INDEPENDENCE POLYNOMIALS AND ALEXANDER-CONWAY POLYNOMIALS OF PLUMBING LINKS

This is preprint. I would be grateful for any comments and corrections.

A. Stoimenow, with a contribution of S. Baader

Abstract. We use the Chudnovsky-Seymour Real Root Theorem for independence polynomials to obtain some statements about the coefficients and roots of the Alexander and Conway polynomial of some types of plumbing links, addressing conjectures of Fox, Hoste and Liechti.

Keywords: independence polynomial, line graph, polynomial root, alternating knot, Alexander polynomial, arborescent link, 2-bridge link

AMS subject classification: 57M25, 26C10 (primary), 05C31, 13F20, 12D10 (secondary)

1 Introduction

The Alexander polynomial $\Delta$, and its equivalent version due to Conway $\nabla$, remains of profound importance as an invariant of knots and links in 3-space. Many features of the polynomial have been studied over the years in a variety of contexts, including surface homeomorphisms, knot groups, Lehmer’s question on the existence of a Mahler measure minimizing polynomial, knot homology, etc. (See [Ro] for a classical treatise and [LMu, St6] for more recent references, beyond those to follow below.)

Alternating knots are a class of knots well-studied, and enjoying many special properties. The question to characterize their Alexander polynomials has sparked extensive efforts. Beyond the classical theorem of Crowell-Murasugi, there are several further-going conjectures on the appearance of the Alexander polynomial for alternating links (sometimes more commonly known for knots only). This includes Fox’s Trapezoidal conjecture on the coefficients of the polynomial, and Hoste’s conjecture on its roots. There are various refinements and similar conjectures for the Conway polynomial of positive links (see §3).

Various root location problems and some of these conjectures have been addressed in a recent monograph of Hirasawa-Murasugi [HM]. The goal of this note is to show how the work in Chudnovsky-Seymour [CS] on the real-rootedness of independence polynomials of claw-free graphs can be used to give short elegant proofs of these conjectures for some classes of links and to partially recover and extend a main result of Hirasawa-Murasugi (see remark 5.2).

We will explain in §2 in a proper graph-theory context a slightly generalized form of Chudnovsky-Seymour’s theorem and in §4 its relation to the construction of plumbing links. We will apply this relationship then (together with some further-going arguments) to alternating plumbing links in §5.

2 Weighted independence polynomial

Graphs considered here will be finite, i.e., with a finite number of vertices and edges. Multiple edges (cycles of length 2) are allowed, but no loops (edges from a vertex to itself). We will write $E(G)$ for the edge set and $V(G)$ for the vertex set. Write also $v(G) = |V(G)|$ for the number of vertices.

For a graph $G$, its line graph $\Lambda = \Lambda(G)$ is defined to have a vertex for each edge in $G$, and an edge for each pair of edges in $G$ with a common vertex.
The independence polynomial $I(G, x)$ of $G$ is defined as

$$I(G, x) = \sum_{A \subset V(G)} x^{|A|},$$

where $A$ goes over independent vertex sets, i.e., sets with no edge in $G$ between any two vertices in $A$.

Now note that

$$I(\Lambda(G), x) = \mu(G, x) = \sum_{L \text{ matching of } G} x^{|L|}, \tag{1}$$

where a matching of $G$ is a set of edges in $G$ with no common vertex.

A vertex of valence 1 is called a leaf. A tree with no vertices of valence $\geq 3$ (i.e., all vertices are of valence 2 except for two leaves) is a path. The tree of three edges which is not a path is the claw. A graph is claw-free, if no induced subgraph is a claw.

**Theorem 2.1** (Chudnovsky-Seymour [CS]) If $G$ is claw-free, then all roots of $I(G, x)$ are real. In particular, this holds for $G = \Lambda(G')$ for any graph $G'$.

We will also call below a polynomial with all roots real to be real-rooted.

It will be somewhat useful to extend this theorem to weighted independence polynomials. Consider a weighted graph, i.e., for which each vertex $v$ has a weight $l_v$. For now we will assume $l_v$ is a positive real number (while occasionally making remarks for more general weights).

The weighted independence polynomial $I(G, x)$ of a weighted graph $G$ is defined as

$$I(G, x) = \sum_{A \subset V(G)} x^{|A|} \prod_{v \in A} l_v,$$

where again $A$ goes over independent vertex sets (cf. (1.8) on page 413 and (2.11–2.14) on page 416 in [La]).

**Theorem 2.2** If $G$ is claw-free and weighted (with $l_v > 0$), then all roots of $I(G, x)$ are real.

For the proof, one just needs to modify the argument of Chudnovsky-Seymour [CS] for the generalized case. However, we will need to work this out in a way that makes later exposition clearer. A reviewer has pointed out that theorem 2.2 (and generalizations of it) is well treated in the article [La]. The proof sketched here turned out to be essentially the one attributed there to Engström [En].

**Proof.** Let $G$ be a graph with vertices $V(G) = \{v_1, \ldots, v_n\}$ and edge multiplicities $m_{ij}$ between $v_i$ and $v_j$. And let $v = (l_1, \ldots, l_n)$ a vector of positive integers $l_i \in \mathbb{Z}_+.$

We define the blowup graph $G_v$. Vertices of $G_v$ are

$$V(G_v) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq l_i\}.$$

And edge multiplicities are defined as

$$m(v_{i,j}, v_{i',j'}) = \begin{cases} m_{i,j} & \text{if } i \neq i' \\ 1 & \text{if } i = i' \end{cases}$$

Pictorially, this means replacing in $G$ each vertex with label $l_i$ by a clique of size $l_i$ (each vertex connected to the same vertices outside the clique).

It is easy to see that when $G$ is claw-free, so is $G_v$.

Note also that if $G = \Lambda(G')$ then $G_v = \Lambda(G'_{[v]})$, where we identify $v = (l_e : e \in E(G'))$ and $G'_{[v]}$ means that in $G'$ each edge $e \in E(G')$ is replaced by $l_e$ parallel edges.
Now if a graph \( G \) is weighted with positive integer weights \( v = (l_v : v \in V(G)) \), applying theorem 2.1 gives that
\[
I(G,v) = I(G,v)
\]
has only real roots. (Note that on the left we have a weighted independence polynomial, while on the right an ordinary
one.)
Also if \( G \) is weighted, and \( \lambda G \) is \( G \) with weights multiplied by \( \lambda > 0 \), then
\[
x \in \mathbb{C} \text{ is a root of } I(G,x) \text{ if and only if } x/\lambda \text{ is a root of } I(\lambda G,x).
\]
If now \( G \) is rational weighted, finitely many rational numbers have a common denominator, so \( \lambda G \) is integer weighted
for some \( \lambda \in \mathbb{Z}_+ \). Then the claim follows for rational weighted \( G \).
Finally, one obtains the result for real weighted graphs by approximation. For example, approximate coefficients by rational numbers, assume the limit has a genuine complex root, use uniform convergence on a (genuine complex)
neighborhood, and [St2, Lemma 2] to get a contradiction. \( \square \)
In fact, in our work below we will not need irrational weights, but we felt it useful to add this short remark.

Remark 2.1 The relationship (2) holds for any real \( \lambda \neq 0 \) and setting \( \lambda = -1 \) one immediately sees theorem 2.2 to hold
if all weights of \( G \) are negative. It becomes much more interesting to examine what occurs for weights of either sign.
(It is obvious why zero weights are redundant.) Unpublished work of Hirasawa-Murasugi (with the added ‘translation’
here) implies that theorem 2.2 may hold for some such graphs, but they are very restricted and, along with the methods
of proof, their graph theoretic significance is unclear. Extensions may be the subject of future investigations.

3 Link polynomial conjectures

Let \( \Delta \) be the 1-variable Alexander polynomial and \( \nabla \) the Conway polynomial of a link \( L \):
\[
\nabla(L)(t^{1/2} - t^{-1/2}) = \Delta(L)(t).
\]
The polynomial \( \nabla \) satisfies the skein relation
\[
\nabla(\includegraphics{crossing_right}) - \nabla(\includegraphics{crossing_left}) = z \nabla(\includegraphics{loop}),
\]
which defines it alternatively (up to a factor, fixed by demanding \( \nabla(\bigcirc) = 1 \)). We will call the leftmost crossing
right-handed (or positive), the other one left-handed (or negative).

It is well known that \( \nabla(L) \) for an \( n \) component link \( L \) has only even/odd degree terms when \( n \) is odd/even, and that
\[\text{mindeg } \nabla(L) \geq n - 1.\]
Consequently, \( \Delta(t) \) has terms \( t^k \) only if \( k \) is an integer for odd number of components and half-
integer for even number of components; we will call such \( k \) admissible. Let \( \text{maxdeg } \Delta \) and \( \text{mindeg } \Delta \) be the maximal
and minimal degrees. Then \( \text{maxdeg } \Delta = -\text{mindeg } \Delta \). We have also the symmetry property
\[
\Delta(t) = (-1)^{n-1} \Delta(1/t)
\]
for a link \( L \) of \( n \) components. It is also well known that when a link is split (i.e., have a disconnected planar projection),
then its polynomials are zero. We will thus consistently assume links we consider are non-split. (A knot has 1
component, and is therefore never split.)

A classical theorem of Crowell-Murasugi [Cw, Mu] asserts that when \( L \) is a (non-split) alternating link, then \( 2 \text{maxdeg } \Delta = 1 - \chi(L) \) for the Euler characteristic \( \chi(L) \) of \( L \), and in particular for an alternating knot \( K \), we have \( \text{maxdeg } \Delta = g(K) \) for the genus \( g(K) \) of \( K \). Moreover, \( \Delta(L) \) is an alternating polynomial, i.e., \( \Delta_k \Delta_{k+1} < 0 \) for admissible \( k \) of \( 0 \leq 2k < -\chi(L) \), where \( \Delta_k \) denotes the coefficient of \( \Delta \) in degree \( k \).

There are several further-going conjectures on the appearance of the polynomial for alternating links \( L \). (We state the
versions for links, although some conjectures are more commonly known for knots only.)
Conjecture 3.1 (Fox’s Trapezoidal conjecture) If $L$ is alternating, there is a number $0 \leq 2m \leq 1 - \chi(L)$ such that for $\Delta[k] := |\Delta_k|$ we have for all admissible $k$,

$$
\begin{align*}
\Delta[k] &= \Delta[k-1] & \text{for } 0 < |k| \leq m, \\
\Delta[k] &< \Delta[k-1] & \text{for } m < |k| \leq 1 - \chi(L).
\end{align*}
$$

(6)

This conjecture was verified for 2-bridge knots (Hartley) and some more arborescent (algebraic) knots (Murasugi [Mu2]).

We will call an alternating polynomial with the conjectured property also trapezoidal. (With symmetry (5) of the Alexander polynomial, this is a special case of a unimodal polynomial.) In case $m = 0$ for links of an odd number of components, and $m = 1/2$ for even number, we call the polynomial triangular. (Note that for even components, symmetry implies that $m \geq 1/2$.)

Call a polynomial $X$ log-concave, if its sequence of coefficients $|X|_k$ (in degree $k$) are log-concave, i.e.,

$$
|X|^2_k \geq |X|_{k+1}|X|_{k-1} \geq 0
$$

(7)

for all $k \in \mathbb{Z}$. (The right non-negativity condition is technically added to restrict us to positive and alternating polynomials.)

Conjecture 3.2 (log-concavity conjecture [St2]) If $K$ is an alternating knot/link, then $\Delta_K(t)$ is log-concave.

It is easily seen as an extension of the Trapezoidal conjecture (see [St2]). The log-concavity conjecture was verified for knots $K$ of genus $g(K) \leq 4$ [St3].

A class exhibiting multiple connections (not only here) to alternating links are the positive links. A link diagram is positive if all its crossings are right-hand. A positive link is a link with such a diagram.

Note that by [St], if $L$ is a positive link, the (simplified) Conway polynomial $\tilde{\nabla} = \nabla(L)(\sqrt{z})$ is a positive polynomial, i.e., $|\tilde{\nabla}|_k > 0$ for $\min \deg \tilde{\nabla} \leq k \leq \max \deg \tilde{\nabla}$ with $\max \deg \tilde{\nabla} - k$ an integer.

Conjecture 3.3 (log-concavity conjecture II [St2]) If $L$ is a non-split positive link, then the simplified Conway polynomial $\nabla(L)(\sqrt{z})$ is log-concave.

Regarding roots of the polynomials, Hoste, based on computer verification, raised the following question (now known as a conjecture) about 15 years ago.

Conjecture 3.4 (Hoste’s conjecture) If $t \in \mathbb{C} \setminus \{0\}$ is a root of the Alexander polynomial $\Delta$ of an alternating knot/link, then $\Re t > -1$.

This is known from Murasugi’s work if $L$ is special alternating. It is also checked for knots $K$ if $g(K) \leq 4$ [St3], and for 2-bridge links $L$ [St5, I]. Hoste’s conjecture is essentially independent from the Trapezoidal and log-concavity conjectures [St4]. For some very recent updates on this conjecture, see [HIS].

Liechti proposed a meaningful analogue of Hoste’s conjecture for positive links. Here is a slightly modified statement from Liechti’s original one, in particular not restricting to positive braid links.

Conjecture 3.5 (Liechti’s conjecture [L]) If $t \in \mathbb{C} \setminus \{0\}$ is a root of the reduced Alexander polynomial $\nabla_L(t)$ of a positive (non-split) link $L$ of $n$ components, then $\Re t < 1$.

(The multiplicity of the root 1 of $\Delta$ can be easily deduced from Hoste’s formulas for $|\nabla|_{n-1}$ [Ht].)
Remark 3.1 It should be noted that the Hoste resp. Liechti conjectures are true for an alternating resp. positive link whenever all roots of $\nabla(L)(\sqrt{z})$ are real, or equivalently, all roots of $\Delta(L)$ are real or lie on $S^1$ (the set of unit norm complex numbers). This is because of our previous remarks that $\nabla(L)$ is positive when $L$ is positive and $\Delta$ is alternating when $L$ is alternating.

Furthermore, we draw attention also to the following fact discussed in [St2].

Lemma 3.1 If a positive polynomial is real-rooted, it is log-concave.

Thus, e.g., the real-rootedness of $\nabla(L)(\sqrt{z})$ for a positive link implies log-concavity conjecture II. Although the proof was only referenced in [St2], this lemma is not hard to see directly; it follows mainly from lemma 5.2 below by rescaling and induction over the degree, noting that all (real) roots must be negative. Obviously, the statement will hold then for an alternating (real-rooted) polynomial as well (with the roots then being positive).

4 Tree plumbing and the independence polynomial

4.1 Construction of plumbing links

We consider a link $L$ obtained by plumbing ordinary positive Hopf bands along a tree $G$. This means that the vertices of $G$ depict Hopf bands and an edge connects two vertices if the corresponding Hopf bands are plumbed. To formalize this construction, and for later reference as well, we will follow (among multiple accounts) the framework of [HM, §6].

Consider an oriented circle in the plane. Let $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_m\}$ be collections of chords. We will understand chords not necessarily as straight, but just as an identification of two (distinct) points on the circle. We will assume, though, that all $2(m+n)$ chord endpoints are distinct.

We say two chords $C, D$ intersect if their endpoints are in cyclic order $CDCD$. (This corresponds under (8) to Hopf bands which are plumbed.) We assume none of $A_i$ intersect, and similarly none of $B_j$, but we do allow some $A_i$ to intersect with $B_j$. However, for the purpose of building link diagrams, it will be better to think of chords in $B$ drawn outside the circle, so that no chords geometrically intersect in the plane.

We build the intersection graph $G(A, B)$ by vertices $V(G(A, B)) = A \cup B$ and an edge between intersecting chords. Obviously, $G(A, B)$ is bipartite, but many bipartite graphs do not arise this way. However, it is important to note that every tree does. A tree can be built from a vertex by adding leaves (valence-1 vertices), and it is easy to see how to install the corresponding chords.

Also, we will endow chords in $C \in A \cup B$ with a non-zero integer label $l_C$.

The corresponding link $L(A, B)$ and its diagram $D(A, B)$ is obtained from $A \cup B$ by the replacement of a (labelled) chord by a (generalized) Hopf band. (The solid line is a piece of the circle; the dotted line a chord.)
The number of crossings $2|bc|$ of a Hopf band coming from chord \( C \in A \cup B \) must be even. They should be right-handed (positive) if \( bc > 0 \) and left-handed (negative) if \( bc < 0 \). We will call \( L \) a bipartite plumbing link. If \( G(A, B) \) is a tree, we will call \( L \) a tree plumbing link. This construction is well-known; in some cases, e.g., [HL], one considers only \( bc = \pm 1 \) (giving ordinary Hopf bands); we will strive to release ourselves from that constraint here.

If \( bc \neq 0 \) for all \( C \in A \cup B \), it is also well understood that for \( L = L(A, B) \) and \( G = G(A, B) \),

\[
1 - \chi(L) = 2 \max \deg \Delta(L) = \max \deg \nabla(L) = \nu(G) = |A \cup B|
\]

(9)
is the number of Hopf bands plumbed.

We will call \( L(A, B) \) a positive plumbing link of \( bc > 0 \) for all \( C \in A \cup B \) and an alternating plumbing link if \( bc > 0 \) for \( C \in A \) and \( bc < 0 \) for \( C \in B \) (which is focused on in [HM]). It is obvious that \( D(A, B) \) is a positive resp. alternating diagram under the stated conditions.

We will mostly consider the case that \( G = G(A, B) \) is a tree, and write \( L(G) = L(A, B) \). This begs the thought why \( L(G) \) depends on \( G = G(A, B) \) only. In fact, it does not! This is because multiple leaves attached to the same vertex have no \textit{a priori} order in \( G \), but this order does matter in \( L(G) \). There are ways to fix this problem, e.g., by considering \( G \) to be a planar rooted tree. But we will see that our arguments apply for all possible \( L(G) \) coming from the same \( G \) (these links are mutants, and all have equal polynomials), so that this point will not lead to a problem.

Note that in particular if \( G = G(A, B) \) is a labelled path, then \( L \) is a \textit{2-bridge (rational) link}. Conversely, it is well known that all 2-bridge links arise in such a way. A general tree plumbing link is \textit{arborescent} (though by far not all arborescent links arise by such plumbing).

### 4.2 Polynomials of positive plumbing links

The line graph \( \Lambda = \Lambda(G) \) of the plumbing tree \( G \) of a link \( L = L(G) \) with labels \( +1 \) has independence polynomial \( I(\Lambda, z) \). Then

\[
\nabla(L) = I(\Lambda, 1/z^2) \cdot z^{\nu(G)},
\]

with \( \nu(G) = 1 - \chi(L) \) from (9).

This observation, which I owe to S. Baader, is the fundamental connection between knot and graph theory in the paper. With \( \mu \) from (1), we can write thus:

**Lemma 4.1**

\[
\nabla(L)(z) = \mu(G, 1/z^2)z^{1-\chi(L)}.
\]

**Proof.** The skein relation (4) gives

\[
\nabla \left( \begin{array}{c} \vline \\ \hline \vline \vline \vline \vline \\ \hline \end{array} \right) = \nabla \left( \begin{array}{c} \hline \vline \vline \vline \vline \\ \hline \vline \vline \vline \vline \\ \hline \vline \vline \vline \vline \\ \hline \vline \vline \vline \vline \\ \hline \end{array} \right) + z \nabla \left( \begin{array}{c} \vline \\ \hline \vline \vline \vline \vline \\ \hline \vline \vline \vline \vline \\ \hline \vline \vline \vline \vline \\ \hline \vline \vline \vline \vline \\ \hline \end{array} \right)
\]

(11)

if we apply this \textit{at a leaf} \( v \) of \( G \), we have with the terminology of [CS, 2.1]

\[
\nabla(L(G)) = z^\nu(L(G \setminus \{v\})) + \nabla(L(G \setminus N[v])).
\]

Recall that, according to [CS], \( G \setminus \{v\} \) means the graph induced by \( V(G) \setminus \{v\} \), and \( G \setminus N[v] \) is the graph induced by \( V(G) \setminus \{v\} \setminus \{\text{vertices adjacent to } v\} \).

Now, there is a graph polynomial \( \hat{I}(G, x) \) satisfying

\[
\hat{I}(G, x) = x\hat{I}(G \setminus \{v\}, x) + \hat{I}(G \setminus N[v], x).
\]

(12)

Namely,

\[
\hat{I}(G, x) = \sum_{\text{A matching of } G} x^{\nu(G \setminus V(A))}.
\]
where $V(A)$ is the set of vertices of edges in $A$, with the recursion (12) coming from distinguishing whether $v \in V(A)$ or not. Now recall the relationship (1), and formula (10) follows.

Now consider a link $L$ with a plumbing tree $G$ of Hopf bands with any positive number of full twists. Again, to indicate the number of twists, we label each vertex $v$ of $G$ by a positive integer $l_v$.

**Theorem 4.1** If $L$ is a positive tree plumbing link, then all roots of $\nabla(L)(\sqrt{z})$ are real (and so $\nabla(L)$ is log-concave).

For ordinary (one positive full twist) Hopf bands, this is an old result of A’Campo [A]. L. Liechti has informed that he obtained this generalization using A’Campo method in his Ph.D. thesis [L2].

**Proof.** We need to clarify how to modify (11) when vertices $w$ have labels $m_w \neq 1$. We have

$$\nabla \left( \frac{m_w}{z} \right) = \nabla \left( \frac{1}{z} \right) + m_w \nabla \left( \frac{1}{z} \right).$$

Thus

$$\nabla(L(G)) = m_w z \nabla(L(G \setminus \{w\})) + \nabla(L(G \setminus N[w])).$$

With (1) in mind, we see that the recursive property [CS, 2.1] becomes

$$\tilde{I}(G, x) = m_w x \tilde{I}(G \setminus \{w\}, x) + \tilde{I}(G \setminus N[w], x),$$

which defines

$$\tilde{I}(G, x) = \sum_{A \text{ matching of } G} x^{|A|} \prod_{w \in V(G) \setminus V(A)} m_w.$$

To express $\tilde{I}(G, x)$ for a labelled graph $G$ by an independence polynomial, do the following.

We defined the weighted independence polynomial of a labelled graph $\Lambda$ (with vertices $w$ labelled by $m_w > 0$) by

$$I(\Lambda, x) = \sum_{A \subset V(\Lambda)} x^{|A|} \prod_{w \in A} m_w.$$

For a labelled graph $G$, define its line graph $\Lambda(G)$ as a labelled graph, with the label of $w \in V(\Lambda(G)) = E(G)$ being $m_w = 1/(l_v l_{v'})$, where $w$ connects $v$ and $v'$ in $G$.

Now we note that for a link $L = L(G)$ with a labelled plumbing tree $G$, we have

$$I(\Lambda(G), x) \cdot \prod_{v \in V(G)} l_v = \tilde{I}(G, x),$$

thus

$$\nabla(L)(z) = \prod_{v \in V(G)} l_v \cdot I(\Lambda(G), 1/z^2) \cdot z^{1-\chi(L)}.$$

And now to prove that all roots of $\nabla(L)(\sqrt{z})$ are real (and so $\nabla(L)$ is log-concave by remark 3.1), one just needs to consult the explanation for theorem 2.2.

**Corollary 4.1** Both the log-concavity conjecture II and Liechti’s conjecture hold for positive tree plumbing links $L$.

We should perhaps stress that working at a leaf of $G$ is what limits the graph-link connection to trees $G$. For general bipartite plumbings it seems, unfortunately, not so clear how to exhibit a relation to a graph invariant.
5 Alternating plumbing links

5.1 Links of alternating trees

Corresponding to a positive plumbing link \( L = L(A, B) \), there is the alternating plumbing link \( L' = L(A, -B) \) (with \( l_C > 0 \) for \( C \in A \cup B \) and the obvious meaning of \( -B \)). We will continue assuming throughout that vertex labels are integers. The following consideration addresses these links, which were also treated by Hirasawa-Murasugi [HM].

Let first \( L' = L(T') \) be the result of an alternatingly labelled plumbing tree \( T' \) (adjacent vertices have labels of opposite sign). As noted, an alternating plumbing tree gives an alternating diagram, so that \( L' \) is an (arborescent) alternating link. Again, write \( L = L(T) \) for the corresponding positive labelled tree (switch in \( T' \) sign of all negative vertex labels, so that all labels become positive). We will retain this notation throughout this section. Then we have the following relationship.

\[
\nabla_{L'}(z) = \nabla_L(i z) \cdot (-i)^{v(T')} \cdot (-1)^{v_-(T')},
\]

with \( v(T') \) the number of vertices of \( T' \), and \( v_-(T') \) the number of negative labelled vertices.

This says that every second coefficient turns its sign around, while we know what should be the sign of the leading coefficient. Compare also with remark 2.1.

**Proof.** It is easy to see recursively using the skein relation (13) that all terms adding have coefficients of the same sign in the same degree. For \( L \) these coefficients will be all positive, and for \( L' \) they will alternate. Then one propagates (14) through skein induction. □

This argument works again for Hopf bands with any number of full twists (not just \( \pm 1 \); see further §5.3).

**Corollary 5.1** If \( L' \) be an alternating tree plumbing link, then all roots of \( \nabla_{L'}(\sqrt{z}) \) are real. Thus Hoste’s conjecture holds for \( L' \).

**Proof.** It follows from (14) and theorem 4.1. □

**Corollary 5.2** Let \( L' \) be an alternating tree plumbing link. Then \( \nabla_{L'}(\sqrt{z}) \) is an alternating log-concave polynomial. Thus \( \Delta_{L'}(t) \) is triangular, with the stronger property that for \( M = \lfloor \max \deg \Delta \rfloor \) and \( \kappa = \lfloor k \rfloor \) (denoting greatest integer),

\[
\frac{|\Delta_{k-1}|}{|\Delta_k|} > \frac{M + \kappa}{M - \kappa + 1}
\]

in all admissible degrees \( k \) with \( 2 \leq 2k \leq 1 - \chi(L) \).

**Proof.** The property of \( \nabla_{L'} \) follows because \( \nabla_{L'}(\sqrt{z}) \) is known to be a positive polynomial for a positive link \( L \) (see [St]). Regarding \( \Delta_{L'} \), use the conversion (3), and observe the contribution to \( \Delta_{L'} \) of each monomial of \( \nabla_{L'} \) is triangular. (Note also that the alternation of \( \Delta_{L'} \) follows from that of \( \nabla_{L'} \), without needing to invoke Crowell-Murasugi.)

By looking more carefully at the mentioned contributions, we can obtain the stronger stated inequality. The top degree \( z \)-term in \( \nabla \) gives the lowest ratio, which is \( \frac{(M - \kappa + 1)^2 M}{(2M)^2} \).

We remind that usual trapezoidality of \( \Delta_{L'}(t) \) is known from [Mu2] for a larger class of arborescent alternating links. For ordinary (\( \pm 1 \) full twist) Hopf bands, see also Hironaka and Liechti [HL]. We can now work out a little more.

**Theorem 5.1** Let \( L' \) be an alternating tree plumbing link. Then the polynomial \( \Delta_{L'}(t) \) is log-concave.
**Proof.** With the remark in the preceding proof, we can change signs of coefficients. It is enough to prove that when $P \in \mathbb{Z}[z]$ is positive and log concave, then $P(t^{-1} + 2 + t)$ and $(t^{-1/2} + t^{1/2})P(t^{-1} + 2 + t)$ are log concave. The latter case is handled using the easy

**Lemma 5.2** If $Q(t)$ is positive and log concave, so is $Q(t) \cdot (t + 1)$.

**Proof.** Let $a, b, c, d$ be four consecutive coefficients of $Q$. Then we check that $ac \leq b^2$ and $bd \leq c^2$ imply $(a + b)(c + d) \leq (b + c)^2$. □

For the theorem it thus remains to prove the below lemma.

**Lemma 5.3** If $P(z)$ is positive and log concave, then so is $Q(t) = P(t^{-1} + 2 + t)$.

**Proof.** We use induction on $M = \max \deg P$. The initial cases are easy, so do the induction step. Let $a_i = |P_i| > 0$. By induction assumption, the case $a_0 = 0$ follows from applying lemma 5.2 twice. Now when changing $a_0 = 0$ to any positive number, only $|Q_0|$ is changed (upward). Thus the only inequality which can be ruined is $|Q_0|Q_2 \leq |Q_1|^2$; and it is enough to check this inequality. In terms of the $a_i$, it can written as

$$\left(\sum_{m=0}^{M} \frac{2m}{m} a_m \right) \left(\sum_{n=2}^{M} \frac{2n}{n+2} a_n \right) \leq \left(\sum_{m=1}^{M} \frac{2m}{m+1} a_m \right) \left(\sum_{n=1}^{M} \frac{2n}{n+1} a_n \right).$$

(16)

This can proved by expanding. For the coefficient of $a_0 a_m$, it is enough to see

$$\frac{2n}{n} \frac{2m}{m+2} \sum_{n=2}^{M} \frac{2n}{n+2} a_n \leq 2 \frac{2n}{n+1} \frac{2m}{m+1} a_n.$$

First assume $n, m > 0$. By expanding binomial coefficients in terms of factorials and clearing denominators, this can be rearranged to

$$(n + 2)(n + 1)m(m - 1) + (m + 2)(m + 1)n(n - 1) - 2nm(n + 2)(m + 2),$$

which is easy to see by expanding using $2nm^2 + 2m^2 \leq 4nm^2$ (and its version for $n \leftrightarrow m$). Since in (16), $n > 0$, it remains to account for the terms on the left of (16) for $m = 0$. For this note that on the right of (16) terms for $n = 1$ occur, while there are none on the left. To settle (16), it is enough to prove

$$\sum_{n=2}^{M} \frac{2n}{n+2} a_0 a_n \leq \sum_{n=1}^{M} \frac{2n}{n+1} a_1 a_n = \sum_{n=1}^{M+1} \frac{2n}{n} a_1 a_{n-1}.$$

This easily follows from $a_0 a_n \leq a_1 a_{n-1}$, which can be seen by multiplying $a_{p+1} a_{p-1} \leq a_p^2$ for all $1 < p < n$ and removing common factors. □

It is easy to make the inequalities strict (and show strict log concavity), but we leave this to the reader.

**Remark 5.1** It follows from corollary 5.1 also for $\Delta \psi(t)$ that all roots $t$ must have $t + t^{-1} - 2$ to be (real) positive, whence $t$ is (real) positive. From here one can obtain theorem 5.1 also using lemma 3.1; in contrast, lemma 5.3 gives a direct connection of log-concavity without looking at roots. See also remark 5.2 below.

### 5.2 Excessive alternating trees

For some alternating tree plumbing links one can show a little bit more.

Let $T'$ be an alternatingly labelled tree. If in each vertex $v$ of valence $\psi$, the label $l_v \in \mathbb{Z}$ satisfies $|2l_v| \geq \psi_v$, then call $T'$ excessive. (This notion leans on, but is slightly different from, the definition in [Mu2].) Note that in particular this property always holds if $T'$ is a labelled path, i.e., $L' = L(T')$ is a 2-bridge link. (However, keep in mind that a general 2-bridge link arises from an arbitrarily integer labelled path, not just an alternatingly labelled one.)
Proposition 5.1 If $L' = L(T')$ is an excessive alternating tree plumbing link, then all roots of $\nabla_L(\sqrt{\tau})$ are in $[0,4]$. Formally, all roots of $\Delta_L$ (are real and) in $[3 - \sqrt{8}, 3 + \sqrt{8}]$.

Proof. One can show that the positive plumbing link $L$ is alternating, by arguing with the Conway notation. This is a standard procedure (it appears, inter alia, in some form in [Mu2]), and we do not like to elaborate on the details, beyond the below brief explanation. (Conway notation and changing it for arborescent tangles is well treated, e.g., in Adams’ book [Ad].)

Fix a root $w$ of $T$, which is a leaf (valence $\psi_w = 1$). Order the vertices w.r.t. distance from the root. Then each $v \in V(T) \setminus \{w\}$ has $\psi_v - 1$ edges to a higher distance vertex. This ordering gives a Conway notation of $L$. Manipulate the tangles from vertices in top-down distance order. Since $|2l_v| > \psi_v$, at least one half-twist of the $|2l_v|$ will always remain after the tangle corresponding to $v$ and all of higher distance that hangs on it is made alternating. At the very last stage we will need $|2l_v| > \psi_v$ for $v = w$, but this is no problem either when $\psi_w = 1$.

Thus $L$ is alternating. A positive link which is alternating is special alternating, and all roots of $\Delta_L$ lie on $S^1$ (see [St2]). Thus all roots of $\nabla_L(\sqrt{\tau})$ are on $[-4,0]$. It follows that all roots of $\nabla_L(\sqrt{\tau})$ are in $[0,4]$, so all roots $t$ of $\Delta_L$ satisfy $t^{-1} - 2 + t \in [0,4]$. □

Corollary 5.3 Let $L'$ be an alternating tree plumbing link of $n$ components. Then the signature $\sigma(L')$ satisfies $|\sigma(L')| \leq n - 1$ (in particular, $\sigma(K') = 0$ if $K'$ is a knot).

Proof. It follows from Remark 5.1 (for an excessive link also from the above proposition) that $\Delta$ has no root on $S^1 \setminus \{1\}$. The rest follows from the jump behavior of the signature. (For example see [GL] or the appendix of [L].) □

This corollary and corollary 5.2 conform also to the slight addition to the Trapezoidal conjecture made in [St3, §9] that, for knots $K$, the number $m$ in (6) satisfies $2m \leq |\sigma(K)|$. (This inequality was rediscovered in [HM, Conjecture 5.11]. The triangular property in the case $\sigma(K) = 0$, which occurs here, was independently conjectured by Murasugi in private communication.)

5.3 Weakly positive/alternating plumbing

Our last piece of work is a generalization of the relationship (14). This leads into a more knot-theoretical area, but it seems appropriate to accomodate here this short digression.

There is no obstacle to define $L(A, B)$ when some $C \subseteq A \cup B$ has $l_C = 0$. We avoided such trivial bands, because they are exceptional in many ways. (Among others, (9) does no longer hold.) However, we will need zero-labelled chords here, so let us say that if $|l_C| \geq 0$ for $C \subseteq A \cup B$ we call the plumbing link $L = L(A, B)$ weakly positive and $L' = L(A, -B)$ weakly alternating. (For these links, all equalities in (9) still hold, except for ‘max deg $\nabla(L) = \nu(G)$’, which becomes ‘max deg $\nabla(L) \leq \nu(G)$’. Also, the diagrams $D(A, B)$ and $D(A, -B)$ are still positive resp. alternating.)

For a treatise of Murasugi atoms, see [QW, St7].

Theorem 5.2 For an arbitrary weakly positive bipartite plumbing link $L = L(A, B)$ and its corresponding weakly alternating plumbing link $L' = L(A, -B)$, we have

$$\nabla_{L'}(z) = \nabla_L(iz) \cdot (-i)^{-\chi(L)} \cdot (-1)^{m_-},$$

where $\chi(L) = \chi(L')$ is the Euler characteristic, and $m_-$ is the number of negative Murasugi atoms of an even number of components of the diagram $D(A, -B)$.

Proof. The proof of lemma 5.1 can be adapted thus. We again do recursive skein resolution of non-trivial bands. Note that, if $w$ is not a leaf, the middle term in (13) leads to a diagram which cannot be obviously simplified to be a positive bipartite plumbing $L(A, B)$ as we defined before. Thus we must work recursively over diagrams $L(A, B)$ in which we allow labels $l_C = 0$, leading to weakly positive or weakly alternating plumbing links $L$.

This recursion does work well, though, and in fact shows the relationship (18) for this even larger class of diagrams/links, but with the proper modification.
In some diagrams in the recursive skein calculation by successively applying (13), twists coming from different Hopf bands will join. To formalize what crossings ‘join’, we introduced and studied extensively the notion of $\sim$-equivalence. (E.g., it was used over the length of almost the entire book [St3]; we thus prefer not to take space and get into these details again here.) It is important to remark that when starting with a positive diagram, all diagrams obtained in the skein resolution will be positive, and similarly with alternating. This ascertains that when $\sim$-equivalent crossings join, all have the same sign, so that none ever cancel.

Thus, use induction over the number of positive $\sim$-equivalence classes. If this number is zero, we have a negative alternating link bounding a planar surface, so its $\nabla$ is a monomial. Its Murasugi atoms are connected sum factors. Mirroring changes the sign as stated. Then check (it does require some thought) that signs fit in (13).

Thus we see:

**Corollary 5.4** If $L'$ is a weakly alternating bipartite plumbing link, then $\nabla_{L'}$ is alternating and inequality (15) holds. □

**Remark 5.2** A result of Hirasawa-Murasugi [HM, Theorem 6.4] asserts that $\Delta_{L'}$ is real-rooted for a (non-weakly) alternating bipartite plumbing link $L'$. Of course, we recovered and complemented this result for the restricted class of alternating tree plumbing links. On the opposite side, one can start from Hirasawa-Murasugi’s result and use our arguments unrelated to independence polynomials to recover and generalize most results in the last two sections. (For instance, one can then see that $\nabla_{L'}$ in corollary 5.4 is also log-concave – in the non-weak case.) Related work has been done by Liechti in his thesis [L2, Theorem 5.4 and Corollary 5.5]. In such an approach, the connection to graphs remains unclear, though.

This circumstance vaguely raises motivation to seek extensions of Hirasawa-Murasugi’s result to weakly alternating bipartite plumbing links. This may be the topic of a separate (longer and, as noted at the end of §4.2, hardly graph theory related) study.

**Acknowledgement.** I would like to thank to S. Baader for proposing to me this piece of research, and to L. Liechti and the reviewers for their helpful remarks.

**References**


