal crossing almost alternating diagrams.

In §7 we construct infinitely many homogeneous knots which are not alternative. This gives a different way of refuting a conjecture of Kauffman about the equivalence of alternative and pseudo-alternating links [Si]. We add another such counterexample showing that alternativeness is not mutation-invariant.

In §4, we add some preliminaries about diagram genus, and the generator-twisting method, which is (mostly, but not only) needed in §7. More background and motivation will be provided at the beginning of the respective sections.

Other uses of these formulas are for instance [St2] (for non-triviality of the Q polynomial of 3-braid links) or [St6] (for the classification of 3-braid links with partial inversion symmetries), but mainly [St8] (for the construction of odd crossing number amphicheiral knots).

2. Basic preliminaries

2.1. Kauffman bracket, Jones polynomial and semiadequacy

The Jones polynomial is useful to define here via Kauffman's state model; we follow the setting of [St3], which is the main reference throughout. Recall, that the Kauffman bracket $\langle D\rangle$ of a link diagram D is a Laurent polynomial in a variable A, obtained by summing over all states S the terms

$$A^{\#A(S)-\#B(S)}\left(-A^2-A^{-2}\right)^{|S|-1}, \tag{1}$$

where a *state* is a choice of *splicings* (or *splittings*) of type A or B for any single crossing (see figure 1), $\#A(S)$ and $\#B(S)$ denote the number of type A (resp. type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in S.

We call the *A-state* the state in which all crossings are A-spliced, and *B-state* is defined analogously. We call a trace a in the A-state *dual* to a trace b in the B-state, if a and b correspond to the same crossing as in figure 1.

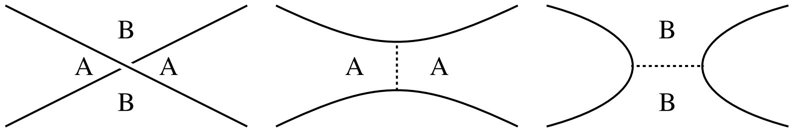


Figure 1: The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing. It is useful to put a “trace” of each splitted crossing as an arc connecting the loops at the splitted spot.

The Jones polynomial of a link L can be specified from the Kauffman bracket of some diagram D of L by

$$V_L(t) = \left(-t^{-3/4}\right)^{-w(D)} \langle D\rangle\Big|_{A=t^{-1/4}}\,, \tag{2}$$

with $w(D)$ being the *writhe* of D.

Let S be the A-state of a diagram D and S' a state of D with exactly one B-splicing. If $|S| > |S'|$ for all such S', we say that D is *A-adequate*. Similarly one defines a B-adequate diagram D. See [LT, Th]. Then we set a diagram to be

$$\begin{aligned} \text{adequate} &= \text{A-semiadequate } \textit{and} \text{ B-semiadequate}\,, \\ \text{semiadequate} &= \text{A-semiadequate } \textit{or} \text{ B-semiadequate}\,, \\ \text{inadequate} &= \textit{neither} \text{ A-semiadequate } \textit{nor} \text{ B-semiadequate}\,. \end{aligned}$$

(Note that inadequate is a stronger condition than not to be adequate.)

A link is called (A or B-)adequate, if it has an (A or B-)adequate diagram. A link is *semiadequate* if it is A- or B-adequate. A link is *inadequate*, if it is neither A- nor B-adequate.

As noted, semiadequate links are a much wider extension of the class of alternating links than adequate links. For example, only 3 non-alternating knots in Rolfsen’s tables [Ro, appendix] are adequate, while all 55 are semiadequate.

Definition 2.1 When

$$V_K = a_0 t^k + V_1 t^{k+1} + \ldots + a_d t^{k+d} \tag{3}$$

with $a_0 \neq 0 \neq a_d$ is the Jones polynomial of a knot or link K, we will write for d the *span* $\operatorname{span} V_K$ of V, for k the *minimal degree* $\min\deg V_K$ and for $k+d$ the *maximal degree* $\max\deg V_K$. We will use throughout the paper the notation $V_i = V_i(K) = a_i$ and $\bar{V}_i = \bar{V}_i(K) = a_{k-i}$ for the the $i+1$-st or $(i+1)$-last coefficient of V (since these terms will occur often, and to abbreviate the clumsier alternative $[V]_{\min\deg V+i}$ resp. $[V]_{\max\deg V-i}$).

2.2. HOMFLY and Kauffman polynomials

We include a description of the other polynomials that will make appearance in the paper, at least to clarify conventions.

The *skein (HOMFLY) polynomial* P is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

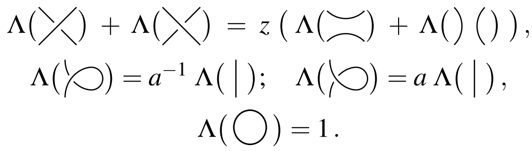
$$l^{-1}\,P\big(\nwarrow\!\!\!\!\nearrow\big) + l\,P\big(\nwarrow\!\!\!\!\nearrow\big) = -m\,P\big(\,)\,(\,\big)\,. \tag{4}$$

This convention uses the variables of [LMi], but differs from theirs by the interchange of l and l^{-1}. We call the three diagram fragments in (4) from left to right a *positive* crossing, a *negative* crossing and a *smoothed out* crossing (in the skein sense).

A diagram is called *positive*, if all its crossings are positive. A(n oriented) link is positive, if it admits a positive diagram (see for example[1] [Cr2, O]).

The *Kauffman polynomial* [Ka2] F is usually defined via a regular isotopy invariant $\Lambda(a,z)$ of unoriented links.

We use here a slightly different convention for the variables in F, differing from [Ka2, Th] by the interchange of a and a^{-1}. Thus in particular we have the relation $F(D)(a,z) = a^{w(D)}\Lambda(D)(a,z)$, where $w(D)$ is the writhe of a link diagram D, and $\Lambda(D)$ is the writhe-unnormalized version of the polynomial. Λ is given in our convention by the properties



Note that for P and F there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity.

The Jones polynomial V can be obtained from P and F (in our conventions) by the substitutions (with $i=\sqrt{-1}$; see [LMi] or [Ka2, §III])

$$V(t) \quad = \quad P(-it, i(t^{-1/2}-t^{1/2})) \quad = \quad F(-t^{3/4}, t^{1/4}+t^{-1/4}), \tag{5}$$

which in particular yields the Jones skein relation as a special form of (4),

$$t^{-1}V_+ - tV_- = (t^{1/2}-t^{-1/2})V_0\,. \tag{6}$$

By $[P]_M$ we denote the coefficient of the monomial $M = l^x m^y$ in the polynomial in P. If P has a single variable, then we use the exponent rather than the whole monomial for M. (For example, $[V]_3 = [V]_{t^3}$ for $V \in \mathbb{Z}[t^{\pm 1}]$.)

We also use the terms $\max\deg_m P$ for the *maximal degree* of P w.r.t. the variable m, and $\max\mathrm{cf}_m P$ for the *leading coefficient* (of P in the m-variable, this coefficient itself being a polynomial in l).

2.3. Crossing numbers and critical line polynomials

Let $c(D)$ be the crossing number of a diagram D, and $c(K)$ the crossing number of a knot or link K,

$$c(K) := \min\{\, c(D) : D \text{ is a diagram of } K\,\}\,.$$

We call D *minimal* if $c(D) = c(K)$.

For reasons that will become clear immediately, let us also define $c_\pm(D)$ to be the number of positive resp. negative crossings of a diagram D, and set

$$c_\pm(K) := \min\{\, c_\pm(D) : D \text{ is a diagram of } K\,\}\,.$$

Obviously, $c(K) \ge c_+(K) + c_-(K)$, but knots like Perko's with minimal diagrams of different writhe show that the inequality can be strict. Nevertheless, in general one can gain good (and often sharp) estimates of $c(K)$ by estimating $c_+(K)$ and $c_-(K)$.

Thistlethwaite proved in [Th] that (with our convention for Λ) for a link diagram D of $c(D)$ crossings we have $[\Lambda(D)]_{z^l a^m} \ne 0$ only if $l + |m| \le c(D)$, and that D is A resp. B-adequate iff such a coefficient does not vanish for some l and m with $l - m = c(D)$ resp. $l + m = c(D)$. These properties imply most of his results, incl. the main one, the crossing number minimality of adequate diagrams.

[1] Occasionally, the name is assigned to positive *braid* links, but we frown upon this obsolete (and confusing) convention here.

The coefficients of $\Lambda(D)$ for which $l \pm m = c(D)$ form the "critical line" polynomials $\phi_{\mp}(D)$. Thistlethwaite expresses these polynomials in terms of some graph invariants, so they clearly encode combinatorial information of the diagram. Unfortunately, we do not know how to interpret most of this information, i.e., to say what tangible features of the diagram it measures.

Still the minimal and maximal degree of $\phi_{\pm}(D)$ do have a "visual" meaning. In §3 of [St5] we translated this meaning to our present context. One degree can be expressed also in terms of the writhe and Jones polynomial, thus giving a new obstruction to semiadequacy (proposition 5.3 below). However, for semiadequate links this obstruction (consequently vanishes and) gives no new information. In contrast, the other degree (see proposition 3.2 below) gives a new invariant for semiadequacy. This will be used crucially in the last sections, together with our work of "decoding" in such a visual way the Jones polynomial coefficients.

Let us, though, better specify $\phi_{\pm}$ directly in terms of $F(L)$. (For the opposite signs, mirror the link.) Let

$$a_-(L) = \max\{\, m - l : [F(L)]_{a^l z^m} \neq 0\,\}/2\,, \tag{7}$$

and

$$\phi_+(z) = \sum_{i=0}^{\max\deg_z F(L)} z^i \cdot [F(L)]_{z^i a^{i-2a_-(L)}}\,.$$

(Note the sign switch between ϕ_+ and a_-; this is not a typo, but our tribute to the incommensurability of all mnemonic and related historical notation.) Then Thistlethwaite proves that $c_-(D) \geq a_-(D)$ for every diagram D of L, and equality holds if and only if D is an A-adequate diagram. In that case $\phi_+(z)$ is a non-negative polynomial.

2.4. Knot numbering

Knots of ≤ 10 crossings will be denoted according to Rolfsen's tables [Ro, appendix], and for ≥ 11 crossings according to Hoste and Thistlethwaite's program KnotScape [HT]. However, I reorganize the KnotScape tables so that non-alternating knots are appended after alternating ones (of the same crossing number), instead of using "a" and "n" super/subscripts. For example, there are 367 prime alternating knots of 11 crossings. Thus a knot commonly named 11_{a111} in the table will be written as 11_{111}, while 11_{n111} as 11_{478} (for $111+367=478$).

Knot diagrams of 17 and 18 crossings were generated by the algorithm (and software, temporarily, but unfortunately no longer available) of [RFS], and are applied for some computational checks, but a numbering is never used. Up to 19 crossings, tables are now available in [Br].

The obverse (mirror image) of K is denoted by $!K$. Fixing a mirroring convention of the table knots will not be very relevant.

To save space, many knot diagrams are displayed in terms of their DT notation [DT], in the way used in [HT]. This means that the DT sequence is preceded by two integers. The first one is the crossing number. The second entry, knot identifier, is an artifact of the way examples were tracked down and has no real significance. It is not to imply any relation to knot tables, etc.

3. Jones polynomial of (semi)adequate links

The following is a brief summary of the formulas and methods developed in [St3] that will be needed later. (See there for full proofs.)

3.1. The second coefficient

We consider the bracket [Ka] (rather than Tutte) polynomial. The A-state of D, the state with all splicings A, is denoted by $A(D)$. (Occasionally, we omit the argument D in this notation, if no ambiguity arises.) For us a state is always understood as a planar picture of loops (solid lines) and traces connecting these loops (dashed lines). Then it is clear that and how to reconstruct D from $A(D)$.

Definition 3.1 One fundamental object exploited in this paper is the *A-graph* $G(A) = G(A(D))$ of D. It is defined as the planar graph with vertices given by loops in the A-state of D, and edges given by crossings of D. (The trace of each crossing connects two loops.) The analogous terminology is set up also for the B-state.

Clearly the A-state determines the A-graph, but not conversely. Their distinction is relevant in some situations. However, $G(A(D))$ (including its planar embedding) determines $A(D)$ if D is alternating; then sometimes $G(A(D))$ is called the Tait graph of D. Note also that, for alternating D, the duality of crossing traces between $A(D)$ and $B(D)$ corresponds to the duality (in the usual graph-theoretic sense) of edges in the planar graphs $G(A(D))$ and $G(B(D))$.

If D is a connected diagram, then $G(A(D))$ is also connected. If D is positive, then $G(A(D))$ is the *Seifert graph*, and its vertices correspond to the Seifert circles of D.

Let $v(G)$ and $e(G)$ be the number of vertices and edges of a graph G. Let G' be G with multiple edges removed (so that a simple edge remains). We call G' the *reduction* of G.

We will write sometimes

$$s_+(D) = v(G(A(D))) = v(G(A(D))'), \qquad s_-(D) = v(G(B(D))) = v(G(B(D))').$$

The definition of A-adequate can be restated saying that $G(A(D))$ has no edges connecting the same vertex. For B-splicings the graph $G(B(D))$ and the property B-adequate are similarly defined (and what is stated below proved).

In the following, we shall explain the second and third coefficient of the Jones polynomial in semiadequate diagrams.

Proposition 3.1 If D is A-adequate connected diagram, then in the representation (3) of V_D we have $V_0 = \pm 1$, $V_1 V_0 \leq 0$, and

$$|V_1| = e(G(A(D))') - v(G(A(D))') + 1 = b_1(G(A(D))') \tag{8}$$

is the first Betti number of the reduced A-graph.

The formula (8) was also explained in [DL2].

Lemma 3.1 If G is a planar simple graph (no multiple edges), then $b_1(G) \leq \left\lfloor \frac{2}{3} e(G) \right\rfloor - 1$. This inequality is sharp for proper G when $e(G) > 2$.

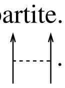
(Here and below $\lfloor x \rfloor$ is the largest integer not greater than x.)

3.2. Conditions for positivity

Even although alternating knots are not generally positive, in the following positivity arguments will be essential. It is well-known that if a diagram D is positive, then it is A-adequate: the A-state of D is just the Seifert picture of D (and the A-state loops are the Seifert circles). This point should be kept in mind throughout.

Since A-adequacy is an unoriented condition, D would remain A-adequate even if we alter orientation of some components. We say that an unoriented diagram D *admits a positive orientation*, or is *positively orientable*, if it arises from such a diagram by forgetting orientation. We observed in [St3] the following lemma, which specifies which A-adequate diagrams are positively orientable. Since the lemma will be of considerable importance, its short argument is reproduced as well.

Lemma 3.2 Let D be A-adequate. Then D is positively orientable iff $G(A(D))$ is a bipartite graph.

Proof. If D is positively orientable, its graph $G(A(D))$ is the Seifert graph of a (positive) diagram, and hence bipartite. Conversely, if $G(A(D))$ is bipartite, it is possible to orient the loops in $A(D)$ so that each trace looks locally like . Then it is clear that it is possible to extend this loop orientation to an orientation of D, and with that orientation D becomes positive. ☐

Corollary 3.1 Let L be an A-adequate link, with an A-adequate connected diagram D, then

$$1-\left\lfloor\frac{2}{3}c(D)\right\rfloor \leq V_0V_1 \leq 0.$$

If $V_1=0$, then D admits a positive orientation, and (with this orientation) L is fibered.

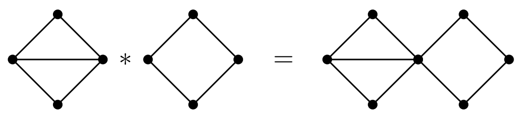
This gives a new semiadequacy test. Beside that Thistlethwaite's condition on the positive critical line coefficients [Th] involves the Kauffman polynomial, which is considerably slower to calculate, the Jones polynomial features sometimes prove essential, as we showed by some examples.

The minimal number $e(n)=e_n$ of edges needed for a planar simple graph to have given $b_1=n$ is by lemma 3.1

$$e_n = \begin{cases} \frac{3}{2}(n+1) & n \text{ odd} \\ \frac{3}{2}n+2 & n \text{ even} \end{cases}.$$

For later discussion, it is useful to define some properties of graphs.

Definition 3.2 The *join* (or *block sum*, *cf.* [Cr2]) '$*$' of two graphs is defined by



This operation depends on the choice of a vertex in each one of the graphs.

We call v a *cut vertex* of a graph G, if G gets disconnected when deleting all edges incident to v and *additionally* v itself. (When we delete an edge, we understand that a vertex it is incident to is *not* to be deleted, too.)

Every connected non-trivial (i.e., with at least one edge) graph G can be written as a join $G_1*\ldots*G_n$ for some non-trivial connected graphs G_i, such that no G_i has a cut vertex. We call G_i the *join factors* of the graph G. The number $a(G)=n$ of join factors of G is called *atom number* of G. (This should not be confused with the critical line quantities $a_\pm(L)$ in (7).)

We set $a(A(D))=a(G(A(D)))$.

Proposition 3.2 ([St5]) We have $\max\deg_z\phi_+(D)=c(D)-a(A(D))$ if D is connected and A-adequate.

Keep in mind that if D is positive, then the atoms are the *Cromwell blocks* [Cr2] (or Murasugi atoms [QW]) D_i of D.

Corollary 3.2 If L is non-split and $e(|V_1|)>\operatorname{span}V(L)$, then L is non-alternating.

Below we will modify this reasoning for a *bipartite* planar simple graph (see lemma 6.1).

Note that in corollary 3.1 the case $V_1=0$ poses strong additional restrictions to A-semiadequacy. The change of component orientation alters V only by a positive unit [LMi], so that several positivity criteria still apply. Among others, with $n(L)$ the number of components of L, we have $V_0=(-1)^{n(L)-1}$ (*cf.* proposition 5.3 below). For knots positively orientable is the same as positive, and we have a series of further properties. For example $\min\deg V_L=g(L)$, the genus of L, which is in particular always positive.

The next corollary determines asymptotically the minimal value of V_0V_1 for positive link diagrams.

Corollary 3.3 Let L be a positive $n(L)$-component link, with a positive connected diagram D. Then

$$-\frac{1}{2}c(D) \leq (-1)^{n(L)-1}V_1 \leq 0,$$

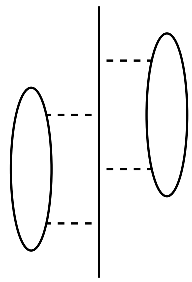
and the left inequality is asymptotically sharp (for large $c(D)$).

3.3. The third coefficient

The third coefficient in semiadequate diagrams can still be identified in a relatively self-contained form. It depends, however, on more than just $G(A(D))'$ or $G(A(D))$.

Definition 3.3 We call two edges $e_{1,2}$ in $G(A(D))'$ *intertwined*, if the following 3 conditions hold:

1. $e_{1,2}$ have a common vertex v.
2. The loop l of v in the A-state of D separates the loops $l_{1,2}$ of the other vertices $v_{1,2}$ of $e_{1,2}$.
3. $e_{1,2}$ correspond to at least a double edge in $A(D)$, and there are traces of four crossings along l that are connected in cyclic order to l_1, l_2, l_1, l_2.



It should be made clear that intertwinedness of edges in $G(A(D))'$ still depends on the *state* $A(D)$, rather than just the graph $G(A(D))'$ or $G(A(D))$: in the graphs the intertwining information becomes lost.

Definition 3.4 A *connection* in $A(D)$ is the set of traces between the same two loops, i.e., an edge in $G(A(D))'$. A connection e in $A(D)$ is said *multiple*, if it consists of at least two crossing traces. More generally we can define the *multiplicity* of a connection as the number of its traces.

Then we can speak also of intertwined (multiple) connections. We will later occasionally relax terminology even more and speak just of intertwined loops M and N, when their connections to a third loop L are intertwined. This is legitimate, because one can determine L from (M,N) uniquely.

Call below a loop in $A(D)$ *separating* if it is connected by crossing traces from either side. So if connections (M,L) and (N,L) are intertwined, then L would be separating, and connected by M and N from opposite sides. Clearly there can be only one such L for given M and N. (Moreover, almost throughout where we will apply this terminology below, there will be in fact only one separating loop in $A(D)$.)

Definition 3.5 Define the *intertwining graph* $IG(A(D))$ to consist of vertices given by multiple connections in $A(D)$ (or multiple edges in $G(A(D))$), and edges connecting pairs of intertwined connections.

This is the second graph associated to D, which plays a fundamental role in the whole paper. Note that this is a simple graph (no multiple edges), but not necessarily planar or connected. It may also be empty. For example, for an alternating diagram this graph has no edges, so is a (possibly empty) set of isolated vertices. With the preceding remark, $IG(A(D))$ is determined by the state $A(D)$, but not (in general) by the graph $G(A(D))$.

Proposition 3.3 If D is A-adequate, then

$$V_0V_2 = \binom{|V_1|+1}{2} - \triangle G(A(D))' + \chi(IG(A(D))) = \binom{|V_1|+1}{2} + e_{++}(A(D)) - \triangle G(A(D))' - \delta A(D)',$$

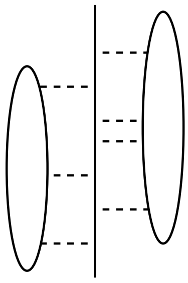
where $\triangle G(A(D))'$ is the number of triangles (cycles of length 3) in $G(A(D))'$, $\delta A(D)' = e(IG(A(D)))$ the number of intertwined edge pairs in $G(A(D))'$, and $e_{++}(A(D)) = v(IG(A(D)))$ the number of multiple connections in $A(D)$.

This formula was obtained independently and simultaneously by Dasbach-Lin [DL2]. Their proof is longer, but it unravels the underlying combinatorics completely, while in [St3] we helped ourselves with some skein theoretic arguments about positive braid links.

Remark 3.1 Note that in alternating diagrams $\delta A(D)' = 0$, since there are no separating loops, while in positive diagrams $\triangle G(A(D))' = 0$ because $G(A(D))'$ is bipartite. Note also that a pair of intertwined edges does not occur in a triangle.

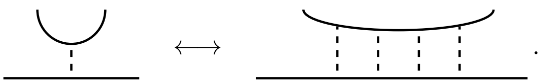
3.4. Legs and traces

Let the *intertwining index* of an edge pair in $G(A(D))$ be half the number of interchanged connections from either side of l. For example the intertwining index of



is 3, and edges are intertwined iff their intertwining index is ≥ 2. (We will from now on, to save space, draw in diagrams only a part of l that contains its basepoints. So the straight line, that represents l, is understood to be closed up.)

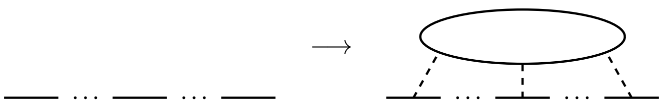
We assume in the following that an *edge* in $A(D)$ stands for a possible (but non-empty) collection of parallel traces:



(9)

(The term "edge" thus assumes a clear separation between the state $A(D)$ and the graph $G(A(D))$.) Traces between loops a and b are *parallel* if between their basepoints on both a and b no traces connecting a or b to other loops occur. With this convention we identify, and do not display several, parallel traces in diagrams. A connection in $A(D)$, which is the set of all traces or edges that connect the same two loops, in general decomposes into several edges.

An edge obviously has a *multiplicity* by the number of faces it contains. Accordingly it is *even* or *odd*. In the case of the A-state of a positive diagram D of a (positive) fibered link, one can obtain $A(D)$ by starting with some single loop, and then attaching new loops with all their traces.



(10)

Since $G(A(D))'$ is a tree, we can assume that we attach the traces of the new loop to fragments of the *same* previous loop. This condition of 'attachment' will be assumed consistently below.

Every loop l which is not in the outer cycle is attached to a single previous loop m. This means that all traces of crossings, that connect l at one end connect to m on the other. The parallel equivalence classes of such traces are called below *legs*. So legs are edges in $A(D)$ connecting an attached loop. As edges, they can also be labeled even or odd.

The number of legs of a loop l is called the *valence* of l. As in (9), and unless stated clearly otherwise, we group parallel traces into a single one (with multiplicity indicated, or explained from the context) when drawing loop diagrams. So a dashed line in a diagram starting from a loop attached by (10) (usually) stands for a leg.

It is clear that the legs of each attached loop form a multiple connection of traces. So the existence of attachments forces the intertwining graph $IG(A)$ to be non-empty (i.e., have at least one vertex). Then the condition $\chi(IG) = 0$ means that we must have a cycle in $IG(A)$. The existence of this cycle will be helpful at several places below.

In our case we will attach a loop to another atom, but we can maintain the above terminology of (even/odd) legs.

4. Genus, generators and twisting

We write $s(D)$ for the *number of Seifert circles* of a knot diagram D, and

$$g(D) = \frac{c(D) - s(D) + 1}{2},$$

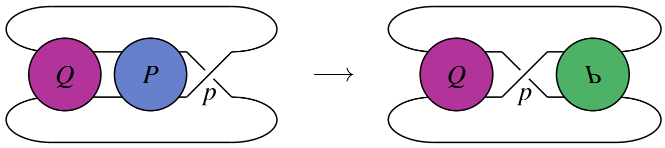
for its *genus*. If D is a link diagram, we write $\chi(D) = s(D) - c(D)$ for the *Euler characteristic*. By Morton [Mo],

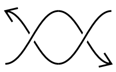
$$\max \deg_m P(D) \le 1 - \chi(D),$$

and we set

$$P_{\max}(D) := [P(D)]_{m^{1-\chi(D)}}. \tag{11}$$

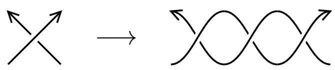
A *flype* is the move



Recall that in [St10] we called two crossings $\sim$*-equivalent*, if after a sequence of flypes they can be made to form a *reverse clasp*  ; it is an exercise to check that this is an equivalence relation. Similarly two crossings are $\underset{*}{\sim}$-*equivalent* if they form a parallel clasp (see the right of (15)) after flypes. We say that two crossings are *twist equivalent* if they are $\sim$- or $\underset{*}{\sim}$-equivalent.

The equivalence classes under these relations will be called $\sim$*-class,* $\underset{*}{\sim}$*-class*, and *twist class*, respectively. We call a $\sim$-class *trivial* if it has only one element (crossing), and *reduced* if it has at most two. Thus a generator is a diagram (or an alternating knot represented by such a diagram) in which all $\sim$-classes are reduced.

Now, we consider the move we call a $\bar{t}'_2$ *twist* or $\bar{t}'_2$ *move*. Up to mirroring this is given by

 . (12)

We call diagrams that cannot be reduced by flypes and inverses of the move (12), and their alternating knots, *generating* or *generators*.

Theorem 4.1 ([St10, St11]) There are only finitely many knot generators of diagrams of fixed genus.

Since the number of generators grows rapidly with the genus g, this was practically possible so far only up to genus 4. Below, we will need the list for genus 3, which has 4017 knots (see [St11]).

It is important, as in Figure 2, to distinguish the orientation of a flyping crossing. Note that $\bar{t}'_2$-twist commute with type B flypes, but in general not with type A.

We call a Seifert circle *separating* is it has crossings attached from either side. A diagram without separating Seifert circles is *special*. For special *prime* diagram the properties alternating and positive/negative are equivalent. See, e.g., [Cr2] for further details.

Finally, any diagram D decomposes under *diagrammatic Murasugi sum* $*$ into special diagrams. We need the following multiplicativity property:

Theorem 4.2 ([MP]) $P_{\max}(D_1 * D_2) = P_{\max}(D_1) P_{\max}(D_2)$.

Note that '$*$' is very much related to Definition 3.2, except here it operates on the level of Seifert graphs, rather than A-state graphs. We will use (and emphasize), though, that when D is a positive diagram, then the two types of graphs identify.

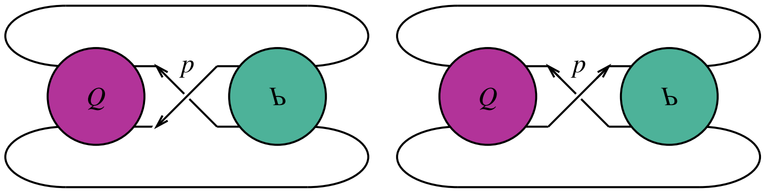


Figure 2: A flype of type A and B

5. The minimal braid length and minimal minimal braid length of a knot

5.1. The examples

Braids are now inseparably intertwined with links, and are used for a variety of applications of or to link theory. In some laborious project, a set of useful braid representatives of prime knots up to 13 crossings were compiled in [St9].

A braid is minimal if it realizes the braid index, and minimal length if it has shortest length word (in the standard Artin generators) among all braid representatives of its closure.

There is an a priori *upper* bound on the length of a braid representative for a given knot in terms of its crossing number: it is quadratic, proved by Vogel [Vo]. It is kind of conjecture that quadratic is the best possible, and examples are twist knots, but the problem is we have no good *lower* bounds (except the crossing number itself).

For most applications minimal width matters more than length, and that is what was focused on in [St9]. It is clear that minimal words may not necessarily be realized on a minimal number of braid strands.

We write for $\mathrm{mmbl}(K)$ the minimal minimal braid length and $\mathrm{mbl}(K)$ the minimal braid length of K. Let us also adopt the convention that braid words are denoted as sequences of non-zero integers, where $\pm i$ (with $i > 0$) stands for $\sigma_i^{\pm 1}$, and σ_i is the i-th Artin generator.

As indicated on my page [St9], I had known about 10_{136}, as example where the minimal width and minimal length braids are not the same. Perhaps this feature was first known to Gittings [Gi], and was further studied by van Cott [vC]. One main question there was how to systematically exhibit such knots. Some other examples had been previously published in Figure 7 of [St4], in the context of positive braid representatives.

Recently, Boden and Shimoda [BS] raised this issue with more emphasis, in relation to simple walks used in calculation of the colored Jones polynomial.

In this section we observe two related facts.

Proposition 5.1 There exist infinitely many prime knots with a positive braid representative but not minimal such one.

Proposition 5.2 There exist infinitely many prime knots where minimal minimal braid length and minimal braid length are not the same.

One can extend the second example in Figure 7 of [St4], $16_{1223549}$, to an infinite family K_k, given by the 4-braids

$$-3\ 2\ 3^{2k}\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 3\ 2\ 2\ 1\ 1\ 2\,.$$

(Here i stands for σ_i if $i > 0$ and for σ_{-i}^{-1} otherwise.)

We call, as in [St3], a braid β to be A-adequate, if it admits a word so that the diagram $\hat{\beta}$ is A-adequate. (Similarly can be done with B-adequate and adequate.) By Thistlethwaite's work [Th], this must a minimal length word in Artin's generators (even up to conjugacy), and if one minimal word is A-adequate, all other such will be. One reason for introducing this concept was to prove and use that all 3-braids are semiadequate, which in particular gives a solution of the minimal conjugacy word problem in the 3-braid group B_3.

Without constantly repeating this below, there are straightforward analogues of every concept/statement under mirror image.

The proofs of proposition 5.1 is just a minor shortening of the one of proposition 5.2. We will use the language of the latter one for contextual reasons.

Proof. Setting $k = 1$ gives $16_{1223549}$. More generally all $15+2k$ crossing 4-braids give a knot diagram which simplifies to a $14+2k$-crossing positive 5-braid diagram.

$$1\ 2\ 3\ 3\ 4\ 2\ 2\ 3\ 1\ 2\ 4\ 2\ 3\ 3\ 2\ 4^{2k-1}$$

The only point to essentially prove is the following.

Lemma 5.1 The crossing number of K_k is $14+2k$.

Assume the lemma is proved.

First, the braid index of K_k is 4. This can be seen by direct calculation of the skein polynomial and the MFW inequality, but another argument is thus. If $b(K_k) \le 3$, and as K_k is a closed positive braid, then by [St2], K_k is a closed positive 3-braid, and by looking at the genus, we see that this 3-braid must have $12+2k$ crossings, a contradiction to lemma 5.1.

Given the lemma 5.1, we have $\mathrm{mmbl}(K_k) = 14+2k$, but a 4-braid with knot closure must have odd crossing number, so $\mathrm{mbl}(K_k) = 15+2k$. By [Cr, O], the knots are also prime. (It is far easier to construct composite examples by connected sums.) □

Proof of lemma 5.1. For lemma 5.1, we use the following tool. We write $c_-(D)$ for the number of negative crossings of a diagram D, and $c_-(K)$ for the the minimal number of negative crossings of a diagram of K.

Following Thistlethwaite, we defined in [St5] a quantity $a(K) = a_-(K)$ from the Kauffman polynomial, with the property that $c_-(K) \ge a_-(K)$ and equality holds if and only if K is A-adequate. Similarly let $a_-(K) = a_+(!K)$ for the mirror image $!K$ of K. We also considered the critical line polynomial $\phi_\pm(D) = \phi_\pm(K)$. (Note that with our signing $a_-(K) = c_-(D)$ if and only if $\phi_+(D) \ne 0$.) And V_L is the Jones polynomial.

Proposition 5.3 ([St5, Corollary 3.10]) If a link L is non-split and A-adequate, then we have

$$2 \min\deg V_L = \min\deg_z \phi_+(L) - 3a(L)$$

and the trailing coefficient of V_L is

$$\min\mathrm{cf}\, V_L = (-1)^{\min\deg_z \phi_+(L)}\,.$$

This gives a fairly handy semiadequacy test. For instance, it rules out one of the two adequacies for 33 of the 38 non-alternating (prime) 10 crossing knots. (3 of the 42 knots are adequate, and the Perko knot has both semiadequacies without being adequate.) Unfortunately, 10_{136} is among the remaining 5.

The test does work for (or rather against) B-adequacy of $16_{1223549}$, which is the starting point of the argument. Since K_k are positive, obviously $a_-(K) = 0$. We calculate $a_+(K) = 13+2k$ from some easy recursive calculation of the Kauffman polynomial. Thus $c(K_k) = 13+2k$ and if equality holds, K_k would be B-adequate.

By direct calculation, we find proposition 5.3 obstructing B-adequacy, so the proof is complete. Proposition 5.3 does require an iterated calculation of the Kauffman polynomial under parallel (double) clasping (15), but although technical, this can be considered straightforward. □

Note that if proposition 5.3 fails, the work in [St3] gives enough insight, in principle, to rule out B-adequacy for general k. But we would have to go through some fairly laborious process, as outlined in [St7] and carried out in [St8] for the odd crossing number amphicheiral knots. (Such complexity will hardly be warranted by the importance of our examples.) On the opposite end, it is obvious that many similar constructions are possible with the introduced tool.

5.2. The conjecture

Originally we thought that the following is a construction of knots where the difference of minimal minimal braid length and minimal braid length can get arbitrarily large.

We write $w(\beta)$ for the writhe (exponent sum) of a braid β.

Proposition 5.4 Let K admit a B-adequate braid β and a minimal braid β' with $w(\beta') < w(\beta)$. Then the iterated connected sum $K_k = \#^k K$ satisfies $\mathrm{mmbl}(K_k) - \mathrm{mbl}(K_k) \geq k$.

Proof. Let us write $c(\beta)$ for the word length of β in Artin's generators (and inverses). Also write $c_-(\beta)$ for the minimal number of inverse Artin's generators in a word for β.

By B-adequacy $a_+(K) = c_+(\beta) \leq c_+(\beta')$ by Thistlethwaite for whatever braid word β' with $\hat{\beta}' = K$. Since $w(\beta')$ is unique by the proof of the Jones conjecture [LM, DP] for any word of a minimal braid β', we have $c_-(\beta') > c_-(\beta)$. Thus $c(\beta') = c_+(\beta') + c_-(\beta') > c_+(\beta) + c_-(\beta) = c(\beta)$. By choosing β' with $\mathrm{mmbl}(K) = c(\beta')$, we have $\mathrm{mmbl}(K) \geq c(\beta) + 1 \geq \mathrm{mbl}(K) + 1$.

If we take K_k, then $w(\beta'_k) = kw(\beta')$ for a minimal braid β'_k of K_k using [BM], and $a_+(K_k) = ka_+(K)$ by B-adequacy, so $\mathrm{mmbl}(K_k) - \mathrm{mbl}(K_k) \geq k$. □

Example 5.1 10_{136} (properly mirrored) admits a B-adequate braid β of 5 strands and writhe -2. It also admits a (minimal) 4-braid, but of writhe $-1 > -2$, not $-3 < -2$.

The below problem, which we like to highlight, is the product of our continuous failure to find examples subject to proposition 5.4. Here, we say a braid is *smaller* if it has fewer strands.

Conjecture 5.1

1. If β is B-adequate, then β is not Markov equivalent to a smaller braid of smaller writhe.
2. If β is A-adequate, then β is not Markov equivalent to a smaller braid of larger writhe.
3. If β is adequate, then β is minimal.

Obviously parts 1 and 2 are equivalent and (either of) both imply part 3. (Unlike for knots, A- and B-adequate together imply adequate for braids, for the writhe is a braid word invariant.)

In fact 'smaller braid' can be replaced by 'minimal braid'. (Then 1 and/or 2 would not quite imply 3.) But this begs additional insight into the braid index, which is a separate difficulty.

The easiest way to test the claim is by sharpness of the right MFW inequality [Mo, FW]:

$$\max \deg_l P(\hat{\beta}) = w(\beta) + n - 1\,. \tag{13}$$

(Keep in mind that the skein polynomial P is used with the variables of [LMi] but exchanging l and l^{-1}; see §2.2.) This implies that an n'-braid Markov equivalent to β for $n' < n$ has $w(\beta') \geq w(\beta) + n - n' > w(\beta)$.

We compiled the list of B-adequate (prime) braids with knot closure up to 18 crossings and tested (13). It failed only in a handful of cases. Most of these B-adequate braids could be proved minimal by 2cMFW (see e.g. [St]). Only for 18 crossings some 5-braids can reduce to 4-braids, but these are minimal and of the "wrong" writhe (as occurs for 10_{136}).

A few known results fit into the conjecture.

Positive braids are A-adequate and (by Bennequin's or Morton's inequality e.g.) do not admit Markov equivalent smaller braids of larger writhe. Part 3 of the conjecture is true for alternating braids ([Mu]) and positive braids without trivial syllables (by an argument essentially due to Nakamura [Na]; see also [St2, §5]).

There is also a possible weaker version:

Conjecture 5.2

1. If β is B-adequate, then β is not conjugate to a positive stabilization.
2. If β is A-adequate, then β is not conjugate to a negative stabilization.
3. If β is adequate, then β is irreducible.

The problem of detecting stabilization up to conjugacy is a fundamental problem in braid-link theory. This will be discussed in a separate paper. But in either form, the conjecture remains more a speculation at this point. Still it highlights a distinct possibility of a connection between the combinatorics of adequacy and braid-theoretic features, which is quite elusive at present.

6. Minimal crossing almost alternating diagrams

Almost alternating diagrams were introduced by Adams et al. [A+]. While many links have such diagrams, their construction usually requires one to go up in crossing number. Thus the question was raised about minimal crossing almost alternating diagrams [Bu]. By inspection, such diagrams do not occur for knots up to 12 crossings, but there are a few of 13 crossings. One is

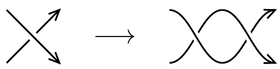
`13 235 4 8 14 2 -16 18 20 6 22 24 26 10 12` (14)

of 13_{5491}. However, the verification of minimality essentially depends on enumerating lower crossing diagrams, which again leads to the problem of finding diagrams of this type of higher (and arbitrarily high) crossing number.

This turns out not so easy. Here we introduce a test that, although at first sight somewhat involved, is successfully applicable. Kauffman, HOMFLY, and Jones, all are needed. From some (sufficiently many, but by far not all) of the examples we give, infinite families can be constructed by iterated double (positive) parallel clasping (15). This yields:

Proposition 6.1 For every crossing number $n \geq 13$, there exists a minimal crossing almost alternating diagram of a knot.

For the main idea behind the description of diagrams, one applies moves, which replace a positive crossing by a parallel positive clasp:


. (15)

We call this procedure below a *(parallel) clasping*. (It is enough to restrict ourselves to positive crossings here.) A diagram D of n components can be transformed into a knot diagram D', with $1-\chi(D') = n-\chi(D)$, by repeatedly applying clasping on crossings of two different components. Note that in $A(D)$ a clasping doubles a trace (replaces it with two parallel ones).

Again, we write $b_1 = e - v + 1$ for the Betti number of a connected graph of e edges and v vertices. The following is a matter of simple counting (and using that all cycles have length at least 4).

Lemma 6.1 When G is a planar bipartite connected graph (no multiple edges), then $b_1 \leq v-3$ and $e \leq 2v-4$. □

We present now the following test (with the caveat of remark 6.1); nothing less involved succeeded in collecting practical examples.

Theorem 6.1 Assume D is a diagram of a knot K, satisfying the following conditions:

1. $a_-(D) = 0$ and $a_+(D) = c(D) - 2$ and proposition 5.3 obstructs B-adequacy.
2. $\max\deg_z \phi_+(D) = c(D) - 3$ (keep in mind our signing of $\phi_\pm$).
3. $V_1(D) \neq 0$.

Then

1. If $c(D)-2\min\deg_t V(D)-|V_1(D)|<4$ then D is minimal
2. If
$$c(D)-2\min\deg_t V(D)-|V_1(D)|=4 \tag{16}$$
then
   (a) If $|V_1|\geq 2\min\deg_t V$ or $|V_2|\leq \binom{|V_1|+1}{2}$, then D is minimal.
   (b) Assume D is not minimal. Then there exists a special alternating prime knot diagram D' with the following properties:
      i. (one can choose which of the two crossing numbers)
$$c(D')=c(D)-2 \tag{17}$$
or
$$c(D')=2c(D)-4\min\deg_t V(D)-6+|V_1(D)| \;=\; 3|V_1(D)|+2, \tag{18}$$
      ii. $\min\deg_t V(D')=\min\deg_t V(D)+\mu$ with
$$\mu=\frac{c(D')-c(D)}{2}+1$$
      iii. $|V_1(D')|=|V_1(D)|$
      iv. $|V_2(D')|>\binom{|V_1(D')|+1}{2}$ if (17), or $1\leq|V_2(D')|-\binom{|V_1(D')|+1}{2}\leq|V_1(D)|$ if (18)
      v. $\max \mathrm{cf}_m P(D')=\max \mathrm{cf}_m P(D)\cdot l^{2\mu}$

The alternative (17) works better for smaller $c(D)$, while for increasing $c(D)$, the option (18) becomes (far) more viable. A crucial reason why this test is often stable under parallel clasping is that (16) remains valid, and that (18) does not increase. Often do the other invariants involved not change either (as long as we do not switch number of components).

If we had alternating *link* tables available, we could content ourselves with lower-crossing number link diagrams D' instead.

Example 6.1

$$D = \texttt{15 61262 6 10 16 30 4 22 26 2 8 24 28 14 20 12 -18}$$

This satisfies conditions 1 and 2. It goes in case 2: we have $\min\deg_t V=3$, $c(D)=15$ and $|V_1|=5$. Then consider test 2b. We have $\mathrm{span}\, V(D)=12$, so (18) and parity gives $c(D')=13$.

We then test 13 crossing prime special alternating knots of genus 3, and $\max \mathrm{cf}_m P(D)=l^6-5l^8+4l^{10}-l^{12}$ does not occur as leading P-term. Thus D is minimal (it belongs to 15_{100431}).

This argument is stable after claspings, since (18) gives $c(D')=17$ and that $l^4\max \mathrm{cf}_m P(D)$ does not occur in genus 5 special alternating knot diagrams of 17 crossings. (See the proof of proposition 6.1.)

Example 6.2 For
$$D = \texttt{16 1497 6 14 22 26 30 -16 4 24 28 8 2 12 20 32 10 18}$$
we have (after doing all previous necessary checks) that $\max \mathrm{cf}_m P(D)=l^6-6l^8+10l^{10}-3l^{12}$. Testing $c(D')=c(D)-2$, we find that this $\max \mathrm{cf}_m P(D)$ occurs among special alternating genus 3 prime 14 crossing knots only for 14_{16368}. But for that knot $|V_1|=6$ and $21=|V_2|=\binom{|V_1|+1}{2}$, so that condition 2(b)iv fails, and D is minimal. In fact, D depicts 16_{894574}. (Since (18) yields $c(D')=20$, this special argument will likely fail after claspings on D.)

The following is a list of almost alternating diagrams (not claimed all to represent mutually distinct knots), which can be proved minimal.

On the 16 crossing diagrams also test 2a applies. This test will likely remain restricted under claspings for these examples. We do not know if test 1 can ever apply.

```
14 526 6 12 16 22 26 4 20 2 24 -8 14 28 10 18
14 647 6 12 22 14 26 4 -20 28 24 8 2 16 10 18
14 648 6 12 22 14 26 4 20 28 24 -8 2 16 10 18
14 694 6 12 22 28 14 4 18 24 10 26 2 16 -8 20
14 721 6 12 24 14 18 4 -22 28 10 26 8 2 16 20
14 722 6 12 24 14 18 4 22 28 10 26 -8 2 16 20
14 731 6 12 26 -16 22 4 20 24 28 10 14 8 2 18

16 3514 6 10 16 32 4 26 20 2 8 24 28 14 30 22 12 -18
16 4531 6 12 18 32 26 4 30 22 2 -10 28 14 20 8 24 16
16 1136 6 12 22 14 28 4 20 32 26 -8 2 16 30 10 18 24
16 1806 6 12 22 28 -14 4 20 26 32 10 2 18 30 8 16 24
16 2822 -6 12 24 16 30 4 26 20 32 8 28 2 10 22 14 18
16 88 6 12 26 32 16 4 30 22 8 14 28 18 2 -10 20 24
16 544 -6 12 28 16 22 4 26 20 32 8 14 30 10 2 24 18
16 877 6 12 30 -16 24 4 20 26 32 10 28 14 8 22 2 18
16 59 6 14 20 24 -30 16 4 28 10 2 26 8 32 12 18 22
16 109 6 14 20 26 16 30 4 24 8 2 28 -10 18 32 12 22
16 1497 6 14 22 26 30 -16 4 24 28 8 2 12 20 32 10 18

15 3689 4 8 14 2 18 26 6 -30 22 12 28 16 20 10 24
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15 4872 4 8 14 2 26 18 6 -30 24 10 16 28 20 12 22
15 7320 4 8 16 2 26 18 22 6 -30 24 14 28 10 20 12
15 20589 4 10 -18 26 2 20 28 8 24 30 14 6 16 22 12
15 61262 6 10 16 30 4 22 26 2 8 24 28 14 20 12 -18
15 61266 6 10 16 30 4 24 20 2 8 26 12 28 14 22 -18
15 66178 6 12 16 22 28 4 20 2 -10 26 14 30 8 18 24
15 66783 -6 12 18 22 28 4 20 26 2 10 16 30 8 14 24
15 67274 6 12 20 14 28 4 -18 24 8 2 26 16 30 10 22
15 67679 -6 12 20 24 14 4 18 28 10 2 26 8 30 16 22
15 68163 6 12 20 30 24 4 28 22 26 2 -10 16 8 14 18
15 68457 6 12 22 14 28 4 -20 30 10 26 2 16 8 18 24
15 68457 6 12 22 14 28 4 20 30 -10 26 2 16 8 18 24
15 68610 -6 12 22 16 28 4 20 26 30 10 2 18 8 14 24
15 68610 6 12 22 -16 28 4 20 26 30 10 2 18 8 14 24
15 69290 6 12 22 30 16 4 28 24 8 26 2 -10 18 14 20
15 69989 6 12 24 30 16 4 20 26 8 14 28 2 -10 18 22
15 70360 6 12 28 14 18 4 -26 20 10 24 30 16 8 2 22
15 70360 6 12 28 14 18 4 26 20 10 24 30 16 -8 2 22
15 72845 6 14 20 26 16 30 4 24 8 2 -12 28 18 10 22
15 74824 6 14 24 30 16 20 4 28 10 26 8 2 -12 18 22
15 77643 6 16 24 30 18 26 20 4 28 12 8 2 -14 10 22
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Proof of theorem 6.1. When we want to study almost alternating diagrams, they are neither A- nor B-adequate. By [Th], if D is almost special alternating, then $a_-(D) < c_-(D) = 1$, so $a_-(D) = 0$ and $a_+(D) < c_+(D) = c(D) - 1$, so $a_+(D) \le c(D) - 2$.

When $a_+(D) = c(D) - 2$ and proposition 5.3 obstructs B-adequacy, then we know that $c_+(D) > a_+(D) \ge c(D) - 1$, so a diagram $\tilde{D}$ of fewer crossings than D must have $c(D) - 1$ crossings and be positive (and no diagram of K has fewer than $c(D) - 1$ positive crossings). So the question is to find practically working way to exclude such a diagram $\tilde{D}$. It is essential to remember throughout that, since $\tilde{D}$ is positive, A-state loops coincide with Seifert circles, and we can use skein arguments as well (§4).

We assume throughout for simplicity that $\tilde{D}$ is a knot diagram. We have $g(\tilde{D}) = \min\deg_t V(\tilde{D})$, so

$$s(\tilde{D}) = c(\tilde{D}) - 2\min\deg_t V + 1 = c(D) - 2\min\deg_t V$$

is the number of Seifert circles of $\tilde{D}$.

Now $G(A(\tilde{D}))$ is the Seifert graph.

Furthermore $c(\tilde{D}) - \max\deg_z \phi_+(\tilde{D}) = 2$, so the Seifert graph has 2 Murasugi atoms. We will assume there is one separating Seifert circle s_0. (The case $\tilde{D}$ is a connected sum can be easily ruled out, but it will be enough to remedy it by attaching the connected sum factors on opposite site of the Seifert circle. The arguments that follow are valid for that situation as well.)

We need a graph $G(A(\tilde{D}))'$ of $b_1 = |V_1|$ with $s(\tilde{D})$ vertices and one cut vertex. If $s(\tilde{D}) < |V_1| + 4$, there is no way to create b_1 cycles with the number of vertices. This explains the (so far fictitious) test 1.

So assume $s(\tilde{D}) = |V_1| + 4$, in the test 2. Then the only option is there is a single Seifert circle s_1 attached (say) outside s_0 as like in (10).

To explain condition 2a, note that, since in $\tilde{D}$ at least the connection between s_0 and s_1 is obviously multiple, we have

$$|V_1(D)| = |V_1(\tilde{D})| = b_1(A(\tilde{D})') < b_1(A(\tilde{D})) = 1 - \chi(\tilde{D}) = 2\min\deg_t V(\tilde{D}) = 2\min\deg_t V(D)$$

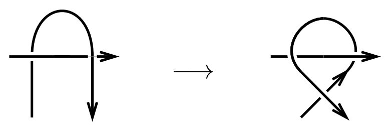
Vertices of $IG(A(\tilde{D}))$ are always pairs of loops in $A(\tilde{D})$ (but only pairs of loops with at least 2 traces between them, i.e., multiple connections). Also as the connection between s_0 and s_1 is multiple, (s_0,s_1) is a vertex of $IG(A(\tilde{D}))$. (We will below label vertices of $IG(A(\tilde{D}))$ by pairs of loops this way.) So $IG(A(\tilde{D}))$ has at least one vertex.

If $\chi(IG(A(\tilde{D}))) = 0$, then it must contain a cycle, but it is easy to see that with only s_1 on one side of s_0, this is not possible. Note that an edge in $IG(A(\tilde{D}))$ is between intertwined (multiple) connections. Two connections inside s_0 can never be intertwined, because they are not on opposite sides of a separating loop. (Because the inside of s_0 is an atom, there are no separating loops inside s_0.) An edge in $IG(A(\tilde{D}))$ can exist only between (s_0,s_1) and (s_0,s'), where s' is a loop inside s_0 (because s_1 is outside s_0, and is the only such loop). This means that (s_0,s_1) is a vertex of every possible edge in IG(A(D)), so $IG(A(\tilde{D}))$ is a "star" plus possible other isolated vertices, but cannot have a cycle, and $\chi(IG(A(\tilde{D}))) > 0$.

This explains conditions 2a.

Now move to test 2b. The idea now is that one can detach s_1 from s_0 at the cost of making some edges inside s multiple, thus yielding the claimed diagram D'. (Thus the move between D and $\tilde{D}$ is an isotopy of the knot K, but in D' the knot type will generally change.) By Murasugi-Przytycki [MP], see Theorem 4.2, the process of detaching a leg of s_1 (as long as other legs remain) changes $\max \mathrm{cf}_m P$ only up to units.

By a trivial skein relation argument, the same holds under changing the multiplicity of a leg of s_1 as well as for any edge inside s_0 (as long its multiplicity remains non-zero). Alternatively, one can use Theorem 4.2, with one positive Hopf (de)plumbing, and the move



(19)

A bit care is needed how to make D' a knot (rather than a multi-component link) diagram.

Obviously s_1 has an odd (multiplicity) leg, otherwise $\tilde{D}$ is not a knot diagram. Also s_1 has at least 2 traces (connected to s_0), otherwise $\tilde{D}$ can reduce in number of positive crossings, in contradiction to $c_+(K) = c(\tilde{D}) = c_+(\tilde{D})$.

Let use first treat the alternative (17). To obtain a diagram D' of $c(D') = c(D) - 2$,

1. detach all even legs of s_1 and add their traces to any edge inside s_1 (this is the effect of positive parallel claspings),
2. every time you detach an odd leg, find a proper edge inside s_1 where the leg (i.e., the set of its traces) can be added so that the result is still a knot (this is possible!),
3. detach this way all odd legs except the last,
4. for the last odd leg, detach all traces except one, and add them to some edge inside s_0,
5. and finally delete s_1 with its last trace.

Since s_1 has more than one trace connected to s_0, (otherwise $c_-(\tilde{D}) > c_-(K)$; cf. the beginning of the proof), obviously at least one clasping is applied inside s_1, so that (the case of (17) in) 2(b)iv holds.

To obtain a diagram D' of fewer crossings, we need $\le 2v-4$ crossings for the atom inside s_0, where

$$v = v(G(A(D'))) = v(G(A(D'))') = s(D') = s(\tilde{D}) - 1$$

(because s_1 is discounted), i.e., at most

$$2(c(D) - 2\min\deg_t V(D) - 1) - 4\,.$$

So make all edges inside s_0 of multiplicity 1. Every time a leg of s_1 is removed, add no trace to any edge inside s_0.

Write below

$$b_1 = b_1(G(A(\tilde{D}))') = |V_1(\tilde{D})| = |V_1(D)|\,.$$

Now the diagram may not a knot diagram. But it has at most b_1+1 components. By b_1 claspings one can get back to a knot diagram. Break up a cycle inside all faces of $G(A(\tilde{D}))'$.

Thus we can choose D' to have at most

$$2c(D) - 4\min\deg_t V(D) - 6 + |V_1| \tag{20}$$

crossings, which leads to (18). Also $s(D') = s(\tilde{D}) - 1$, of the same parity of $s(D)$, so to have D' being a knot diagram, we need $c(D') - c(D)$ even. But this is ascertained because $|V_1|$ has same parity to $c(D)$ by (16).

We applied claspings at at most b_1 edges, in $G(A(D'))$ at most b_1 edges are multiple. So with $IG(A(D'))$ having no edges, and $\triangle G(A(D'))' = 0$, we have

$$0 \le V_2(D') - \binom{|V_1(D')|+1}{2} = e_{++}(G(A(D'))) = \chi(IG(A(D'))) \le b_1 = b_1(G(A(\tilde{D}))')\,.$$

If

$$c(D') < (20)$$

(the difference is even) add double claspings at edges which are already multiple. If all edges are simple, do claspings at some edge until $c(D') = (20)$. This will not augment the number of multiple edges inside s_0 unless all edges are simple. Then it will augment 0 to 1. But we excluded the case $b_1 = |V_1| = 0$ (it is not very practically relevant). Thus the number of multiple edges $= \chi(IG(A(D')))$ is at most b_1.

Also since $b_1 > 0$ and $c(D') = (20)$, we have $c(D') > 2v-4$, not all edges inside s_0 can be simple, so $\chi(IG(A(D')) \ge 1$. Then (the case of (18) in) 2(b)iv holds. □

Remark 6.1 The test can be (probably) applied on many almost positive diagrams. It is just our interest in almost alternating diagrams that led us to apply it on almost positive-alternating diagrams (or almost special alternating diagrams). What the test can in general do is to prove that a certain almost positive knot diagram is minimal. It is not claimed that the test rules out the knots in question from being positive. Even another minimal crossing diagram could potentially be positive. It is also known that some positive knots have only non-minimal positive diagrams.

Remark 6.2 If

$$c(D) - 2\min\deg_t V(D) - |V_1(D)| < 4 \tag{21}$$

and $\max\mathrm{cf}_z P$ does not factor non-trivially into (non-strictly) alternating polynomials, then similar argument works, except the right equality in (18) should be revised. (The alternating property of $\max\mathrm{cf}_z P$ translates under convention change from Cromwell's positivity [Cr2, Corollary 4.3].) I have not experimented with this idea, though.

Proof of proposition 6.1. For $n = 13$ we have to serve ourselves upon some auxiliary example like (14).

For $n = 15+2k$, one family is obtained by iterated double clasping at (one of the crossings of) the parallel clasp in the diagram of example 6.1.

For $n = 14+2k$, do the same in the first diagram given above. This gives the sequence 14_{41015}, $16_{1177477}$, etc. □

The question how to deal with minimality of non-special almost alternating diagrams remains to be considered.

7. Homogeneous, alternative, and pseudo-alternating knots

7.1. Infinite families

Following a definition of Kauffman [Ka3], a(n orientated link) diagram is *alternative* if all crossings in a connected component of the complement of the Seifert circles have the same sign. Another way of saying this is that there should be no crossings of opposite signs attached to the same Seifert circle from the same side.

For *Murasugi atoms*, see [QW]; these are the minimal pieces D_i under which one can decompose a diagram D under (diagrammatic) connected and *Murasugi sum* ($*$-product). Note that all these atoms D_i are special diagrams, and

D being *homogeneous* [Cr2] is defined by demanding that all D_i are (positively or negatively) special alternating. In [Cr2] their Seifert graphs are called *blocks*.

The difference between 'homogeneous' and 'alternative' is subtle: only if a connected component of the complement of the Seifert circles is a diagrammatic connected sum, its different summands in a homogeneous diagram need not be special alternating of the same sign. In particular, the classes of alternating and positive diagrams are alternative as well. The below proof gives an example of a homogeneous knot diagram which is not alternative.

The properties 'alternative' and 'homogeneous' are extended to knots and links defined by admitting the corresponding type of diagram.

All properties of homogeneous diagrams (and links) apply to alternative as well. In particular, an alternative diagram of a positive link is positive. (See [Cr2, Theorem 4].)

One difference is that homogeneity of a diagram is preserved by flypes, whereas alternativeness in general is not (neither by type A nor by type B). Below, this non-invariance will create many caveats to pay attention to.

Kauffman had conjectured that alternative links are equivalent to *pseudo-alternating* ones, which are defined as the boundary of (not necessarily diagrammatic) Murasugi sums of special alternating surfaces. By [Cr2, §1], this equivalence would also imply equivalence to homogeneous links. Kauffman's conjecture was disproved in [Si] (see there for further details on the definitions), by finding a pseudo-alternating knot 10_{145} which is non-homogeneous. There are also some *ad hoc* examples that show that the inclusion of alternative links in homogeneous ones is also proper. Here we systematize this.

Theorem 7.1 There exist infinitely many pseudo-alternating knots which are not homogeneous.

Proof. We consider K_k to be the closures of the 4-braids $1^{2k-1}(1\ 1\ 2\ 2\ 3\ -2)^4$.

Since these braids are products of the braids (1), (2) and $(2\ 3\ -2)$, we see that K_k are pseudo-alternating (where the Murasugi sum is applied at the disk of the second braid strand). They are also strongly quasipositive and (since the Murasugi summands are $(2,n)$-torus link surfaces) fibered.

It can be checked by recursive application of (6) that

$$V_0(K_k)=1\,,\quad V_1(K_k)=0\,,\quad V_2(K_k)=2\,. \tag{22}$$

Now assume some K_k were homogneous. By an observation of Baader [Ba], this would imply that K_k is (fibered and) positive. Let D be a positive diagram of this K_k. It is easy to see that $G(A(D))$ must be a tree, and then $V_2(D)>1$ implies that $IG(A(D))$ is disconnected. But then $D=D_1\#D_2$ would be a (diagrammatic) connected sum of (non-trivial) positive diagrams D_i.

There are several ways to see that this cannot occur, e.g., by showing that K_k are prime, but here we choose a different argument. Let

$$\mathrm{MFW}(D):=1+\frac{1}{2}\,\mathrm{span}_l\,P(D)$$

be the Morton-Franks-Williams bound. Since K_k is the closure of a 4-braid, by [Mo, FW],

$$\mathrm{MFW}(D)=\mathrm{MFW}(D_1)+\mathrm{MFW}(D_2)-1\le 4\,.$$

Also, if $\mathrm{MFW}(D_i)=1$, then from [LMi, Proposition 21] the trivializing substitution

$$1=P(l,-l^{-1}-l) \tag{23}$$

would yield $P(D_i)=1$, but no non-trivial positive (diagram and) knot has trivial skein polynomial. Thus $\mathrm{MFW}(D_i)\in\{2,3\}$.

Since D_i represent fibered positive links,

$$P_{\max}(D_i)=(lm)^{1-\chi(D_i)} \tag{24}$$

is a single monomial. Also $\min\deg_l P(D_i) = 1-\chi(D_i)$. (See Theorems 4 and 5 in [Cr2] and keep in mind Theorem 4.2.)

Now, it can be seen from (23) and $\mathrm{MFW}(D_i) \le 3$ that in $P(D_i)$, the term $m^{-1-\chi(D_i)}l^{5-\chi(D_i)}$ cannot occur. Since $P(D) = P(D_1)P(D_2)$, because of (24) the same is true for D: with $1-\chi(D) = \max\deg_m P(D)$, the coefficient of $m^{\max\deg_m P(D)-2}l^{\max\deg_m P(D)+4}$ in $P(D)$ must be zero.

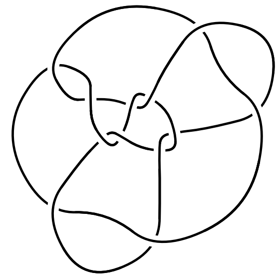
But one can use (5) (similarly as (6) for (22)) to recursively check that this term always appears in $P(K_k)$. This gives the sought contradiciton and concludes the proof. □

Theorem 7.2 There exist infinitely many homogeneous knots which are not alternative.

Proof. It was known from [St11] that all homogeneous knots of genus 1 and 2 are positive or alternating (and hence alternative), thus the simplest examples must have genus 3. We resume the setup of §4.

Consider the following diagram of 12_{1644} on the right. It is homogeneous (and not alternative), and since

$$V_0 = \bar{V}_0 = 2\,, \tag{25}$$



12_{1644} is inadequate.

We use proposition 3.1 in the easiest possible way. (There are more options, but unnecessary to complicate matters at this point.) One can find a $\sim$-class, $\bar{t}'_2$-twisting at which will give infinitely many distinct Jones polynomials all with (25). (Not all classes will do, but some.) We claim that these knots, call them K_i, are examples of the desired type.

Obviously, K_i are inadequate. In particular, they are neither alternating nor positive. Assume D_i is an alternative diagram of K_i, and we choose D_i to have minimal crossing number among all such diagrams. We will below gradually rule out all possibilities for D_i.

If D_i were composite, then $D_i = D'_i \# D''_i$, where $g(D'_i) = 1$ and $g(D''_i) = 2$. But all homogeneous knots of genus 1 (that may are represented by D'_i) are alternating (see [St10]) and of genus 2 (for D''_i) are alternating or positive (see[1] [St11, §4]). Thus K_i would be semiadequate, which it is not. This means that we can assume D_i is prime, and or genus 3. (There are ways to show primeness of K_i directly, but they require more writing.)

We will further need the following property of the HOMFLY polynomial of K_i, which can be seen also with the help of Theorem 4.2:

$$\max\deg_m P(K_i) = 6 \text{ and } \max\mathrm{cf}_m P(K_i) \text{ is an alternating polynomial in } l^2 \text{ with coefficients } \pm 1. \tag{26}$$

This property will be used as follows.

Lemma 7.1 Assume D is a homogeneous diagram with $\max\mathrm{cf}_m P(D)$ having a(n l-)coefficient of absolute value at least 2. Then D, and all diagrams obtained from it using flypes and $\bar{t}'_2$ twists, cannot represent any K_i.

Proof. This essentially follows from the argument behind [Cr2, Corollary 4.3]. Using Theorem 4.2, work with a special alternating diagram D. Then $P_{\max}(D)$ from (11) is an alternating polynomial in l (keep in mind our different convention from P from Cromwell's).

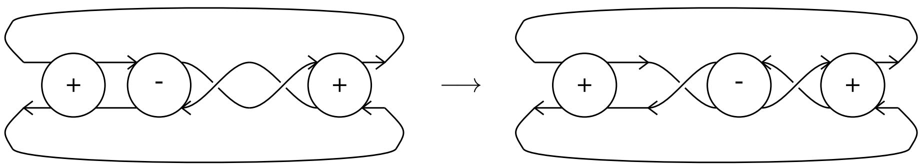
It follows then from that corollary that the absolute value of the coefficients of $P_{\max}(D)$ never decreases under $\bar{t}'_2$-twisting. The same then holds under multiplication according to Theorem 4.2. Thus (26) cannot be reinstated once spoilt. □

From here, we write down the step-by-step investigation for the potential alternative diagrams D_i.

[1] In that paper, there is a mixup of 'alternative' and 'homogeneous', as was later pointed out in [Si]. However, the special case of alternative, which is what was checked there, is enough here. Moreover, the check extends to the (correct) homogeneous case without many difficulties.

1. Take the genus 3 generators with at least 3 Murasugi atoms. (For the others all homogeneous diagrams will be alternating or positive/negative.) 604 of the 4017 knots remain.
2. Flype the diagrams (in all possible ways).
3. Switch twist classes in either way (up to mirror image). It is very easy to argue that in any homogeneous diagram twist equivalent crossings must have the same sign. This gives a list of about 1.4 million diagrams.
4. Choose the homogeneous diagrams among them; they are 30,012.
5. Remove all diagrams that are alternating or positive/negative; this leaves 19,366.
6. Type-B-flype reduce these diagrams, which gives a list of 1801. We know now that all homogeneous (in particular, alternative) diagrams of genus 3 which are not alternating or positive can be obtained from this list, we call it $\mathcal{D}$, of 1801 by type-B flyping and $\bar{t}'_2$-moves (which commute).
7. We build now the sublist $\mathcal{D}_0$ of $\mathcal{D}$ after discarding semiadequate diagrams and those for which Lemma 7.1 applies. (Note that flypes and $\bar{t}'_2$ moves preserve semiadequacy.) These are 99 diagrams.
8. We build now a list $\mathcal{D}_1$ by twisting once at a $\sim$-class of a diagram in $\mathcal{D}_0$, type-B flype-reducing to remove duplicates, and discarding diagrams to which Lemma 7.1 applies. (It turns out that no new semiadequate diagrams are generated.) These are 518 diagrams.
9. We repeat this process one more time with $\mathcal{D}_1$, obtaining a family $\mathcal{D}_2$, of equal size 518. We also check that no diagram in $\mathcal{D}_2$ has more than one unreduced $\sim$-class. I.e., if one twists in $\mathcal{D}_0$ at two different $\sim$-classes, the resulting diagrams will always fall to Lemma 7.1. This explains why $|\mathcal{D}_1| = |\mathcal{D}_2|$: diagrams in $\mathcal{D}_2$ are obtained by one $\bar{t}'_2$ twist in the unique non-reduced $\sim$-class of a diagram in $\mathcal{D}_1$.

   Now we know that if K_i has an alternative diagram D_i, then D_i must be obtained from $\mathcal{D}_0$ or $\mathcal{D}_1$ under type B flypes, or from $\mathcal{D}_2$ under type B flypes and $\bar{t}'_2$-moves at its single non-reduced $\sim$-class.

   (27)

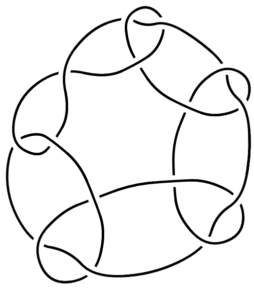
   Figure 3: Type-B flyping atoms in a homogeneous diagram to make it alternative

   The reason we should test $\mathcal{D}_1$ and $\mathcal{D}_2$ extra is illustrated in Figure 3. There is the possibility that additional crossings introduced after $\bar{t}'_2$-twists allow for flypes making a homogeneous diagram alternative, which was not possible before the $\bar{t}'_2$-twist. A $\bar{t}'_2$-twist creates additional Seifert circles which can be used to accommodate Murasugi atoms so as to avoid oppositely signed ones lying in the same Seifert circle. Here $g(D_i) = 3$, which means that there are at most 6 Murasugi atoms, and one can easily see that after at most two $\bar{t}'_2$-twists, enough Seifert circles are created for such possible extra alternative diagrams to be constructable.
10. Now we flype all diagrams in $\mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$, test whether they become alternative, and remove diagrams containing a tangle T like on the right of (19). Such diagrams can always be simplified to smaller crossing number alternative diagrams. (Keep in mind that the reverse clasp in T is a reduced $\sim$-class for a diagram in $\mathcal{D}_2$, and no $\bar{t}'_2$ twisting is allowed at it, by the above check with Lemma 7.1.) We are left with 3 diagrams:

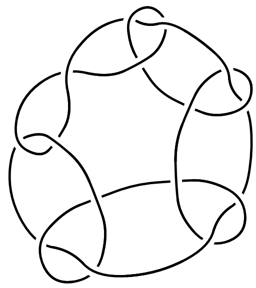
   ```
   10 52 4 12 -10 16 -14 2 -20 18 8 -6
   11 53 4 12 -10 18 -16 14 2 -22 20 8 -6
   12 55 4 14 -12 10 20 -18 16 2 -24 22 8 -6
   ```
11. Now one can see that all these 3 diagrams are rational diagrams. This will not change if a $\bar{t}'_2$-twist is applied at a non-trivial (in particular, non-reduced) $\sim$-class, and thus all these diagrams depict rational (2-bridge) knots, which clearly K_i are not. This completes the exclusion process for the D_i. □

7.2. Mutants

Example 7.1 One can use the idea (27) of flyping Murasugi atoms into Seifert circles of $\sim$-classes to show that alternative links may have non-alternative mutants. The following picture shows two 14 crossing Montesinos knots (in 15 crossing diagrams, to visualize the reasoning better). The flyping argument works for 14_{25183} to produce an alternative diagram, but not for 14_{25182}, where the Murasugi atoms are signed "in the wrong order".



14_{25183}: alternative



14_{25182}: homogeneous, non-alternative

This still does not prove that 14_{25182} is not alternative, but it can be checked. Since 14_{25182} is fibered of genus 5, one needs to find all fibered genus-5 generators (which have at most 20 crossings), flype and switch twist classes. This can be done within a few hours on a(n old) desktop machine. We partly used the resources [RFS, Br] (see §2.4), and optimized the test using the Goeritz matrix for the determinant, using Gauß sums for the Vassiliev invariants of degree 2 and 3, and the MFW inequalities. Further (technical) details are omitted, though. We just mention that we found the pair has two other mutants, 14_{25178} and 14_{25190}. The first is also alternative, while the second is not (albeit homogeneous), as can be verified simultaneously with 14_{25182}.

The question arises, which seems far harder:

Question 7.1 Is homogeneity mutation-invariant?

By Menasco's work, this is known for alternating knots. Using the tabulations given in [St9] (and some further amount of rather laborious effort), we were able to confirm that homogeneity (and positivity) are invariant for prime mutant knots up to 15 crossings.

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