SOME MINIMAL DEGREE VASSILIEV INVARIANTS NOT REALIZABLE BY THE HOMFLY AND KAUFFMAN POLYNOMIAL

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. We collect some examples showing that some Vassiliev invariants are not obtainable from the HOMFLY and Kauffman polynomials in the real sense, namely, that they distinguish knots not distinguishable by the HOMFLY and/or Kauffman polynomial.

Keywords: Vassiliev invariants, HOMFLY polynomial, Kauffman polynomial, chirality.

1. Introduction

Briefly after the introduction of Vassiliev knot invariants [Va], it was discovered [BL], that a lot of such invariants can be ontained from knot polynomials [H, J, Ka], basically by taking some coefficient in a version of the polynomial where some variable was replaced by an exponential power series. In a subsequent series of papers, several authors (see [MR] and *loc. cit.* and [Da]) independently computed the number of Vassiliev invariants of given degree arising in this way from the HOMFLY and Kauffman polynomials. This, however, is *a priori*, only a lower bound for the number of Vassiliev invariants of given degree ontainable from the knot polynomials as it is not clear that no more can be generated in some other way. For example, one could take some long polynomial expression of higher degree Vassiliev invariants coming from the polynomial and little seems to be known about how to control from below the degree of the resulting Vassiliev invariant. Of course, one of the intrinsics of Vassiliev theory (see [BN]) is that Vassiliev invariants form a symmetric (i.e., commutative polynomial) algebra over a (primitive) subspace of them, but trying to prove general degree bounds by looking for the expression of a given Vassiliev invariant as a polynomial in some fixed primitive basis is practically impossible, because already the question of the number of primitive Vassiliev invariants of given degree is far from being solved and one of the most intriguing in the whole area (see [CD, Da2, Kn, Za]). See, however, [St, §3].

Given this subtlety, in this note we collect some examples showing that some Vassiliev invariants are not ontainable from the HOMFLY and Kauffman polynomials in the real sense, namely, that they distinguish knots not distinguishable by the HOMFLY and/or Kauffman polynomial. We hope these examples to be interesting at least because (to the best of my knowledge) they never appeared explicitely before in this context. All these examples are in a way related to problems of detecting chirality with the knot polynomials.

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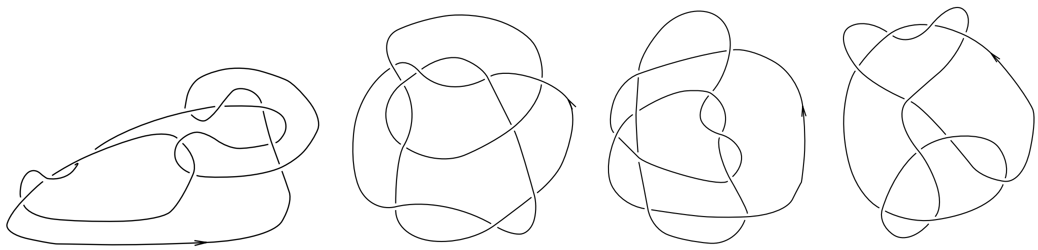
2. A degree 5 Vassiliev invariant not obtainable from the HOMFLY polynomial

It is known that the HOMFLY polynomial can be used to obtain all Vassiliev invariants of degree at most 4 and two of the three primitive Vassiliev invariants of degree 5.

There are 4 knots with 10 crossings with self-conjuagate HOMFLY polynomial but non-self-conjuagate Kauffman polynomial, see figure 1. Each one of them together with its obverse provide an example of two knots not distinguishable by the HOMFLY polynomial but distinguishable by a degree 5 Vassiliev invariant, showing that the third primitive Vassiliev invariant of degree 5 is not obtainable from the HOMFLY polynomial.



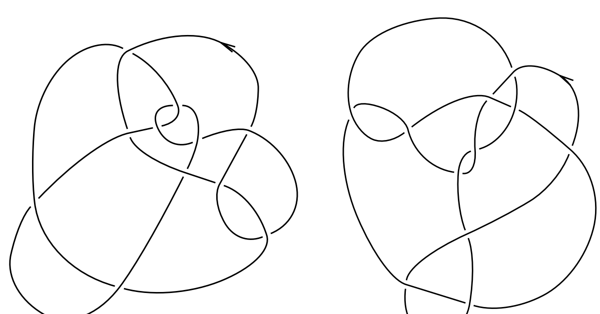
10_{48} 10_{91} 10_{104} $!10_{125}$

Figure 1: The 4 knots with 10 crossings with self-conjuagate HOMFLY polynomial but non-self-conjuagate Kauffman polynomial.

3. A degree 6 Vassiliev invariant not obtainable from the Kauffman polynomial

P. Johnson [Jo] used the tables [St2] to obtain all Vassiliev invariants of degree at most 5 from the Kauffman polynomial and also 4 of the 5 primitive Vassiliev invariants of degree 6, but not the fifth.

The example that the Kauffman polynomial does not contain a degree 6 Vassiliev invariant is given by the two 11 crossing alternating knots with equal Kauffman polynomials but different Conway polynomials, see figure 2. In Thistlethwaite's notation [HT] the knots are 11_{30} and $!11_{189}$. They Conway polynomials are $\nabla(11_{30}) = -2x^6 + x^4 - x^2 + 1$ and $\nabla(!11_{189}) = x^8 + 2x^6 + x^4 - x^2 + 1$, which differ at the coefficient of x^6, which is a degree 6 Vassiliev invariant [BN]. This shows that the Kauffman polynomial misses a degree 6 Vassiliev invariant, which is contained in the Conway polynomial and hence also in the HOMFLY polynomial. Thus, the HOMFLY and Kauffman polynomials together exhaust all Vassiliev invariants up to degree 6.



11_{30} $!11_{189}$

Figure 2: The pair of 11 crossing knots with equal Kauffman polynomials but different Conway polynomials.

Remark 3.1 The pair of knots 11_{30} and $!11_{189}$ was already known to Lickorish at the 1986 Santa Cruz conference [K, remark on p. 472 top], probably from the 11 crossing knot (polynomial) tabulation, but has been reconstructed in a more elucidative way by Kanenobu [K, theorem 5]. Kanenobu observed that these knots can be used to provide a counterexample to Lou Kauffman's hope [Ka, p. 428] that the Kauffman polynomial always detects chirality when the HOMFLY polynomial does so. Consider the knot $11_{30}\#11_{189}$. Then it has self-conjuagate Kauffman polynomial but non-self-conjuagate HOMFLY polynomial. Note also, that whenever the signature of a knot is not divisible by 4, it has negative determinant $\Delta(-1)$, which is clearly exhibited by the HOMFLY polynomial. So the HOMFLY polynomial detects chirality in all such cases even if it is self-conjuagate, as for 9_{42} (where the Kauffman polynomial is also self-conjuagate) and 10_{125}, although not being able to distinguish between the knot and its obverse. Therefore, care should be taken that the formulations "K has self-conjuagate polynomial" and "the polynomial does not detect chirality of K" are not quite equivalent!

4. A degree 7 Vassiliev invariant not obtainable from the Kauffman and HOMFLY polynomial

Soon after the discovery of the three knot polynomials capable of detecting chirality of knots, the knot 9_{42} became prominent by hiding its chirality to all of them. A less noted relative of 9_{42} is 10_{71}, which has a crossing more but is more interesting because it has additionally zero signature (contrarily to 9_{42}).

Morton and Short [MS] developed a program allowing to exhibit 9_{42}'s chirality by showing non-symmetry of the HOMFLY polynomial of its untwisted 2-cable. An attempt to apply the same procedure on 10_{71} meets some (resource requirement) difficulties, as 10_{71} has braid index 5 (see [J, appendix]). Hence, *a priori*, one has only a 10-braid representation of its untwisted 2-cable. This was revealed to de indeed minimal by the Morton-Williams-Franks inequality (see [J]) and the result of the calculation shown on figure 3. For its interpretation, if not self-explaining, we refer to [MS].

```
braid            : -2-3-1-287986576-4-5-3-4-4-5-3-421326576-4-5-3-48798-2-3-1-26576-4-5-3-4-6-7-5-68798

algebraic cross no:  0

       -9         -7         -5         -3         -1          1          3          5          7          9
    ____________________________________________________________

        1         -7         21        -39         53        -53         39        -21          7         -1     | -1
       10        -57        146       -245        330       -365        308       -181         64        -10     |  1
       50       -246        540       -759        919      -1059        990       -645        260        -50     |  3
      112       -566       1182      -1467       1506      -1723       1789      -1294        573       -112     |  5
      113       -703       1552      -1823       1576      -1742       2042      -1606        704       -113     |  7
       54       -468       1226      -1447       1069      -1135       1525      -1238        468        -54     |  9
       12       -165        562       -716        461       -474        730       -563        165        -12     | 11
        1        -29        144       -208        120       -121        209       -144         29         -1     | 13
                  -2         19        -32         17        -17         32        -19          2                | 15
                              1         -2          1         -1          2         -1                           | 17
```

Figure 3: (A part of) the output of the program of Morton and Short on the untwisted 2-cable of 10_{71}.

Now, non-symmetry of the HOMFLY polynomial can be shown in terms of Vassiliev invariants by evaluating at $i=\sqrt{-1}$ derivations of the polynomial given by the rows of the output, i. e. the coefficients of the z powers (in the convention of [MS]). Then the a-th v-derivation at $v=i$ of the coefficients of z^b is a Vassiliev invariant of degree at most $a+b$ (by arguments analogous to [BL]). The difference between the polynomial and its conjuagate comes out as a Vassiliev invariant of degree 7 (e.g., for $b=1$ and $a=6$), which is therefore not contained neither in the HOMFLY nor in the Kauffman polynomial, but in the HOMFLY polynomial of the untwisted 2-cable.

Remark 4.1 Przytycki conjectures that all Vassiliev invariants of degree at most 10 are contained in the HOMFLY and Kauffman polynomials and their untwisted 2-cables. A counterexample to this conjecture would require a pair of knots not distinguishable by all of them. Such pairs are mutants [Co, LL, P]. However, there is no Vassiliev invariant known distinguishing mutants below degree 11 (see [MC]). Are there more such pairs?

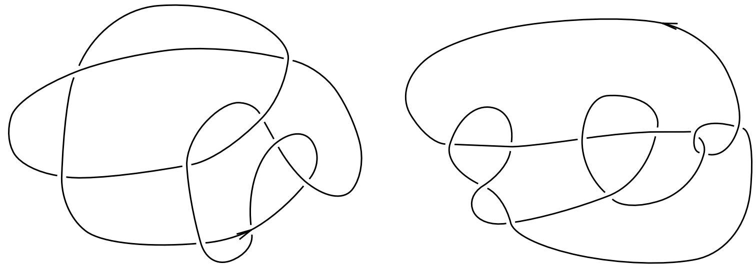


Figure 4: The 2 chiral knots with at most 10 crossings with self-conjuagate HOMFLY and Kauffman polynomials.

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