ON THE NUMBER OF LINKS AND LINK POLYNOMIALS

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. We use a recent method of Sundberg–Thistlethwaite to improve Welsh's upper bound on the rate of growth of the number of links of given crossing number and to show that the number of polynomials of alternating links drops off exponentially in the crossing number compared to the number of such links.

Keywords: link polynomials, crossing number, tangle, enumeration

AMS subject classification: 57M25

1. Introduction and results

Let $L(n)$ denote the number of prime links L of crossing number $c(L)=n$ and $A(n)$ the number of prime alternating links of crossing number n.

After the appearance of the Jones polynomial and the crossing number theorem for alternating links [Ka, Mu, Th], Ernst and Sumners [ES] showed that $A(n)\geq(2^{n-2}-1)/3$ for $n\geq 4$.

This result was used in [W], where Welsh proved that

$$4\leq\liminf_{n\to\infty}\sqrt[n]{L(n)}\leq\limsup_{n\to\infty}\sqrt[n]{L(n)}\leq\frac{27}{2}$$

and

$$4\leq\liminf_{n\to\infty}\sqrt[n]{A(n)}\leq\limsup_{n\to\infty}\sqrt[n]{A(n)}\leq\frac{27}{4}.$$

The rate of growth of the number of alternating links was recently established by Sundberg and Thistlethwaite [ST] to be

$$\lambda=\lim_{n\to\infty}\sqrt[n]{A(n)}=\frac{\sqrt{21001}+101}{40}\approx 6.1479\dots \tag{1}$$

Here we use the Sundberg–Thistlethwaite method to improve Welsh's upper bound on the rate of growth of the number of arbitrary prime links. We have

Theorem 1

$$\limsup_{n\to\infty}\sqrt[n]{L(n)}\leq\frac{\sqrt{13681}+91}{20}\approx 10.39829\dots \tag{2}$$

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Then we address the following question:

Question 1 Is

$$\frac{\#\{\ I_L : c(L) = n\ \}}{\#\{\ L : c(L) = n\ \}} < C^n \tag{3}$$

for some $C < 1$, where I_L denotes some polynomial invariant of L?

Using elementary arguments based on the skein relations, one can give an estimate on the number of different polynomials admissible by links of given crossing number, but such an estimate would be too crude to be of any interest, and it is not clear (to me) how to enhance the method in this direction. Here we use the Sundberg–Thistlethwaite method to obtain the following further inequality.

Let

$$\tilde{A}(n) := \#\left(\{\ L : c(L) = n,\ L \text{ alternating }\ \} \big/ \text{mutation}\right)$$

be the number of mutation equivalence classes of alternating prime links of crossing number n. That is, links are considered equivalent if they are interconvertible by a sequence of mutations (for mutations, see [Ad, Co]).

Theorem 2

$$\limsup_{n\to\infty} \sqrt[n]{\tilde{A}(n)} \le \frac{109417 + \sqrt{24584873929}}{43334} \approx 6.1432742\ldots \tag{4}$$

(Note, that by the additivity of the crossing number of alternating links under connected sum, following from [Ka, Mu, Th], and the subexponential growth of the number of partitions, considering prime or composite alternating links has no effect on the rate of growth.)

This theorem, together with (1), gives a positive answer to question 1 for alternating links, at least for the polynomial invariants invariant under mutation. This includes the most important polynomial invariants like the skein (HOMFLY) [H] P and Kauffman [Ka2] F polynomial, and their well-known specializations, the Jones polynomial V [J], Alexander polynomial Δ [Al] and Brandt-Lickorish-Millett-Ho polynomial Q [BLM, Ho]. There are also further, more exotic, examples with this property, like the Kuperberg [Ku] polynomial.

However, beside for these polynomials, (3) holds (in the alternating case) also for any other mutation invariant, like 2-cables of the HOMFLY polynomial and the Kauffman polynomial [LL], arbitrary cables of the Alexander polynomial [BZ, proposition 8.23(b)] and the Jones polynomial [MTr], and also some geometric invariants like 2-fold branched covers, hyperbolic volume [Ru] etc. (see [Ad, LM]).

In the same way as in theorem 2 we can obtain an estimate for mutation equivalence classes of arbitrary prime links.

Theorem 3 Let

$$\tilde{L}(n) := \#\left(\{\ L : c(L) = n,\ L \text{ prime }\} \big/ \text{mutation}\right)$$

be the number of mutation equivalence classes of prime links of crossing number n. Then

$$\limsup_{n\to\infty} \sqrt[n]{\tilde{L}(n)} \le \frac{197167 + \sqrt{64100413969}}{43334} \approx 10.392477\ldots \tag{5}$$

It should be noted, that, as in [ST], all tangles and links are considered *unoriented*. Mutation, as originally defined by Conway (and explained in [LM]), does use the orientation of the tangles, but the orientation of the mutant is uniquely determined up to reversal of orientation of *all* components. Thus mutation can be understood also for unoriented tangles and links, when identifying them with the unordered set of all their oriented versions (up to reversal of orientation of all components).

Also some remark must be made on the orientation sensitivity of the invariants. Some of them, as hyperbolic volume or the Q polynomial, are independent on the orientation of *any* component of the link, and thus can be considered as invariants of unoriented links. Some others, like the Jones and Kauffman polynomial, depend only on the writhe of the diagram, and hence considering the polynomials of all oriented versions multiplies the number of polynomials only by a linear factor in the crossing number, which has no effect on the rate of growth of the number of such polynomials. However, for the other invariants, like the HOMFLY and Alexander polynomial, which behave unpredictably under reversal of orientation of a single component, the above convention must be made that we consider the polynomial of an unoriented link as the unordered set of the polynomials of all its oriented versions (up to reversal of orientation of all components of the link).

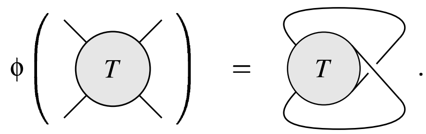
Since the number of possible orientations of a link depends on the number of its components, an attempt to modify the enumeration process to oriented links would require to refine it to enumerate links of a given number of components. This does not seem easily feasible at present, though.

2. Preliminaries for the proofs

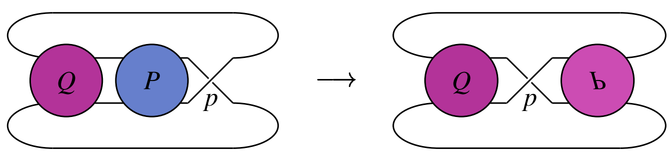
The proofs of our results heavily depend on the proof of (1). Therefore, it is appropriate to start with a (more or less detailed) overview of the Sundberg–Thistlethwaite method [ST].

To prove (1), Sundberg and Thistlethwaite start with the proof of the same property for the number $a(n)$ of alternating *tangle types* with *prime* alternating diagrams of n crossings. Here tangle types means tangles up to isotopy fixing the 4 endpoints of the tangle, and a tangle diagram is called prime if its plane curve is connected (as subset in $\mathbb{R}^2$), and if any circle meeting it in two points encloses a single arc (with no crossings).

Then they consider a map ϕ given by



The new crossing is chosen so that ϕ maps n-crossing alternating prime tangle diagrams to $(n+1)$-crossing prime alternating link diagrams. From [MT] we have that two alternating tangle/link diagrams represent the same tangle/link type, iff they are transformable by a sequence of *flypes*.



Using [Me], it follows then that ϕ descends to a map from tangle types with n-crossing alternating prime tangle diagrams to $(n+1)$-crossing prime alternating link types.

Clearly, ϕ is surjective. Then it is shown ([ST, §3]) that $\#\phi^{-1}(L) = O(c(L))$. Thus the rate of growth of the number of tangles and links is the same, and it suffices to prove (1) for tangles.

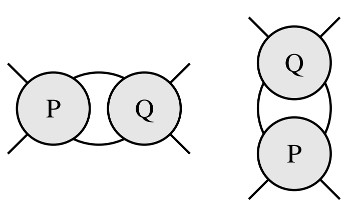
Henceforth in this section by "tangles" we mean "tangle types with alternating prime tangle diagrams".

Let

$$w(z) = \sum_{n=1}^{\infty} a(n) z^n$$

be the generating function of the number of tangle types with alternating prime tangle diagrams.

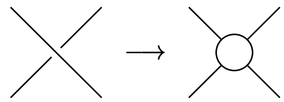
To prove (1) for tangles, the method of building diagrams of [Co] is adapted to tangles and slightly refined. In [Co], Conway defined the operations of (*horizontal* and *vertical*) *tangle sums*



(6)

An *algebraic* tangle diagram he called a diagram which can be obtained from the one-crossing diagram by iterated tangle sums. By cutting out algebraic tangle diagrams one can simplify any link diagram to one in which no two crossings are joined by two arcs. Such diagrams he called *basic polyhedra*.

To formalize and adapt Conway's idea to tangle diagrams, we define a *template* T to be a tangle diagram D with some of its crossings replaced by *slots*.



We say that T is *derived* from D. In particular, D can be considered as a template derived from itself. The operations of flypes and tangle sums extend naturally to templates. The function of slots is to enable the substitution of further templates. If a template T_2 is obtained from another template T_1 by slot substitutions, then we say that T_2 is *associated* to T_1.

We call a template T *algebraic* if it is derived from an algebraic tangle diagram. If T has no crossings, and one slot, it is the *trivial algebraic template*. We call a template *basic polyhedral*, if it has no crossings, at least 2 slots, is not the (horizontal or vertical) sum of two tangles (in a non-trivial way, with both P and Q in (6) having at least one crossing), and has no "Conway circle", that is, any circle meeting the template in 4 points encloses either the whole template, or a single slot.

Then, by a crucial observation, the generating function $q(z)$ of the number of basic polyhedral templates of given slot number is related to Tutte's enumeration of rooted c-nets [Tu], thereby obtaining the expression

$$q(z) = \frac{1}{2(z+2)^3}\left(\sqrt{(1-4z)^3} + (2z^2-10z-1)\right) - \frac{2}{1+z} - z + 2. \tag{7}$$

The link between algebraic and basic polyhedral templates is now the observation that any tangle diagram D can be obtained *in a unique way* by repeatedly inserting templates into slots of templates, such that any template substituted into (a slot of) an algebraic template is a basic polyhedral template.

Let a *Conway circle* be a circle meeting the tangle diagram in 4 points, such that it does not enclose the whole diagram, or a single crossing. Call a Conway circle C *non-algebraic*, if the tangle diagram it encloses is not the non-trivial horizontal or vertical sum of two tangle diagrams. Call C *maximal*, if it is not enclosed by another Conway circle.

To find the above described construction of D, decompose it along the maximal among its non-algebraic Conway circles C_i. Removing their interior, one obtains the "maximal" algebraic template to which D is associated. The tangle diagrams D_i in the interiors of C_i are associated to basic polyhedral templates T_i. To obtain T_i from D_i, decompose D_i along its maximal Conway circles.

In this hierarchy, a flype of a tangle diagram passes on to a flype in a unique algebraic template used to build up the diagram.

When denoting (we use here 'ξ' instead of 'ζ', as in [ST])

$$\alpha(z,\xi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} z^m \xi^n$$

the generating function of the number of flype-equivalence classes of algebraic templates with a given number m of crossings and n of slots, the above observation translates into the identity

$$w(z) = \alpha(z, q(w(z)))\,. \tag{8}$$

To determine α we define

$$\gamma = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n} z^m \xi^n$$

to be the generating function of the set of (i) flype-equivalence classes of algebraic templates (non-trivially) decomposable as horizontal tangle sum, and (ii) the trivial algebraic template. Since algebraic templates decomposable as horizontal tangle sum bijectively correspond to such decomposable as vertical tangle sum, we have

$$\alpha(z,\xi) = 2\gamma(z,\xi)+z-\xi,$$

z accounting for the single crossing tangle, and $-\xi$ for the twice counted trivial algebraic template.

Consider now an algebraic template τ of m crossings and n slots, which is a horizontal tangle sum. If one of the summands is the 1-crossing tangle, then by flypes we may assume that it is the rightmost summand. Since the other summands form an algebraic template of $(m-1)$ crossings and n slots, the contribution to γ of this type of τ's is $z\alpha = z(2\gamma+z-\xi)$. Otherwise, the (at least 2) summands of τ are all either the trivial algebraic template, or a vertical tangle sum, and thus also enumerated by γ. The contribution to γ of this type of τ's is $\frac{\gamma^2}{1-\gamma}$. Hence

$$\gamma = z(2\gamma+z-\xi) + \frac{\gamma^2}{1-\gamma} + \xi. \tag{9}$$

Now from (8) we have

$$w(z) = \alpha(z,q(w(z))) = 2\gamma(z,q(w(z)))+z-q(w(z)). \tag{10}$$

Thus

$$\gamma(z,q(w(z))) = \frac{1}{2}\left(w(z)-z+q(w(z))\right).$$

Putting this into the quadratic equation for γ we obtain from (9), we arrive at

$$F(z,w) = w(1+z)-w^2-(w+1)q(w)-z-\frac{2z^2}{1-z} = 0. \tag{11}$$

Then the authors obtain

$$r_0 := \frac{\sqrt{21001}-101}{270}$$

using (7) as the smaller norm solution of $F(z,\frac{1}{4}) = 0$, the number $^1\!/_4$ coming from the radius of convergence of q evident from (7).

The rest of the proof is analytical and consists in showing that r_0 is indeed the radius of convergence of w, thereby establishing (1), since $\lambda = {}^1\!/_{r_0}$.

3. The proofs

Proof of theorem 1. Our main aim is to modify the procedure of building tangles, which has consequences to the identities satisfied by the generating functions. Thus the main part of our modification of the Sundberg–Thistlethwaite proof will concern its combinatorial part. As for the analytical part, we will be brief, since only a few changes need to be made.

The idea to prove theorem 1 is to consider now prime non-alternating tangle diagrams modulo flypes.

The main difference between alternating and non-alternating tangle diagrams is that there is no longer a canonical choice of sign of every crossing imposed. Thus instead of one tangle diagram of one crossing, we have two.

The first place where this difference comes out is the relation between α and γ. (10) now turns into

$$\alpha = 2\gamma+2z-\xi.$$

Also, non-alternating tangle diagrams may not be of minimal crossing number, and our aim is to discard such diagrams which are reducible (in crossing number).

To do this, as a next step, it is appropriate to rewrite the recursion in (9) in the alternating case. For this we reconsider the case, when τ has a summand with just one crossing. Instead of considering this single crossing, we consider all such n crossings ($n \geq 1$), which can be flyped to the right. The remaining summands form a sequence S of tangles enumerated by γ. All combinations of $n \geq 1$ single crossings and sequences S are allowed, except the case $n = 1$ and S being the empty sequence (in this case the tangle is a single crossing). Thus the contribution to $g_{m,n}$ from such sums can be written as

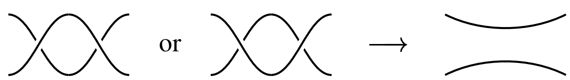
$$\frac{z}{1-z}\frac{1}{1-\gamma} - z\,.$$

It is straightforward to verify that the resulting identity for γ,

$$\left[\frac{z}{1-z}\frac{1}{1-\gamma} - z\right] + \frac{\gamma^2}{1-\gamma} + \xi = \gamma \tag{12}$$

is equivalent to (9). Equation (12), however, is more convenient when translating it to non-alternating tangles.

We consider only the simplest possible crossing number reduction given by the resolution of a trivial clasp.



(13)

The situation where diagrams with trivial clasps come about are those of τ having two summands, which are 1-crossing diagrams of different sign. Thus we can adapt (12) to non-alternating tangles, by replacing 'z' by '$2z$', *except* in the denominator:

$$\left[\frac{2z}{1-z}\frac{1}{1-\gamma} - 2z\right] + \frac{\gamma^2}{1-\gamma} + \xi = \gamma. \tag{14}$$

Then (11) turns into

$$F(z,w) = w(1+2z) - w^2 - (w+1)q(w) - 2z - \frac{4z^2}{1-z} = 0\,,$$

and the smaller norm solution r_0 of $F(z,\frac{1}{4}) = 0$ is found to be

$$r_0 = \frac{\sqrt{13681} - 91}{270}\,,$$

which is the inverse of the r.h.s. of (2).

Then, the analytic part of the proof needs to be verified. Luckily, the calculations in [ST] are kept very general, and the special value of r_0 is needed only very few times. We indicate these places, so as to point to what needs to be re-checked.

First, lemma 4.2.1 must be reproved. This happens in the same way (only in the first line replace 'x' by '$2x$'). Then proposition 4.3 needs to be checked. The calculation is similar; only the first occurrence of 'z' on the right of the equation array must be replaced by '$2z$' in lines 1 to 4, and '$-r_0$' by '$-2r_0$' in lines 5 and 6.

Then the formula for $V(z)$ on the last line of p. 350 needs to be made more explicit. There, the actual value of r_0 was inserted; the form for general r_0 is

$$V(z) = \frac{5}{4(1-z)}\left(\frac{3+5r_0}{5-5r_0} + z\right). \tag{15}$$

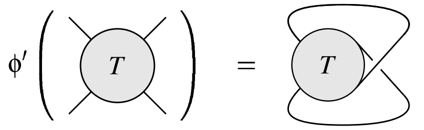
Now, our new value of r_0 needs to be substituted into (15), and $V(r_0)$ in [ST, (13)] needs to be recalculated and shown to be non-zero. However, $V(r_0) \neq 0$ is equivalent to

$$r_0 \neq 1 \pm \sqrt{\frac{8}{5}}, \tag{16}$$

which is trivially satisfied. The other condition $U(0,r_0) \neq 0$ is stated in [ST, (12)] to be equivalent to $r_0 \neq -\frac{32}{81}$, which is also trivially satisfied.

In [ST, (21)], c_1 needs to be recomputed, but as we are interested only in the rate of growth, we do not need to do this (we have not stated the new value for c_1 in the theorem).

Finally, we must take the step from tangle diagrams to link diagrams. For this just consider the two maps ϕ and ϕ', where the latter is given by



Clearly, any prime link diagram has a preimage under ϕ or ϕ'. Thus

$$\#\{\,\text{prime } n\text{-crossing link diagrams} \big/ \text{flypes}\,\} \leq 2\,\#\{\,\text{prime } (n-1)\text{-crossing tangle diagrams} \big/ \text{flypes}\,\}\,.$$

Thus the rate of growth of the number of prime link diagrams up to flypes is less than or equal to the growth rate of the number of prime tangle diagrams up to flypes, and the proof is complete. □

The other two theorems require a little more work. The difference in the proofs for both theorems is minimal, so it is more appropriate to carry them out in parallel. We will be briefer with the explanations, since they are very analogous to the previous proof.

Proof of theorem 2 and theorem 3. We need now to incorporate a certain type of mutation into the construction of tangles. We consider the special type of mutation which permutes the summands of an algebraic template (counted by γ), so as to have all summands, which are trivial algebraic templates, on the right of the others. This results in replacing $\frac{1}{1-\gamma}$ in (12) and (14) by

$$\frac{1}{1-\xi}\frac{1}{1-\gamma+\xi}$$

and $\frac{\gamma^2}{1-\gamma}$ by

$$\frac{1}{(1-\xi)(1-\gamma+\xi)}-1-\gamma.$$

Then we have

$$\left[\frac{z}{1-z}\frac{1}{(1-\xi)(1-\gamma+\xi)}-z\right]+\left[\frac{1}{(1-\xi)(1-\gamma+\xi)}-1-\gamma\right]+\xi=\gamma \tag{17}$$

for theorem 2 and

$$\left[\frac{2z}{1-z}\frac{1}{(1-\xi)(1-\gamma+\xi)}-2z\right]+\left[\frac{1}{(1-\xi)(1-\gamma+\xi)}-1-\gamma\right]+\xi=\gamma \tag{18}$$

for theorem 3.

Equation (17) is equivalent to

$$\frac{1}{1-z}+\left[1-\gamma+\gamma\xi-\xi^2\right](-z-1-2\gamma+\xi)=\frac{1}{1-z}+\left[1-\gamma(1-\xi)-\xi^2\right](-z-1-2\gamma+\xi)=0\,,$$

and (18) is equivalent to

$$\frac{1+z}{1-z}+\left[1-\gamma+\gamma\xi-\xi^2\right](-2z-1-2\gamma+\xi)=\frac{1+z}{1-z}+\left[1-\gamma(1-\xi)-\xi^2\right](-2z-1-2\gamma+\xi)=0\,.$$

Setting $\alpha=2\gamma+z-\xi=w$ for the alternating case (with $w(z)=\sum a_n z^n$, and the a_n here being just upper bounds on the number of n crossing tangles up to mutation), respectively $\alpha=2\gamma+2z-\xi=w$ for the non-alternating case, we get

$$\frac{1}{1-z}+\left[1-\gamma(1-\xi)-\xi^2\right](-w-1)=0 \tag{19}$$

for theorem 2 and with $\alpha=2\gamma+2z-\xi=w$

$$\frac{1+z}{1-z}+\left[1-\gamma(1-\xi)-\xi^2\right](-w-1)=0 \tag{20}$$

for theorem 3.

We have, substituting $\xi = q(w)$ and $2\gamma = w+\xi - z = w + q(w) - z$,

$$F(z,w) \;=\; (q-1)(w+1)\frac{2+q-w+z}{2} + \frac{1}{1-z} = 0$$

for theorem 2 and with $2\gamma = w+\xi-2z = w+q(w)-2z$

$$F(z,w) \;=\; (q-1)(w+1)\frac{2+q-w+2z}{2} + \frac{1+z}{1-z} = 0$$

for theorem 3 (with $q = q(w)$).

Setting $w = w(r_0) = \frac{1}{4}$ and $\xi = q(w(r_0)) = q(\frac{1}{4}) = \frac{1}{540}$, we obtain for theorem 2

$$\gamma = \frac{1}{2}(w+\xi-z) = \frac{68}{540} - \frac{z}{2},$$

and from (19)

$$\frac{1}{1-z} + \left[1 - \left(\frac{68}{540} - \frac{z}{2}\right)\frac{539}{540} - \frac{1}{540^2}\right]\cdot\left(-\frac{5}{4}\right) = 0\,.$$

Multiplying this by $1-z$, we obtain a quadratic equation for z. The absolutely smallest root is

$$r_0 = \frac{\sqrt{24584873929} - 109417}{291060}$$

for theorem 2. By a similar calculation from (20) we find

$$r_0 = \frac{\sqrt{64100413969} - 197167}{582120}$$

for theorem 3, the inverse of the r.h.s. of (5).

For the proof of lemma 4.2.1 we need to reorganize slightly the estimates. We have with $z \in [0, r_0]$, $w \in [0, 1/4]$, and $q = q(w) = q(w(z))$ for theorem 2

$$\frac{\partial F}{\partial z} = \frac{(q-1)(w+1)}{2} + \frac{\partial}{\partial z}\left[\frac{1}{1-z}\right] > \frac{-w-1}{2} + 1 > 0\,.$$

For theorem 3

$$\frac{\partial F}{\partial z} = (q-1)(w+1) + \frac{\partial}{\partial z}\left[\frac{1+z}{1-z}\right] > (-w-1) + 2 > 0\,.$$

Then, using $q(w) \le q(\frac{1}{4}) = \frac{1}{540}$ and $q'(w) \le q'(\frac{1}{4}) = \frac{167}{2025}$, and that $2+q-w+2z > 0$, we have for theorem 3

$$\begin{aligned}
\frac{\partial F}{\partial w} &= q'\cdot(w+1)\frac{2+q-w+2z}{2} + (q-1)\frac{2+q-w+2z}{2} + (q-1)(w+1)\frac{q'-1}{2}\\
&\le \frac{167}{2025}\cdot\frac{5}{4}\cdot\frac{2+\frac{1}{540}+2r_0}{2} + \left(\frac{1}{540}-1\right)\frac{2+\frac{1}{540}+2r_0}{2} + \frac{1}{2}\cdot\frac{5}{4}\\
&< 0\,.
\end{aligned}$$

For theorem 2, replace ‘$2z$’ by ‘z’ and ‘$2r_0$’ by ‘r_0’.

For proposition 4.3, to estimate $\dfrac{(q(w)-1)(q'(w)-1)}{2}$ for $|z| \le r_0$ and $|w| \le \frac{1}{4}$, replace the second last term $\dfrac{1}{2}$ on the second line of the above equation array by

$$\frac{1}{2}\left(1+\frac{1}{540}\right)\left(1+\frac{167}{2025}\right);$$

the expression remains negative for both theorems.

As (16) still holds in both cases, we are through. □

4. Problems

It is striking that accounting for one of the simplest types of crossing number reduction, the resolving a trivial clasp, has a sensible (≈ 10.4 vs. $2\lambda \approx 12.3$ or Welsh's bound 13.5) effect on the estimate of the rate of growth of non-alternating links. Thus one can expect to achieve significant further reduction by incorporating further simplifications. However, the difficulty is that the proof here (and in [ST]) heavily depends on the property of knowing the generating function w (whose coefficients estimate above the number of tangle types) by an equation of the form $F(z,w(z)) = 0$. Unfortunately, this does not seem to happen except in very special cases.

For the same reason we were forced to consider also a very restricted type of mutations. It is obvious that the reduction of the base we achieved by this special kind of mutations is very small, and one can ask in how far we can do better by considering more mutations. Unfortunately, this becomes technically difficult.

Flipping the tangles requires us to consider basic polyhedral templates up to various symmetries. One of them (vertical flip) is compatible with the identification with rooted c-nets (for which this symmetry may be non-trivial to take account of for itself), but the others aren't. If we ignore the symmetries of rooted c-nets and use q as in (7), we can use the full symmetry of the summands in the tangles τ. By a theorem of Cayley [Ca] the generating function $\tilde{\gamma}$ of "multisets" of primitive bi-graded objects (here the summands of the algebraic template graded by the number of crossings and slots) with generating function γ is

$$\tilde{\gamma}(z,w) = \exp\left(\sum_{i=1}^{\infty} \frac{\gamma(z^i,w^i)}{i}\right) \tag{21}$$

(see also [PR, §1, (1.11)] and [Wi, §3.17]), and then $\frac{1}{1-\gamma}$ in (12) and (14) can be replaced by $\tilde{\gamma}$ and $\frac{\gamma^2}{1-\gamma}$ by $\tilde{\gamma}-1-\gamma$. However, the above method is no longer applicable to such a function. For example, F then depends not only on two, but on infinitely many variables, since it involves all $w(z^i)$, $i \geq 1$.

Nonetheless, an upper estimate on possible improvement can be obtained by using instead of $\tilde{\gamma}$ just e^{γ} (we ignore the reduction of symmetries coming from identical summands). The same sort of calculation as above (but solving for r_0 only numerically) then shows that we cannot improve the bases using our method by more than ≈ 0.1, so that such attempts do not seem worthwhile. This should be understood as a reflection of the fact that mutations, although occurring to a larger extent at increasing crossing numbers, still are rather exceptional phenomena in general.

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