Mathematical entertainment and a combinatorial complexity problem

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1 Introduction

Rather than a paper this is a piece of mathematical entertainment (or, one might think, nonsens). However, in the background of this there are some interesting mathematical problems concerning complexity theory. The main aim of this note is to point to that problems and to make an interested reader think a while about them. There are surely some more profound results in this direction but, as the author is not sufficiently acquainted with the subject his observations lead to, as usual, he would be very grateful for each hint, suggestion, reference …

2 Motivation

In Bulgaria a car number plate contains always exactly 4 digits. Each such combination is possible except 0000. A number is called to be "lucky", if the sum of his first two digits is equal to the sum of the last two. It is not difficult to find out, that in each series of 9999 combinations there are exactly 669 "lucky" ones.

Once I came to ask myself whether it is not possible to generalize the situation by the following question.

Problem 2.1 *Given 4 digits, find a way of putting mathematical symbols between them, preserving their order and obtaining a correct equality.*

It is a very reasonable question to ask which operations we should allow (beside '='). There are many variations of this assumption, and the results are quite different. Here are 2 examples of what can happen.

1. Allow only binary operators, e. g. $\{+,-,*,/\}$[1]. This case is interesting to be dealt with by hand.
   But combinatorially trivial as far as there are only finitely many expressions which can be obtained out of the 4 digits using exclusively binary operations, so it is algorithmically solvable.

2. Allow also unary operators, e. g. $\sqrt{\ }$. Now things become much more interesting, since such operators can be arbitrarily iterated. However, we should not be too disrestrictive with respect to our choice of operators. If we allow an operator such as [ ] (integer part) to be used, then again everything is not very exciting, since for each digit $n \geq 1$ we have $\left[\sqrt{\sqrt{n}}\right] = 1$, and with 0's and 1's it is not hard to build a desired equality. And [ ] is idempotent (i. e., $[\,]^2 = [\,]$), so using it without $\sqrt{\ }$ we land in point 1.

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[1] Latter two are supposed to denote multiplication and division.

With these both counts in mind, the operator set $\{+,-,*,/,\sqrt{\ }\}$ turns out to be a good (because non-trivial) choice.

Of course, in such a case terms can become arbitrarily complicated and we can write arbitrarily messy equalities such as

$$\underbrace{\sqrt{\dots}}_{20\text{ times}}\sqrt{2} \;=\; \underbrace{\sqrt{\dots}}_{20\text{ times}}\sqrt{3+5-6}\ .$$

But it is somehow intuitively clear that beyond a certain point such a complexity is redundant and we can generate a much simpler equation out of the one above. To describe this more rigorously, first introduce a naive notion of complexity.

Definition 2.1 *A complexity function is a function*

$$\mathcal{K}\ :\ \Big\{\ \text{mathematical terms}\ \Big\}\ \longrightarrow\ \mathbf{N}_+\ .$$

What I wouldn't like to do here is to discuss which properties such a function should have. For the following statements one could e. g. have in mind the simple function

$$\mathcal{K}\Big(\ \text{term}\ \Big)\ :=\ \#\text{ of math. symbols in term}\ ,$$

which appears reasonable, but for describing the problem the concrete definition of $\mathcal{K}$ is practically irrelevant.

One would really be lucky if the statement below is true in general and a proof in a big generality would be an ingenious result, but the hope expressed appears to be in this fashion very strong. However, I would really like to know about a result in a special (but non-trivial) case.

Conjecture 2.1 *For each operator set $\mathcal{O}$ and each $n \in \mathbf{N}_+$ there is a number $\omega_{\mathcal{O},n} \in \mathbf{N}_+$ such that for each n-word w of digits there holds: If there exists a mathematically correct equality term built up of w by operators in $\mathcal{O}$, then there exists also such one of complexity $\leq \omega_{\mathcal{O},n}$.*

With such a powerful tool in hand we could in good spirits let the stupid but diligent computers do the rest of the work …

There is one (and perhaps a myriad more) problem(s) similar to problem 2.1, and in that case practically everything said by the present point can be translated without change.

Problem 2.2 *Given a digit word ω, an operator set $\mathcal{O}$ and a number $x \in \mathbf{R}$, find a mathematical term built from ω by operations in $\mathcal{O}$, whose value is x.*

3 About the list of terms

Before I include the complete list of solutions to problem 2.1, just some words about how it came about.

First I started with operators $+,-,*,/$, the binary root operator $\sqrt{\ }(a,b) := \sqrt[a]{b}$ and the unary root operator $\sqrt{\ }$ with up to 5-fold iteration. For the above mentioned reasons I generally excluded [ ] from the very beginning. The program I used was not designed to produce the most simple solution, so I got some funny suggestions!

It found a solution except in ≈ 300 cases. But, as I don't know anything about conjecture 2.1, I don't know whether there does not exist a solution with, say, 55 $\sqrt{\ }$'s for one or the other of these cases.

To deal with them, I looked for solutions with factorials $n!$, which reduced the non-solvable cases to 42. Then I saw myself forced to take some more special notations of number theory (I don't know in how much these notations are common in mathematics in general), such as the primorial (or prime factorial) $n\#$ and the multifactorials $n\underbrace{!\ldots!!}_{k\text{ copies}}$ up to $k \le 4$.

For those, who don't know them, their definitions are

$$n\# \;:=\; \prod_{\substack{p \text{ prime} \\ 1 \le p \le n}} p$$

and

$$n\underbrace{!\ldots!!}_{k} \;:=\; \prod_{i=0}^{\left\lfloor \frac{n-1}{k} \right\rfloor} (n - ik) \;\; .$$

Among the last 42 cases there were some rather tricky ones (such as 2667), so I didn't find a better solution than the one presented in the list. If someone has one, I am open to suggestions …

I also tried to vary the number of digits, but 3 digits are too restrictive (i. e., it is decisively up to conjecture 2.1 to prove non-solvability in many cases), and with 5 digits there always exists a solution with $\{ +, -, *, /, \sqrt{\ } \}$.

Now here is the final result.

4 The table

In this version of the paper the table has been omitted for size reasons. If you are interested, a version with the table is available at request from the author.