STIRLING NUMBERS, EULERIAN IDEMPOTENTS AND A DIAGRAM COMPLEX

ALEXANDER STOIMENOW

This is a preprint / draft. Any correction, comment, hint is welcome to the author.

ABSTRACT. We give a homological explanation of the ideals of the Eulerian elements in the ring of the symmetric group in terms of subspaces of relations in the Kohno algebra of singular braid diagrams and discuss some known and new properties of Eulerian elements.

Keywords: Pure braid chord diagrams, Eulerian idempotents, Drinfel'd asscociator, Hochschild cohomology, Harrison cohomology, shuffles.

CONTENTS

Date: Current version: April 11, 1997 First version: January 10, 1996 .
This paper is also available from the WWW site `http://www.informatik.hu-berlin.de/~stoimeno`.

1. Introduction

The Eulerian idempotents $e_n^{(i)}$ are certain elements in the ring $\mathbf{F}S_n$ of the symmetric group S_n over a field $\mathbf{F} \supset \mathbb{Q}$, which came up in the consideration of the Poincaré-Birkhoff-Witt theorem [Re] and were later found to split Hochschild (co)homology [GS, Lo, Lo3]. Several properties of these elements are well-known.

The present paper has 2 aims. The one is to provide more direct combinatorial proofs for some known and new properties of these elements using Stirling numbers [Ri]. The second and main aim is to give a homological interpretation of the ideals $\mathbf{F}S_n \cdot e_n^{(i)}$ of these elements in the symmetric group ring as representations of the symmetric group. Such interpretation was quested by Hanlon [Hl] after work of Orlik and Solomon [OS].

The complex we consider is a resolution of the space of singular pure braid chord diagrams, which is considered in the Vassiliev theory for knots [BL, BN2, BS, Va, Vo] and braids [BN3]. My personal motivation and starting point for considering this space was a conjecture by Drinfel'd on the Harrison cohomology of the space of braid chord diagrams. We will explain this conjecture, however, we will not treat it extensively here. For a detailed discussion and a more topological interpretation see [St]. This paper is an excerpt of this monography.

Although our situation is somewhat different from that of Orlik and Solomon, there appears to be a close connection. Note, that by Vassiliev's approach, there is an interpretation of braid chord diagrams as homology classes within the space of braid immersions. However, this paper fails to give a strongly desired precise explanation of this connection. The author would be more than grateful for a hint how to find one.

1.1. General notations and definitions.

From now on, we will use the following notation.

By $|A|$ we will from now on denote the cardinality of a set A. The set of the first n natural numbers $\{1, \ldots, n\}$ we will denote N_n.

S_n will be the symmetric group of n elements, generated by the transpositions $\tau_i = (i\ i+1)$. Each permutation $\sigma \in S_n$ we will denote by the sequence $\sigma(1)\sigma(2)\ldots\sigma(n)$. The sign of σ, which we will write $sgn(\sigma)$ or $(-1)^\sigma$, can be linearly extended to a map of the group ring of the symmetric group

$$sgn : \mathbf{F}S_n \longrightarrow \mathbf{F}$$

By $\mathrm{dc}(\sigma)$ we will denote the descent of σ (cf. [Lo]), defined by

$$\mathrm{dc}(\sigma) = |\{ 1 \leq i \leq n-1 : \sigma(i) > \sigma(i+1)\}|.$$

The binomial coefficients will be denoted as usual by

$$\binom{n}{k} = \frac{n\,!}{k\,!\,(n-k)\,!}.$$

By $\mathrm{cf}_x^i(\text{expression})$ will be denoted the coefficient of x^i in 'expression' expanded as a polynomial in the formal variable x.

We shall also introduce some handy multiindex notations: For a multiindex $p = (p_i)_{i=1}^k \in \mathbb{N}_+^k$ set the length of p to be $\mathrm{len}(p) = k$ and its norm to be $|p| = \sum_{i=1}^k p_i$.

Let $\mathcal{P}(A)$ be the power set (the set of all subsets) of a given set A which forms a lattice with the inclusion relation.

Accoring to the referee's comment, the following remark is in place: an "exercise" is a statement the author has, but does not include a proof of, for space resons on the one hand and on the other hand because he hopes that it can be deduced by the reader by (maybe messy but) elementary means, or found in the references. The author is always open to remarks or questions concerning the difficulty or style of these exercises.

1.2. **Acknowledgements.** I would like to thank DROR BAR-NATAN, whose papers [BN] – [BN4] offered me a comprehensive introduction to the theory of VASSILIEV invariants for the many interesting discussions.

I would also like to thank TH. FRIEDRICH and A. NESTKE from Humboldt University and E. VOGT from Freie Universität Berlin for their support during my studies and supervising my work and V. TURAEV and C. KASSEL for supporting my application in France and for inviting me to Strasbourg to give a talk. There I also had some very helpful discussions with C. REUTENAUER and J.-L. LODAY.

M. LÜDDE from the Graduiertenkolleg "Geometrie und nichtlineare Analysis" gave me some useful hints and pointed out some mistakes in [St].

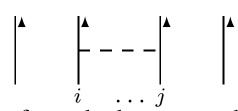
2. THE ALGEBRA OF BRAID CHORD DIAGRAMS

2.1. **The complexes A^* and A^*_{nc}.** The n strand pure braid diagram algebra A^n is the quotient of the free (non-commutative) algebra A^n_{nc} a of $\binom{n}{2}$ generators $\{t_{ij},\ 1 \le i \ne j \le n\}$ with $t_{ij} = t_{ji}$ by factoring out the following relations

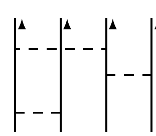
$$\begin{array}{rll} R_{ijk} := [t_{ij} + t_{ik}, t_{jk}] & |\{i,j,k\}| = 3 & \text{(}4T\text{-relation)} \\ R_{ijkl} := [t_{ij}, t_{kl}] & |\{i,j,k,l\}| = 4 & \text{(isotopy relation)} \end{array} \tag{2.1}$$

Both algebras are graded by setting each t_{ij} to have degree 1. Let A^n_m and $A^n_{m,nc}$ denote their degree-m-pieces.

The algebra A^n was first considered by Kohno [K] and later by Drinfel'd [Dr] and Bar-Natan [BN, BN3]. It has a well-known diagram interpretation, used in the theory of Vassiliev invariants for braids [BN3]. Set

$t_{ij} \quad =$  ,

and denote diagrammatic multiplication from the bottom to the top, e. g.

$t_{12}t_{34}t_{13} \ =$  $\in \ A^4_3$.

These algebras carry a comsimplicial structure [Lo2].

We have a natural strand permuting (left) action of S_n on A^n. We will denote for $\sigma \in S_n$ and $\xi \in A^n$ the result of the action $\sigma(\xi)$ by ξ^σ. In terms of the t_{ij}'s the action of $\sigma \in S_n$ on

A^n is the algebra endomorphism defined by $t_{ij} \mapsto t_{\sigma(i)\sigma(j)}$, which is obviously graduation-preserving.

The tautological embedding $i^n : A^n \hookrightarrow A^{n+1}$, defines for each $\tilde{\sigma} : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, n+1\}$ an embedding (denoted by the sequence $\sigma(1)\sigma(2)\ldots\sigma(n)$)

$$(.)^{\tilde{\sigma}(1)\ldots\tilde{\sigma}(n)} : A_m^n \hookrightarrow A_m^{n+1}$$

by

$$(.)^{\tilde{\sigma}(1)\ldots\tilde{\sigma}(n)} := \sigma \circ i^n ,$$

where $\sigma \in S_{n+1}$ is determined by $\sigma\Big|_{\{1,\ldots,n\}} = \tilde{\sigma}$.

The doubling maps $\Delta^i : A_m^n \hookrightarrow A_m^{n+1}$, $1 \le i \le n$ are given as algebra morphisms on t_{jk} $(j < k)$ by

$$\Delta^i(t_{jk}) = \begin{cases} t_{j+1,k+1} & i < j \\ t_{j,k+1} + t_{j+1,k+1} & i = j \\ t_{j,k+1} & j < i < k \\ t_{jk} + t_{j,k+1} & i = k \\ t_{jk} & i > k \end{cases} .$$

A_m^* becomes a *cosimplicial algebra* with coface maps

$$d_{(i)}^n : A_m^n \to A_m^{n+1} \quad 0 \le i \le n+1,$$

where

$$d_{(0)}^n = (.)^{2\ldots n+1}, \quad d_{(n+1)}^n = (.)^{1,\ldots,n},$$

$$d_{(i)}^n = \Delta^i \quad 1 \le i \le n ,$$

and codegeneracy maps $s_{(i)}^n : A_m^n \to A_m^{n-1}$ for $1 \le i \le n$, given on t_{jk} $(j < k)$ by

$$s_{(i)}^n(t_{jk}) = \begin{cases} t_{j-1,k-1} & i < j \\ 0 & i = j \\ t_{j,k-1} & j < i < k \\ 0 & i = k \\ t_{jk} & i > k \end{cases} .$$

Recall [Lo2], that each cosimplicial vector space becomes a cochain complex with the coboundary operator $d^n : A_m^n \to A_m^{n+1}$ given by

$$d^n = \sum_{i=0}^{n+1} (-1)^i \, d_{(i)}^n .$$

2.2. **Hodge decomposition and Harrison cohomology.** The diagram complexes A^* and A_{nc}^* have a certain quasi-HOCHSCHILD cochain complex structure [BN], so in the following it will be useful to recall some facts of standard HOCHSCHILD cohomology which have their direct translation to our context.

Look at a HOCHSCHILD cochain complex C^*. Then one has a left action of S_n on C^n, so C^n becomes a (left) $\mathbb{Q}[S_n]$ module.

There is a well known family of orthogonal projectors

$$e^n = e_{(1)}^n, e_{(2)}^n \ldots, e_{(n)}^n \in \mathbb{Q}[S_n]$$

also called EULERian idempotents (cf. [Lo]), which can be defined by

$$e^n_{(i)} \;:=\; \sum_{\sigma\in S_n} (-1)^\sigma \sigma^{-1}\,\mathrm{cf}^i_t \left[\binom{t+n-\mathrm{dc}(\sigma)-1}{n}\right] . \tag{2.2}$$

The following properties of these elements are well-known.

Lemma 2.1. *The* EULER*ian elements satisfy*

$$e^n_{(i)}e^n_{(j)} \;=\; \begin{cases} e^n_{(i)} & i=j \\ 0 & i\neq j \end{cases} \qquad \textit{and} \tag{2.3}$$

$$d^n e^n_{(i)} \;= e^{n+1}_{(i)} d^n \qquad 1\le i\le n\,.$$

We will omit the subscription of the first idempotent, because it will appear often in the following. However, this can be motivated also by a historical background, since e^n was discovered years before its "relatives". It was constructed already in [Ba], whereas the first appearance of the whole family is (as far as I know) in [GS].

In fact, here we use the dualization of all EULERian elements $e^{(i)}_n$ constructed in the various above cited papers on HOCHSCHILD homology, and our $e^n_{(i)}$'s are obtained from them by the linear extension of the map $\sigma\mapsto\sigma^{-1}$ which is an algebra antiautomorphism of $\mathbf{F}S_n$ (reversing the multiplication order).

We will later need a further variation of the EULERian elements, so let us introduce the following notation.

Definition 2.2. *There is an automorphism of* $\mathbf{F}S_n$ *as algebra, given by* $\sigma\mapsto(-1)^\sigma\sigma$ *for* $\sigma\in S_n$*. Let us denote this automorphism by " $\bar{\ }$ ".*

Exercise 2.3. Show that

$$e^n_{(1)} = \frac{1}{n}\sum_{\sigma\in S_n}(-1)^\sigma\sigma^{-1}(-1)^{\mathrm{dc}(\sigma)}\binom{n-1}{\mathrm{dc}(\sigma)}^{-1} \tag{2.4}$$

and that

$$e^n_{(n)} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\sigma \;=:\; \varepsilon_n$$

is the totally antisymmetric element in $\mathbb{Q}[S_n]$.

In the following we will introduce a set of integers appearing e. g. in combinatorial enumeration problems, called STIRLING numbers [Ri] of the first kind $S^{(m)}_n$. These numbers can be defined by the polynomial equation in x

$$x(x-1)\cdot\cdots\cdot(x-n+1) \;=\; \sum_{m=1}^{n} x^m S^{(m)}_n\,,$$

or alternatively by

$$S^{(m)}_n \;=\; (-1)^{n-m}\,|\{\,\sigma\in S_n : \sigma \text{ has exactly } m \text{ cycles }\}|\,. \tag{2.5}$$

In particular, we have $S^{(1)}_n = (-1)^{n-1}(n-1)\,!$, $S^{(n)}_n = 1$ and $S^{(n-1)}_n = -\binom{n}{2}$.

Exercise 2.4. Generalize the formula (2.4) by showing that

$$e_n^{(i)} = \frac{1}{n!} \sum_{\sigma} (-1)^{\sigma} \sigma^{-1} \sum_{j=1}^{i} |S_{n-\mathrm{dc}(\sigma)}^{(j)}| \, S_{\mathrm{dc}(\sigma)+1}^{(i+1-j)} .$$

The properties of the EULERian elements in lemma 2.1 imply that for

$$C_{(i)}^n := Im \, e_{(i)}^n$$

$C_{(i)}^*$ form subcomplexes of C^* and as vector spaces they give a complete direct decomposition of C^*. This yields a decomposition of the HOCHSCHILD cohomology $H^* = H^*(C^*)$ into $H_{(i)}^* = H^*(C_{(i)}^*)$, called HODGE decomposition,

$$H^n = \bigoplus_{i=1}^{n} H_{(i)}^n .$$

To define HARRISON cohomology we introduce the notion of a shuffle.

Definition 2.5. *For a multiindex $p = (p_1, \ldots, p_k)$ with $|p| = n$ a permutation $\sigma \in S_n$ is called a p-shuffle, if*

$$\sigma(i) < \sigma(i+1), \text{ for all } i \in N_n \setminus \Big\{ \sum_{i=1}^{j} p_i \Big\}_{j=1}^{k} .$$

Definition 2.6. *Let us define the following shuffle operators in $\mathbb{Q}[S_n]$ (cf.* [BN, Ba] *)*

$$s_{p_1,\ldots,p_k} := \sum_{\sigma (p_1,\ldots,p_k)-\textit{shuffle}} (-1)^{\sigma} \sigma \tag{2.6}$$

$$\omega_{p_1,\ldots,p_k} := \sum_{\sigma (p_1,\ldots,p_k)-\textit{shuffle}} (-1)^{\sigma} \sigma^{-1} \tag{2.7}$$

The original definition of the HARRISON subcomplex $C_{\mathrm{Harr}}^* \subset C^*$ [Ba] can be written in our notation

$$C_{\mathrm{Harr}}^n := \Big\{ \xi \in C^n : \quad \omega_{pq} \xi = 0 \quad p + q = n \quad p, q \geq 1 \Big\}$$

and its cohomology is called HARRISON cohomology.

Exercise 2.7. Prove the following formula

$$\omega_{pq}(d^n \xi) = (d^{p-1} \otimes 1_q)(\omega_{p-1,q} \xi) + (-1)^p \, (1_p \otimes d^{q-1})(\omega_{p,q-1} \xi) , \tag{2.8}$$

where the “twisted” codifferentials are defined by

$$d^p \otimes 1_q = \sum_{i=1}^{p} (-1)^{i-1} d_{(i)}^{p+q} \qquad \text{and}$$

$$1_p \otimes d^q = \sum_{i=p+1}^{p+q} (-1)^{i-p-1} d_{(i)}^{p+q}$$

and deduce from this formula that C_{Harr}^* is really a subcomplex of C^*.

Remark 2.8. The formula (2.8) appeared already in BARR’s paper [Ba], there stated for homology (see proposition 2.2.).

Already BARR [Ba] represented C^*_{Harr} as the image sequence of a projector family e^*. However, the shape of e^* was not quite clear since they have been constructed inductively (over n). Later one found (see [Lo, GS]) that $e^n_{(1)}$ is exactly BARR's old e^n, and so the first component in the HODGE decomposition is the same as the HARRISON subcomplex.

$$C^n_{\text{Harr}} \cong C^n_{(1)}$$

This comes from the following statement

Lemma 2.9. *In $\mathbb{Q}[S_n]$ we have*

$$Im\, e^n_{(1)} = \bigcap_{p+q=n} \ker \omega_{pq} \tag{2.9}$$

There is already an inductive proof over n of (2.9) in [Ba]. However, it is possible to give a direct combinatorial proof, which I will just sketch here.

Proof sketch. For the inclusion '$\subset$' in (2.9) you have to show that $\omega_{pq} e^n_{(1)} = 0$.

Expand $\omega_{pq} e^n_{(1)}(\xi)$ using (2.4) and (2.7) into the form $\sum_{\sigma\in S_n} c_\sigma(\xi)\xi^\sigma$. Using lemma 2.10 and the following identity

$$\sum_{k=0}^{n-1} \binom{n-1-k}{q-1}\binom{q}{k}(-1)^k = 0 \qquad \text{for } 1 \le q \le n-1\,,$$

show that $c_{\text{Id}}(\xi) = 0$ and then deduce the vanishing of all other coefficients from the formula

$$c_{\tilde\sigma}(\xi^\varrho) = (-1)^\varrho \cdot c_{\varrho\circ\tilde\sigma}(\xi)\,.$$

For more details, see [St].

To show the inclusion '$\supset$', define the following elements in $\mathbb{Q}[S_n]$

$$\begin{aligned}
\lambda_0 &:= Id \\
\lambda_1 &:= \sum_{i=1}^{n-1} \omega_{i,n-i} \\
\lambda_2 &:= \sum_{1\le i_1<i_2\le n-1} (\omega_{i_1,i_2-i_1} \otimes 1_{n-i_2}) \cdot \omega_{i_2,n-i_2}
\end{aligned}$$

a. s. o. Prove that

$$\lambda_i = \sum_{\sigma\in S_n} (-1)^\sigma \sigma^{-1} \binom{n-1-\text{dc}(\sigma)}{i-\text{dc}(\sigma)}\,.$$

Now you have

$$\sum_{i=0}^{n-1} a_i\lambda_i = n e^n_{(1)}\,,$$

where (by comparison of coefficients on the Euler partition)

$$\left(\binom{n-1-d}{i-d}\right)_{d,i=0}^{n-1} (a_i)_{i=0}^{n-1} = \left(\frac{(-1)^i}{\binom{n-1}{i}}\right)_{i=0}^{n-1}.$$

Invert the matrix of binomial coefficients and get $a_0 = n$. So $e^n_{(1)} - Id$ lies in the right ideal of the ω_{pq}'s, and all elements killed by all ω_{pq}'s are fixed by $e^n_{(1)}$, especially they lie in its image.

Joining both parts, the lemma is proven. Additionally we see, that $e^n_{(1)}$ fixes all elements in its image, i. e. it is an idempotent. □

Doing the previous proof I found the following lemma, which is also interesting for itself.

Lemma 2.10. *There are exactly* $\binom{p}{k}\binom{q}{k}$ *pq-shuffles* σ *with* $\mathrm{dc}(\sigma^{-1}) = k$.

Proof. Let $m^k_{p,q}$ be the number of all pq-shuffles σ with $\mathrm{dc}(\sigma^{-1}) = k$. Denoting as usual σ by $\sigma(1)\dots\sigma(n)$ and sorting all σ's as above by the position of the '1' in this notation one has

$$\begin{aligned} m^k_{p,q} &= m^k_{p-1,q} + m^{k-1}_{p-1,q-1} + m^{k-1}_{p-1,q-2} + \dots + m^{k-1}_{p-1,0} \\ &= m^k_{p-1,q} + \sum_{l=0}^{q-1} m^{k-1}_{p-1,l} \end{aligned} \tag{2.10}$$

and $m^0_{p,q} = 1$ und $m^k_{0,q} = 0$ for $k \geq 1$. The j-th summand on the right hand-side of (2.10) is exactly the number of pq shuffles σ with $\mathrm{dc}(\sigma^{-1}) = k$ where '1' appears at position j.

Recursively applied (2.10) gives

$$m^k_{p,q} = \begin{cases} \sum_{l=0}^{q-1}\sum_{n=0}^{p-1} m^{k-1}_{l,n} & k > 0 \\ 1 & k = 0 \end{cases}$$

from which the assertion follows inductively over k using the triangular property of binomial coefficients. □

Looking again at our special case $C^* = A^*$, another easy property of elements in A^*_{Harr} could be verified in the following

Exercise 2.11. (see [St]) For $n > 2$ and for all

$$\xi = \sum_{\alpha \in \binom{N_n}{2}^m} a_\alpha t_\alpha \in A^n_{m,\mathrm{Harr}}$$

there holds

$$\sum_{i \in \binom{N_n}{2}^m} a_i = 0\,.$$

Hint: Look at

$$\mathrm{cf}^p_x\Big(\underbrace{(x-1)(x+1)(x-1)\dots(x+(-1)^n)}_{n \text{ factors}}\Big)$$

and deduce from it a formula for

$$sgn(\bar{\omega}_{pq}) \;=\; sgn(\bar{s}_{pq}) \;=\; \sum_{\sigma\ pq\text{-shuffle}} (-1)^\sigma$$

in terms of binomial coefficients. Show especially that this sum vanishes only if both p and q are odd. If you would like to, you could try to generalize the formula for $sgn(\bar{s}_{pq})$ for *all* multiindices p and show that it vanishes exactly if p has at least 2 odd components.

Finally, let us recall another usual construction in HOCHSCHILD cohomology.

There is a subcomplex in $\tilde{I}_m^*$ called the *symmetric* subcomplex which is the image of the family of projectors

$$s_n \;:=\; \frac{1-(-1)^{n(n+1)/2}\omega_n}{2}\,,$$

where $\omega_n \in S_n$ is the transposition element, $\omega_n(i) = n+1-i$, $i = 1,\ldots,n$. It is straight forward to show using the shape of $e_{(1)}^n$ in (2.4), that latter is symmetric, i. e.

$$s_n \cdot e_{(1)}^n \;=\; e_{(1)}^n \cdot s_n \;=\; e_{(1)}^n\,,$$

and so the HARRISON subcomplex of HOCHSCHILD cohomology is a subcomplex also of the symmetric subcomplex.

Note, that because of the orthogonality properties (2.3) we have a decomposition of the group algebra $\mathbf{F}S_n$ into the principal right ideals

$$\mathbf{F}S_n \cdot e_n^{(i)} \;=:\; \mathbf{F}S_{n,i}$$

of the EULERian idempotents

$$\mathbf{F}S_n \quad=\quad \bigoplus_{i=1}^{n} \mathbf{F}S_{n,i}\,,$$

which also gives a decomposition of the regular representation of S_n. The resulting representations $\mathbf{F}S_{n,i}$ have been studied by HANLON in [Hl]. In particular, he derived a formula for their dimensions. This formula is however originally due to REUTENAUER [Re]. REUTENAUER's result we can write in the following way.

Theorem 2.12 (see [Hl, corollary 5.13.])**.**

$$\dim \mathbf{F}S_{n,i} \;=\; (-1)^i S_n^{(n-i)} \;=\; \dim \tilde{I}_{n-i}^n\;. \tag{2.11}$$

Although the definitions are quite explicite, it was not yet possible to solve a profound problem pointed out by Drinfel'd

Conjecture 2.13 (Drinfel'd [Dr], see also [BN])**.**

$$H_{Harr}^4\,(A^*) \;=\; 0\,.$$

This conjecture came about at the attempt to construct homologically a Drinfel'd associator [Ca, Pi, Dr] in A^3. The known solutions of Drinfel'd [Dr] use analysis and a purely homological construction yet works only in an unfredlier-looking extension of A^3 [BN, BN4, BS]. See [St] for more details on this conjecture.

3. THE SPACES $\tilde{I}_m^n$

The kernel I_m^n of the map $A_{m,nc}^n \to A_m^n$ is a biideal in $A_{m,nc}^n$ whose degree-m-piece decomposes into the (non-direct) sum of $m-1$ spaces.

$$I_m^n \;=\; \sum_{j=1}^{m-1} I_{m,j}^n\,, \;\text{ where } \quad I_{m,j}^n := A_{nc,j-1}^n I_2^n A_{nc,m-j-1}^n\,. \tag{3.1}$$

Henceforth we will consider the spaces

$$\tilde{I}^n_m := \bigcap_{j=1}^{m-1} I^n_{m,j} \; .$$

In the following we will first identify them with the Eulerian ideals and later motivate the consideration of such spaces by giving them an interpretation as relations in a resolution of A^n.

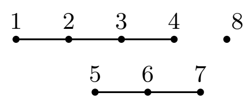
For former, we need some graph theory. The most terms from graph theory used in the following are standard (see e.g. [Ha, Cm]). We will make precise only two of them.

Definition 3.1. *The* restriction[1] *of the graph $D = (G, E)$ on $G' \subset G$ is the graph*

$$D\Big|_{G'} := (G', E \cap \mathcal{E}(G')) \; ,$$

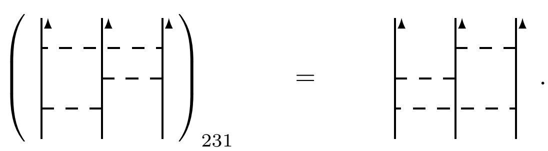
where $\mathcal{E}(\Lambda)$ denotes the set of 2-element subsets in Λ.

Definition 3.2. *A tree graph (G, E) is called a* path graph*, if and only if all vertices in G have a valence ≤ 2, or equivalently if for each connected component G_i of G there is a path[2] in G_i which covers all the edges in G_i. There is a set of standard path graphs which will be denoted as follows. For a multiindex p with $|p| = n$ and* $\mathrm{len}(p) = k$ *let γ_p be the graph consisting of paths between $1, \ldots, p_1$; $p_1 + 1, \ldots, p_1 + p_2$; … ; $n - p_k + 1, \ldots, n$. E. g. we have*

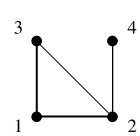
$\gamma_{4,3,1} =$ 

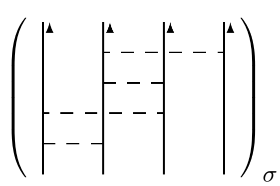
(3.2)

3.1. **The graph representation of $\tilde{I}^n_m$.** S_m acts on A^n_m by vertically permuting the chords or permuting the various t_{ij} within the monomials. This action we will denote by subscribing the permutation $(.)_\sigma$ for $\sigma \in S_m$, e. g.



Define a graph $G = (N_n, E)$ with $E \subset \mathcal{E}(N_n)$ and $|E| = m$ to represent the vertical antisymmetrization in F^n_m of the lexicographically increasingly ordered monomial corresponding to the edges in G (where the edge between i and j is sent to t_{ij}).



$\simeq \sum_{\sigma \in S_4} (-1)^\sigma$ 

.

[1] This is calssically called *induced subgraph*.
[2] In the sense that no vertex is passed twice!

The strand permuting action of S_n on graphs can be described as follows.

$$D^\sigma \;=\; \sigma(D)\,\cdot\,sgn(\mathcal{E}(\sigma))\,, \tag{3.3}$$

where $\sigma(D)$ is D with all vertices i replaced by $\sigma(i)$ and $\mathcal{E}(\sigma)$ is the map induced by σ on $\mathcal{E}(N_n)$. Its sign is taken with respect to the lexicographic ordering by the lower element (i. e. $\{1,4\}<\{2,3\}$), which we will fix from now on on $\mathcal{E}(N_n)$.

Example 3.3. For $p+q=n$ the "full" pq-shuffle $v_{pq}\in S_n$ given by $i\;\mapsto\; i+q$ for $i\le p$ sends the graph γ_{pq} to $(-1)^{(p-1)(q-1)}\gamma_{qp}$.

In the same way do the following

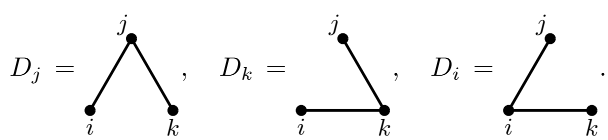
Exercise 3.4. Show using (3.3) that the transposer ω_n acts on γ_n by multiplication with $(-1)^{(n-1)(n-2)/2}$.

Evidently, $\tilde{I}^n_m$ is vertically totally antisymmetric, and so we can describe it by graphs. Note, that because of the total antisymmetry we have no diagrams with 2 copies of one and the same chord, so our graphs are always simple (no double edges).

Elements in $\tilde{I}^n_m$ satisfy some additional linear relations which come from the fact that commutators of chords between non-disjoint pairs of strands appear pairwise. We will call these relations $\triangle$-relations, because of what they say in terms of graphs. Let us first introduce the following notation.

Definition 3.5. *For a graph $D=(G,E)$ and a triple $T=\{i,j,k\}\subset E$ such that $D\Big|_T$ is trivial (empty) denote for each $a\in T$ by D_a the graph obtained from D by adding the two edges from a to the other two vertices in T.*

Using the same convention as in the case of knots and depicting only the differing parts of the three graphs we will denote



Now the $\triangle$-relation can be written for $i<j<k$ as [St]

$$(-1)^{d_D(ij,jk)}D_j = (-1)^{d_D(jk,ki)}D_k + (-1)^{d_D(ki,ij)}D_i\,, \tag{3.4}$$

where for a graph $D=(N_n,E)$ and for $\alpha,\beta\in\mathcal{E}(N_n)$ the number $d_D(\alpha,\beta)$ is given by[3]

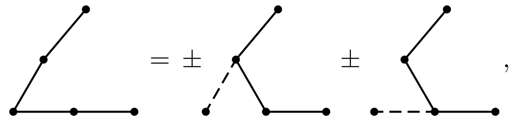
$$d_D(\alpha,\beta)\;:=\;|\{(\alpha,\beta)\cap E\}|\;=\;|\{\,g\in E\;:\;\;\alpha<g<\beta\,\}|\,.$$

Although calculating with the graphs becomes quite messy because of the signs, they are helpful in explaining certain facts.

Lemma 3.6. *In the graph representation of each $g\in\tilde{I}^n_m$ all diagrams have graphs without a cycle.*

[3]Remember, that despite of our convention the coefficients in (3.4) *do* depend on what happens in D outside the three vertices drawn !

Proof. It is an immediate consequence of the $\triangle$-property. Consider the coefficient of g of a graph containing the following part of a cycle. One has



and the assertion follows by induction over the length of the cycle. $\square$

An immediate consequence of the lemma is

Corollary 3.7.

$$\tilde{I}_k^n = 0 \quad \textit{for} \quad n \leq k. \tag{3.5}$$

3.2. **On the dimension of the spaces $\tilde{I}_m^n$.** Beside the vanishing property (3.5) there is a more general formula about the dimension of the spaces $\tilde{I}_m^n$.

Theorem 3.8. *The dimension of the space $\tilde{I}_m^n$ is the absolute value of the Stirling number $S_n^{(n-m)}$.*

Proof. Note that $\tilde{I}_m^n$ for $n > m$ consists of linear combinations of graphs with exactly $n-m$ components. It is not quite correct that in the identification of diagrams with graphs as introduced above the $\triangle$-relation acts within connected graph components separately. But there is another way of identifying diagrams with graphs (sort first the chords by the connected component of the resulting graph they appear in and order them lexicographically *within* this partition) which allows us to make the identification

$$\bigoplus_{\substack{\mathrm{len}(p) = n-m \\ |p| = n}} \bigotimes_{i=1}^{n-m} \tilde{I}_{p_i - 1}^{p_i} \quad \simeq \quad \tilde{I}_m^n \,. \tag{3.6}$$

So if we look only at the dimensions of the spaces, we see that it suffices to study the case of one connected component, which is $\tilde{I}_m^{m+1}$. The dimension of this vector space in turn can be computed recursively using the following

Theorem 3.9. *There is an isomorphism*

$$\tilde{I}_m^{m+1} \simeq \bigoplus_{i=0}^{m-1} \tilde{I}_i^m \,.$$

Using theorem 3.9 and (3.6) one obtains immediately a numeric recursion for the numbers $\dim \tilde{I}_m^n$, and then it is not hard to deduce that the numbers satisfying this recursive property are exactly those appearing in theorem 3.8. $\square$

Proof sketch of theorem 3.9. The isomorphism

$$\Phi \,:\, \tilde{I}_m^{m+1} \longrightarrow \bigoplus_{i=0}^{m-1} \tilde{I}_i^m$$

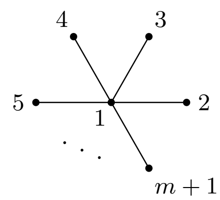
can be given as follows. It is sufficient to say what is Φ of a graph G.
Sort the connected components of $G\Big|_{\{2,\ldots,m+1\}}$ by the lowest vertex number they contain and enumerate them in this order $G_1, \ldots, G_{\mathrm{val}_G(1)}$. Since G does not contain cycles '1' is in G connected to exactly one vertex in each of the G_i's. Enumerate the G_i's once

again as $G^1, \ldots, G^{\mathrm{val}_G(1)}$, but this time by the increasing order of their vertex connected to '1' in G. Then set

$$\Phi(G) := sgn\begin{pmatrix} G_1, \ldots, G_{\mathrm{val}_G(1)} \\ G^1, \ldots, G^{\mathrm{val}_G(1)} \end{pmatrix} \cdot G\Big|_{\{2,\ldots,m+1\}}.$$

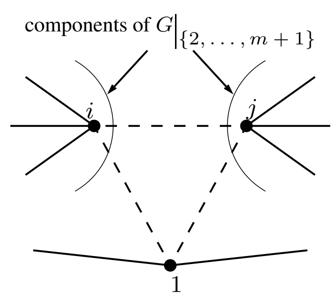
The proof that this is an isomorphism is for the technical part a tedious calculation with the signs appearing in (3.4). So we should rather outline only the idea.

We have to show that we always find for a $g \in \bigoplus_i \tilde{I}_i^m$ a unique preimage in $\tilde{I}_m^{m+1}$. We construct it inductively on decreasing valence of '1'. Start with the graph with maximal valence of '1', i. e. the star



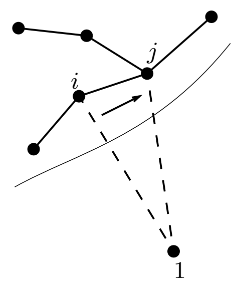
Fix its coefficient in $\Phi^{-1}(g)$ to be the same as the coefficient of the empty graph in g. This is the only choice for this coefficient we can make. Then assume we have constructed all coefficients of graphs G in $\Phi^{-1}(g)$ where $\mathrm{val}_G(1) > k$ and that all $\triangle$-properties which can be written using only such graphs are fixed to hold in $\Phi^{-1}(g)$.

Now look for those of the graphs G (trees with vertex set N_{m+1}) where $\mathrm{val}_G(1) = k$. Consider the following picture



where without the dashed edges '1' has valence $k-1$, and let G_a for $a \in \{i, j, 1\}$ be the graph where exactly the 2 dashed edges emitting from a are added. We know by induction what is the coefficient c_{G_1} of the graph G_1 in $\Phi^{-1}(g)$ (because there '1' has valence $k+1$). If we want to fix the $\triangle$-property to hold for c_{G_1}, c_{G_i} and c_{G_j}, we immediately obtain a relation between c_{G_i} and c_{G_j}. Applying the idea repeatedly we can "slide" (as shown on figure 1) the connections of 1 in G along the connected components of $G\Big|_{\{2,\ldots,m+1\}}$ at the cost of some (already constructed) coefficients of $\Phi^{-1}(g)$.

In this manner we can express each coefficient of a graph obtained in this way from certain G as $\pm c_G$ and some known number (a messy signed sum of coefficients of graphs with a higher valence of '1'). On the other hand the coefficient of $G\Big|_{\{2,\ldots,m+1\}}$ in g is by definition of Φ the (properly signed) sum of the coefficients of exactly those graphs in



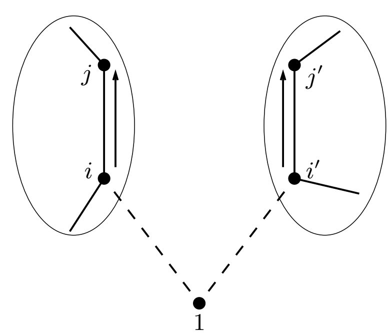
timestimes**Figure 1.** Sliding one step (from i to j) an edge emitting from '1' and connecting one of the components in $G\big|_{\{2,\ldots,m+1\}}$.

$\Phi^{-1}(g)$ which can be obtained from G by the "sliding" as explained above, so it can be written as a linear term in the coefficient c_G of G in $\Phi^{-1}(g)$.

To determine latter now we have to solve a linear equation in one variable. The only dangerous thing that can happen is that the linear term in this equation vanishes. But this is avoided by this strange looking choice of the signs in the definition of Φ. So, doing this for all graphs G with $\mathrm{val}_G(1) = k$ we have constructed Φ a step further, and the choice of the new coefficients is unique.

We should not forget to check two things.

First of all there are many ways of obtaining a certain graph G' from G by the "sliding". We should ensure that the linear dependency of $c_{G'}$ by c_G does not depend on the concrete sliding we have chosen (else we might obtain further restrictions on our coefficient c_G we are about to construct and could stay without no solution at all). This is, we have to verify that sliding 1 step in two components commutes (see figure 2). One can show that this is a consequence of the $\triangle$-properties assumed for $\Phi^{-1}(g)$ on graphs with higher $\mathrm{val}_G(1)$.



timestimes**Figure 2.** Sliding of the edge emitting from 1 in two components in the rest of G

The other thing is, we have per ansatz fixed the $\triangle$-property to hold only for all such triples of vertices which contain the vertex labeled by '1'. So we still have to verify all the others. They can be deduced from the $\triangle$-property of the coefficients in g.

This completes the induction step and the proof sketch. $\square$

For an impression, table 1 shows the dimensions of the first $\tilde I^n_m$. We have in particular $\dim \tilde I^{m+1}_m = m!$ and $\dim \tilde I^n_1 = \binom{n}{2}$.

timestimes**Table 1.** The STIRLING numbers appearing as dimensions of the spaces $\tilde I^n_m$ for $n, m \leq 10$.

n/m	1	2	3	4	5	6	7	8	9
2	1								
3	3	2							
4	6	11	6			$\dim \tilde I^n_m$			
5	10	35	50	24					
6	15	85	225	274	120				
7	21	175	735	1,624	1,764	720			
8	28	322	1,960	6,769	13,132	13,068	5,040		
9	36	546	4,536	22,449	67,284	118,124	109,584	40,320	
10	45	870	9,450	63,273	269,325	723,680	1,172,700	1,026,576	362,880

Exercise 3.10. Analogously to the previous considerations, show that the dimension of the *non-degenerate* part $\tilde I^n_{m,nd}$ of the space $\tilde I^n_m$, that is, the space containing graphs with no isolated vertices (valence $= 0$), is the number $\tilde S^{(n-m)}_n$, which is defined by

$$\tilde S^{(m)}_n \;:=\; |\{\, \sigma \in S_n : \sigma \text{ has exactly } m \text{ cycles and no fixelements } \}| \,. \tag{3.7}$$

The obvious way of calculating $\tilde S^{(m)}_n$ out of $S^{(m)}_n$ is by grouping permutations in S_n with m cycles by the number i of fixelements. Then one gets

$$\tilde S^{(m)}_n \;=\; \sum_{i=0}^{m} (-1)^i \binom{n}{i} |S^{(m-i)}_{n-i}| \;.$$

One can observe that

$$\tilde S^{(m)}_n \;=\; \mathrm{cf}^m_x\, Q_n(x)\,, \tag{3.8}$$

where the polynomials Q_n are given recursively by $Q_0 := 1$, $Q_1 := 0$ and

$$Q_n \;:=\; (n-1)\,[\,Q_{n-1} + xQ_{n-2}\,]$$

for $n > 1$, i. e. we have for $n > 1$ the equality

$$\tilde S^{(m)}_n \;=\; (n-1)\,\left[\,\tilde S^{(m)}_{n-1} + \tilde S^{(m-1)}_{n-2}\,\right] \;. \tag{3.9}$$

Exercise 3.11. Show the property (3.9).

3.3. **The complexes $\tilde I^*_m$ and the group ring $\mathbf{F}S_n$.** Remember that our linear spaces of diagrams $\tilde I^n_m$ can be represented as linear spaces of graphs of n vertices and m edges with the $\triangle$-relation imposed.

The following lemma just says that using $\triangle$ we can always "resolve" in $\tilde I^*_m$ (on the level of coefficients) all ≥ 3valent vertices in a graph. First let

$$\hat I^n_m := \mathcal{L}in\{ \text{ path graphs } (G,E) \text{ with } G = \{1,\dots,n\} \text{ and } |E| = m \;\} \;.$$

Then we have the following

Lemma 3.12. *The obvious map*

$$\phi^n_m \,:\, \tilde{I}^n_m \longrightarrow \hat{I}^n_m$$

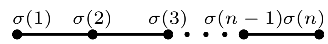
which sends each non-path graph to 0, is an injection.

Note also, that since the $\triangle$-relations are S_n invariant (that is, $\sigma(\triangle_{ijk}) = \triangle_{\sigma(i)\sigma(j)\sigma(k)}$), so is ϕ^n_m, i. e. $\sigma \circ \phi^n_m = \phi^n_m \circ \sigma$.

However, the linear dependency of coefficients of graphs with a $\geq$ 3valent vertex of such of path graphs is not all which is implied by the $\triangle$-relation. There are some additional relations in $\hat{I}^n_m$ (i. e. relations between the coefficients of path graphs) on the image of ϕ^n_m caused by the $\triangle$-condition in $\tilde{I}^n_m$. To see what happens, first take $n = m+1$, i. e. consider path graphs of only one component.

Since $\hat{I}^n_{n-1}$ is a left module over $\mathbf{F}S_n$ we have a module homomorphism $\iota \,:\, \mathbf{F}S_n \rightarrow \hat{I}^n_{n-1}$ given by

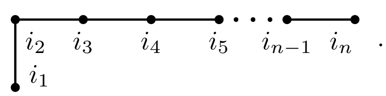
$$\iota(\sigma) \,:=\, \sigma(\gamma_n) \,=\, \begin{smallmatrix}\text{some}\\ \text{sign}\end{smallmatrix} \cdot \tag{3.10}$$



Using exercise 3.4 we see that s_n acts trivially on $\hat{I}^n_{n-1}$, and, since both spaces have dimension $\frac{n!}{2}$, ι descends to an isomorphism of left S_n modules

$$\mathbf{F}S_{n,\text{sym}} \,\simeq\, \hat{I}^n_{n-1}\,.$$

Now look at the following picture



Using $\triangle$ we can "slide" the edge emitting from i_1 from i_2 to i_3, i_4, and finally to i_n getting in each step a graph in which i_1 appears somewhere between the other i_j's. This procedure gives a linear relation between the coefficient of $\sigma = i_1 \dots i_n$ and those of permutations of the form $\sigma\tau^{-1}$ where τ is a $(1, n-1)$-shuffle. The signs turn out to be all equal to $(-1)^\sigma$ and so we have the condition that on $\iota(\tilde{I}^n_m)$ the right multiplication with $\tilde{\omega}_{1,n-1}$ gives 0. To see this, check it explicitly for $i_j = j$ and use the S_n invariance of the $\triangle$-relation to deduce it for all other path graphs. One can slide in a similar way p points ($p \leq \left[\frac{n}{2}\right]$) of the graph obtaining $\tilde{\omega}_{pq} \equiv 0$ for $q = n-p$. This means that under the identification $\hat{I}^n_{n-1} \simeq \mathbf{F}S_{n,\text{sym}}$ $\quad \phi^n_{n-1}(\tilde{I}^n_{n-1}) \subset \hat{I}^n_{n-1}$ is taken into $\mathbf{F}S_{n,\text{Harr}}$.

But now by theorem 2.11 we see that both spaces have the same dimension. So in fact $\phi^n_{n-1}(\tilde{I}^n_{n-1})$ turns out to be *equal* to $\mathbf{F}S_{n,\text{Harr}}$, and now the injectivity and S_n invariance of ϕ imply that

Lemma 3.13. *ι descends to an isomorphism of left S_n modules*

$$\mathbf{F}S_{n,\text{Harr}} \,\simeq\, \tilde{I}^n_{n-1}\,.$$

Note that the dimension equality (2.11) holds for *all* other n and m. We could ask whether $\tilde{I}^n_{n-i} \;\cong\; \mathbf{F}S_{n,i}$ as left $\mathbf{F}S_n$ modules. However, looking at the case $i = n$ we find that we have the trivial and the alternating representation, respectively. So we need a slight modification of the statement. To describe it, we shall introduce some notations.

Definition 3.14. *For $\sigma \in S_m$ and $\tau \in S_n$ let their "tensor" product $\sigma \otimes \tau \in S_{m+n}$ be defined by*

$$(\sigma \otimes \tau)(i) \;=\; \left\{ \begin{array}{ll} \sigma(i) & i \leq m \\ \tau(i-m)+m & i > m \end{array} \right\}.$$

This is a natural notation if one thinks of S_* as acting on a graded cotensor algebra (cf. [Lo2, Appendix A.6]). We will later need this model for a further statement about the Eulerian idempotents (proposition 3.22).

Denote for a multiindex p with $\mathrm{len}(p) = k$ and $|p| = n$ the "tensored" first Eulerian idempotent $e_p \in S^n$ by

$$e_p \;:=\; e^{(1)}_{p_1} \otimes \ldots \otimes e^{(1)}_{p_k}.$$

Definition 3.15. *We will call an* ordered partition (or splitting) $\mathcal{P}$ *of* N_n *a* vector $\mathcal{P} = (P_1, \ldots, P_k) \in (\mathcal{P}(N_n))^k$ *with* $\dot{\bigcup}_{i=1}^{k} P_i \;=\; N_n$*,* $\emptyset \neq P_i$ *and* $P_i \cap P_j = \emptyset$ *for* $i \neq j$*. The* length $|\mathcal{P}|$ *of* $\mathcal{P}$ *is then* k*.*

An unordered *partition* $\bar{\mathcal{P}}$ *is a* subset $\bar{\mathcal{P}} \subset \mathcal{P}(N_n)$ *with* $\dot{\bigcup}\bar{\mathcal{P}} = N_n$ *and* $\emptyset \notin \mathcal{P}$*.*

There is an obvious order-forgetting map

$$\bar{\cdot} \;:\; \{\text{ ordered part. }\} \;\longrightarrow\; \{\text{ unordered part. }\}$$

Definition 3.16. *For an ordered partition $\mathcal{P} = (P_i)$ let the multiindex $p = (|P_i|)^k_{i=1}$ be the norm $||\mathcal{P}||$ of $\mathcal{P}$. Define for multiindices p as above the order-forgetting map $\bar{\cdot}$ by mapping p to the decreasingly sorted multiindex, e. g.* $\overline{(1,4,2,3)} \;=\; (4,3,2,1)$.

Define for a $\bar{\mathcal{P}}$ its norm $||\bar{\mathcal{P}}||$ as a decreasingly ordered multiindex, such that the diagram

$$\begin{array}{ccc} \mathcal{P} & \longmapsto & \bar{\mathcal{P}} \\ \big\downarrow & & \big\downarrow \\ ||\mathcal{P}|| & \longmapsto & ||\bar{\mathcal{P}}|| \end{array}$$

commutes.

Definition 3.17. *For an unordered partition $\bar{\mathcal{P}}$ of N_n define $\hat{I}^n_{\bar{\mathcal{P}}}$ and $\tilde{I}^n_{\bar{\mathcal{P}}}$ as subspaces of $\tilde{I}^n_{n-|\bar{\mathcal{P}}|}$ and $\hat{I}^n_{n-|\bar{\mathcal{P}}|}$ respectively by requiring the vertex sets of the components of the corresponding graphs to form exactly the partition $\bar{\mathcal{P}}$. For an ordered $\mathcal{P}$ set $\hat{I}^n_{\mathcal{P}} := \hat{I}^n_{\bar{\mathcal{P}}}$ and $\tilde{I}^n_{\mathcal{P}} := \tilde{I}^n_{\bar{\mathcal{P}}}$.*

Definition 3.18. *For each ordered multiindex p of $|p| = n$ and* $\mathrm{len}(p) = k$ *we will denote by $\mathcal{C}_p$ the standard (ordered) splitting of N_n*

$$(\{1, \ldots, p_1\}, \{p_1+1, \ldots, p_1+p_2\}, \ldots, \{n-p_k+1, \ldots, n\}).$$

Note, that there is a bijection between

$$\left\{ \begin{array}{c} \text{ordered } k\text{-comp.} \\ \text{splittings of } N_n \end{array} \right\} \;\longleftrightarrow\; \left\{ \begin{array}{c} p\text{-shuffles } \sigma \text{ for some } p, \\ |p| = n \text{ and } \mathrm{len}(p) = k \end{array} \right\}, \tag{3.11}$$

given by

$$\Sigma_\sigma := \sigma(\mathcal{C}_p) \;\longleftrightarrow\; \sigma.$$

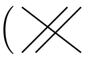
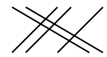
Then Σ_σ is the ordered partition determined by the shuffle σ. Conversely,

$$(\Sigma_\star)^{-1}(\mathcal{P}) \;=:\; \sigma_{\mathcal{P}}$$

is the partition specific shuffle for the partition $\mathcal{P}$.

Let for σ, σ' with $\overline{\Sigma_\sigma} = \overline{\Sigma_{\sigma'}}$ the permutation $\delta_{\sigma\sigma'} \in S_k$ be defined by $\delta_{\sigma\sigma'}(\Sigma_\sigma) = \Sigma_{\sigma'}$, where $k = |\Sigma_\sigma|$.

Define the cabling operations Δ^p on permutations by treating a permutation as a diagram and replacing strand i (on the bottom and counted from left to right) by a bundle of p_i strands, e. g.

$\Delta^{212}(231) \;=\; \Delta^{212}\Big($  $\Big) \;=$  $=\; 34512\,.$ (3.12)

As a further example, we have that $(\Delta^p(\delta_{\sigma\sigma'}))(\mathcal{C}_p) \;=\; \mathcal{C}_{p'}$, where σ and σ' are p- and p'-shuffles with $\overline{\Sigma_\sigma} = \overline{\Sigma_{\sigma'}}$.

Now, define a new action of S_n in $\tilde{I}^n_m$, which we will call "*-action" (by "action" we will henceforth mean the standard action used so far).

Definition 3.19. *Let $\tau \in S_n$ *-act on a graph G in $\tilde{I}^n_m$ by the action of $(-1)^{\sigma_{\mathcal{P}}\sigma^{-1}_{\tau(\mathcal{P})}}\tau$, where $\mathcal{P}$ is a fixed ordered partition of the connected components of G. Equivalently, a transposition τ_i *-acts on G by the action of τ_i if i and $i+1$ are in the same component of G and by $-\tau_i$ otherwise.*

Exercise 3.20. Check that this is indeed an action.

Now we have

Proposition 3.21. *As a *-module over $\mathbf{F}S_n$, $\tilde{I}^n_{n-i}$ is isomorphic to $\mathbf{F}S_{n,i}$.*

Proof. We have

$$\tilde{I}^n_{n-k} \;=\; \bigoplus_{\substack{\bar{\mathcal{P}} \text{ unordered} \\ \text{splitting of} \\ N_n,\, |\bar{\mathcal{P}}| = k}} I^n_{\bar{\mathcal{P}}} \;=\; \bigoplus_{\substack{\mathcal{P} \text{ ordered} \\ \text{splitting}}} I^n_{\mathcal{P}} \Big/ I^n_{\mathcal{P}} \;=\; I^n_{\mathcal{Q}} \text{ for } \bar{\mathcal{P}} = \bar{\mathcal{Q}}\cdot \tag{3.13}$$

Let

$$S_p := \bigotimes_{i=1}^{\mathrm{len}(p)} S_{p_i}$$

be the s. c. YOUNG group and for an ordered partition $\mathcal{P}$ define

$$S_{\mathcal{P}} := \sigma_{\mathcal{P}} S_p \sigma_{\mathcal{P}}^{-1}\,.$$

Identifying for a fixed ordered partition $\mathcal{P}$ of N_n with $||\mathcal{P}|| = p$ a permutation $\sigma \in S_{\mathcal{P}}$ by $\sigma\bar{\sigma}_{\mathcal{P}}(\gamma_p) \in \tilde{I}^n_{\mathcal{P}}$ we get a linear isomorphism between

$$\mathbf{F}S_{\mathcal{P}}\bar{\sigma}_{\mathcal{P}}e_p \;\longleftrightarrow\; I^n_{\mathcal{P}}\,.$$

So we have an isomorphism of modules

$$\tilde{I}^n_{n-k} \;\longleftrightarrow\; \dot{\bigoplus_{\substack{|p| = n,\, \mathrm{len}(p) = k \\ \sigma\ p\text{-shuffle}}}} \mathbf{F}S_{\Sigma_\sigma}\bar{\sigma}\, e_p\,, \tag{3.14}$$

where the r.h.s. is by definition

$$\left\{ \left(\tau_{\sigma,p} \in \mathbf{F}S_{\Sigma_\sigma}\right)_{\sigma,p} \;\middle|\; \tau_{\sigma,p}\bar{\sigma} = \tau_{\sigma',p'}\bar{\sigma}'\bar{\Delta}^p(\delta_{\sigma\sigma'}) \quad \text{for} \quad \overline{\Sigma_\sigma} = \overline{\Sigma_{\sigma'}} \right\}. \tag{3.15}$$

Note that for each $\sigma \in S_n$ the *-action of σ on $\tilde{I}^n_{n-k}$ produces the same element in all components on the r.h.s. of (3.14), since for both $\sigma_{\mathcal{P}}$ and $\Delta^p(\delta_{\sigma\sigma'})$ their action and *-action differ by their sign.

The action of $\varrho \in S_n$ on the right of (3.14) is given for

$$\tau_{\sigma,p}\bar{\sigma}e_p \in \mathbf{F}S_{\Sigma_\sigma}\bar{\sigma}e_p \tag{3.16}$$

the element

$$\tau'_{\sigma',p'}\bar{\sigma}'e_{p'} \in \mathbf{F}S_{\Sigma_{\sigma'}}\bar{\sigma}'e_{p'}\,,$$

where

$$\varrho\tau_{\sigma,p}\bar{\sigma} = \tau'_{\sigma',p'}\bar{\sigma}'\,, \tag{3.17}$$

and σ' is a p'-shuffle with $\bar{p}' = \bar{p}$ and $\tau'_{\sigma',p'} \in \mathbf{F}S_{\Sigma_{\sigma'}}$. Note that we have $k!$ possible choices to choose the decomposition on the right hand-side of (3.17) (one for each permutation of $\Sigma_{\sigma'}$), which however are all equivalent because of the relation imposed. Also, because of this relation it doesn't matter which of the $k!$ equal representations we take in (3.16).

Now notice that on the right hand-side of (3.14) we could replace S_{Σ_σ} both times by S_n simply setting $\tau \circ \bar{\sigma}e_p$ (the superposition of permutations) also to be $\tau(\bar{\sigma}e_p)$ (the module action) and generalizing the relation between the τ's (that is, restating the same, but this time for $\tau_{\sigma,p} \in \mathbf{F}S_n$ instead of $\mathbf{F}S_{\Sigma_\sigma}$).

Now we have an injective S_n-module homomorphism

$$\bigoplus_{\substack{\text{all classes}\\ [\sigma]=\overline{\Sigma_\sigma}}} \mathbf{F}S_n \sum_{\substack{\text{all representatives of } [\sigma], \text{ i e.}\\ \text{all } p\text{-sh. } \sigma' \text{ for } |p|=n,\\ \text{len}(p)=k \text{ with } \overline{\Sigma_\sigma}=\\ \overline{\Sigma_{\sigma'}}.}} \bar{\sigma}'e_p \hookrightarrow \dot{\bigoplus_{p,\sigma}} \mathbf{F}S_n\,\bar{\sigma}e_p \tag{3.18}$$

given by

$$\sum_{[\sigma]} \tau_{[\sigma]} \Big(\sum_{\sigma' \in [\sigma]} \bar{\sigma}'e_p \Big) \longmapsto \left(\tau_{[\sigma]}\,\bar{\sigma}^{-1}\right)_{p,\sigma}.$$

On the other hand, there is an obvious surjection from the left hand-side of (3.18) to

$$\sum_{\sigma,p} \mathbf{F}S_n\bar{\sigma}e_p \supset \mathbf{F}S_n \sum_{\sigma,p} \bar{\sigma}e_p = \mathbf{F}S_n \underbrace{\sum_{|p|=k} \tilde{\omega}_p e_p}_{\varrho_n^{(k)}}\,. \tag{3.19}$$

The proof is now completed by the

Proposition 3.22.

$$\varrho_n^{(k)} = k!\,e_n^{(k)}\,. \tag{3.20}$$

This implies proposition 3.21 because combining (3.13) – (3.19) we have that

$$\dim \mathbf{F}S_n \cdot \varrho_n^{(k)} \leq \ \ldots \ \leq \dim \tilde{I}^n_{n-k}\,,$$

and now the dimension equality (2.11) implies that the surjection from the l.h.s. of (3.18) to the l.h.s. of (3.19) and inclusion in (3.19) are in fact isomorphism and equality. □

With the HOPF algebra structure of a cotensor algebra [Lo2, Appendix A.6] the statement (3.20) says that

$$[e_*]^{*k} := \mu^{k-1} \circ e_* \circ \Delta^{k-1} = k\,! \cdot e_*^{(k)}\,, \tag{3.21}$$

i. e., the k-th Eulerian element (regarded as a graded homomorphism) arises from the first one by k-fold convolution, or equivalently, that Eulerian elements are additive in the sense that

$$e_*^{(p)} * e_*^{(q)} \;=\; \binom{p+q}{p} e_*^{(p+q)}\,.$$

This property is well known (for a good description see [Lo3], and also [Lo2]), but we will outline an independent proof of proposition 3.22 in the next subsection by using elementary properties of the Stirling numbers.

Note, that $\tilde{I}^*_m$ inherits a cochain complex structure from A^*_m, as the coboundary maps relations to relations. Now we can go even a step further and ask about the cochain complex structure of $\tilde{I}^*_m$ under the identification with Eulerian ideals.

Question 3.23. *Is there any reasonable explanation of the coboundary map $d^n : \tilde{I}^n_m \to \tilde{I}^{n+1}_m$ under the identification $\tilde{I}^n_m \simeq \mathbf{F}S_n \cdot e^n_{(n-m)}$?*

3.4. **The convulotory additivity of the Eulerian elements.** As already announced, in this subsection we will be concerned with proving proposition 3.22.

However, let us first begin with (re)introducing some notations and recalling some facts which will turn out useful. In order to omit the accents, we will apply ‘ ¯ ’ on both hand-sides of (3.20) and prove this equivalent identity.

Remember that

$$\bar{e}_n^{(i)} \;=\; \sum_{\sigma \in S_n} c^{(i)}_{n,\mathrm{dc}(\sigma)}\,\sigma\,,$$

where (see exercise 2.4)

$$\begin{aligned} c^{(i)}_{n,d} \quad &:= \quad \mathrm{cf}^i_t\Big[\binom{n+1-d+t}{n}\Big] \\ &= \quad \frac{1}{n!}\sum_{j=1}^{i} |S^{(j)}_{n-d}|\, S^{(i+1-j)}_{d+1} \qquad \text{for } 0 \leq d \leq n-1. \end{aligned}$$

There is a well known property of the Stirling numbers, namely

$$S^{(j)}_n + (n-1)S^{(j)}_{n-1} \;=\; S^{(j-1)}_{n-1}\,, \tag{3.22}$$

which we will just use to show a triangular property for the numbers $c^{(i)}_{n,d}$:

Lemma 3.24.

$$c^{(i)}_{n,d} \;=\; c^{(i)}_{n,d-1} + c^{(i)}_{n-1,d} \qquad 0 \leq d \leq n-1\,. \tag{3.23}$$

Proof. Use the definition of $c^{(i)}_{n,d}$ to expand (3.23) into

$$\sum_{j=1}^{i} |S^{(j)}_{n-d}| S^{(i+1-j)}_{d+1} \; = \; \sum_{j=1}^{i} |S^{(j)}_{n-d-1}| \, [S^{(i+1-j)}_{d+2} \; + \; nS^{(i+1-j)}_{d+1}\,] \;\; .$$

Now take from the n copies of the product in the bracket on the right $n-d-1$ on the left and (taking care of the signs) get

$$\sum_{j=1}^{i} |S^{(j)}_{n-d}+(n-d-1)S^{(j)}_{n-d-1}| S^{(i+1-j)}_{d+1} \; = \; \sum_{j=1}^{i} |S^{(j)}_{n-d-1}| [\, S^{(i+1-j)}_{d+2} + (d+1) S^{(i+1-j)}_{d+1}\,] \; .$$

But by (3.22) latter equation turns out to be equivalent to the trivial equality

$$\sum_{j=1}^{i} |S^{(j-1)}_{n-d-1}| S^{(i+1-j)}_{d+1} \; = \; \sum_{j=1}^{i} |S^{(j)}_{n-d-1}| S^{(i-j)}_{d+1} \; . \quad \square$$

Now let's come back to our proposition.

Proof of proposition 3.22. First we shall prove a partial case of it – we will look only at the coefficient of Id of both hand sides of (3.21). Since $c^{(i)}_{n,0} = |S^{(i)}_n|$, this equality can be reformulated as

$$k\,! \, \cdot |S^{(k)}_n| \quad = \quad \sum_{\substack{|p| \,=\, n \\ \mathrm{len}(p) \,=\, k}} \; \prod_{i=1}^{k} (p_i - 1)\,! \; .$$

To see this recall the property of $S^{(i)}_n$ (2.5) and observe, that the number on the right is exactly the number of possibilities, for each $\sigma \in S_n$ with k cycles to write down a cycle decomposition of σ by all the $k\,!$ possible sequences to order its cycles.

Now look at the general case of proposition 3.22. We will make a nested induction.

First of all observe, since the proposition for $i = 1$ is trivial (and $*$ is associative), that the equality for the coefficient of $\sigma \in S_n$ in (3.21) would follow by induction over i from the following formula

$$c^{(i)}_{n,d} \; = \; \frac{1}{i} \sum_{p=1}^{n-1} c^{(1)}_{p,d_p} c^{(i-1)}_{n-p,d-d_{p+1}} \, , \tag{3.24}$$

where $d_p = \mathrm{dc}(\sigma\Big|_{\{1,\ldots,p\}}),\;\; 1 \le p \le n$ and $d = d_n = \mathrm{dc}(\sigma)$. Note, that

$$\mathrm{dc}(\sigma\Big|_{\{p+1,\ldots,n\}}) = d - d_{p+1} \, .$$

For such a $\sigma \quad \bar{d} = (d_p)^n_{p=1}$ is a monotonous sequence with $0 = d_1$ and $d_i \le d_{i+1} \le d_i + 1, \;\; i < n. \quad$ (3.25)

Now forget about permutations and consider (3.24) for all sequences of the kind (3.25)[4].

Prove (3.24) inductively over n. The case $n = 1$ is trivial. Now fix some $n \in \mathbb{N}$. Each $\bar{d}$ can be obtained from the special vector $(0, 0, \ldots, 0)$ (for which we have verified (3.24) by the previous considerations) by a sequence of two kinds of steps:

1. change d_p to $d_p + 1$ where $d_{p-1} = d_p = d_{p+1} - 1, \quad 1 \le p < n$, and
2. change d_n to $d_n + 1$, if $d_{n-1} = d_n$.

[4] In fact, we obtain each such sequence $\bar{d}$ by a certain permutation σ, so both views on (3.24) are equivalent.

It suffices to show that (3.24) is preserved by both types of procedures. In the first case we have to show that

$$0 \;=\; c^{(1)}_{p-1,d_p}\big[\, c^{(i-1)}_{n-p+1,d-d_p-1} - c^{(i-1)}_{n-p+1,d-d_p} \,\big] \;+\; \big[\, c^{(1)}_{p-1,d_p+1} - c^{(1)}_{p-1,d_p} \,\big]\, c^{(i-1)}_{n-p,d-d_p-1}\,,$$

which is an immediate consequence of lemma 3.24.

In the second case we are left with

$$c^{(i)}_{n,d+1} - c^{(i)}_{n,d} \;=\; \frac{1}{i}\sum_{p=1}^{n-1} c^{(1)}_{p,d_p}\big[\, c^{(i-1)}_{n-p,d-d_{p+1}+1} - c^{(i-1)}_{n-p,d-d_{p+1}} \,\big]\,.$$

Now again, by lemma 3.24 latter equality is the same as

$$-c^{(i)}_{n-1,d} \;=\; -\frac{1}{i}\sum_{p=1}^{n-1} c^{(1)}_{p,d_p} c^{(i-1)}_{n-1-p,d-d_{p+1}}\,,$$

which is the induction assumption. □

4. Constructing a free resolution of A^*

To have a good understanding of A^n, it appears reasonable to ask for a free graded resolution of A^n as a vector space. A certain result in this direction was obtained by HUTCHINGS [Hu] who gave generators of the relations among relations in A^*.

We will denote a resolution in the following way:

$$\cdots \longrightarrow A^{n,1}_m \longrightarrow A^{n,0}_m \longrightarrow A^n_m \longrightarrow 0\,, \tag{4.1}$$

where $A^{n,k}_m \;=\; \mathbf{F}\cdot B^{n,k}_m$ for certain formal sets $B^{n,k}_m$, whose elements we will call "relations of depth m"or "m-relations". We can understand elements in $B^{n,k}_m$ also as their image in $A^{n,k-1}_m$ by "saying what the relation represents".

Let us consider a *minimal* resolution of A^n_m, i. e. we pick the free generators $B^{n,i}_m$ of $A^{n,i}_m$ to be linearly independent (treated as elements in $A^{n,i-1}_m$). A (more or less natural) choice of such a minimal resolution would begin by $B^{n,0}_m = \{$ monomials $\}$ and

$$B^{n,1}_m \;=\; \left\{ \begin{array}{cc} \alpha R_{hkjl}\beta, & \\ \alpha R_{hjk}\beta & h,j<k<l \end{array} \;\middle|\; \begin{array}{cc} \alpha \in F^n_{m-i-1}, & \\ \beta \in F^n_{i-1} & i=1,\dots,m-1 \end{array} \right\}.$$

Note, that we have the linear identity

$$R_{ijk} + R_{jki} + R_{kij} \;=\; 0,$$

so we have to remove one sort of all the R_{ijk}'s. Since in such a resolution a relation of degree m must have at least $m+1$ chords, we see that the graded resolution (4.1) breaks up.

$$0 \longrightarrow A^{n,m-1}_m \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} A^{n,1}_m \xrightarrow{\partial_1} A^{n,0}_m \longrightarrow A^n_m \longrightarrow 0\,.$$

Note, that the fact that $A^{n,m}_m = A^{n,m+1}_m = \cdots = 0$ has a certain similarity with equation (3.5).

In the general case of an algebra resolution there is a natural procedure of generating relations of higher degrees from such of lower ones by a sort of "multiplication". If $\gamma \in B^{n,k_1}_{m_1}$ and $\delta \in B^{n,k_2}_{m_2}$ are k_1- and k_2-relations in degree m_1 and m_2, build a k_1+k_2 relation $\gamma * \delta \in B^{n,k_1+k_2}_{m_1+m_2}$ by

$$\partial_{m_1+m_2}(\gamma * \delta) \;:=\; \partial_{m_1}(\gamma) * \delta \;-\; \gamma * \partial_{m_2}(\delta)\,.$$

Let us include the usual multiplication of a relation by a monomial into this sort of procedure by treating monomials as relations of degree 0 and setting $\partial_0 \equiv 0$.

There is a way how to see in certain relations again elements in our spaces $\tilde{I}^n_m$. Note, that using the short exact sequence

$$0 \longrightarrow U \cap V \xrightarrow{i} U \oplus V \longrightarrow U + V \longrightarrow 0\,,$$

where i is the embedding $u \mapsto (u, -u)$, we have an interpretation of $U \cap V$ as a space of relations between U and V. So, inductively, $\tilde{I}^n_m$ has an interpretation as a space of relations of depth $m-1$ of A^n_m. Set

$$A^{n,m-1}_m := \tilde{I}^n_m \quad \text{and} \quad A^{n,m-k}_m := \bigoplus_{\substack{\sum m_i = m-k \\ \sum n_i = n}} \mathop{*}_{i=1}^{k} \tilde{I}^{n_i}_{m_i}$$

Conjecture 4.1. *With this definition of $A^{n,k}_m$, (4.1) is a free resolution.*

This is a very optimistic hope to obtain an easily describable free resolution of at least the most simple of the Vassiliev diagram spaces [BN2, BN4, BN3]. The other ones are much harder, is not hopeless, to understand.

Note, that Orlik and Solomon considered and proved acyclic a very similar complex constructed out of lattices [OS, (2.18)]. However, I don't know how to carry their argument preciccely over to this case.

5. Open Problems

Beside conjectures 2.13 and 4.1, the considerations of this paper open 2 principal problems: On the one hand, what are the connections of the idempotent structures of dagram spaces to the ones considered by Orlik and Solomon? On the other hand, do they point to a more profound connection between Vassiliev braid theory and homological algebra? Is there something more serious behind this identification and question 3.23?

Without having an answer to any of these questions, I hope that finding one will be fertile for both mathematical areas.

References

[Ba] M. Barr, *Harrison Homology, Hochschild Homology and Triples*, J. Algebra **8**, 314-323 (1968).

[BN] D. Bar-Natan, *Non-associative tangles*, to appear in "The Georgia Inter. Topology Conference proceedings".

[BN2] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34**, 423-472(1995).

[BN3] D. Bar-Natan, *Vassiliev and quantum invariants for braids*, Harvard University preprint, November 1994.

[BN4] D. Bar-Natan, *Vassiliev Homotopy String Link Invariants*, Jour. of Knot Theory and its ramifications **4** (1995) 13–32.

[BN5] D. Bar-Natan, *Bibliography of Vassiliev invariants*, available via `ftp` from `ftp.ma.huji.ac.il`, subdirectory `drorbn`.

[BS] D. Bar-Natan and A. Stoimenow, *The Fundamental Theorem of Vassiliev invariants*, Proceedings of the 1995 Summer School on Geometry and Physics, to appear.

[BL] J. S. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993) 225–270.

[Ca] P. Cartier, *Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds*, C. R. Acad. Sci. Paris **316** Série I (1993) 1205–1210.

[Cm] P. J. Cameron, *Combinatorics*, Cambridge Univ. Press, 1994
[Dr] V. G. Drinfel'd, *On quasitriangular Quasi-Hopf algebras and a group closely connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829–860.
[GS] M. Gerstenhaber, S. D. Schack, *A Hodge-type decomposition of commutative algebra cohomology*, J. Pure Appl. Algebra **48** (1987), 229–247.
[Ha] F. Harary, *Graph theory*, Addison-Wesley, Reading, 1969.
[Hl] P. Hanlon, *The action of S_n on the components of the Hodge decomposition of Hochschild homology*, Michigan Math. J. **37** (1990), 105–124.
[Hu] M. Hutchings, *Integration of singular braid invariants and graph cohomology*, Harvard University preprint, May 1995.
[K] T. Kohno, *Vassiliev Invariants and De-Rham Complex on the Space of Knots*, Contemp. Math. **179** (1994) 123–138.
[Ko] M. Kontsevich, *Vassiliev's knot invariants*, Adv. in Sov. Math., **16(2)** (1993), 137–150.
[Lo] Jean-Louis Loday, *Opérations sur l'homologie cyclique des algèbres commutatives*, Inv. Math., **96** (1989) 205–230.
[Lo2] J. L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, vol. **301**, Springer, Berlin, 1992.
[Lo3] J. L. Loday, *Série de Hausdorff, idempotents eulériens et algèbres de Hopf*, Exp. Math. **12** (1994), 165-178.
[OS] P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Inv. Math. **56** (1980), 167–184.
[Pi] S. Piunikhin, *Combinatorial expression for universal Vassiliev link invariant*, Commun. Math. Phys. **168** (1995), 1-22.
[Re] C. Reutenauer, *Theorem of Poincaré-Birkhoff-Witt, logarithm and symmetric group representations of degrees equal to Stirling numbers*, Springer Lecture Notes in Math. **1234** (1986), 267-284.
[Ri] J. Riordan, *An Introduction to combinatorial analysis*, Wiley, New York, 1958.
[RT] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990) 1-26.
[St] A. Stoimenow, *On Harrison cohomology and a conjecture by Drinfel'd*, Humboldt University Berlin, 1996.
[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.
[Vo] P. Vogel, *Algebraic structures on modules of diagrams*, Université Paris VII preprint, July 1995.

HUMBOLDT UNIVERSITY, INSTITUTE OF MATHEMATICS, ZIEGELSTR. 13A, 10099 BERLIN

E-mail address: stoimeno@informatik.hu-berlin.de