Mutant links distinguished by degree 3 Gauß sums

A. Stoimenow

e-mail: stoimeno@informatik.hu-berlin.de,
WWW: http://www.informatik.hu-berlin.de/~stoimeno

Current version: April 27, 1998 First version: December 2, 1997

Abstract

We give examples showing that the Fiedler solid torus degree 3 Gauß sum invariants can be used to detect mutation of links.
Keywords: mutation, Gauß sums *AMS subject classification:* 57M25

1 Introduction

The notion of a mutant was introduced by Conway [Co]. Mutants have been later intensively studied and received much attention especially because of the difficulties they provide to distinguish them [P, P2, LL, LM, MC, MT, MR, MS, CD]. Beside the problem of detecting orientation they are the largest class of links left indistinguishable by the knot polynomials [H, J, Ka]. Although the knot polynomials and the related quantum theory are not completely useless for detecting mutation, the applicable methods result in calculations, which are (almost) infeasible already for the simplest examples, see [MR, MC, MS].

A conceptually new approach to finding invariants was initiated by Fiedler for braids [Fi2] and later by Viro-Polyak [PV] for invariants of finite degree [BN, BL, Va, Vo]. Recently, Fiedler [Fi] generalized the approach to invariants of degree 3 for knots in the solid torus [Go] and more generally in orientable S^1- and line bundles over surfaces. These invariants can be used to study two component links $K \cup T$ in $\mathbb{R}^3$ with an unknotted component T by the fact, that for isotopic links $K_1 \cup T_1 \sim K_2 \cup T_2$ in $\mathbb{R}^3$ it holds $K_1 \sim K_2$ in the solid torus $S^3 \setminus T_1 \simeq S^3 \setminus T_2$.

Now if $K_1 \sim K_2$ are isotopic knots in $\mathbb{R}^3$, then $K_i \cup T_i$, $i = 1,2$ are isotopic links, where T_i is the meridian of K_i (K_i may be replaced by some satellite around itself). This rendered it possible to examine knots in $\mathbb{R}^3$ by the new invariants in this way. Unfortunately, it turned out [St2], that so the Fiedler invariants do not give more than the usual degree 3 Vassiliev invariant for knots in $\mathbb{R}^3$, explaining the preceeding series of disappointing experimental results [St], in particular the failure to distinguish mutants. Further experiments with mutants suggested that the situation might not be better, when we place the trivial component somewhere else than to be the meridian of the knotted component (or of its companion). This opened the possibility, that the Fiedler invariants do not detect mutation at all.

Here we present examples of mutations in which the rotated tangle contains parts of both components. Computation with the program of [St] showed, that in this case the mutants could be distinguished.

It is worth remarking that this method to distinguish mutants will not work with the various solid torus link polynomial invariants coming from Markov traces on Hecke algebras of type B [HK, La, GL] as they also satisfy skein relations, and so the same arguments which apply for the HOMFLY polynomial on mutants (see [Li]) similarly apply for its analogues in the solid torus as well. For the polynomials of generalized Hecke algebras of [La2] the skein arguments do not apply in full generality, but concrete examples of distinguished mutants are not known (at least to me) due to lack of practical computations. In any case, however, the arguments apply, when the geometric linkning number in the mutated tangle of the strands belonging to T and to K is ± 2 and the strand of T has no self-crossings, as in our examples below.

But anyway, there is never an absoulte certainty that computers really do what we tell them or we really tell them what we want, so we encourage an independent verification of our result.

The spirit of this paper is intended to be similar to this of [MR]. We would, however, like to convince the reader, that the new invariants are simpler than quantum invariants and appear as a promising alternative.

The concept of these new invariants was outlined in [St]. However, no one of the formulas given there serves in our case, so, to make the result verifiable, we need to make new definitions. But then, also to encourage more experiments, we felt it would be better to give a complete list of all invariants.

2 Gauß diagrams and Gauß sums

We start by briefly recalling the definition of our invariants. See [St] for details.

Consider a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ (S^1 and $\mathbb{R}^3$ oriented). Decompose $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ so that the projection (henceforth called *knot diagram*) of K into $\mathbb{R}^2$ is generic. To this projection we can assign a *Gauß diagram* (GD), a circle with oriented chords, by connecting points in S^1 mapped to a crossing and orienting the chord from the preimage of the undercrossing to the preimage of the overcrossing. See [PV].

Figure 1 shows the knot 6_2 in its standard projection and the corresponding Gauß diagram.

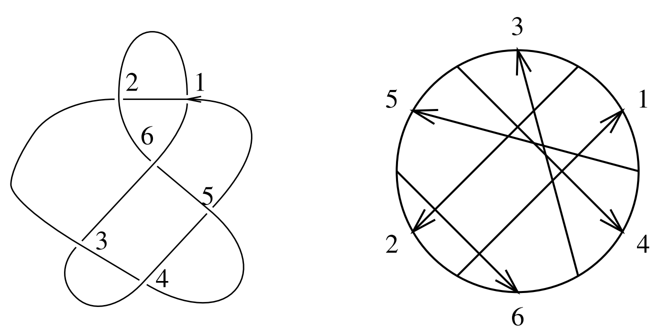


Figure 1: The knot 6_2 and its Gauß diagram.

A *Gauß sum* of degree k is a term assigned to a knot diagram, which is of the following form

$$\sum_{\substack{\text{ordered choices of } k \text{ crossings of the} \\ \text{knot diagram, whose arrows in the GD} \\ \text{form a given subdiagram}}} \text{function}\big(\text{ data, assigned to the crossings }\big)\,.$$

Each summand we will call *weight* and the function *weight function*. We will denote the summation by the subdiagram itself, which we will also call *configuration*.

Now we need to specify the data assigned to the crossings.

Definition 2.1 *The* winding index *of a plane curve $C \subset \mathbb{R}^2 = \mathbb{C}$ around a point $p \notin C$ is*

$$w(C,p) := \frac{1}{2\pi i}\oint_C \frac{1}{z-p}\,dz.$$

Pictorially it measures how many times the curve "walks" around p, counting reverse walk negatively.

Definition 2.2 *The* Whitney index $n(C)$ *of a plane curve C is the degree of the map*

$$\frac{C'}{||C'||} : S^1 \longrightarrow S^1 .$$

The Whitney index of a knot diagram is the Whitney index of its underlying plane curve.

Definition 2.3 *The* writhe $w(D)$ *of a knot diagram D is the sum of the writhes of all crossings (see figure 2).*

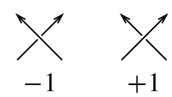
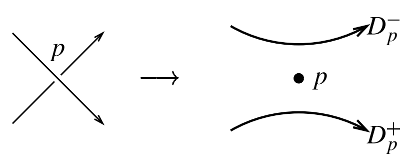


Figure 2: The writhe of a crossing.

Example 2.1 The standard projection of 6_2 on figure 1 has Whitney index 1 and writhe -2.

Definition 2.4 *A* smoothing *of a crossing is the procedure*



,

where D_p^+ denotes the component, where the under-crossing is smoothed to the over-crossing. Note, that beside the link diagram resulted after this operation, we have the "trace" of p in its complement.

Apart from its writhe w_p, for each crossing p we have two more data:

$$i_p^{\pm} := w(D_p^{\pm}, p) .$$

Here by p we mean the trace of p in the complement of $D_p^{\pm}$, as described above. Set

$$i_p := i_p^+ + i_p^- , \qquad \delta_p := i_p^+ - i_p^- .$$

Now consider a two component link $K \cup T$ in S^3 where T is the trivial knot (unknot). Let K, T, S^3 be oriented. Deform $K \cup T$ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ so that $\infty \in T$. This isotopy is unique up to isotopy. Such a link we can represent choosing an appropriate projection $\mathbb{R}^3 \to \mathbb{R}^2$ as knot with a point in its complement, on which T projects, assuming the orientation of T to be from the sheet of paper to the reader's eye.

Definition 2.5 *The* type *of a crossing p is* $w(D_p^+, T)$ mod 2.

All invariants are regular isotopy invariants. A regular isotopy is an ambient isotopy preserving the writhe w and Whitney index n of K.

We distinguish two cases according to the parity of linking number $lk(K,T) = w(K,T)$.

For the definition we need the following atomary terms. Here near any arrow in the configuration its name and the type of the crossing is indicated by '0', '1' or by '*', if both types are allowed. A chord denotes an arrow whose orientation does not matter. The default weight function is $w_p w_q w_r$, which is not written.

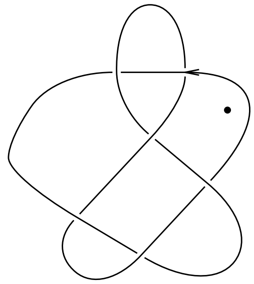
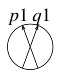


Figure 3: The knot 6_2 with its meridian depicted in our favourable way.

$A_1 =$ 

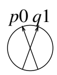
$w_p \cdot w_q \cdot (i_p^+ + i_p^- - i_q^+ - i_q^-)$

$A_2 =$ 

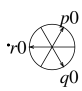
$w_p \cdot w_q \cdot (i_p^+ + i_p^- - i_q^+ - i_q^-)$

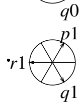
$A_3 =$ 

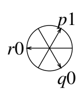
$w_p \cdot w_q \cdot (i_p^+ + i_p^- - i_q^+ - i_q^-)$

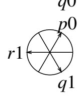
$A_4 =$ 

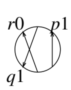
$w_p \cdot w_q \cdot (i_p^+ + i_p^- - i_q^+ - i_q^-)$

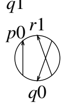
$A_5 = \frac{1}{3} \cdot$ 

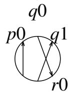
$A_6 = \frac{1}{3} \cdot$ 

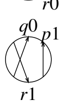
$A_7 =$ 

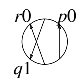
$A_8 =$ 

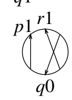
$A_9 =$ 

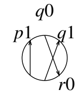
$A_{10} =$ 

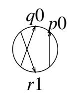
$A_{11} =$ 

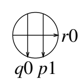
$A_{12} =$ 

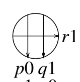
$A_{13} =$ 

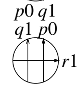
$A_{14} =$ 

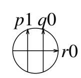
$A_{15} =$ 

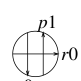
$A_{16} =$ 

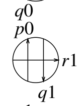
$A_{17} =$ 

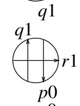
$A_{18} =$ 

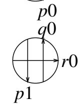
$A_{19} =$ 

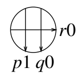
$A_{20} =$ 

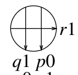
$A_{21} =$ 

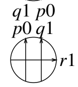
$A_{22} =$ 

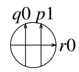
$A_{23} =$ 

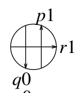
$A_{24} =$ 

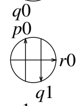
$A_{25} =$ 

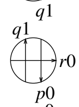
$A_{26} =$ 

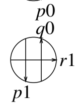
$A_{27} =$ 

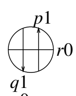
$A_{28} =$ 

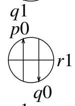
$A_{29} =$ 

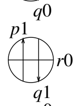
$A_{30} =$ 

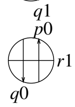
$A_{31} =$ 

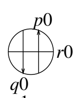
$A_{32} =$ 

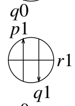
$A_{33} = \frac{1}{2} \cdot$ 

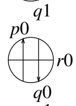
$A_{34} = \frac{1}{2} \cdot$ 

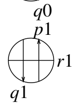
$A_{35} = \frac{1}{2} \cdot$ 

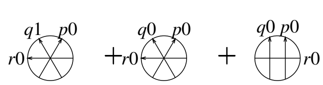
$A_{36} = \frac{1}{2} \cdot$ 

$A_{37} = \frac{1}{2} \cdot$ 

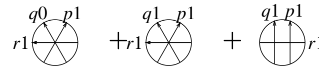
$A_{38} = \frac{1}{2} \cdot$ 

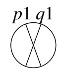
$A_{39} = \frac{1}{2} \cdot$ 

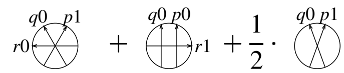
$A_{40} = \frac{1}{2} \cdot$ 

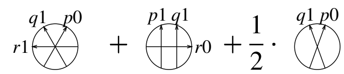
$A_{41} =$ 

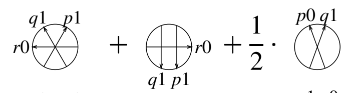
$+\frac{1}{4} \cdot$  $(w_p + w_q)$

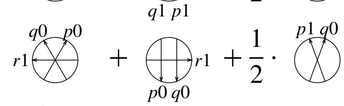
$A_{42} =$ 

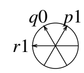
$+\frac{1}{4} \cdot$  $(w_p + w_q)$

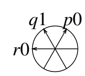
$A_{43} =$  w_p

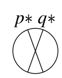
$A_{44} =$  w_p

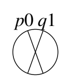
$A_{45} =$  w_p

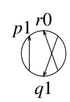
$A_{46} =$  w_p

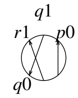
$A_{47} =$ 

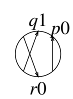
$A_{48} =$ 

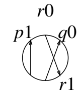
$A_{49} = \frac{1}{4} \cdot$  $w_p \cdot w_q \cdot (i_p^+ + i_p^- + i_q^+ + i_q^-)$

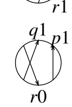
$A_{50} = \frac{1}{2} \cdot$  $w_p \cdot w_q \cdot (i_p^+ + i_p^- + i_q^+ + i_q^-)$

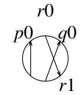
$A_{51} =$ 

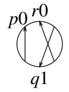
$A_{52} =$ 

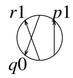
$A_{53} =$ 

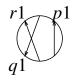
$A_{54} =$ 

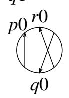
$A_{55} =$ 

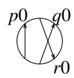
$A_{56} =$ 

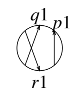
$A_{57} =$ 

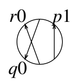
$A_{58} =$ 

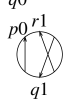
$A_{59} =$ 

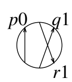
$A_{60} =$ 

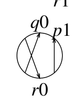
$A_{61} =$ 

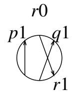
$A_{62} =$ 

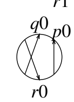
$A_{63} =$ 

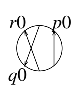
$A_{64} =$ 

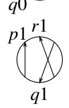
$A_{65} =$ 

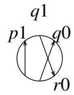
$A_{66} =$ 

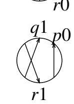
$A_{67} =$ 

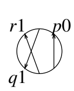
$A_{68} =$ 

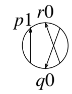
$A_{69} =$ 

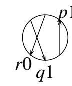
$A_{70} =$ 

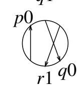
$A_{71} =$ 

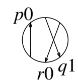
$A_{72} =$ 

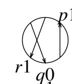
$A_{73} =$ 

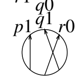
$A_{74} =$ 

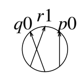
$A_{75} =$ 

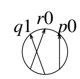
$A_{76} =$ 

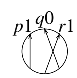
$A_{77} =$ 

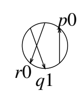
$A_{78} =$ 

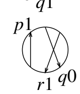
$A_{79} =$ 

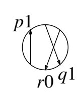
$A_{80} =$ 

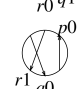
$A_{81} =$ 

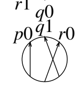
$A_{82} =$ 

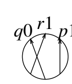
$A_{83} =$ 

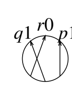
$A_{84} =$ 

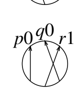
$A_{85} =$ 

$A_{86} =$ 

$A_{87} =$ 

$A_{88} =$ 

$A_{89} =$ 

$A_{90} =$ 

2.1 Even linking number invariants

For the case $w(K,T) \equiv 0 \pmod 2$ we obtained 39 invariants. They are given below.

$$
\begin{aligned}
In_1 &= A_{56} - A_{57} - A_{87} + A_{90}, \\
In_2 &= A_{55} - A_{58} - A_{59} + A_{62} - A_{88} + A_{89}, \\
In_3 &= A_{52} - A_{53} - A_{83} + A_{86}, \\
In_4 &= A_{51} - A_{54} - A_{67} + A_{70} - A_{84} + A_{85}, \\
In_5 &= A_{14} - A_{15} - A_{67} + A_{70} - A_{79} + A_{82}, \\
In_6 &= A_{13} - A_{16} - A_{80} + A_{81}, \\
In_7 &= -A_9 + A_{12} - A_{59} + A_{62} - A_{75} + A_{78}, \\
In_8 &= -A_{10} + A_{11} - A_{76} + A_{77}, \\
In_9 &= -A_1 + A_6 + A_7 + A_9 + A_{14} - A_{17} + 2 \cdot A_{21} + A_{23} + A_{25} + A_{29} + A_{51} \\
&\quad + A_{58} + A_{59} + A_{63} + A_{70} + A_{74}, \\
In_{10} &= A_1 - 2 \cdot A_6 - 2 \cdot A_7 - A_8 + A_{10} + A_{13} + A_{17} - 2 \cdot A_{21} - A_{23} - A_{25} - A_{29} + A_{52} \\
&\quad + A_{57} + A_{64} + A_{73}, \\
In_{11} &= A_1 - 2 \cdot A_6 - 2 \cdot A_7 - A_8 + A_{10} + A_{16} + A_{17} - 2 \cdot A_{21} - A_{23} - A_{25} - A_{29} + A_{53} \\
&\quad + A_{57} + A_{64} + A_{72}, \\
In_{12} &= -A_1 + A_6 + A_7 + A_9 + A_{15} - A_{17} + 2 \cdot A_{21} + A_{23} + A_{25} + A_{29} + A_{54} \\
&\quad + A_{58} + A_{59} + A_{63} + A_{67} + A_{71}, \\
In_{13} &= -A_5 + A_{60} + A_{69}, \\
In_{14} &= -A_5 + A_{60} + A_{68}, \\
In_{15} &= -A_9 + A_{12} + A_{55} - A_{58} - A_{59} + A_{62} - A_{63} + A_{66}, \\
In_{16} &= -A_{10} + A_{11} + A_{56} - A_{57} - A_{64} + A_{65}, \\
In_{17} &= -A_{60} + A_{61}, \\
In_{18} &= -A_7 + 2 \cdot A_{17} + A_{18} - A_{19} - A_{21} - A_{24} + A_{48}, \\
In_{19} &= 2 \cdot A_6 + 2 \cdot A_7 + A_8 + A_{19} + A_{21} + A_{23} + A_{24} + 2 \cdot A_{25} + A_{26} + A_{29} + 2 \cdot A_{43} + A_{47}, \\
In_{20} &= -A_1 + A_{18} - A_{19} + A_{21} - A_{24} - A_{43} + A_{46}, \\
In_{21} &= -2 \cdot A_1 - A_3 + 2 \cdot A_6 + 2 \cdot A_7 - 2 \cdot A_{17} - A_{18} + 2 \cdot A_{19} + 2 \cdot A_{21} + A_{23} + 4 \cdot A_{25} + A_{26} \\
&\quad + A_{29} - 2 \cdot A_{33} + 2 \cdot A_{43} + 2 \cdot A_{45}, \\
In_{22} &= -4 \cdot A_1 - A_3 + 2 \cdot A_6 + 2 \cdot A_7 - 2 \cdot A_{17} + A_{18} + 2 \cdot A_{21} - 2 \cdot A_{22} + 3 \cdot A_{23} + 4 \cdot A_{25} + A_{26} \\
&\quad + A_{29} - 2 \cdot A_{33} + 2 \cdot A_{43} + 2 \cdot A_{44}, \\
In_{23} &= 4 \cdot A_1 + A_3 - A_6 + A_8 + 2 \cdot A_{17} + A_{19} - 2 \cdot A_{21} + A_{22} - 2 \cdot A_{23} + 2 \cdot A_{24} - 2 \cdot A_{25} + A_{26} \\
&\quad - A_{29} + 2 \cdot A_{33} + A_{42}, \\
In_{24} &= A_5 - A_7 + 2 \cdot A_{17} + A_{18} - A_{19} - A_{21} - A_{24} + A_{41}, \\
In_{25} &= 2 \cdot A_1 + A_3 + 2 \cdot A_6 + 2 \cdot A_8 + 2 \cdot A_{17} + A_{18} - A_{23} - 2 \cdot A_{25} - A_{26} + A_{29} + 2 \cdot A_{33} + 2 \cdot A_{40}, \\
In_{26} &= 2 \cdot A_5 + A_{39}, \\
In_{27} &= 4 \cdot A_1 + A_3 + 2 \cdot A_6 + 2 \cdot A_8 + 2 \cdot A_{17} - A_{18} - 2 \cdot A_{21} + 2 \cdot A_{22} - A_{23} + 2 \cdot A_{24} - 2 \cdot A_{25} + A_{26} \\
&\quad - A_{29} + 2 \cdot A_{33} + 2 \cdot A_{38}, \\
In_{28} &= 2 \cdot A_5 + A_{37}, \\
In_{29} &= -2 \cdot A_1 - A_3 + 2 \cdot A_6 + 4 \cdot A_7 + 2 \cdot A_8 - 2 \cdot A_{17} - A_{18} + 4 \cdot A_{21} + A_{23} + 2 \cdot A_{25} + A_{26} \\
&\quad + 3 \cdot A_{29} + 2 \cdot A_{36},
\end{aligned}
$$

$$
\begin{aligned}
In_{30} &= -3\cdot A_1 - A_3 - 2\cdot A_{17} + A_{21} - A_{22} + A_{23} - A_{24} + 2\cdot A_{25} - A_{33} + A_{35},\\
In_{31} &= -2\cdot A_1 + A_3 + 2\cdot A_6 + 4\cdot A_7 + 2\cdot A_8 - 2\cdot A_{17} + A_{18} + 4\cdot A_{21} + 3\cdot A_{23} + 2\cdot A_{25} - A_{26}\\
&\quad + A_{29} + 2\cdot A_{34},\\
In_{32} &= -A_{21} + A_{24} - A_{29} + A_{32},\\
In_{33} &= A_1 + A_{17} - A_{21} - A_{25} + A_{31},\\
In_{34} &= A_1 + A_{17} - A_{21} + A_{22} - A_{23} - A_{25} + A_{30},\\
In_{35} &= 2\cdot A_1 - A_{18} + A_{19} - A_{21} + A_{22} - A_{23} + A_{24} - A_{25} + A_{28},\\
In_{36} &= -2\cdot A_1 + A_{18} - A_{19} + A_{21} - A_{22} + A_{23} - A_{24} - A_{26} + A_{27},\\
In_{37} &= -A_{17} - A_{18} + A_{19} + A_{20},\\
In_{38} &= 2\cdot A_1 + A_3 + A_4,\\
In_{39} &= -A_1 + A_2
\end{aligned}
$$

2.2 Odd linking number invariants

For the case $w(K,T) \equiv 1 \pmod 2$ we obtained 36 invariants. They are given in terms of the expressions

$$
\begin{aligned}
In_{1} &= A_{56} - A_{57} - A_{60} + A_{61} - A_{87} + A_{90},\\
In_{2} &= A_{55} - A_{58} - A_{59} + A_{62} - A_{88} + A_{89},\\
In_{3} &= A_{52} - A_{53} - A_{68} + A_{69} - A_{83} + A_{86},\\
In_{4} &= A_{51} - A_{54} - A_{67} + A_{70} - A_{84} + A_{85},\\
In_{5} &= -A_{67} + A_{70} - A_{79} + A_{82},\\
In_{6} &= -A_{68} + A_{69} - A_{80} + A_{81},\\
In_{7} &= -A_9 + A_{12} - A_{59} + A_{62} - A_{75} + A_{78},\\
In_{8} &= -A_{10} + A_{11} - A_{60} + A_{61} - A_{76} + A_{77},\\
In_{9} &= 3\cdot A_1 + A_3 - 2\cdot A_5 - 6\cdot A_6 - 4\cdot A_7 - 4\cdot A_{13} + 4\cdot A_{14} + A_{17} + A_{19} - 6\cdot A_{21} + 4\cdot A_{22} - A_{23}\\
&\quad + A_{24} - 2\cdot A_{25} - 5\cdot A_{29} + 3\cdot A_{30} - A_{49} + 8\cdot A_{51} + 8\cdot A_{70} + 8\cdot A_{74},\\
In_{10} &= 3\cdot A_1 + A_3 - 6\cdot A_5 - 2\cdot A_6 - 4\cdot A_8 + 4\cdot A_{13} - 4\cdot A_{14} + A_{17} + A_{19} + 2\cdot A_{21} - 4\cdot A_{22} - A_{23}\\
&\quad + A_{24} - 2\cdot A_{25} + 3\cdot A_{29} - 5\cdot A_{30} + A_{49} + 8\cdot A_{52} + 8\cdot A_{69} + 8\cdot A_{73},\\
In_{11} &= 3\cdot A_1 + A_3 - 6\cdot A_5 - 2\cdot A_6 - 4\cdot A_8 + 4\cdot A_{13} - 4\cdot A_{14} + A_{17} + A_{19} + 2\cdot A_{21} - 4\cdot A_{22} - A_{23}\\
&\quad + A_{24} - 2\cdot A_{25} + 3\cdot A_{29} - 5\cdot A_{30} + A_{49} + 8\cdot A_{53} + 8\cdot A_{68} + 8\cdot A_{72},\\
In_{12} &= 3\cdot A_1 + A_3 - 2\cdot A_5 - 6\cdot A_6 - 4\cdot A_7 - 4\cdot A_{13} + 4\cdot A_{14} + A_{17} + A_{19} - 6\cdot A_{21} + 4\cdot A_{22} - A_{23}\\
&\quad + A_{24} - 2\cdot A_{25} - 5\cdot A_{29} + 3\cdot A_{30} - A_{49} + 8\cdot A_{54} + 8\cdot A_{67} + 8\cdot A_{71},\\
In_{13} &= -3\cdot A_1 - A_3 + 2\cdot A_5 - 2\cdot A_6 - 4\cdot A_7 + 8\cdot A_{12} + 4\cdot A_{13} + 4\cdot A_{14} - A_{17} - A_{19} - 2\cdot A_{21}\\
&\quad + 4\cdot A_{22} + A_{23} - A_{24} + 2\cdot A_{25} - 3\cdot A_{29} + 5\cdot A_{30} + A_{49} + 8\cdot A_{55} + 8\cdot A_{62} + 8\cdot A_{66},\\
In_{14} &= -3\cdot A_1 - A_3 - 2\cdot A_5 + 2\cdot A_6 - 4\cdot A_8 + 8\cdot A_{11} + 4\cdot A_{13} + 4\cdot A_{14} - A_{17} - A_{19} + 6\cdot A_{21}\\
&\quad - 4\cdot A_{22} + A_{23} - A_{24} + 2\cdot A_{25} + 5\cdot A_{29} - 3\cdot A_{30} - A_{49} + 8\cdot A_{56} + 8\cdot A_{61} + 8\cdot A_{65},\\
In_{15} &= -3\cdot A_1 - A_3 - 2\cdot A_5 + 2\cdot A_6 - 4\cdot A_8 + 8\cdot A_{10} + 4\cdot A_{13} + 4\cdot A_{14} - A_{17} - A_{19} + 6\cdot A_{21}\\
&\quad - 4\cdot A_{22} + A_{23} - A_{24} + 2\cdot A_{25} + 5\cdot A_{29} - 3\cdot A_{30} - A_{49} + 8\cdot A_{57} + 8\cdot A_{60} + 8\cdot A_{64},\\
In_{16} &= -3\cdot A_1 - A_3 + 2\cdot A_5 - 2\cdot A_6 - 4\cdot A_7 + 8\cdot A_9 + 4\cdot A_{13} + 4\cdot A_{14} - A_{17} - A_{19} - 2\cdot A_{21}\\
&\quad + 4\cdot A_{22} + A_{23} - A_{24} + 2\cdot A_{25} - 3\cdot A_{29} + 5\cdot A_{30} + A_{49} + 8\cdot A_{58} + 8\cdot A_{59} + 8\cdot A_{63},\\
In_{17} &= A_1 + A_3 - 4\cdot A_5 - A_{17} - A_{19} + 2\cdot A_{21} + A_{23} - A_{24} - 2\cdot A_{25} + A_{29} - 3\cdot A_{30} - 4\cdot A_{43}
\end{aligned}
$$

$$
\begin{aligned}
& & -2\cdot A_{47}+2\cdot A_{48}, \\
In_{18} &=& -A_1+A_{18}-A_{19}+A_{21}-A_{24}-A_{43}+A_{46}, \\
In_{19} &=& -A_1-A_3+2\cdot A_5+2\cdot A_6+A_{17}+A_{19}-A_{23}-A_{24}+2\cdot A_{25}+A_{29}+A_{30} \\
& & +2\cdot A_{43}+2\cdot A_{45}, \\
In_{20} &=& -3\cdot A_1-A_3+2\cdot A_5+2\cdot A_6+A_{17}+2\cdot A_{18}-A_{19}-2\cdot A_{22}+A_{23}-A_{24}+2\cdot A_{25} \\
& & +A_{29}+A_{30}+2\cdot A_{43}+2\cdot A_{44}, \\
In_{21} &=& -A_1+A_3-2\cdot A_6+2\cdot A_7+A_{17}+2\cdot A_{18}-A_{19}+2\cdot A_{21}-2\cdot A_{22}+A_{23}+A_{24} \\
& & +2\cdot A_{25}-A_{29}-A_{30}+2\cdot A_{42}, \\
In_{22} &=& -A_1+A_3-2\cdot A_5+2\cdot A_8+A_{17}+2\cdot A_{18}-A_{19}+3\cdot A_{23}-A_{24}+2\cdot A_{25}-A_{29} \\
& & -A_{30}+2\cdot A_{41}, \\
In_{23} &=& 3\cdot A_1+A_3+4\cdot A_6+2\cdot A_7+2\cdot A_8+A_{17}+A_{19}+2\cdot A_{21}-A_{23}+A_{24}-2\cdot A_{25}+3\cdot A_{29} \\
& & -A_{30}+4\cdot A_{40}, \\
In_{24} &=& 3\cdot A_1+A_3+4\cdot A_5+2\cdot A_7+2\cdot A_8+A_{17}+A_{19}-2\cdot A_{21}+4\cdot A_{22}-A_{23}+A_{24}-2\cdot A_{25} \\
& & -A_{29}+3\cdot A_{30}+4\cdot A_{39}, \\
In_{25} &=& -A_1+A_3+4\cdot A_6+2\cdot A_7+2\cdot A_8+A_{17}-3\cdot A_{19}+2\cdot A_{21}+3\cdot A_{23}+A_{24}+2\cdot A_{25} \\
& & -A_{29}-A_{30}+4\cdot A_{38}, \\
In_{26} &=& -A_1+A_3+4\cdot A_5+2\cdot A_7+2\cdot A_8-3\cdot A_{17}+A_{19}+2\cdot A_{21}+3\cdot A_{23}+A_{24}+2\cdot A_{25} \\
& & -A_{29}-A_{30}+4\cdot A_{37}, \\
In_{27} &=& -5\cdot A_1-3\cdot A_3+4\cdot A_5+2\cdot A_7+2\cdot A_8+A_{17}-3\cdot A_{19}+2\cdot A_{21}-A_{23}-3\cdot A_{24}+2\cdot A_{25} \\
& & +3\cdot A_{29}+3\cdot A_{30}+4\cdot A_{36}, \\
In_{28} &=& -5\cdot A_1-3\cdot A_3+4\cdot A_6+2\cdot A_7+2\cdot A_8-3\cdot A_{17}+A_{19}+2\cdot A_{21}-A_{23}-3\cdot A_{24}+2\cdot A_{25} \\
& & +3\cdot A_{29}+3\cdot A_{30}+4\cdot A_{35}, \\
In_{29} &=& 3\cdot A_1+A_3+4\cdot A_5+2\cdot A_7+2\cdot A_8+A_{17}+A_{19}-2\cdot A_{21}+4\cdot A_{22}-A_{23}+A_{24}-2\cdot A_{25} \\
& & -A_{29}+3\cdot A_{30}+4\cdot A_{34}, \\
In_{30} &=& 3\cdot A_1+A_3+4\cdot A_6+2\cdot A_7+2\cdot A_8+A_{17}+A_{19}+2\cdot A_{21}-A_{23}+A_{24}-2\cdot A_{25}+3\cdot A_{29} \\
& & -A_{30}+4\cdot A_{33}, \\
In_{31} &=& -A_{21}+A_{24}-A_{29}+A_{32}, \\
In_{32} &=& -A_{22}+A_{23}-A_{30}+A_{31}, \\
In_{33} &=& 2\cdot A_1-A_{18}+A_{19}-A_{21}+A_{22}-A_{23}+A_{24}-A_{25}+A_{28}, \\
In_{34} &=& -A_{25}+A_{27}, \\
In_{35} &=& 2\cdot A_1-A_{18}+A_{19}-A_{21}+A_{22}-A_{23}+A_{24}-A_{25}+A_{26}, \\
In_{36} &=& -A_{17}-A_{18}+A_{19}+A_{20}, \\
In_{37} &=& -A_{13}+A_{16}, \\
In_{38} &=& -A_{14}+A_{15}, \\
In_{39} &=& 2\cdot A_1+A_3+A_4, \\
In_{40} &=& -A_1+A_2
\end{aligned}
$$

From these terms all In_i with $i<9$ or $i>16$ and In_i+In_{i+1} for $i=9,11,13,15$ are invariants.

3 The examples

Figure 4 shows 2 link mutants of an unknot and a positive trefoil. The mutation "interchanges" the two components.

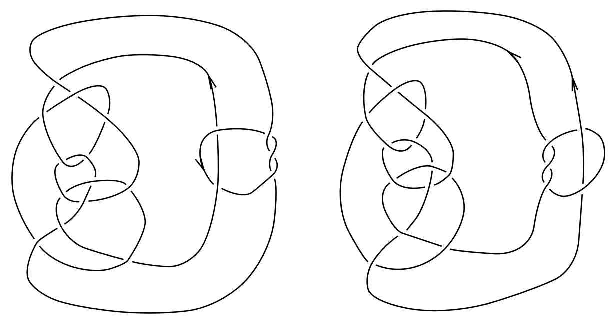


Figure 4: The link mutants with $lk = 1$.

The same links are transformed into our favourable diagram and shown on figure 5.

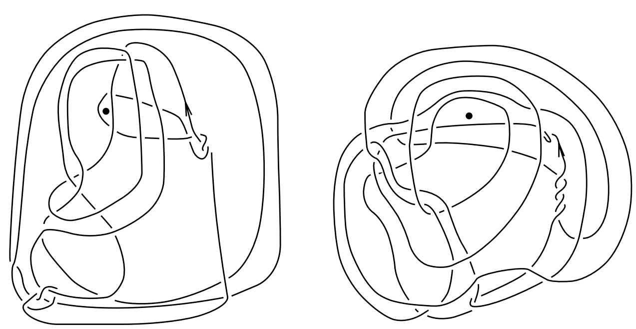


Figure 5: The link mutants of figure 4 depicted in our favourable way.

They have the same linking number $lk = w(K,T) = 1$ (as should be), and are adjusted K to have the same n and w. We verified that the degree 3 Vassiliev invariant vt_3 of [St] on K is in both cases 4 (as for the positive trefoil). The computer calculation took just about 2 minutes and gave the following result:

```
file: k-mut1
lk(K,T)=1
w=0
n=3
invt     k

 1       0
 2       0
 3       0
 4       0
 5       0
 6       0
 7       0
 8       0
 9       -18
10       -34
11       -34
12       -18
```

```
file: k-mut2a
lk(K,T)=1
w=0
n=3
invt     k

 1       0
 2       0
 3       0
 4       0
 5       0
 6       0
 7       0
 8       0
 9       -1
10       -19
11       -19
12       -1
```

13	10	13	-7
14	10	14	-5
15	10	15	-5
16	10	16	-7
17	-2	17	-2
18	0	18	0
19	2	19	2
20	2	20	2
21	2	21	2
22	2	22	2
23	2	23	2
24	2	24	2
25	2	25	2
26	2	26	2
27	2	27	2
28	2	28	2
29	2	29	2
30	2	30	2
31	0	31	0
32	0	32	0
33	0	33	0
34	0	34	0
35	0	35	0
36	0	36	0
37	0	37	0
38	0	38	0
39	0	39	0
40	0	40	0

If the mutants were the same, both diagrams would have to be isotopic and would have the same invariants. However, $In_{13} + In_{14}$ is once 20 and another time -12.

Figure 6 shows 2 link mutants of an unknot and a positive trefoil with $lk = 0$. The mutation "interchanges" again the two components.

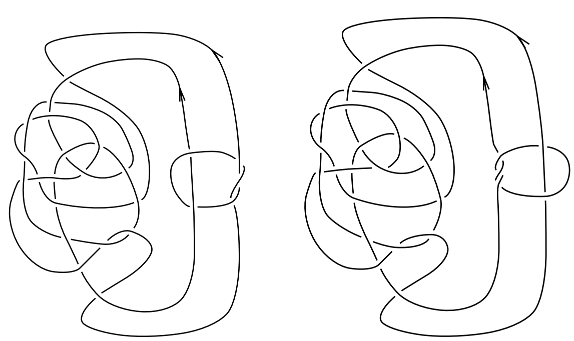


Figure 6: The link mutants with $lk = 0$.

The same links are transformed into our favourable diagram and shown on figure 7, and are adjusted K to have the same n and w. We verified again that vt_3 on K is in both cases 4.

The computer calculation gave the following result:

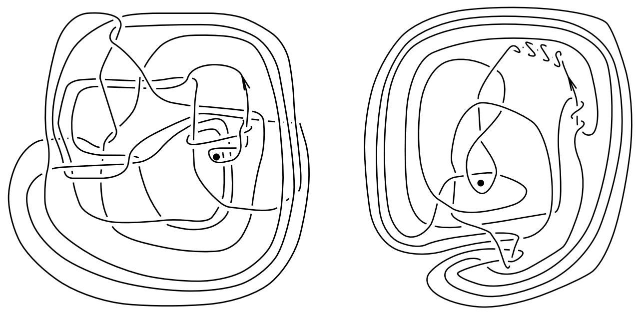


Figure 7: The link mutants of figure 6 depicted in our favourable way.

```
file: k-mut3
mod=2
w=7
n=2
invt     k
         lk=0

 1       0
 2       0
 3       0
 4       0
 5       0
 6       0
 7       0
 8       0
 9       214
10       -168
11       -168
12       214
13       -43
14       -43
15       0
16       0
17       0
18       0
19       624
20       0
21       624
22       624
23       -188
24       -432
25       -188
26       -216
27       -188
28       -216
29       624
30       0
31       624
32       0
```

```
file: k-mut4
mod=2
w=7
n=2
invt     k
         lk=0

 1       0
 2       0
 3       0
 4       0
 5       0
 6       0
 7       0
 8       0
 9       2
10       -4
11       -4
12       2
13       5
14       5
15       0
16       0
17       0
18       0
19       4
20       0
21       4
22       4
23       0
24       0
25       0
26       0
27       0
28       0
29       4
30       0
31       4
32       0
```

33	0
34	0
35	0
36	0
37	0
38	0
39	0

33	0
34	0
35	0
36	0
37	0
38	0
39	0

Here the difference is even clearer.

Acknowledgement. I would wish to thank to T. Fiedler and S. Lambropoulou for their helpful remarks.

4 References

[BN] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423–472.

[BL] J. S. Birman and X-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993) 225–270.

[CD] S. Chmutov, S. Duzhin, *Kontsevich integral*, Preprint of Max-Planck-Institut für Mathematik (Bonn, August 1997); available via anonymous file transfer from `pier.botik.ru`, subdirectory `pub/local/zmr`, file name `ki.ps.gz`.

[Co] J. H. Conway, *On enumeration of knots and links*, in "Computational Problems in abstract algebra" (J. Leech, ed.), 329–358. Pergamon Press, 1969.

[Fi] T. Fiedler, *New knot invariants*, UPS Toulouse preprint, to appear.

[Fi2] ——— ” ———, *A small state sum for knots*, Topology **32 (2)** (1993), 281–294.

[GL] M. Geck and S. Lambropoulou, *Markov traces and knot invariants related to Iwahori–Hecke algebras of type B*, J. reine angew. Math. **482** (1997), 191–213.

[Go] V. Goryunov, *Finite Order Invariants of Framed Knots in a solid torus and in Arnold's J^+-Theory of plane curves*, "Geometry and Physics", Lecture Notes in Pure & Appl. Math. **184**, M. Dekker, New York, 1996, 549–556.

[HK] J. Hoste and M. Kidwell, *Dichromatic link invariants*, Trans. Amer. Math. Soc. **321 (1)** (1990), 197–229.

[H] P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–246.

[J] V. F. R. Jones, *A polynomial invariant of knots and links via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[Ka] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[La] S. Lambropoulou, *Solid torus links and Hecke algebras of B-type*, Proceedings of the Conference on Quantum Topology (D. N. Yetter, ed.), World Scientific, 1994, 225–245.

[La2] ——— ” ——— , *Knot theory related to generalized and cyclotomic Hecke algebras of type B*, Mathematica Gottingensis Schriftenreihe, Heft **25** (1997).

[Li] W. B. R. Lickorish, *Linear skein theory and link polynomials*, Top. Appl. **27** (1987), 265–274.

[LL] ——— ” ——— and A. S. Lipson, *Polynomials of 2-cable-like links*, Proc. Amer. Math. Soc. **100** (1987), 355–361.

[LM] ——— ” ——— and K. C. Millett, *A polynomial invariant for oriented links*, Topology **26 (1)** (1987), 107–141.

[MS] H. R. Morton and H. B. Short, *The 2-variable polynomial of cable knots*, Math. Proc. Camb. Philos. Soc. **101** (1987), 267–278.

[MC] ——— ” ——— and P. R. Cromwell, *Distinguishing mutants by knot polynomials*, Jour. of Knot Theory and its Ramifications **5(2)** (1996), 225–238.

[MT] ——— ” ——— and P. Traczyk, *The Jones polynomial of satellite links around mutants*, In ‘Braids’, ed. Joan S. Birman and Anatoly Libgober, Contemporary Mathematics **78**, Amer. Math. Soc. (1988), 587–592.

[MR] ——— ” ——— and H. J. Ryder, *Mutants and $SU(3)_q$ invariants*, preprint 1997.

[PV] M. Polyak and O. Viro, *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Notes **11** (1994) 445–454.

[P] J. Przytycki, *Search for different links with the same Jones’ type polynomials: Ideas from graph theory and statistical mechanics*, Panoramas of Mathematics, Banach Center Publications, Vol. **34**, Warszawa 1995.

[P2] ——— ” ———, *Equivalence of cables of mutants of knots*, Canad. J. Math., **41 3(2)**, 1989, 250–273.

[St] A. Stoimenow, *New knot and link invariants*, preprint, April 1997.

[St2] ——— ” ———, *Gauss sum invariants, Vassiliev invariants and braiding sequences*, preprint, May 1997.

[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Vo] P. Vogel, *Algebraic structures on modules of diagrams*, Université Paris VII preprint, June 1996.