

ON CABLED KNOTS AND VASSILIEV INVARIANTS (NOT) CONTAINED IN KNOT POLYNOMIALS

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Abstract. It is known that the Brandt-Lickorish-Millett-Ho polynomial Q contains Casson's knot invariant. Whether there are (essentially) other Vassiliev knot invariants obtainable from Q is an open problem. We show that this is not so up to degree ≤ 9 . We also give the (apparently) first examples of knots not distinguished by 2-cable HOMFLY polynomials, which are not mutants. Our calculations provide evidence to a negative answer to the question whether Vassiliev knot invariants of degree ≤ 10 are determined by the HOMFLY and Kauffman polynomial and their 2-cables, and for the existence of algebras of such Vassiliev invariants not isomorphic to the algebras of their weight systems.

Keywords: Vassiliev invariant, Conway polynomial, Brandt-Lickorish-Millett-Ho polynomial, Kauffman polynomial, HOMFLY polynomial, cable link, mutant, Casson invariant, hyperbolic volume, braid index.

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1 Introduction and historical motivation

The standard definition of a Vassiliev invariant [BL, BN, BN2, BS, Va, Vo] of degree at most d is to be an invariant vanishing on $d + 1$ -singular knots. Vassiliev invariants are a class of knot invariants, which can be associated in many ways to polynomials. One such analogy is to think of singularity resolutions as a way to differentiate a knot invariant, and in this setting the Vassiliev invariants are (as polynomials) functions with a vanishing derivative. An extension of this idea is the approach of braiding sequences and braiding polynomials. It was initiated in a special case in [Tr] and later developed in [St]. It provides a method to study Vassiliev invariants via their polynomial behaviour on certain sequences of knots. This approach works directly on knots and so it is a counterpart to the classical approach of chord diagrams. Another relation to polynomials was conjectured by Lin and Wang [LW]. It asserts that (the values of) Vassiliev invariants are polynomially bounded in the crossing number of knots. The first substantial application of the approach of braiding sequences, in [St7], was to give a new proof of the statement conjectured by Lin and Wang. (It was proved previously by Bar-Natan [BN3], and also by Stanford [S].) In [St2] later this proof was extended to Vassiliev invariants of links of arbitrary number of components. Recently, a paper of Eisermann [Ei2] appeared, which mainly (apart from the application to $S^1 \times S^2$ it discusses) covers some initial part of our braiding sequence theory in [St, St7, St2]. This also explains that braiding sequences are a natural concept.

How to obtain Vassiliev invariants from the link polynomials (or polynomials of cables) was explained in [BL]. Since this procedure is *a priori* not exhaustive, it is not straightforward to prove that some Vassiliev invariant v is actually *not* obtainable from the link polynomials (or cables). The only way is to find knots not distinguished by the polynomials (or cables), but by v , as in [K4, St6]. Unfortunately, in particular for cables, coincidences of polynomials are rare, and this makes the task difficult. It was known from [LL] that mutants [Co] have equal 2-cable skein (or HOMFLY) P [F&, LM] and Kauffman F [Ka2] polynomials, and that they are not distinguished by Vassiliev invariants of degree ≤ 10 [Mr]. This led to the question whether all such invariants are determined by the skein and Kauffman polynomials and their 2-cables.

A different suggestive problem with Vassiliev invariants is to decide for a given invariant whether it is such or not. Usually, so far either a knot invariant is a Vassiliev invariant, or it can be excluded from being such by rather elementary means (as far as the Vassiliev invariant part of the argument goes). See e.g. [De, Tr, Bi, Ei]. However, in [St2] we introduced a certain type of invariants that satisfy similar polynomial behaviour, but in some weaker sense than Vassiliev invariants. We called such invariants extended Vassiliev invariants. As an extended Vassiliev invariant behaves polynomially on braiding sequences, it becomes difficult to recognize it as not of finite degree. The first class of examples of such invariants, given in [St2], are the derivatives of the Brandt-Lickorish-Millett-Ho polynomial Q [BLM, Ho] evaluated at -2 . Kanenobu had been studying the values $Q^{(k)}(-2)$ before. For knots $Q(-2) \equiv 1$, and by his result [K] we have $Q'(-2) = V''(1)$, with V the Jones polynomial [J], which is the Vassiliev invariant of degree 2. (A similar statement holds for links, which we do not discuss here, since in this case the further terms occurring are products of linking numbers, which are Vassiliev invariants of degree 1.) In [K2], Kanenobu found a formula (Theorem 1), expressing the Q polynomial of a rational (2-bridge) knot by its Jones polynomial. A consequence of this formula is that $Q^{(k)}(-2)$ on rational knots equals a polynomial of degree $\leq 2k$ in

the derivatives of $V(t)$ at $t = 1$ (where the n -th derivative is taken to be of degree n). Hence the restriction of $Q^{(k)}(-2)$ to rational knots is a Vassiliev invariant of degree $\leq 2k$.

To examine the finite degree property for $Q^{(k)}(-2)$ on arbitrary knots turns out to be rather difficult. Apparently they are not Vassiliev invariants (see §3.3). However, as also independently observed by Kanenobu, the previous easy arguments will not suffice to show this. The problem whether $Q^{(k)}(-2)$ are Vassiliev invariants (and of which degree, in the unlikely event that they are) remains open.

The actual origin for the considerations in [St2] was the search for a way to obtain Vassiliev invariants out of the Q polynomial. The polynomials V , P and F , and the Alexander/Conway polynomial ∇ [Al, Co] have been treated in [BL, BN], but apparently Q received little attention. Unfortunately, as the previous remarks already suggest, beyond degree 2 the question whether (or how) one can obtain Vassiliev invariants from Q seems rather difficult. Our aim here will be to provide a negative answer up to degree 9. This problem was investigated independently in a recent paper by Choi, Jeong, and Park [CJP].

This paper has two main parts. In the first part, comprising §3, we explain how to show that Q determines no low degree Vassiliev invariants, and settle degree up to 7. To that extent the problem is treated with a more detailed argument and mainly in its own right. Then, in the second part, comprising §4, for degrees 8 and 9 we are led to consider invariants of 2-cable knots and links. Here the application to the problem requires more of an explanation of our computation. This computation has also other noteworthy implications. In particular, it provides some evidence that not all Vassiliev knot invariants of degree ≤ 10 are determined by the HOMFLY and Kauffman polynomial and their 2-cables. It also turns up the (apparently) first examples of knots not distinguished by 2-cable HOMFLY polynomials, which are not mutants (because distinguished by 2-cable Kauffman polynomials and by hyperbolic volume), and determines the braid index of prime knots up to 12 crossings.

We should mention that some of our calculations are related to work of Meng [Me] and Lieberum [Li], and extend similar previous calculations in degree ≤ 6 due to Kanenobu [K4]. We will make some remarks that put these, and other, results into our context. For the computations various programs, written in C++ and MATHEMATICA™, were used, as well as some tools included in the program KnotScape [KS].

2 Notations and basic terminology

2.1 General notations

\mathbf{Z} , \mathbf{N} , \mathbf{N}_+ , \mathbf{Q} , \mathbf{R} and \mathbf{C} denote the integer, natural, positive natural, rational, real and complex numbers, respectively.

For a set S , the expressions $|S|$ and $\#S$ are equivalent and both denote the cardinality of S . In the sequel the symbol ' \subset ' denotes a not necessarily proper inclusion.

An expression containing an asterisk '*' *subscript* is meant to denote the union of all expressions in which the asterisk is replaced by all values that make sense, including omission. Contrarily, an asterisk as *super-script* is meant to denote the dual of a space.

Let $[Y]_{t^a} = [Y]_a$ be the coefficient of t^a in a polynomial $Y \in \mathbf{Z}[t^{\pm 1}]$. For $Y \neq 0$, let $C_Y = \{a \in \mathbf{Z} : [Y]_a \neq 0\}$ and

$$\min \deg Y = \min C_Y, \quad \text{and} \quad \max \deg Y = \max C_Y,$$

be the minimal and maximal degree of Y , respectively. Similarly one defines for $Y \in \mathbf{Z}[x_1, \dots, x_n]$ the coefficient $[Y]_X$ for some monomial X in the x_i , and $\min \deg_{x_i} Y$ etc.

The encoded notation for 1-variable polynomials we use is the one of [St3]: if the absolute term occurs between the minimal and maximal degrees, then it is bracketed, else the minimal degree is recorded in braces before the coefficient list. It is the same notation as the one of [Ad, appendix], or the one of [LM, appendix] for the m -coefficients of P , whichever of both is shorter.

2.2 Knots and knot diagrams

The *crossing number* $c(L)$ of a link L is the minimal number of crossings $c(D)$ of all diagrams D of L (cf. [Ka]). The *braid index* $b(L)$ of L is the minimal number of strands of a braid, whose closure is L (cf. [Mo, FW]).

The diagram on the right of figure 1 is called *connected sum* $A\#B$ of the diagrams A and B . If a diagram D can be represented as the connected sum of diagrams A and B , such that both A and B have at least one crossing, then D is called *composite*, else it is called *prime*. A knot K is *prime* if, whenever $A\#B$ is a composite diagram of K , one of A and B (but not both) represents an unknotted arc. Otherwise, if K is not the unknot, K is called *composite*. K is the connected sum $K = K_1\#K_2$ of K_1 and K_2 , with K_1 represented by A and K_2 by B .

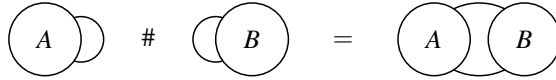


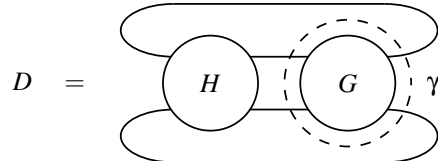
Figure 1

Prime knots are denoted according to [Ro, appendix] for up to 10 crossings and according to [HT] for ≥ 11 crossings. We number non-alternating knots after alternating ones. So for example $11_{216} = 11_{a216}$ and $11_{484} = 11_{n117}$.

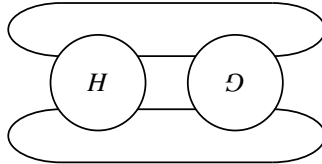
$!K$ is the *obverse* (mirror image) of K . If $K = !K$, then K is called *achiral*. For a knot invariant v , define the invariant $v!$ by $v!(K) = v(!K)$. v is called *symmetric* (resp. *antisymmetric*) if $v = v!$ (resp. $v = -v!$).

A knot K is called *rational* (*2-bridge*) if it has a diagram on which the one (planar) coordinate has exactly two local minima (or two local maxima) [Sh].

Given a knot diagram D and a closed curve γ intersecting D in exactly four points, γ defines a *tangle decomposition* of D .



A *mutation* of D is obtained by removing one of the tangles in some tangle decomposition of D and replacing it by a rotated version of it by 180° along the axis vertical to the projection plane, or horizontal or vertical in the projection plane. For example:



(To make the orientations compatible, possibly the orientation of either H or G must be altered.) Then γ is called the *Conway circle* for this mutation. If some knots $K_{1,2}$ have diagrams differing by a mutation, then $K_{1,2}$ are called *mutants* [Co]. We call K an (*iterated*) *mutant* of K' , if there are knots $K = K_1, K_2, \dots, K_n = K'$ with K_i and K_{i+1} being mutants. In the following, we will abuse the word ‘iterated’ when referring to mutants but assume it implicitly.

2.3 Link polynomials

As for knot invariants, our notation is also the usual one. $\Delta(t)$ denotes the Alexander [Al], $\nabla(z)$ the Conway [Co], $V(t)$ the Jones [J] (see also [Ka]), $P(l, m)$ the HOMFLY (skein) [LM, F&], $F(a, z)$ the Kauffman [Ka2], and $Q(z)$ the Brandt-Lickorish-Millett-Ho polynomial [BLM, Ho]. In our convention the skein and Kauffman polynomials are conjugate (that is, obtained by replacing a by a^{-1} in F and l by l^{-1} in P) to those in [LM] and [Ka2]. The local relations in this convention will be given below. Δ is normalized so that $\Delta(1) = 1$ and $\Delta(t^{-1}) = \Delta(t)$. For V and Q the conventions (also used here) are fairly standard.

The *skein (HOMFLY) polynomial* $P(l, m)$ is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1}P(\text{crossing}) + lP(\text{crossing}) = -mP(\text{crossing}). \quad (1)$$

We call the crossings in the first two fragments resp. *positive* and *negative*. The sum of the signs (± 1) of the crossings of a diagram D is called *writhe* of D and written $w(D)$. The writhe is invariant under simultaneous reversal of orientation of *all* components of the diagram, so is in particular well-defined for unoriented *knot* diagrams.

The *Conway polynomial* ∇ [Co], given by $\nabla(z) = P(\sqrt{-1}, \sqrt{-1}z)$, is well-known to be equivalent to the (1-variable) *Alexander polynomial* Δ by a variable substitution: $\Delta(t) = \nabla(t^{1/2} - t^{-1/2})$. Another well-known property of ∇ is that for any link L we have $[\nabla_L(z)]_{z^i} = 0$, if i has the same parity as the number $n(L)$ of components of L , and that $z^{n(L)-1} \mid \nabla_L(z)$. For a knot K we always have $[\nabla_K(z)]_{z^0} = 1$.

For the *Kauffman polynomial* F in our convention we have the relation $F(D)(a, z) = a^{w(D)}\Lambda(D)(a, z)$, where $w(D)$ is the writhe of D , and $\Lambda(D)$ is the writhe-unnormalized version of F . Λ is given in our convention by the properties

$$\begin{aligned} \Lambda(\text{crossing}) + \Lambda(\text{crossing}) &= z(\Lambda(\text{crossing}) + \Lambda(\text{crossing})), \\ \Lambda(\text{crossing}) &= a^{-1}\Lambda(\text{crossing}); \quad \Lambda(\text{crossing}) = a\Lambda(\text{crossing}), \\ \Lambda(\text{circle}) &= 1. \end{aligned}$$

Thus the positive (right-hand) trefoil has $\min \deg_a F = 2$.

The *Brandt-Lickorish-Millett-Ho polynomial* is given by $Q(z) = F(1, z)$, and the *Jones polynomial* by

$$V(t) = F(-t^{3/4}, t^{1/4} + t^{-1/4}) = P(-\sqrt{-1}t, \sqrt{-1}(t^{-1/2} - t^{1/2})).$$

(See [Ka2, §III] and [LM].)

Q and ∇ (and hence Δ) are symmetric *knot* invariants, i.e. coincide on K and $!K$ for any knot K . (Q is symmetric also for links, while ∇ is symmetric or antisymmetric depending on the parity of the number of components.) V , P and F differ on mirror images under conjugation of a variable:

$$V!(t) = V(t^{-1}), \quad (2)$$

$$F!(a, z) = F(a^{-1}, z), \quad (3)$$

$$P!(l, m) = P(l^{-1}, m). \quad (4)$$

All polynomials $X \in \{F, P, Q, V, \Delta\}$ are multiplicative under connected sum: $X(K_1 \# K_2) = X(K_1)X(K_2)$.

By $\text{vol}(L)$ we denote the (finite) volume of the (unique if it exists) hyperbolic structure on the complement $S^3 \setminus L$ of a link L in S^3 (that is, a representation $S^3 \setminus L = H^3/\Gamma$, where H^3 is the 3-dimensional hyperbolic space, and Γ is a properly discontinuously acting discrete group of isometries of H^3). We write $\text{vol}(L) = 0$ if $S^3 \setminus L$ has no hyperbolic structure.

3 Vassiliev invariants

3.1 Generalities

Consider the linear space \mathcal{V} , (freely) generated by all the (isotopy classes of) knot embeddings. Let \mathcal{V}^d be the space of singular knots with exactly d double points \times (up to isotopy). \mathcal{V}^d can be identified with a linear subspace of \mathcal{V} by resolving the singularities into the difference of an over- and an undercrossing via the rule

$$\begin{array}{c} \diagup \diagdown \\ \times \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}, \quad (5)$$

where all the rest of the knot projections are assumed to be equal. This yields a filtration of \mathcal{V}

$$\mathcal{V} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \mathcal{V}^2 \supset \mathcal{V}^3 \supset \dots \quad (6)$$

There exists a combinatorial description of the graded vector space,

$$\bigoplus_{d=0}^{\infty} \left(\mathcal{V}^d / \mathcal{V}^{d+1} \right) \quad (7)$$

associated to this filtration, namely

$$\mathcal{A}_d := \mathcal{V}^d / \mathcal{V}^{d+1} \simeq \mathcal{L}in \{ \text{chord diagrams of degree } d \} / \begin{array}{l} 4T \text{ relation} \\ FI \text{ relation} \end{array}, \quad (8)$$

where $\mathcal{L}in$ denotes linear span, and the *chord diagrams (CDs)* are objects like this (an oriented circle with finitely many dashed chords in it up to isotopy)



and are graded by the number of chords. The *4T (4 term) relations* have the form

$$\begin{array}{c} \text{CD}_1 \\ \text{---} \\ \text{CD}_2 \end{array} = - \begin{array}{c} \text{CD}_3 \\ \text{---} \\ \text{CD}_4 \end{array},$$

and the *FI (framing independence) relation* requires, that each CD with an *isolated chord* (i. e., a chord not crossed by any other one) is zero.

The map which yields the isomorphism (8) is a simple way how to assign a CD D_K to a singular knot K . Connect in the parameter space of K (which is an oriented S^1) pairs of points with the same image by a chord. When adding arrows for the crossings of K oriented from the preimage of the undercrossing to the preimage of the overcrossing, we obtain a (singular) *Gauß diagram* (see [St5, PV]).

The *connected sum* of chord diagrams is defined by $D_{K_1} \# D_{K_2} = D_{K_1 \# K_2}$ (well up to the *4T* relation).

We define a knot invariant v to be a *Vassiliev invariant of degree $\leq d$* if, when extended to singular knots via

$$v \left(\begin{array}{c} \diagup \diagdown \\ \times \\ \diagdown \diagup \end{array} \right) = v \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) - v \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right),$$

vanishes on $(d+1)$ -singular knots. The *degree* $\deg v$ of v is (suggestively) the smallest integer d such that v is of degree $\leq d$. Several properties and constructions of Vassiliev (finite degree) invariants were known

from [BL, BN]. In particular, introducing \mathcal{V}_d to be the linear space of Vassiliev invariants of degree $\leq d$, the space $\mathcal{V}_d/\mathcal{V}_{d-1}$ is isomorphic to the dual \mathcal{A}_d^* of the linear space \mathcal{A}_d of chord diagrams of d chords modulo the $4T$ relation. Elements in \mathcal{A}_d^* are called *weight systems* (of degree d). $v \in \mathcal{V}_d$ gives rise to a weight system $W_v \in \mathcal{A}_d^*$ by evaluating it on a d -singular knot representing the chord diagram,

$$W_v(D_K) := v(K).$$

The bijectivity of this assignment is dual to the isomorphism (8), and is established using a *universal Vassiliev invariant*, such as the *Kontsevich-integral* Z [Ko]. The application $W_v \circ Z$ of the weight system of $v \in \mathcal{V}_d$ on the Kontsevich-integral gives back v modulo lower degree invariants:

$$v(K) \equiv (W_v \circ Z)(K) = W_v(Z(K)) \pmod{\mathcal{V}_{d-1}}.$$

If even $v = W_v \circ Z$, we call v *canonical* (see [BG]).

Vassiliev invariants are easily seen to form an algebra with usual addition and multiplication, and the structure of this algebra was known to be the free symmetric (polynomial) graded algebra generated by *primitive* Vassiliev invariants. Such invariants v are given by the additional property that $v(K_1 \# K_2) = v(K_1) + v(K_2)$ for any knots $K_{1,2}$.

3.2 Deterministic sets for Vassiliev invariants

Since the space of Vassiliev invariants of given degree is finite-dimensional, there exist finite sets \mathcal{K}_d of knots, the values on which determine uniquely a Vassiliev invariant of degree $\leq d$. Equivalently, we say

Definition 3.1 A set \mathcal{K}_d of knots is *d-deterministic*, if any Vassiliev invariant of degree at most d vanishing on \mathcal{K}_d vanishes identically. It is called *d-primitive deterministic*, if this property holds for primitive Vassiliev invariants of degree at most d .

In practice, it is desirable to choose a d -deterministic set as small as possible. The minimal size is clearly $\dim \mathcal{V}_d$, and many such sets of this cardinality exist, but no one knows how to find any of them except by computation for a few small values of d . Thus we may try to find larger sets which, however, is provable to be d -deterministic. This problem has been considered (including for links) in several previous papers of mine (see [St7]), and estimates on the crossing number of knots in one particular d -deterministic set \mathcal{K}_d were given. The estimates, however, are not optimal. For our subsequent purposes, we will derive a more efficient estimate for knots. It is formulated in the following lemma, needed to make the later arguments more rigorous.

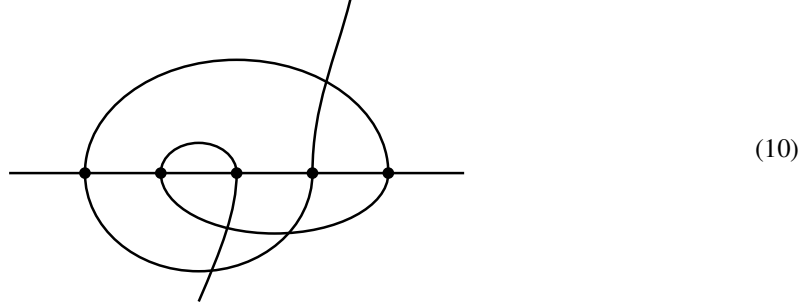
Lemma 3.1 For any $d > 0$, the set of knots with (prime) diagrams of at most $d + 1 + \frac{d(d-2)}{4}$ crossings is d -(primitive) deterministic.

Remark 3.1 Note that knots with prime diagrams may well be composite, and so we do not make any claim as to the primeness of the knots represented by our diagrams.

Proof. We use the result of [CD] that chord diagrams modulo the $4T$ -relation and composite chord diagrams are generated by such with a *special chord* (that is, a chord intersecting all the others). Note (as in [CD]) that such a chord diagram is described by a permutation of the endpoints of the non-special chords. Thus it suffices to consider chord diagrams with a special chord or connected sums of such diagrams. To realize a prime chord diagram with d chords, including a special one, by a singular knot diagram, put $d - 1$ singular crossings on a straight line.



First assume d is even. The other strand must pass through these singular crossings in some (arbitrary) permuted order. Its part above and below the line in (9) consists of $\frac{d}{2} - 1$ arcs joining two singular crossings, and one arc connected to the remaining singular crossing with a “loose end”.



Clearly any two of these $\frac{d}{2}$ arcs can be made to have at most one intersection. Thus the strand can be made to have at most $2 \binom{d/2}{2}$ self-intersections. There remain the d singular crossings. Call the one of the special chord the *special singular crossing*. One additional (non-singular) crossing is needed for the second (generally self-intersecting) strand in (10) to exit the loop made up of the first strand between the two passes of the special singular crossing.

In case d is odd, one side of (9) contains $\frac{d-1}{2}$ arcs joining two singular crossings, and the other $\frac{d-3}{2}$ such arcs, and two arcs with a loose end. Then one has at most

$$d+1 + \binom{(d-1)/2}{2} + \binom{(d-3)/2}{2} + 2 \cdot \frac{d-3}{2} = \frac{(d-1)^2}{4} + d \leq d+1 + \frac{d(d-2)}{4}$$

crossings.

Now with $f(d) := d+1 + \frac{d(d-2)}{4}$, we have

$$f(d) \geq \sum_{i=1}^k f(d_i),$$

when $d_i \geq 2$ and $\sum_{i=1}^k d_i = d$. This shows the assertion of the lemma for arbitrary Vassiliev invariants. Now considering primitive Vassiliev invariants, we can restrict ourselves to chord diagrams which are not connected sums. Thus, we must argue why the (singular) knot diagrams representing prime chord diagrams with a special chord are prime. Now the special chord remains special not only in the chord diagram, but also in the whole (singular) Gauß diagram. Also, it is easy to see that each arrow of a non-singular crossing intersects a chord of a singular crossing. Then the intersection graph of this Gauß diagram is connected, which (see [St5]) is equivalent to the knot diagram being prime. \square

Corollary 3.1 (see also [K4]) A primitive Vassiliev invariant of degree ≤ 4 is determined by its values on rational knots.

Proof. Knots with prime ≤ 7 crossing diagrams are all rational. \square

3.3 Vassiliev invariants derived from the polynomials

From [BN, BL] we know that the Conway, Jones and Kauffman polynomials give rise to Vassiliev invariants. We recall that there is a relation between the Conway Vassiliev invariants $\nabla_i = [\nabla]_{z^i}$ and the Kauffman

Vassiliev invariants (see [K3])

$$F_{i,j}(K) := \sqrt{-1}^{i+j} \frac{d^j}{da^j} \Big|_{a=\sqrt{-1}} [F(K)]_{z^i}. \quad (11)$$

This is a Vassiliev invariant of degree $\leq i+j$ by [BL]. (Since $F_{i,0} \equiv \delta_{i,0}$ is constantly 1 or 0, we can assume $j > 0$.)

We have the identity $F_{1,1} = -2\nabla_2$ coming from the uniqueness of the (symmetric) Vassiliev invariant of degree 2. For higher degree, the evident problem is that the dimension of the space of Vassiliev invariants grows rapidly. The only further relation to the Conway Vassiliev invariants is (see [K3, p. 422])

$$\frac{F_{2,1} + F_{2,2}}{2} - 6F_{3,1} = \nabla_2 - 7\nabla_2^2 + 18\nabla_4. \quad (12)$$

With lemma 3.1 in hand, the verification of such identities (at least in not too high degree) is straightforward.

For $i > 4$, ∇_i cannot be expected to be related to the $F_{i',j'}$. Indeed, ∇_6 is not contained in F , as shown in [K4] and [St6]. That is, there are two distinct knots K_1 and K_2 with $F(K_1) = F(K_2)$ but $\nabla_6(K_1) \neq \nabla_6(K_2)$. For instance, K_1 and K_2 can be taken to be the two 11 crossing knots 11_{30} and $!11_{189}$ with equal Kauffman polynomial, but different Conway polynomial, pointed out by Lickorish [L]. As observed by Kanenobu, for the higher ∇_i the same property then follows by taking the connected sum of the $K_{1,2}$ with trefoils.

The Jones polynomial gives rise to a series of Vassiliev invariants by its values $V^{(n)}(1)$. The skein polynomial P yields Vassiliev invariants in the same way as F : for a link L ,

$$P_{i,j}(L) := \sqrt{-1}^{i+j} \frac{d^j}{dl^j} \Big|_{l=\sqrt{-1}} [P(L)]_{m^i}, \quad (13)$$

is a Vassiliev invariant of degree $\leq i+j$. However, here rather than $j > 0$ we must pose $j \geq 0$ and i of the opposite parity to the number of components $n(L)$ of L , and $i \geq 1 - n(L)$. (Remark that for $j = 0$ we obtain, up to sign, the ∇_i .)

As for Q , the results of Kanenobu, explained in §1, suggest that we consider the values $Q^{(k)}(-2)$ for $k \geq 2$. Sadly here we are less fortunate, and the following is easy to see.

Proposition 3.1 $Q''(-2)$ is not a (global) Vassiliev knot invariant of degree ≤ 4 .

Proof. Assume $v = Q''(-2)$ is a Vassiliev invariant of degree ≤ 4 . Using $Q(-2) \equiv 1$, one can correct v by a multiple of $Q'(-2)^2$ to a Vassiliev invariant \bar{v} that is additive under connected sum, and so primitive. By corollary 3.1, we have that \bar{v} is determined by its values on rational knots, and Kanenobu's formula [K2] shows that on rational knots \bar{v} can be expressed using $V^{(n)}(1)$. Since this expression is also a Vassiliev invariant of degree ≤ 4 , it would extend to all knots. Since also $Q'(-2)$ can be expressed from V using [K], we obtain that v is determined by V (on all knots). Then any pair of knots with equal (or conjugate) V would have equal $Q''(-2)$. But the pair 5_1 and 10_{132} shows that this is not the case. We quote their V and Q polynomials from [St3] using encoded notation:

$$\begin{aligned} V(5_1) &= V(10_{132}) = \{2\} 1 0 1 - 1 1 - 1, \\ Q(5_1) &= [5] - 2 - 6 2 2, \quad Q(10_{132}) = [5] - 18 - 14 38 20 - 24 - 12 4 2. \end{aligned}$$

We thus obtain a contradiction. □

The fact that $Q''(-2)$ is not of degree ≤ 4 was observed by Kanenobu by a similar reasoning. Of course, this argument can only work in low degree, but a more general argument for arbitrary degree and arbitrary derivative is not obvious.

3.4 Braiding sequences

The approach of braiding sequences gives another motivation for the non-triviality of the finite degree property question on the derivatives of the Brandt-Lickorish-Millett-Ho polynomial evaluated at $z = -2$. It also suggests similar phenomena for the evaluations at $z = 2$.

Definition 3.2 ([St]) For some odd $k \in \mathbf{Z}$, a (parallel) k -braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k . A braiding sequence $\mathcal{B}_{D,P}$ (associated to a numbered set P of crossings in a diagram D ; all crossings by default) is a family of diagrams, parametrized by $n = |P|$ odd numbers x_1, \dots, x_n , each one indicating that at the crossing numbered as i an x_i -braiding is done.

Figure 2 shows the parallel -3 -braiding and the antiparallel one. The theory for antiparallel braidings is almost equivalent, but if a reader feels more convenient, he may assume that only parallel braidings are done.

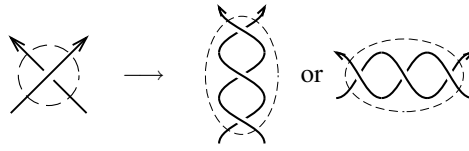


Figure 2: Two ways to do a -3 -braiding at a crossing.

Definition 3.3 If for a knot invariant v and any braiding sequence $\mathcal{B}_{D,P}$ with $|P| = n$, the map

$$\mathcal{P}_{D,P} : (x_1, \dots, x_n) \mapsto v(D(x_1, \dots, x_n))$$

is a polynomial, we call v a *braiding polynomial invariant*. We call $\mathcal{P}_{D,P}$ the braiding polynomial of v on $\mathcal{B}_{D,P}$.

Theorem 3.1 ([St]) A knot invariant v is a Vassiliev invariant of degree $\deg v \leq d$ if and only if it is a braiding polynomial invariant, and all its braiding polynomials have degree $\deg \mathcal{P}_{D,P} \leq d$ for any $\mathcal{B}_{D,P}$. Herein degree is counted in *all* variables altogether, that is, with respect to

$$\deg \prod_{i=1}^n x_i^{l_i} = \sum_{i=1}^n l_i.$$

Definition 3.4 A knot invariant is an *extended Vassiliev invariant of degree $\leq d$* , if it is a braiding polynomial invariant, and for any $\mathcal{B}_{D,P}$ its braiding polynomial has degree $\deg_{x_j} \mathcal{P}_{D,P} \leq d$ in *any single* $x_j \in \{x_1, \dots, x_n\}$ with $|P| = n$, that is, with respect to

$$\deg_{x_j} \prod_{i=1}^n x_i^{l_i} = l_j.$$

Example 3.1 The determinant $\Delta(-1) = V(-1)$ is an extended Vassiliev invariant of degree 1, *if one restricts oneself to braiding sequences of antiparallel braidings only*. The squared determinant $\Delta(-1)^2 = Q(2)$ is an extended Vassiliev invariant of degree 2 (also for parallel braidings).

Theorem 3.2 ([St7]) The invariants $Q^{(k)}(-2)$ are extended Vassiliev invariants of degree $\leq 2k$. The invariants $Q^{(k)}(2)$ are extended Vassiliev invariants of degree $\leq 2k + 2$.

This leads to a suggestive, but not very easy to answer, question:

Question 3.1 Are $Q^{(k)}(\pm 2)$, or polynomial expressions thereof, (ordinary) Vassiliev invariants?

Definition 3.5 A knot invariant v is called *polynomially bounded of degree $\leq d$* if there is a constant $C > 0$ such that $|v(K)| \leq Cc(K)^d$ for any knot K . (Here $c(K)$ is the crossing number of §2.2.)

The following is the polynomial growth conjecture of [LW], proved in [BN3, S] for knots, and in [St7] for links.

Theorem 3.3 Vassiliev invariants of degree $\leq d$ are polynomially bounded of degree $\leq d$.

Since the determinant is not a polynomially bounded invariant, it is not a Vassiliev invariant, and thus extended Vassiliev invariants are a non-trivial notion.

We prove here now the following straightforward but useful criterion

Theorem 3.4 A knot invariant is a Vassiliev invariant (of degree $\leq d$) if and only if it is a polynomially bounded (of degree $\leq d$) and a braiding polynomial invariant.

Proof. The ‘only if’ part follows from our previous results. Now assume v is braiding polynomial. We also assume that $|v(K)| < Cc(K)^d$ for all K , and wish to conclude that $\deg \mathcal{P}_{D,P} \leq d$ for all $\mathcal{B}_{D,P}$. Now assume that for some $\mathcal{B}_{D,P}$ we have $d_{D,P} := \deg \mathcal{P}_{D,P} > d$. Let $Q_{D,P} = [\mathcal{P}_{D,P}]_{d_{D,P}} \neq 0$ be the homogeneous degree- $d_{D,P}$ -part of $\mathcal{P}_{D,P}$. There are odd (say, even positive) numbers k_1, \dots, k_n with $Q_{D,P}(k_1, \dots, k_n) \neq 0$. Then consider the diagrams $D_p := D(k_1 p, \dots, k_n p)$ for odd $p \rightarrow \infty$. (By proper choice of sign of k_i one can, if one likes, achieve that D_p is alternating.) The map $p \mapsto v(D_p)$ is a polynomial in p of degree $d_{D,P} > d$, and the crossing number of D_p is linearly bounded in p , so that for the knots K_p represented by D_p we have $|v(K_p)|$ growing faster than $c(K_p)^d$, a contradiction. \square

Corollary 3.2 The invariants $Q^{(k)}(\pm 2)$, or polynomial expressions thereof, are Vassiliev invariants (of some degree) if and only if they are polynomially bounded (of that degree). \square

3.5 Vassiliev invariants not obtained from the Q polynomial

3.5.1 An example for degree 3 and 4

The first purpose of our investigation is to show the following statement. It explains the method of computation that is later extended to higher degrees.

Proposition 3.2 Q does not contain a Vassiliev knot invariant of degree 3 or 4 that is substantial (i.e. not a linear combination of composite and lower degree ones).

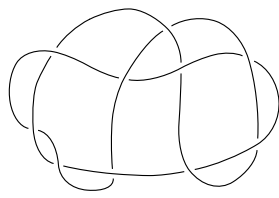
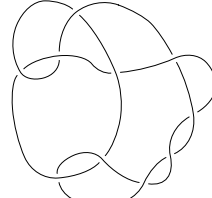
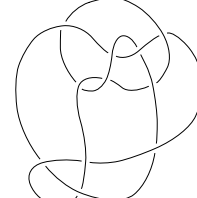
Proof. First we recall that by [St4] it does not make sense to look for a Vassiliev invariant of degree 3 (or any other odd degree), as Q is a symmetric invariant. (Even non-mutually obverse examples with the same Brandt-Lickorish-Millett-Ho polynomial and different Vassiliev invariants of degree 3 are easily found, e.g., 9_{12} and 10_{156} .)

As well-known (see e.g. [BN, KM]), the linear space of primitive Vassiliev invariants of degree 4 (modulo degree ≤ 3) is 2-dimensional and generated by the projections on it of the degree 4 Vassiliev invariants c_4 coming from the Conway/Alexander polynomial and v_4 coming from the Jones polynomial.

As Q contains v_2 and hence v_2^2 , we may waive on primitivity and adjust c_4 and v_4 in which ever way we like, only taking care, that v_4 has no part in degree 3, that is, is symmetric. (Clearly c_4 is so, in whichever way we choose it, as is Δ .) Thus, consider

$$c_4 := \frac{1}{24} \Delta^{(4)}(1) \quad \text{and} \quad v_4 := \frac{1}{12} \left(V^{(4)}(1) + 6V^{(3)}(1) \right)$$

$$Q = [5] - 6 - 20 28 30 - 30 - 26 8 10 2$$

10₁₉10₃₆11₄₅₄

$$V = -1 2 - 3 6 [-7] 8 - 8 7 - 5 3 - 1 \quad V = 1 [-2] 4 - 6 8 - 8 8 - 6 4 - 3 1 \quad V = 1 - 3 5 - 7 8 - 9 8 - 5 4 [-1]$$

$$\Delta = 2 - 7 11 [-11] 11 - 7 2$$

$$\Delta = -3 13 [-19] 13 - 3$$

$$\Delta = 1 - 5 12 [-15] 12 - 5 1$$

Figure 3: Three knots with the same Q polynomial, showing that it cannot contain any interesting Vassiliev invariant of degree 4, and their V and Δ polynomials (all recorded as in [St3]).

(for which $v_4(!K) = v_4(K)$ is straightforwardly checked).

If an invariant of the kind $av_4 + bc_4$ for some $a, b \in \mathbf{R}$ is contained in Q , and for two knots K_1, K_2 we have $Q(K_1) = Q(K_2)$, then

$$b(c_4(K_1) - c_4(K_2)) + a(v_4(K_1) - v_4(K_2)) = 0. \quad (14)$$

Thus, to show that it is not the case for any $(a, b) \neq (0, 0)$, it suffices to find a triplication of Q , that is, knots K_1, K_2 and K_3 with $Q(K_1) = Q(K_2) = Q(K_3)$, such that

$$\det \begin{pmatrix} c_4(K_1) - c_4(K_2) & c_4(K_1) - c_4(K_3) \\ v_4(K_1) - v_4(K_2) & v_4(K_1) - v_4(K_3) \end{pmatrix} \neq 0. \quad (15)$$

Such an example is the triple $10_{19}, 10_{36}$ and 11_{454} . (This is one of the two triplings of Q I found in Hoste-Thistlethwaite's tables [HT] of ≤ 11 crossing prime knots.) We let the reader verify (15), just recording their polynomials (see figure 3). \square

Thus, unfortunately, there seems no easy way, e.g., to show via Vassiliev invariants (as it works for V , see [St5, corollary 7.1]) that the untwisted Whitehead doubles of a positive or almost positive knot have non-trivial Q polynomial. This was my original motivation for a large part of the investigations described in [St2].

3.5.2 Vassiliev invariants up to degree 7

Now we explain how to extend our result. For degree up to 7 we can present a detailed argument and calculation.

Theorem 3.5 The Q polynomial determines no Vassiliev invariants up to degree 7, except those derived (as polynomials of degree at most 3) from v_2 .

Proof. Let v be a Vassiliev invariant of degree ≤ 7 determined by Q . Since v is symmetric, it has even degree [St4]. The space of symmetric invariants of degree up to 6 is generated by the primitive invariants

$$v_2; v_{4,1}, v_{4,2}; v_{6,1}, v_{6,2}, v_{6,3}, v_{6,4}, v_{6,5}, \quad (16)$$

and the composite invariants

$$v_2^2; v_2^3, v_3^2, v_2v_{4,1}, v_2v_{4,2}. \quad (17)$$

Here so far v_i (resp. $v_{i,j}$) denotes the (unique resp. j -th in some arbitrary ordering) primitive Vassiliev invariant of degree i . From now on, call all these (including composite) invariants $v_{i,j}$ (by setting $v_{i,1} := v_i$ for $i = 2, 3$ and assigning such a term for the invariants of degree i in (17), with j above the range in (16)). Concrete expressions for $v_{i,j}$ (with one exception, $v_{6,5}$, and up to symmetric invariants of lower degree) can be found from the Kauffman polynomial. Set $F_{i,j}$ as in (11) for $i \geq 0$, $j > 0$. The property (3) implies that there are numbers $c_{i,j}$ such that

$$\tilde{F}_{i,j} = F_{i,j} + \sum_{k=1}^{j-1} c_{i,k} F_{i,k}$$

is a symmetric invariant for $i + j$ even (and antisymmetric for $i + j$ odd). In fact, one can restrict the k -sum over $1 \leq k < j$ with $j - k$ odd. For instance, one can choose

$$\begin{aligned} \tilde{F}_{d,1} &= F_{d,1} \\ \tilde{F}_{d,2} &= F_{d,2} + F_{d,1} \\ \tilde{F}_{d,3} &= F_{d,3} + 3F_{d,2} \\ \tilde{F}_{d,4} &= F_{d,4} + 6F_{d,3} - 6F_{d,1} \\ \tilde{F}_{d,5} &= F_{d,5} + 10F_{d,4} - 60F_{d,2} \\ \tilde{F}_{d,6} &= F_{d,6} + 15F_{d,5} - 300F_{d,3} + 360F_{d,1} \end{aligned}$$

The $\tilde{F}_{i,j}$ are not primitive, but a test on a few knots (see below) shows that most of them are linearly independent. Thus one can obtain (some) primitive Vassiliev invariants $v_{i,j}$ from the $\tilde{F}_{i,j}$ by linear combinations (possibly including products). Even more, since the $\tilde{F}_{i,j}$ exceed the dimension of the space of primitive (symmetric) invariants for $i + j \leq 6$, there are linear dependencies.

A first easy observation shows that

$$\tilde{F}_{0,2} = 4\tilde{F}_{1,1},$$

which is also a multiple of $v_2 = \nabla_2$, so that we can discard $\tilde{F}_{1,1}$. Then turn to degree ≤ 4 . Consider the few thousand (including composite) knots of up to 13 crossings. (They can be generated from the tables of [HT].) A test of $\tilde{F}_{0,2}$, $\tilde{F}_{0,2}^2$, $\tilde{F}_{0,4}$, $\tilde{F}_{1,3}$, $\tilde{F}_{2,2}$, $\tilde{F}_{3,1}$ on these knots shows the linear relations

$$\begin{aligned} 31\tilde{F}_{0,2} + 5\tilde{F}_{0,4} - 16\tilde{F}_{1,3} + 16\tilde{F}_{2,2} - 4\tilde{F}_{0,2}^2 &= 0 \\ 3\tilde{F}_{0,2} + \tilde{F}_{0,4} - 8\tilde{F}_{1,3} + 48\tilde{F}_{2,2} - 192\tilde{F}_{3,1} &= 0 \end{aligned}$$

Thus one can eliminate $\tilde{F}_{2,2}$ and $\tilde{F}_{3,1}$. Then a test in degree ≤ 6 of

$$\tilde{F}_{0,2}, \tilde{F}_{0,2}^2, \tilde{F}_{0,2}^3, \tilde{F}_{0,3}^2, \tilde{F}_{0,4}, \tilde{F}_{1,3}, \tilde{F}_{0,2}\tilde{F}_{0,4}, \tilde{F}_{0,2}\tilde{F}_{1,3}, \tilde{F}_{0,6}, \tilde{F}_{1,5}, \tilde{F}_{2,4}, \tilde{F}_{3,3}, \tilde{F}_{4,2}, \tilde{F}_{5,1}$$

shows the relations

$$\begin{aligned} -1485\tilde{F}_{0,2} - 135\tilde{F}_{0,4} + 2\tilde{F}_{0,6} + 360\tilde{F}_{1,3} - 24\tilde{F}_{1,5} + 240\tilde{F}_{2,4} - 1920\tilde{F}_{3,3} \\ + 11,520\tilde{F}_{4,2} - 46,080\tilde{F}_{5,1} + 180\tilde{F}_{0,2}^2 &= 0 \\ -4464\tilde{F}_{0,2} - 3564\tilde{F}_{0,4} - 45\tilde{F}_{0,6} + 4410\tilde{F}_{1,3} + 54\tilde{F}_{1,5} + 1296\tilde{F}_{2,4} - 14,688\tilde{F}_{3,3} \\ + 97,920\tilde{F}_{4,2} - 403,200\tilde{F}_{5,1} + 64\tilde{F}_{0,2}^2 + 72\tilde{F}_{0,2}^3 - 48\tilde{F}_{0,2}\tilde{F}_{0,4} + 384\tilde{F}_{0,2}\tilde{F}_{1,3} &= 0 \end{aligned}$$

Thus eliminate $\tilde{F}_{5,1}$ and $\tilde{F}_{4,2}$.

This calculation is justified by lemma 3.1. It also confirms the (previously well-known) fact that F contains both primitive invariants of degree 4 and 4 of the 5 primitive invariants of degree 6. The missing invariant

$v_{6,5} = \nabla_6$ is provided (up to some correction by composite invariants) by the coefficient of z^6 in the Conway polynomial $\nabla(z)$, as explained in [St6] from the example of Lickorish (and recalled above in §3.3).

Now assume $Q(z) = F(1, z)$ determines $v = \sum_{i=2,4,6} \sum_j c_{i,j} v_{i,j}$. First note that if $c_{6,5} \neq 0$, then F determines ∇_6 , a contradiction. Thus assume $c_{6,5} = 0$, and we deal only with the Vassiliev invariants coming from the Kauffman polynomial. Then we can w.l.o.g. replace $v_{i,j}$ by $\tilde{F}_{i',j'}$ (for $i' + j' = i$).

Among prime knots of ≤ 10 crossings [Ro, appendix], Q has 13 duplications. These are the pairs

$$(9_{44}, 8_2), (9_{45}, 8_7), (9_{15}, 10_{159}), (9_8, 10_{131}), (9_5, 10_{134}), (9_{21}, 10_{151}), (9_{12}, 10_{156}), \\ (9_{25}, 9_{26}), (10_{22}, 10_{35}), (10_{14}, 10_{31}), (10_{56}, 10_{33}), (10_{19}, 10_{36}), (10_{43}, 10_{72}).$$

The polynomial of (one knot of) each pair is given in table 1.

8	2	[-7]	0	22	2	-20	-4	6	2		
8	7	[-7]	4	20	-8	-20	2	8	2		
9	5	[1]	-12	2	28	0	-22	-4	6	2	
9	8	[1]	-8	8	22	-12	-22	2	8	2	
9	12	[-3]	-6	10	14	-12	-16	4	8	2	
9	15	[1]	4	-2	-2	-8	-8	6	8	2	
9	21	[-3]	-2	16	4	-26	-12	12	10	2	
9	25	[-7]	0	30	2	-42	-14	18	12	2	
10	14	[1]	-4	-10	20	16	-26	-18	10	10	2
10	19	[5]	-6	-20	28	30	-30	-26	8	10	2
10	22	[1]	0	-4	6	12	-12	-16	4	8	2
10	33	[1]	-16	0	44	4	-48	-16	18	12	2
10	43	[-7]	-4	28	22	-32	-42	0	22	12	2

Table 1: The Q polynomials of the ≤ 10 crossing prime knots occurring in duplications. (Only one knot in each pair is recorded.)

Each such pair gives rise to a linear relation on the $c_{i,j}$ as (14) in the proof of proposition 3.2 on a and b . The $\tilde{F}_{i,j}$ are given in table 2.

By imposing jointly all these 13 conditions on the $c_{i,j}$, we find that the only possible linear combinations $\sum c_{i,j} v_{i,j}$ determined by Q must lie in the span of v_2 , v_2^2 and v_2^3 , as desired. \square

Remark 3.2 As in the proof of proposition 3.2, we could have tried to use a single large group of knots with equal Q polynomial, but different Vassiliev invariants, to find enough relations between the $c_{i,j}$. (Note that a group of n knots can give up to $n - 1$ independent linear relations.) However, among prime knots of up to 16 crossings I found no group, whose linear conditions on the $c_{i,j}$ eliminate anything except polynomials of v_2 . Note that generically, a considerable part of the coincidences of the Q polynomial arises from mutations. But mutations preserve Vassiliev invariants up to degree 6 (see [CDL, CDL2, CM, Mr]), and are useless for our purpose.

4 Vassiliev invariants and 2-cable polynomials

4.1 Calculating invariants

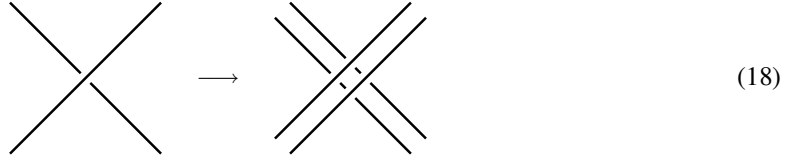
If one likes to extend our result to degrees ≥ 8 , more computation is required. We will present here the outcome that suffices to cover degree 8 and 9. A first task is to find a way to obtain all such Vassiliev invariants. Expectedly, this problem has been encountered before. In particular, a related (and still unsolved) question is

Table 2: This table gives the values of the $\tilde{F}_{i,j}$ (as $F(i, j)$, with $\tilde{F}_{0,3}$ accurate up to sign) on the ≤ 10 crossing prime knots occurring in Q duplications (see table 1). cr and nr give the crossing number and order number in the tables of [Ro, appendix] (that is, denote the knot cr_{nr}).

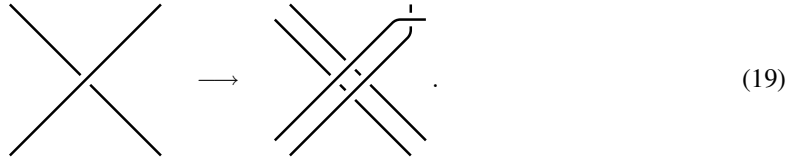
cr	nr	$F(0, 2)$	$F(0, 4)$	$F(0, 6)$	$F(1, 1)$	$F(1, 3)$	$F(1, 5)$	$F(2, 2)$	$F(2, 4)$	$F(3, 1)$	$F(3, 3)$	$F(4, 2)$	$F(5, 1)$	$F(0, 3)$
9	44	0	-192	8640	0	-96	-480	-36	-852	-6	-336	-72	-8	-48
8	2	0	576	-2880	0	288	32160	108	14268	18	1920	136	12	48
9	45	-16	-1872	-54720	-4	-768	-65280	-88	-8184	0	-624	-88	-6	192
8	7	-16	-336	14400	-4	-288	-9120	-88	-2808	-12	-408	-72	-10	96
9	5	-48	-8112	-420480	-12	-3600	-522000	-396	-38940	8	-2424	-592	10	-720
10	134	-48	-5040	455040	-12	-2640	-210000	-396	-40476	-16	-4008	-400	-10	-624
9	8	0	-768	11520	0	-240	-7920	0	-1728	6	-384	-40	2	-96
10	131	0	1536	69120	0	624	80880	144	26448	18	3336	184	6	96
9	12	-8	-1704	-61920	-2	-624	-48240	-60	-7692	2	-1704	-216	0	-144
10	156	-8	-168	7200	-2	-144	-7440	-60	-2316	-10	-48	-40	-16	48
9	15	-16	-2448	-51840	-4	-960	-73920	-100	-8532	2	-1224	-176	0	240
10	159	-16	-912	17280	-4	-480	-33120	-100	-9300	-10	-1104	-128	-16	-144
9	21	-24	-2616	-44640	-6	-1104	-81360	-96	-6624	8	-1128	-188	8	288
10	151	-24	-1080	24480	-6	-624	-25200	-96	-2784	-4	-432	-108	-8	192
9	25	0	-960	-72000	0	-288	-32160	12	-2916	10	-240	-28	2	-48
9	26	0	576	-2880	0	192	8640	12	3996	-2	936	84	-2	-48
10	14	-16	-912	17280	-4	-336	30000	44	21372	20	3648	336	30	-144
10	31	-16	-144	5760	-4	-240	-240	-100	-660	-16	-696	-224	-30	48
10	19	-8	24	-1440	-2	-144	240	-120	-1368	-24	-984	-284	-38	0
10	36	-8	-744	10080	-2	-240	7440	24	9144	12	2112	260	22	-96
10	22	32	288	-11520	8	144	-240	248	5400	58	888	216	48	96
10	35	32	1056	-23040	8	240	240	104	4104	22	672	120	24	-96
10	33	0	0	0	0	-96	1440	-96	-864	-20	-768	-232	-32	0
10	56	0	1536	69120	0	672	97440	192	37824	28	6048	536	32	96
10	43	-16	-336	14400	-4	-240	-240	-40	-1032	-2	-528	-128	-14	0
10	72	-16	-1872	-54720	-4	-720	-48720	-40	2808	10	2136	304	26	-192

Question 4.1 (Przytycki, problem 1.92 (M)(c) of [Ki]) Do all degree ≤ 10 invariants of knots come from the HOMFLY and Kauffman polynomial and their 2-cables?

Recall that a 2-cable K_p of a knot K with framing p is constructed as follows. For even p , (a diagram of) K_p is obtained by applying



to any crossing of a diagram of K of writhe $p/2$. For odd p one applies (18) to a diagram of writhe $(p-1)/2$, except at one crossing, where one performs



K_p is connected (a knot) for odd p and disconnected (a 2-component link) for even p . We write K_{\pm} for $K_{\pm 1}$.

In an attempt to approach Przytycki's problem, we considered the Vassiliev invariants

$$\mathcal{P}_d := \{P_{i,j} : i+j \leq d\}, \quad (20)$$

where $P_{i,j}$ is defined as in (13), and $i, j \geq 0$, with i even. Note that \mathcal{P}_d are all invariants of degree $\leq d$. Similarly are

$$\mathcal{F}_d := \{F_{i,j} : i+j \leq d\}. \quad (21)$$

To obtain a Vassiliev invariant of degree $\leq d$, one can also use products of invariants $P_{i,j}$ and $F_{i,j}$.

The invariants in \mathcal{P}_d and \mathcal{F}_d were considered by Meng [Me] and Lieberum [Li], using their weight systems. Our calculation is supported by some results they obtain. However, it also shows phenomena that point to caution in some tempting conclusions concerning the structure of the algebra generated by Vassiliev invariants of the HOMFLY and Kauffman polynomials.

One can apply \mathcal{P}_d and \mathcal{F}_d also to 2-cables K_p of K with various framings. We denote by $\mathcal{P}_d(K_p)$ and $\mathcal{F}_d(K_p)$ the resulting invariants. If the framing is even, then the 2-cable is disconnected, and then the restriction to i modifies to $i \geq -1$, with i odd for $P_{i,j}$.

For

$$v \in (\mathcal{P}_d \setminus \mathcal{P}_{d-1})(K_*) \cup (\mathcal{F}_d \setminus \mathcal{F}_{d-1})(K_*),$$

let $\widetilde{\deg} v := d$. Note that $\widetilde{\deg} v$ is not *a priori* evident to be the same as the degree of v as a Vassiliev invariant (whence the notational distinction), although clearly $\deg v \leq \widetilde{\deg} v$. In some situations though we have equality, and we clarify why, since the notation and arguments will be of relevance in later explanation. We formulate a statement only with F , letting the reader understand that most subsequent remarks on one of P and F also apply on the other in a similar way.

Lemma 4.1 For odd p and $i, j \geq 0$ with $i+j$ even, and i even or $i=1$, we have $\deg F_{i,j}(K) = \deg F_{i,j}(K_p) = i+j$.

Proof. Let us write for a set $M \subset \mathcal{P}_*(K_*) \cup \mathcal{F}_*(K_*)$ of invariants

$$M_d := \left\{ \prod_{l=1}^k v_l : v_l \in M, \sum_{l=1}^k \widetilde{\deg} v_l \leq d \right\},$$

$\begin{array}{l} \text{deg} \leq \\ \text{dim} \geq \end{array}$	0	1	2	3	4	5	6	7	8	9	10
total	1	1	2	3	6	10	19	33	60	104	184
$P/PP!$	1	1	2	3	6	9	16	24	40	60	95
P_+/P_-	1	1	2	3	6	9	16	24	40	60	95
PP_+/PP_-	1	1	2	3	6	10	19	31	53	85	140
P_+P_-	1	1	2	3	6	9	17	27	46	72	117
PP_+P_-	1	1	2	3	6	10	19	31	53	86	142
$P_+P_-P_0$	1	1	2	3	6	10	19	30	52	82	136
PP_+P_0/PP_-P_0	1	1	2	3	6	10	19	31	54	87	145
$PP_+P_-P_0$	1	1	2	3	6	10	19	31	54	87	145
$PP_+P_-P_3$	1	1	2	3	6	10	19	31	53	86	142
$P_{-2}P_-P_0P_+$	1	1	2	3	6	10	19	30	52	82	136
$PP_{-2}P_-P_0P_+$	1	1	2	3	6	10	19	31	54	87	145
$PP_-P_0P_+P_2P_3$	1	1	2	3	6	10	19	31	54	87	145
F	1	1	2	3	6	10	18	29	49	78	127
F_+/F_-	1	1	2	3	6	10	18	29	49	78	127
$FPP_+P_-P_0$	1	1	2	3	6	10	19	32	57	94	159
$FF_+F_-F_0$	1	1	2	3	6	10	19	33	59	99	168
$FF_+F_-F_0F_{-2}$	1	1	2	3	6	10	19	33	59	99	168
$FF_+F_-F_0F_{-3}$	1	1	2	3	6	10	19	33	59	99	168
$FF_+F_-F_0F_2$	1	1	2	3	6	10	19	33	59	99	168
$FF_+PP_+P_-P_0$	1	1	2	3	6	10	19	33	60	102	176
$FF_0PP_+P_-P_0$	1	1	2	3	6	10	19	33	60	102	176
$FF_+F_-F_0PP_+P_-P_0$	1	1	2	3	6	10	19	33	60	102	177

Table 3: This table contains dimensions of various spaces of Vassiliev invariants for degree ≤ 10 .

The first row gives the total dimension of Vassiliev invariants up to degree deg as calculated by Bar-Natan [BN] and Kneissler [Kn].

The second section of rows gives lower bounds for the dimension of Vassiliev invariants up to degree deg obtainable as polynomial expressions from \mathcal{P}_{deg} of the HOMFLY polynomial P , and its (application on) 2-cables $P_p(K) = P(K_p)$ of twist p . P_{\pm} denotes $P_{\pm 1}$. The product of some of the P symbols denotes that the invariants of these polynomials have been taken together. The slash separates between alternative combinations of polynomials that give, as we explain, the same dimensions (although not necessarily the same linear spaces!).

The last section gives dimensions of invariants derived via \mathcal{F}_{deg} from the Kauffman polynomial F and its applications F_p on 2-cables of twist p , with $F_{\pm} = F_{\pm 1}$. Some combinations of P_* and F_* invariants are also given. They are chosen so as to make evident that the last row's dimensions cannot be increased by adding further invariants.

and consider

$$\hat{F}(h, N) := F\left(\sqrt{-1}e^{-(N-1)h/2}, \sqrt{-1}(e^{h/2} - e^{-h/2})\right).$$

Then we have (extending the notation of coefficients to power series)

$$F_{i,j}(K) \equiv C_{i,j} [\hat{F}(h, N)(K)]_{h^{i+jNj}} \equiv C_{i,j} [\hat{F}(h, N)(K \cup O)]_{h^{i+jNj+1}} \pmod{\mathcal{L}in(\mathcal{F}_*(K))_{i+j-1}}, \quad (22)$$

where $C_{i,j}$ are non-zero numbers, and $K \cup O$ is the split union of K with an unknot. The r.h.s. of the congruence is a canonical Vassiliev invariant of degree $i + j$ by the result of Le-Murakami and Kassel [LMr, LMr2, LMr3, Ks] (compare Proposition 5 in [Li]). Now $[\hat{F}_{K \cup O}(h, N)]_{h^{i+jNj+1}} \neq 0$ is not hard to see. Let K be the unknot, with $F(K \cup O) = (a + a^{-1})/z - 1$. The coefficient of the power series $\hat{F}(K \cup O)$ is easily found to be non-zero for the given i, j ; for $i \neq 1$ it is, up to a factor, a Bernoulli number. (With other K and a similar calculation one can settle more i, j .)

Thus indeed $\deg F_{i,j}(K) = i + j$. Then the same is true for $F_{i,j}(K_p)$ if p is odd. To see this, recall that connected n -cabling of a degree d Vassiliev invariant v applies a dual Adams operation $(\psi^n)^*$ of [BN] on its weight system $W_v \in \mathcal{V}_d/\mathcal{V}_{d-1} \simeq \mathcal{A}_d^*$. That ψ^n is an automorphism of \mathcal{A}_d was stated in Exercise 3.12 of [BN]. In fact, we know that the eigenvalues of ψ^n are powers of n with exponents given by the number of univalent vertices of univalent graphs (see [KSA, MR]). \square

For the calculation of 2-cable polynomials of K it is sufficient (but also, up to algebraic transformations, necessary) to determine the polynomials of a connected cable of K and $!K$. To keep the diagrams as simple as possible, we decided to use the 2-cables with blackboard framing from the diagrams in [HT] and one negative half-twist. For the skein polynomial, this calculation was possible for all prime knots up to 13 crossings (including mirror images). The Kauffman polynomial is technically more difficult. We obtained a set S of 898 prime knots up to 12 crossings (including all ≤ 10 crossing knots, except 10_5), where both Kauffman polynomials could be determined. We used this set S for all subsequent Vassiliev invariant calculations.

4.2 Dimensions

Table 3 gives lower bounds for the dimension of Vassiliev invariants of bounded degree that we calculated for various combinations of $P_{i,j}$ and $F_{i,j}$ applied on knots and various 2-cables. With the previous designation, for example the column d entry of the row PP_+P_- is

$$\dim \mathcal{L}in(\mathcal{P}_*(K_+) \cup \mathcal{P}_*(K_-) \cup \mathcal{P}_*(K))_d \Big|_{K \in S},$$

and S is the set of knots explained above. Clearly, many linear dependencies will occur, but in degree ≥ 7 , they are increasingly difficult to prove rigorously. Contrarily, linear independence is easy to prove if S is large enough. Although some general theory behind table 3 is known, there are many detailed aspects in the calculations it reflects that were never apparently brought clearly to attention. Thus we will list below several features of the table that should be clarified, and point out phenomena and previous results it relates to.

The numbers obtained, given in the table, can be only ensured to represent lower bounds for the dimensions, since it is difficult to rigorously verify that some Vassiliev invariant is identically zero. From the fact that we evaluated enough invariants to obtain the full dimension up to degree 8 we can conclude that the set S we used is d -deterministic, and so our numbers are exact, for $d \leq 8$. However, we do not know about degree 9 or 10. Indeed, non-trivial Vassiliev invariants of increasing degree may vanish on many low-crossings knots (for example the ∇_i). All deterministic sets we know in degree $d > 8$, also the one from §3.2, are too large to allow efficient calculations. One can find smaller sets using a basis of $\mathcal{V}_d/\mathcal{V}_{d-1}$. (Its primitive part would be enough.) But such a basis is itself non-trivial to find, and was apparently never explicitly given (even if likely obtained in the course of the calculations of Bar-Natan [BN] and Kneissler [Kn]). Even if so done, the resulting reduction is still unlikely to be well manageable. Another way to prove a set deterministic is to evaluate the remaining Vassiliev invariants, but this does not seem very efficient either.

At least the comparison of the first and last rows of the table shows that the difference between the numbers in degree 9 resp. 10 and the actual dimension is at most 2 resp. 7.

Once we obtain only lower bounds, it makes sense to reduce invariants modulo a large prime, which we chose to be 9091, in order to keep numbers simple. (In particular in 2-cable Kauffman polynomials the coefficients are large enough to make $F_{i,j}$ exceed machine-size integers. MATHEMATICA™, which bypasses this problem, could not handle well the extent of calculation needed for the upper degrees.)

Some coincidences of rows are easy to explain (even without knowing the absolute accuracy of the numbers in giving the proper dimensions), or at least known. In particular, mirroring the (set of) invariant(s) induces an involution on the space $\mathcal{V}_d/\mathcal{V}_{d-1}$. The injectivity of Ψ^2 was mentioned in the proof of lemma 4.1. The fact that $PP!$ contributes the same linear span of invariants as P is a consequence of the property (4). For that same reason, and because $(!K)_p = !(K_{-p})$, it becomes useless to consider the invariants from the various 2-cable polynomials of $!K$ for $K \in S$.

We obtained also lists of linear independent invariants (omitted here for space reasons), but we have not tried to identify a basis of the primitive part of $\mathcal{V}_d/\mathcal{V}_{d-1}$ that is obtainable. It is very difficult (see the following remarks) to determine the exact degree of the Vassiliev invariants and their primitivity status. One should also be cautioned that the linear relations between such invariants involve up to about-30-digit coefficients, and are much more complicated than insightful.

4.3 “Hidden” Vassiliev invariants

Assume for a moment that the numbers in the table are accurate (rather than just lower bounds), that each new set of \mathcal{P}_d contributes in comparison to \mathcal{P}_{d-1} entirely invariants of degree d , and that all (prime) factors of all composite invariants obtained have been generated for smaller d . Then we find from the various rows of the table the *projected* sequences of primitive Vassiliev invariants of degree exactly d that can be obtained. For example, for the P -row it reads 1,0,1,1,2,2,3,3,4,4,5 and for the F -row 1,0,1,1,2,3,4,5,6,7,8. These sequences appear in Proposition 12 of [Li], and seem the only case studied closer so far. However, the projected sequences may not always be correct. Consider the rows PP_+ , where we obtain 1,0,1,1,2,3,5,6,7,8,9 and PP_-P_+ , where we obtain 1,0,1,1,2,3,5,6,7,9,10. Apparently, adjoining P_- seems to give a new invariant in degree 9. But it is easy to see that $\mathcal{P}_d(K_p)$ gives the same elements in $\mathcal{V}_d/\mathcal{V}_{d-1}$ for any p of a given parity. Thus $\mathcal{P}_d(K_-)$ cannot increase the dimension in degree d . This means that a Vassiliev invariant of degree d may be realizable from some $\mathcal{P}_{d'}$ with $d' > d$, but not from \mathcal{P}_d (of a given set of cables). In particular, the difference between PP_-P_+ and PP_+ in degree $d = 9$ comes from a Vassiliev invariant v_8 in degree < 9 . We know that v_8 can not be obtained from $(\mathcal{P}_*)_8$, since by the remarks in §4.2 our numbers are accurate for $d = 8$. It must have degree 7 or 8, as we have already exhausted all invariants of degrees ≤ 6 with PP_+ . The additional difference in degree 10 must come from a new Vassiliev invariant of degree 7 to 9. But these invariants are immediately lost if we work with the (degree d) weight systems of \mathcal{P}_d . This explains why the weight systems obscure sometimes essential information.

Even if we cannot explicitly observe an instance of this phenomenon, it is in principle even possible that one can obtain a composite Vassiliev invariant from some $\mathcal{P}_d(K_*)$, without being able to obtain some of its factors.

These (possible) peculiarities explained above caution that

- a) *the algebra of some set of Vassiliev invariants may not be isomorphic to the algebra of their weight systems.* This can occur if not all invariants are primitive and have linear independent weight systems (of the appropriate degree). To exclude such possibility, the composite and lower degree Vassiliev invariants must be proved to be generatable from previous degrees. One situation where this is needed is Theorem 3 of [Li]. It requires the result used in (22) that any $F_{i,j}(K)$ can be altered by elements in $\mathcal{L}in(\mathcal{F}_*(K))_{i+j-1}$ so that it becomes canonical (of degree $i+j$), and similarly $P_{i,j}(K)$. For canonical invariants, linear dependencies of the weight systems extend to linear dependencies of invariants.

- b) it is difficult to prove that some Vassiliev invariant v is actually *not* obtainable from HOMFLY (or some cables of it). For P we can deduce from the proof of lemma 4.1 that if a Vassiliev invariant v of degree $d = \deg v$ lies in the algebra generated by $\mathcal{P}_*(K)$, then it lies in $\mathcal{L}in(\mathcal{P}_*(K))_d$. On the opposite side, for cables of P , there is no *a priori* limit on d' in terms of d , whose $\mathcal{P}_{d'}$ we must consider, and possibly not only polynomials, but fractions of polynomials of $\mathcal{P}_{d'}$ must be examined. There may be even other (yet unknown) ways to obtain Vassiliev invariants, not using (only) the \mathcal{P}_* . Thus the only approach is to find knots not distinguished by HOMFLY (or its cables) but by v , as in [K4, St6]. A systematic way to find such examples is unknown.

4.4 Connected and disconnected cables

It is suggestive from the skein relation of P that the Vassiliev invariant v_8 in §4.3 can be obtained from $\mathcal{P}_8(K_0)$. This explains the difference between the PP_+P_- and $PP_+P_-P_0$ rows occurring already in degree 8.

In general one can obtain the P -polynomial of a disconnected n -cable as a linear combination of polynomials of connected n -cables whose coefficients have a power in m between 0 and $1 - n$. This means that

$$\mathcal{L}in \mathcal{P}_d (\text{all } n\text{-cables}) \subset \mathcal{L}in \mathcal{P}_{d+n-1} (\text{connected } n\text{-cables}).$$

However, in general

$$\mathcal{L}in \mathcal{P}_d (\text{all } n\text{-cables}) \neq \mathcal{L}in \mathcal{P}_d (\text{connected } n\text{-cables}).$$

That is, there is a way of obtaining new Vassiliev invariants by disconnectedly cabling invariants of the same degree, not obtainable by connected cablings. This was noticed by Dasbach [Da]. The eigenvalues of the Adams operations (mentioned in the proof of lemma 4.1) show, as observed in [MR], that the space of invariants given by connected n -cablings of an invariant v of degree d stabilizes (modulo lower degree) for $n > d$. In contrast, Dasbach's result roughly means that, by starting from \mathcal{P}_d , one will obtain new invariants of degree d from disconnected n -cables at least up to $n \leq \exp(C \cdot \sqrt{d})$ for some constant $C > 0$ (independent on n and d). Thus, even although polynomials of disconnected cables are linear combinations of polynomials of connected cables, and hence the same is true for their global sets of Vassiliev invariants, the situation is quite different if one restricts oneself to their invariants of bounded degree.

On the other hand, for any connectivity, the relations between cable polynomials allow us to limit the number of cables of that connectivity that suffice to generate all possible Vassiliev invariants from all such cables. In case $n = 2$ we have

Lemma 4.2 For the polynomials P_p of the 2-cables of framing p (connected for p odd and disconnected for p even), we have

$$P_p = -l^4 P_{p-4} - (2l^2 - m^2 l^2) P_{p-2}.$$

Proof. Consider the generating series $f(l, m, z) = \sum_{p=0}^{\infty} P_p(l, m) z^p$ (whose convergence is easy to establish). The skein relation implies $P_{p+2} = -ml P_{p+1} - l^2 P_p$, so that

$$f(l, m, z) = \frac{A(l, m, z)}{1 + mlz + l^2 z^2},$$

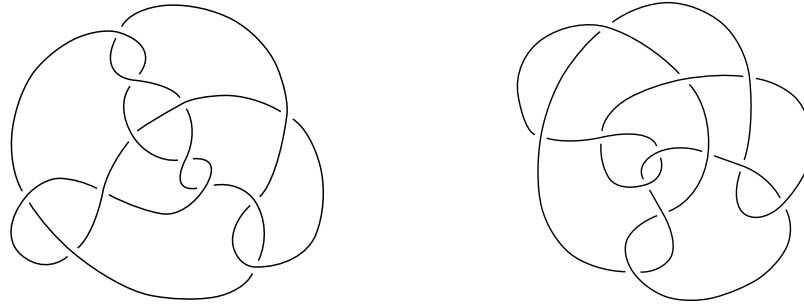
for some $A \in \mathbf{Q}[l, m, z]$. Taking $f(l, m, z) \pm f(l, m, -z)$, we obtain the denominator

$$(1 + mlz + l^2 z^2)(1 - mlz + l^2 z^2) = 1 + l^4 z^4 + 2l^2 z^2 - m^2 l^2 z^2,$$

which leads to the stated relation. \square

This means that for connected/disconnected 2-cables, the invariants of \mathcal{P}_d are exhausted, if we apply them on P_p for two consecutive odd resp. even p . By a similar argument for F , three consecutive p of a given parity suffice. In practice, as the table shows, $p = \pm 1, 0$ already apparently generate all invariants from \mathcal{P}_d and \mathcal{F}_d for $d \leq 10$ (for both parities of p taken together).

4.5 Mutations and non-mutations



12₃₄₁

12₆₂₇

49	341	0	24										
-8	10			5	27	69	99	75	6	-44	-44	-20	-4
-8	10			-60	-315	-755	-1016	-710	-22	467	462	213	42
-8	10			331	1705	3800	4752	3129	33	-2126	-2157	-1040	-206
-8	10			-1011	-5283	-11308	-13286	-8335	-7	5603	5795	2823	537
-8	10			1805	10023	21665	24516	14755	-52	-9503	-9781	-4556	-794
-8	10			-1965	-12201	-27766	-31277	-18030	41	10913	10763	4610	692
-8	10			1325	9768	24362	28057	15528	-11	-8578	-7886	-2968	-355
-8	10			-549	-5129	-14689	-17703	-9448	1	4563	3811	1191	104
-8	10			135	1728	5989	7729	4007	0	-1583	-1179	-285	-16
-8	10			-18	-357	-1602	-2259	-1148	0	338	222	37	1
-8	8			1	41	267	418	210	0	-40	-23	-2	
-6	6				-2	-25	-44	-22	0	2	1		
-4	0					1	2	1					

Figure 4: Two knots with the same 2-cable HOMFLY polynomials (P_+ is displayed), which are not mutants.

Note the difference between the $P_+P_-P_0$ and the $PP_+P_-P_0$ rows. This suggests that HOMFLY may have Vassiliev invariants not contained in its 2-cables. In general almost all knots with different HOMFLY polynomial will also have different 2-cable HOMFLY polynomial. But the Vassiliev invariant observation suggests that it may not always be so. So far the only known examples of knots with equal 2-cable HOMFLY polynomial are mutants [LL]. They also have the same HOMFLY (and Kauffman) polynomial. However, the calculations performed along compiling the above table led to the discovery of some duplications of P_* which are not mutants.

Example 4.1 The knots 12_{1305} and $!12_{1872}$ have the same P , F and 2-cable P . To check the coincidence of P_* , comparing P_{\pm} suffices. Still 12_{1305} and $!12_{1872}$ are not mutants. Most easily this is shown using the result of [Ru], since their hyperbolic volumes differ: $\text{vol}(12_{1305}) \approx 15.483$, while $\text{vol}(!12_{1872}) \approx 15.619$. Another such group is made of the two mutants 12_{1378} , 12_{1423} , and the knot $!12_{1704}$. Again P , F and 2-cable P coincide, but while $\text{vol}(12_{1378}) = \text{vol}(12_{1423}) \approx 15.094$, we have $\text{vol}(!12_{1704}) \approx 14.983$.

Later, after considerable calculation, we found that these pairs of knots have also different 2-cable Kauffman polynomials F_+ , with the difference coming out as a Vassiliev invariant of degree 7. Thus there is a Vassiliev invariant of degree 7 not contained in the HOMFLY, Kauffman and 2-cable HOMFLY polynomial, but in the 2-cable Kauffman polynomial. (Note that P_* exhaust all invariants up to degree 6.)

There is one further pair made up of 12_{341} and 12_{627} (see figure 4). These knots are achiral, and for them comparing P_+ suffices to see that P_* coincide. This time they are distinguished using an invariant of degree 8 of the 2-cable Kauffman polynomials. (Note that the lowest degree of an invariant distinguishing 12_{341} and 12_{627} must be even, since by [St4] odd degree invariants can be changed by invariants of lower degree so that they vanish on achiral knots.)

There has been further work on generalizations of mutations [APR, JR, Tz, HP], but none of this seems to explain the coincidence of the 2-cable HOMFLY polynomial in these examples.

The observed coincidences of P and F also on non-mutants with the same 2-cable HOMFLY polynomials extend to prime ≤ 13 crossing knots and suggest

Question 4.2 Does P_p for some p (or at least for all p taken together) determine P and/or F ?

Note that this question may relate to more than mere curiosity. In [KS] we observed a (conjectural) relation between F and the Whitehead double HOMFLY polynomials, and there is also Yamada's remarkable result [Y] that F determines the 2-cable Jones polynomial.

Remark 4.1 Using Alexander Shumakovitch's database, we found that the new Khovanov polynomial Kh [Kh] coincides on these examples as well, and on all other pairs of prime ≤ 13 crossing knots with equal P_+ . Still Kh is known to distinguish some knots with equal P and F (most interestingly 9_{42} and its mirror image). However, for instance I do not know of an example showing that Kh can distinguish knots with equal F and Murasugi-signatures.

4.6 Braid index

It is known that one can estimate the braid index of a knot K from its P polynomial [Mo, FW]:

$$2(b(K) - 1) \geq \max \deg_l P(K) - \min \deg_l P(K). \quad (23)$$

This estimate is called the Morton-Franks-Williams inequality. Since obviously $b(K_p) \leq 2b(K)$ for any $p \in \mathbf{Z}$, we can estimate $b(K)$ also from the 2-cable P polynomials of K , as done in [MS]. We attempted to use this method to settle the braid index for prime knots of up to 12 crossings. This requires us to find braid representations of the strand number given as (lower) bound from the Morton-Franks-Williams inequality or its application on the 2-cable polynomials. (For a few cases of large bound, one can conclude the existence of such representations from Ohya's inequality [Oh], and for special types of knots from Murasugi's results [Mu2].) We were able to calculate 2-cable P polynomials up to 13 crossings, but were aware of the difficulties of finding braid representations. We know from [HS] of one undecidable 13 crossing knot, and in [St8] we gave a 14 crossing example of failure of the 2-cable Morton-Franks-Williams inequality. On the contrary, we indeed succeeded in finding the desired braid representations for ≤ 12 crossing knots, thereby showing

Proposition 4.1 The 2-cable Morton-Franks-Williams inequality is sharp for prime knots of up to 12 crossings. \square

To summarize the result of our computation, we assume that the calculation of P is easy, so restrict ourselves to the exceptions. The following table 4 gives the 98 prime knots of 12 crossings or less for which the (usual) Morton-Franks-Williams inequality is not sharp, with their braid index. (The unsharpness of (23) is by 2, except for the knots marked with an asterisk, where it is 4.) Note that all these knots are non-alternating, although for higher crossing numbers alternating examples are known at least for links from [Mu].

4.7 Main application

With all possible framings of P and F , we still do not obtain two invariants of degree 9, and expectedly several invariants of degree 10. Thus it seems that Przytycki's question 4.1 is to be negatively answered. However, by the previous remarks, the only way to do so is to find knots not distinguished by the HOMFLY and Kauffman polynomial and their 2-cables. The only such known examples are mutants [LL], but they have the same invariants up to degree ≤ 10 [Mr]. (This result in fact motivated Przytycki's question.) Thus a systematic approach to answer negatively the question seems lacking.

The calculations up to degree 8 now allow us to prove our main result

9_42	4	12_1298	5	12_1499	5	12_1695	4	12_1899	5
9_49	4	12_1313*	5	12_1503	5	12_1702	5	12_1922	5
10_132*	4	12_1333	5	12_1506	5	12_1704	4	12_1929	4
10_150	4	12_1373	5	12_1541	4	12_1712	4	12_1933	5
10_156	4	12_1378	4	12_1542	4	12_1723	5	12_1944	5
11_387	5	12_1382	5	12_1548	5	12_1726	4	12_1946	4
11_391	4	12_1385	5	12_1553	5	12_1737	5	12_1982	4
11_400	5	12_1391	4	12_1598	5	12_1787	5	12_1983	4
11_404	4	12_1396	5	12_1600	5	12_1803	5	12_2008	5
11_437	4	12_1400	5	12_1610	5	12_1804	5	12_2015	5
11_446	5	12_1408	4	12_1628	4	12_1811*	5	12_2016	4
11_449	4	12_1418	5	12_1650	4	12_1825	5	12_2017	4
11_453	4	12_1423	4	12_1652	5	12_1833	5	12_2037	3
11_484	5	12_1430	5	12_1653	5	12_1837	4	12_2053	5
11_491	5	12_1473	4	12_1657	4	12_1839	5	12_2075	4
11_503	4	12_1476	4	12_1672	5	12_1845	5	12_2099	4
11_538	5	12_1486	5	12_1679	5	12_1883	5	12_2122	5
11_547	4	12_1487	4	12_1683	5	12_1884	5	12_2129	5
11_548	5	12_1488	5	12_1684	5	12_1898	4	12_2131	5
12_1295	5	12_1489	5	12_1685	5				

Table 4: Knots with unsharp Morton-Franks-Williams inequality

Theorem 4.1 The Q polynomial determines no Vassiliev knot invariants of degree ≤ 9 which are not polynomials of v_2 .

Proof. By the previous symmetry argument, it suffices to consider degree ≤ 8 . Take the 60 duplications of Q in table 5. We chose them so that the knots are not mutants (which was verified using the hyperbolic volume). We already observed that the invariants of $FF_+PP_+P_-P_0$ generate all invariants up to degree 8. By evaluating these families on the 120 knots in these pairs, we can confirm this. Now consider the matrix obtained by evaluating $v(K_1) - v(K_2)$ for any Vassiliev invariant v of degree ≤ 8 and knots $K_{1,2}$ in a pair (with rows given by a basis of invariants v and columns by pairs of knots). One calculates that this matrix has rank 55, which corresponds to removing from the dimension 60 of Vassiliev invariants of degree ≤ 8 the powers v_2^i for $i = 0, \dots, 4$. (Thus 55 pairs would suffice, but the other 5 are used to have some security in the calculation.) \square

From corollary 3.2 we obtain now

Corollary 4.1 Assume that

$$X \in \mathbf{Q}[x_1, x_2, x_3, \dots, y_0, y_1, y_2, \dots]$$

is an honest polynomial¹. If

$$X(Q'(-2), Q''(-2), \dots, Q(2), Q'(2), Q''(2), \dots)$$

is a polynomially bounded invariant of degree ≤ 9 , then it is *as a knot invariant* a polynomial of degree ≤ 4 in $Q'(-2)$. \square

Note we do not know whether X is a polynomial of degree ≤ 4 in x_1 , since we do not know whether the $Q^{(k)}(\pm 2)$ are algebraically independent invariants. On the opposite side, one can incorporate, with just a bit of reformulation and extra argument, also the values $V^{(k)}(\pm 1)$ into X in a statement of the above type.

¹Note here *not* the completion $\mathbf{Q}[[x_1, x_2, \dots]]$ of $\mathbf{Q}[x_1, x_2, \dots]$ is meant, so that, even if infinitely many variables available, each element has only finitely many monomials, and so also finitely many variables occurring in it.

7_2 - 12_1659	11_415 - 10_8	12_1728 - 12_1668
8_19 - 12_1727	11_431 - 11_395	12_1298 - 12_1295
9_44 - 8_2	10_36 - 10_19	12_1589 - 12_1326
9_45 - 8_7	11_370 - 10_7	11_140 - 12_1770
10_159 - 9_15	11_473 - 12_1823	11_210 - 12_1735
10_131 - 9_8	11_452 - 10_10	11_110 - 12_1468
10_133 - 12_1670	11_388 - 11_371	11_118 - 11_45
11_512 - 10_140	11_491 - 10_38	11_294 - 11_146
10_151 - 9_21	11_374 - 10_30	11_189 - 11_30
10_156 - 9_12	10_72 - 10_43	11_56 - 12_1608
9_26 - 9_25	11_427 - 10_46	11_216 - 11_196
12_1750 - 12_1682	11_546 - 10_71	11_28 - 12_1792
10_35 - 10_22	12_1893 - 12_1556	11_180 - 12_1302
11_492 - 11_435	12_2070 - 12_1337	11_165 - 12_1824
11_434 - 12_1867	12_1789 - 12_1576	11_225 - 12_1630
10_31 - 10_14	12_2105 - 12_1336	11_279 - 12_1913
11_461 - 10_85	12_1458 - 12_1394	11_330 - 11_24
11_453 - 11_385	12_1901 - 12_1709	12_1150 - 12_492
10_56 - 10_33	12_1903 - 12_1652	12_742 - 12_503
11_484 - 10_20	12_1685 - 12_1600	12_882 - 12_212

Table 5: 60 pairs of knots of ≤ 12 crossings with the same Q polynomial, which are not mutants. The comparison of Vassiliev invariants of degree ≤ 8 on them allows to prove theorem 4.1.

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