INDEPENDENCE POLYNOMIALS AND ALEXANDER-CONWAY POLYNOMIALS OF PLUMBING LINKS

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Abstract. We use the Chudnovsky-Seymour Real Root Theorem for independence polynomials to obtain some statements about the coefficients and roots of the Alexander and Conway polynomial of some types of plumbing links, addressing conjectures of Fox, Hoste and Liechti.

Keywords: independence polynomial, line graph, polynomial root, alternating knot, Alexander polynomial, arborescent link, 2-bridge link

AMS subject classification: 57M25, 26C10 (primary), 05C31, 13F20, 12D10 (secondary)

1 Introduction

The Alexander polynomial Δ, and its equivalent version due to Conway ∇, remains of profound importance as an invariant of knots and links in 3-space. Many features of the polynomial have been studied over the years in a variety of contexts, including surface homeomorphisms, knot groups, Lehmer's question on the existence of a Mahler measure minimizing polynomial, knot homology, etc. (See [Ro] for a classical treatise and [LMu, St6] for more recent references, beyond those to follow below.)

A knot or link is *alternating* if it has an alternating diagram, one where every strand passes crossings (alternatingly) over-under. Alternating knots are a class of knots well-studied, and enjoying many special properties. The question to characterize their Alexander polynomials has sparked extensive efforts. Beyond the classical theorem of Crowell-Murasugi, there are several further-going conjectures on the appearance of the Alexander polynomial for alternating links (sometimes more commonly known for knots only). This includes Fox's Trapezoidal conjecture on the coefficients of the polynomial, and Hoste's conjecture on its roots. There are a number of refinements and similar conjectures for the Conway polynomial of positive links (see §3).

Various root location problems and some of these conjectures have been addressed in a recent monograph of Hirasawa-Murasugi [HM]. The goal of this note is to show how the work in Chudnovsky-Seymour [CS] on the real-rootedness of independence polynomials of claw-free graphs can be used to give short elegant proofs of these conjectures for some classes of links and to partially recover and extend a main result of Hirasawa-Murasugi (see remark 5.3).

We will explain in §2 in a proper graph-theory context a slightly generalized form of Chudnovsky-Seymour's theorem and in §4 its relation to the construction of plumbing links. We will apply this relationship then (together with some further-going arguments) to alternating plumbing links in §5.

Acknowledgement. This piece of research was suggested to the author by S. Baader. He also supplied the key resource [CS], whose discovery (in the present context) he credits to L. Liechti. Both have thus provided very substantial motivation for and contribution to this paper. I would like to thank to L. Liechti and the reviewers also for their helpful remarks and numerous corrections.

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2 Weighted independence polynomial

Graphs considered here will be finite, i.e., with a finite number of vertices and edges. Multiple edges (cycles of length 2) are allowed, but no loops (edges from a vertex to itself). We will write $E(G)$ for the edge set and $V(G)$ for the vertex set. Write also $v(G) = |V(G)|$ for the number of vertices.

For a graph G, its *line graph* $\Lambda = \Lambda(G)$ is defined to have a vertex for each edge in G, and an edge for each pair of edges in G with a common vertex.

The *independence polynomial* $I(G,x)$ of G is defined as

$$I(G,x) = \sum_{A\subset V(G)} x^{|A|},$$

where A goes over independent vertex sets, i.e., sets with no edge in G between any two vertices in A.

Now note that

$$I(\Lambda(G),x) = \mu(G,x) = \sum_{\substack{L \text{ matching} \\ \text{of } G}} x^{|L|}, \tag{1}$$

where a matching of G is a set of edges in G with no common vertex.

A vertex of valence 1 is called a *leaf*. A tree with no vertices of valence ≥ 3 (i.e., all vertices are of valence 2 except for two leaves) is a *path*. The tree of three edges which is not a path is the *claw*. A graph is *claw-free*, if no induced subgraph is a claw.

Theorem 2.1 *(Chudnovsky-Seymour [CS]) If G is claw-free, then all roots of $I(G,x)$ are real. In particular, this holds for $G = \Lambda(G')$ for any graph G'.*

We will also call below a polynomial with all roots real to be *real-rooted*.

It will be somewhat useful to extend this theorem to weighted independence polynomials. Consider a *weighted* graph, i.e., for which each vertex v has a *weight* l_v. For now we will assume l_v is a positive real number (while occasionally making remarks for more general weights).

The *weighted independence polynomial* $I(G,x)$ of a weighted graph G is defined as

$$I(G,x) = \sum_{A\subset V(G)} x^{|A|} \prod_{v\in A} l_v,$$

where again A goes over independent vertex sets (*cf.* (1.8) on page 413 and (2.11–2.14) on page 416 in [La]).

Theorem 2.2 *If G is claw-free and weighted (with $l_v > 0$), then all roots of $I(G,x)$ are real.*

One way to prove this theorem is to observe that nothing really obstructs the approach of Chudnovsky-Seymour [CS] for this generalized case. However, we will need to work out an argument in a way that makes later exposition clearer. A reviewer has pointed out that theorem 2.2 (and generalizations of it) is also well treated in the article of Lass [La]. The proof sketched here turned out to be essentially the one attributed (in §1) there to Engström [En].

Proof. Let G be a graph with vertices $V(G) = \{v_1,\dots,v_n\}$ and edge multiplicities m_{ij} between v_i and v_j. And let $\mathbf{v} = (l_1,\dots,l_n)$ a vector of positive integers $l_i \in \mathbb{Z}_+$.

We define the *blowup* graph $G_\mathbf{v}$. Vertices of $G_\mathbf{v}$ are

$$V(G_\mathbf{v}) = \{\, v_{i,j} \,:\, 1 \leq i \leq n, 1 \leq j \leq l_i \,\}.$$

And edge multiplicities are defined as

$$m(v_{i,j}, v_{i',j'}) = \begin{cases} m_{i,i'} & \text{if } i \neq i' \\ 1 & \text{if } i = i' \end{cases}$$

Pictorially, this means replacing in G each vertex with label l_i by a clique of size l_i (each vertex connected to the same vertices outside the clique).

It is easy to see that when G is claw-free, so is $G_{\mathbf{v}}$.

Note also that if $G = \Lambda(G')$ then $G_{\mathbf{v}} = \Lambda(G'_{[\mathbf{v}]})$, where we identify $\mathbf{v} = (l_e : e \in E(G'))$ and $G'_{[\mathbf{v}]}$ means that in G' each edge $e \in E(G')$ is replaced by l_e parallel edges.

Now if a graph G is weighted with positive integer weights $\mathbf{v} = (l_v : v \in V(G))$, applying theorem 2.1 gives that

$$I(G,x) = I(G_{\mathbf{v}},x)$$

has only real roots. (Note that on the left we have a weighted independence polynomial, while on the right an ordinary one.)

Also if G is weighted, and λG is G with weights multiplied by $\lambda > 0$, then

$$x \in \mathbb{C} \text{ is a root of } I(G,x) \text{ if and only if } x/\lambda \text{ is a root of } I(\lambda G,x). \tag{2}$$

If now G is rational weighted, finitely many rational numbers have a common denominator, so λG is integer weighted for some $\lambda \in \mathbb{Z}_+$. Then the claim follows for rational weighted G.

Finally, one obtains the result for real weighted graphs by approximation. For example, approximate coefficients by rational numbers, assume the limit has a genuine complex root, use uniform convergence on a (genuine complex) neighborhood, and [St2, Lemma 2] to get a contradiction. □

In fact, in our work below we will not need irrational weights, but we felt it useful to add this short remark.

Remark 2.1 The relationship (2) holds for any real $\lambda \neq 0$ and setting $\lambda = -1$ one immediately sees theorem 2.2 to hold if all weights of G are negative. It becomes much more interesting to examine what occurs for weights of either sign. (It is obvious why zero weights are redundant.) Unpublished work of Hirasawa-Murasugi (with the added 'translation' here) implies that theorem 2.2 may hold for some such graphs, but they are very restricted and, along with the methods of proof, their graph theoretic significance is unclear. Extensions may be the subject of future investigations.

3 Link polynomial conjectures

Let Δ be the 1-variable Alexander polynomial and ∇ the Conway polynomial of a link L:

$$\nabla(L)(t^{1/2} - t^{-1/2}) = \Delta(L)(t)\,. \tag{3}$$

The polynomial ∇ satisfies the *skein relation*

(4)

which defines it alternatively (up to a factor, fixed by demanding $\nabla(\bigcirc) = 1$). We will call the leftmost crossing *right-handed* (or positive), the other one *left-handed* (or negative).

It is well known that $\nabla(L)$ for an n component link L has only even/odd degree terms when n is odd/even, and that $\min\deg\nabla(L) \geq n-1$. Consequently, $\Delta(t)$ has terms t^k only if k is an integer for odd number of components and half-integer for even number of components; we will call such k *admissible*. Let $\max\deg\Delta$ and $\min\deg\Delta$ be the maximal and minimal degrees. Then $\max\deg\Delta = -\min\deg\Delta$. We have also the *symmetry* property

$$\Delta(t) = (-1)^{n-1}\Delta(1/t) \tag{5}$$

for a link L of n components. It is also well known that when a link is *split* (i.e., have a disconnected planar projection), then its polynomials are zero. We will thus consistently assume links we consider are non-split. (A *knot* has 1 component, and is therefore never split.)

A classical theorem of Crowell-Murasugi [Cw, Mu] asserts that when L is a (non-split) alternating link, then $2\max\deg\Delta = 1-\chi(L)$ for the *Euler characteristic* $\chi(L)$ of L, and in particular for an alternating knot K, we have $\max\deg\Delta = g(K)$ for the genus $g(K)$ of K. Moreover, $\Delta(L)$ is an *alternating* polynomial, i.e., $[\Delta]_k[\Delta]_{k+1} < 0$ for admissible k of $0 \le 2k < -\chi(L)$, where $[\Delta]_k$ denotes the *coefficient* of Δ in degree k.

There are several further-going conjectures on the appearance of the polynomial for alternating links L. (We state the versions for links, although some conjectures are more commonly known for knots only.)

Conjecture 3.1 (Fox's Trapezoidal conjecture) *If L is alternating, there is a number $0 \le 2m \le 1-\chi(L)$ such that for $\Delta_{[k]} := \big|[\Delta_L]_k\big|$ we have for all admissible k,*

$$\begin{array}{ll} \Delta_{[k]} = \Delta_{[k-1]} & \textit{for } 0 < |k| \le m, \\ \Delta_{[k]} < \Delta_{[k-1]} & \textit{for } m < |k| \le 1-\chi(L)\,. \end{array} \tag{6}$$

This conjecture was verified for 2-bridge knots (Hartley) and some more arborescent (algebraic) knots (Murasugi [Mu2]).

We will call an *alternating* polynomial with the conjectured property also *trapezoidal*. (With symmetry (5) of the Alexander polynomial, this is a special case of a *unimodal* polynomial.) In case $m = 0$ for links of an odd number of components, and $m = 1/2$ for even number, we call the polynomial *triangular*. (Note that for even components, symmetry implies that $m \ge 1/2$.)

Call a polynomial X *log-concave*, if its sequence of coefficients $[X]_k$ (in degree k) are log-concave, i.e.,

$$[X]_k^2 \ge [X]_{k+1}[X]_{k-1} \ge 0 \tag{7}$$

for all $k \in \mathbb{Z}$. (The right non-negativity condition is technically added to restrict us to positive and alternating polynomials; see also remark 3.2 below.)

Conjecture 3.2 (log-concavity conjecture [St2]) *If K is an alternating knot/link, then $\Delta_K(t)$ is log-concave.*

It is easily seen as an extension of the Trapezoidal conjecture (see [St2]). The log-concavity conjecture was verified for knots K of genus $g(K) \le 4$ [St3].

A class exhibiting multiple connections (not only here) to alternating links are the *positive links*. A link diagram is *positive* if all its crossings are right-hand. A positive link is a link with such a diagram.

Note that by [St], if L is a positive link, the (simplified) Conway polynomial $\tilde{\nabla} = \nabla(L)(\sqrt{z})$ is a *positive polynomial*, i.e., $[\tilde{\nabla}]_k > 0$ for $\min\deg\tilde{\nabla} \le k \le \max\deg\tilde{\nabla}$ with $\max\deg\tilde{\nabla} - k$ an integer.

Conjecture 3.3 (log-concavity conjecture II [St2]) *If L is a non-split positive link, then the simplified Conway polynomial $\nabla(L)(\sqrt{z})$ is log-concave.*

Regarding roots of the polynomials, Hoste, based on computer verification, raised the following question (now known as a conjecture) about 20 years ago.

Conjecture 3.4 (Hoste's conjecture) *If $t \in \mathbb{C} \setminus \{0\}$ is a root of the Alexander polynomial Δ of an alternating knot/link, then $\Re e\, t > -1$.*

This is known from Murasugi's work if L is special alternating. It is also checked for knots K if $g(K) \le 4$ [St3], and for 2-bridge links L [St5, I]. Hoste's conjecture is essentially independent from the Trapezoidal and log-concavity conjectures [St4]. For some very recent updates on this conjecture, see [HIS].

Liechti proposed a meaningful analogue of Hoste's conjecture for positive links. Here is a slightly modified statement from Liechti's original one, in particular not restricting to positive braid links.

Conjecture 3.5 (Liechti's conjecture [L]) *If $t \in \mathbb{C} \setminus \{0\}$ is a root of the reduced Alexander polynomial*

$$\tilde{\Delta}_L(t) = \frac{\Delta_L(t)}{(t^{1/2} - t^{-1/2})^{n-1}}$$

of a positive (non-split) link L of n components, then $\Re e\, t < 1$.

(The multiplicity of the root $t = 1$ of Δ can be easily deduced from Hoste's formulas for $[\nabla]_{n-1}$ [Ht].)

Remark 3.1 It should be noted that the Hoste resp. Liechti conjectures are true for an alternating resp. positive link whenever all roots of $\nabla(L)(\sqrt{z})$ are real, or equivalently, all roots of $\Delta(L)$ are real or lie on S^1 (the set of unit norm complex numbers). This is because of our previous remarks that $\nabla(L)$ is positive when L is positive and Δ is alternating when L is alternating.

Furthermore, we draw attention also to the following fact discussed in [St2].

Lemma 3.1 *If a positive polynomial is real-rooted, it is log-concave.*

Thus, e.g., the real-rootedness of $\nabla(L)(\sqrt{z})$ for a positive link implies log-concavity conjecture II.

Although the proof was only referenced in [St2], this lemma is not hard to see directly.

Lemma 3.2 *If $Q(t)$ is positive and log-concave, so is $Q(t) \cdot (t+1)$.*

Proof. Let a, b, c, d be four consecutive coefficients of Q. Then we check that $ac \leq b^2$ and $bd \leq c^2$ imply $(a+b)(c+d) \leq (b+c)^2$. □

Lemma 3.1 follows mainly from lemma 3.2 by induction over the degree and rescaling (if $\lambda, \lambda' > 0$, then with $Q(t)$ also $\lambda Q(\lambda' t)$ is log-concave), noting that all (real) roots must be negative. Obviously, Lemma 3.1 will hold then for an alternating (real-rooted) polynomial as well (with the roots then being positive).

Remark 3.2 There seems to be another common notion of log-concavity of a, say positive, polynomial X, namely that $\log X$ is a concave function on $\mathbb{R}_+$, i.e., $(\log X(t))'' \leq 0$ for all $t > 0$. This alternative property is never meant here. Although, for instance, real-rooted polynomials X are easily seen to satisfy that condition as well, one can easily find examples that it neither implies nor follows from (7).

4 Tree plumbing and the independence polynomial

4.1 Construction of plumbing links

We consider a link L obtained by plumbing ordinary positive Hopf bands along a tree G. This means that the vertices of G depict Hopf bands and an edge connects two vertices if the corresponding Hopf bands are plumbed. To formalize this construction, and for later reference as well, we will follow (among multiple accounts) the framework of [HM, §6].

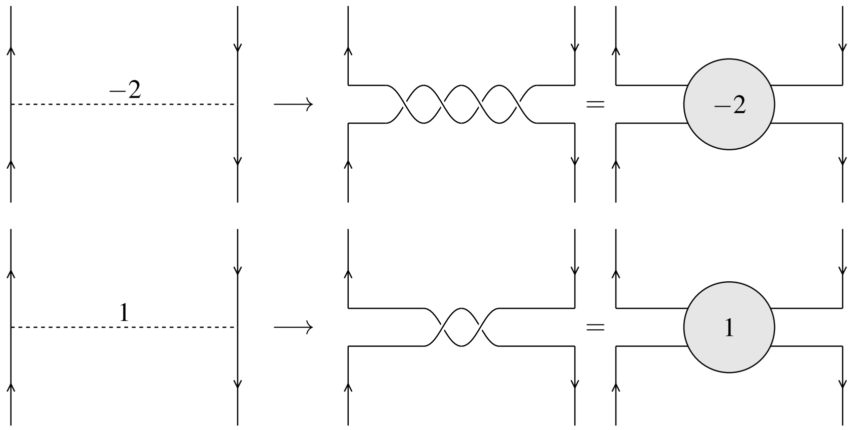
Consider an oriented circle in the plane. Let $\mathbf{A} = \{A_1, \dots, A_n\}$ and $\mathbf{B} = \{B_1, \dots, B_m\}$ be collections of chords. We will understand chords not necessarily as straight, but just as an identification of two (distinct) points on the circle. We will assume, though, that all $2(m+n)$ chord endpoints are distinct.

We say two chords C, D *intersect* if their endpoints are in cyclic order $CDCD$. (This corresponds under (8) to Hopf bands which are plumbed into one another.) We assume none of A_i intersect, and similarly none of B_j, but we do allow some A_i to intersect with B_j. However, for the purpose of building link diagrams, it will be better to think of chords in $\mathbf{B}$ drawn *outside* the circle, so that no chords geometrically intersect in the plane.

We build the *intersection graph* $G(\mathbf{A},\mathbf{B})$ by vertices $V(G(\mathbf{A},\mathbf{B})) = \mathbf{A}\cup\mathbf{B}$ and an edge between intersecting chords. Obviously, $G(\mathbf{A},\mathbf{B})$ is bipartite, but many bipartite graphs do not arise this way. However, it is important to note that every tree does. A tree can be built from a vertex by adding leaves (valence-1 vertices), and it is easy to see how to install the corresponding chords.

Also, we will endow chords in $C \in \mathbf{A}\cup\mathbf{B}$ with a non-zero integer label l_C.

The corresponding link $L(\mathbf{A},\mathbf{B})$ and its diagram $D(\mathbf{A},\mathbf{B})$ is obtained from $\mathbf{A}\cup\mathbf{B}$ by the replacement of a (labelled) chord by a (generalized) Hopf band. (The solid line is a piece of the circle; the dotted line a chord.)



(8)

The number of crossings $2|l_C|$ of a Hopf band coming from chord $C \in \mathbf{A}\cup\mathbf{B}$ must be even. They should be right-handed (positive) if $l_C > 0$ and left-handed (negative) if $l_C < 0$. (Some complete explicit examples appear in §5.3.) We will call L a *bipartite plumbing* link. If $G(\mathbf{A},\mathbf{B})$ is a tree, we will call L a *tree plumbing* link. This construction is well-known; in some cases, e.g., [HL], one considers only $l_C = \pm 1$ (giving ordinary Hopf bands); we will strive to release ourselves from that constraint here.

If $l_C \neq 0$ for all $C \in \mathbf{A}\cup\mathbf{B}$, it is also well understood that for $L = L(\mathbf{A},\mathbf{B})$ and $G = G(\mathbf{A},\mathbf{B})$,

$$1-\chi(L) = 2\max\deg\Delta(L) = \max\deg\nabla(L) = v(G) = |\mathbf{A}\cup\mathbf{B}| \qquad (9)$$

is the number of Hopf bands plumbed.

We will call $L(\mathbf{A},\mathbf{B})$ a *positive plumbing link* of $l_C > 0$ for all $C \in \mathbf{A}\cup\mathbf{B}$ and an *alternating plumbing link* if $l_C > 0$ for $C \in \mathbf{A}$ and $l_C < 0$ for $C \in \mathbf{B}$ (which is focused on in [HM]). It is obvious that $D(\mathbf{A},\mathbf{B})$ is a positive resp. alternating diagram under the stated conditions.

We will mostly consider the case that $G = G(\mathbf{A},\mathbf{B})$ is a tree, and write $L(G) = L(\mathbf{A},\mathbf{B})$. This begs the thought why $L(G)$ depends on $G = G(\mathbf{A},\mathbf{B})$ only. In fact, it does not! This is because multiple leaves attached to the same vertex have no *a priori* order in G, but this order does matter in $L(G)$. There are ways to fix this problem, e.g., by considering G to be a *planar rooted* tree. But we will see that our arguments apply for all possible $L(G)$ coming from the same G (these links are *mutants*, and all have equal polynomials), so that this point will not lead to a problem.

Note that in particular if $G = G(\mathbf{A},\mathbf{B})$ is a labelled path, then L is a *2-bridge (rational) link*. Conversely, it is well known that all 2-bridge links arise in such a way. A general tree plumbing link is *arborescent* (though by far not all arborescent links arise by such plumbing).

4.2 Polynomials of positive plumbing links

The line graph $\Lambda = \Lambda(G)$ of the plumbing tree G of a link $L = L(G)$ with labels $+1$ has independence polynomial $I(\Lambda,x)$. Then

$$\nabla(L) = I(\Lambda, 1/z^2)\cdot z^{v(G)},$$

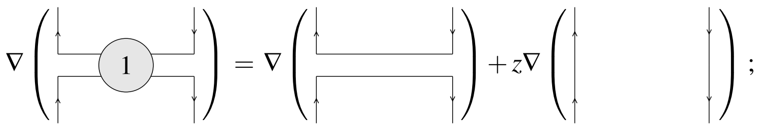
with $v(G) = 1-\chi(L)$ from (9).

This observation, which I owe to S. Baader, is the fundamental connection between knot and graph theory in the paper. With μ from (1), we can write thus:

Lemma 4.1

$$\nabla(L)(z) = \mu(G, 1/z^2)z^{1-\chi(L)}\,. \tag{10}$$

Proof. The skein relation (4) gives



(11)

and if we apply this *at a leaf* v of G, we have with the terminology of [CS, 2.1]

$$\nabla(L(G)) \,=\, z\nabla(L(G\setminus\{v\})) + \nabla(L(G\setminus N[v]))\,. \tag{12}$$

Recall that, according to [CS], $G\setminus\{v\}$ means the graph induced by $V(G)\setminus\{v\}$, and $G\setminus N[v]$ is the graph induced by $V(G)\setminus\{v\}\setminus\{\text{vertices adjacent to } v\}$.

Now, there is a graph polynomial $\hat{I}(G,x)$ satisfying for a tree G and leaf v the property

$$\hat{I}(G,x) \,=\, x\hat{I}(G\setminus\{v\},x) + \hat{I}(G\setminus N[v],x)\,. \tag{13}$$

Namely, this polynomial can be explicitly defined by

$$\hat{I}(G,x) = \sum_{\substack{A \text{ matching}\\ \text{of } G}} x^{|V(G)\setminus V(A)|}\,, \tag{14}$$

where $V(A)$ is the set of vertices of edges in A. (We do not assume A is a perfect matching.) The recursion (13) comes from distinguishing whether $v\in V(A)$ (i.e., whether the matching A has an edge incident from v) or not, giving the two terms on the right of the equation.

This should be illustrated by a very simple example. Let P_n be the path of n edges (and $n+1$ vertices). We have $\hat{I}(P_2,x) = x^3+2x$ by the formula (14). Similarly $\hat{I}(P_1,x) = x^2+1$ and $\hat{I}(P_0,x) = x$. Then (13) gives

$$\hat{I}(P_2,x) = x\hat{I}(P_1,x) + \hat{I}(P_0,x) = x(x^2+1)+x = x^3+2x\,,$$

agreeing with the above.

Now notice that comparing (12) and (13) we have

$$\nabla(L(G))(z) \,=\, \hat{I}(G,z)\,, \tag{15}$$

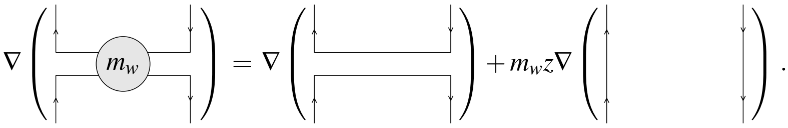
and recall the relationship (1), and formula (10) follows. □

Now consider a link L with a plumbing tree G of Hopf bands with any positive number of full twists. Again, to indicate the number of twists, we label each vertex v of G by a positive integer l_v.

Theorem 4.1 *If L is a positive tree plumbing link, then all roots of $\nabla(L)(\sqrt{z})$ are real (and so $\nabla(L)$ is log-concave).*

For ordinary (one positive full twist) Hopf bands, this is an old result of A'Campo [A]. L. Liechti has informed that he obtained this generalization using A'Campo's method in his Ph.D. thesis [L2].

Proof. We need to clarify how to modify (11) when vertices w have labels $m_w \neq 1$. We have



(16)

Thus for a leaf w of G

$$\nabla(L(G)) \,=\, m_w z\nabla(L(G\setminus\{w\})) + \nabla(L(G\setminus N[w]))\,.$$

With (1) in mind, we see that the recursive property [CS, 2.1] becomes

$$\hat{I}(G,x) = m_w x \hat{I}(G \setminus \{w\},x) + \hat{I}(G \setminus N[w],x),$$

which defines

$$\hat{I}(G,x) = \sum_{\substack{A \text{ matching} \\ \text{of } G}} x^{|V(G)\setminus V(A)|} \prod_{w\in V(G)\setminus V(A)} m_w .$$

To express $\hat{I}(G,x)$ for a labelled graph G by an independence polynomial, do the following.

We defined the weighted independence polynomial of a labelled graph Λ (with vertices w labelled by $m_w > 0$) by

$$I(\Lambda,x) = \sum_{A\subset V(\Lambda)} x^{|A|} \prod_{w\in A} m_w .$$

For a labelled graph G, define its line graph $\Lambda(G)$ as a labelled graph, with the label of $w \in V(\Lambda(G)) = E(G)$ being $m_w = 1/(l_v l_{v'})$, where w connects v and v' in G.

Now we note that we have

$$x^{|V(G)|} \cdot I(\Lambda(G), 1/x^2) \cdot \prod_{v\in V(G)} l_v = \hat{I}(G,x),$$

thus for a link $L = L(G)$ with a labelled plumbing tree G, similarly to (15),

$$\nabla(L(G))(z) = \hat{I}(G,z) = \prod_{v\in V(G)} l_v \cdot I(\Lambda(G), 1/z^2) \cdot z^{1-\chi(L)} .$$

And now to prove that all roots of $\nabla(L)(\sqrt{z})$ are real (and so $\nabla(L)$ is log-concave by remark 3.1), one just needs to consult the explanation for theorem 2.2. □

Corollary 4.1 *Both the log-concavity conjecture II and Liechti's conjecture hold for positive tree plumbing links L.* □

We should perhaps stress that working at a leaf of G is what limits the graph-link connection to trees G. For general bipartite plumbings it seems, unfortunately, not so clear how to exhibit a relation to a graph invariant. On the opposite side, for trees G, some of the further-going study of $I(G,x)$ also becomes applicable; for instance, one can use a formula like [La, (1.7)] to gain lower estimates of $|\nabla(L)(\sqrt{z})|$ in terms of $\Im m\, z$, etc.

5 Alternating plumbing links

5.1 Links of alternating trees

Corresponding to a positive plumbing link $L = L(\mathbf{A},\mathbf{B})$, there is the alternating plumbing link $L' = L(\mathbf{A},-\mathbf{B})$ (with $l_C > 0$ for $C \in \mathbf{A}\cup\mathbf{B}$ and the obvious meaning of $-\mathbf{B}$). We will continue assuming throughout that vertex labels are integers. The following consideration addresses these links, which were also treated by Hirasawa-Murasugi [HM].

Let first $L' = L(T')$ be the result of an *alternatingly labelled* plumbing *tree* T' (adjacent vertices have labels of opposite sign). As noted, an alternating plumbing tree gives an alternating diagram, so that L' is an (arborescent) alternating link. Again, write $L = L(T)$ for the corresponding positive labelled tree T (switch in T' sign of all negative vertex labels, so that all labels become positive). We will retain this notation throughout this section. Then we have the following relationship.

Lemma 5.1

$$\nabla_{L'}(z) = \nabla_L(iz) \cdot (-i)^{v(T')} \cdot (-1)^{v_-(T')}, \tag{17}$$

with $v(T')$ the number of vertices of T', and $v_-(T')$ the number of negative labelled vertices.

This says that every second coefficient turns its sign around, while we know what should be the sign of the leading coefficient. Compare also with remark 2.1.

Proof. It is easy to see recursively using the skein relation (16) that all terms adding have coefficients of the same sign in the same degree. For L these coefficients will be all positive, and for L' they will alternate. Then one propagates (17) through skein induction. □

This argument works again for Hopf bands with any number of full twists (not just ± 1; see further §5.3).

Corollary 5.1 *If L' be an alternating tree plumbing link, then all roots of $\nabla_{L'}(\sqrt{z})$ are real. Thus Hoste's conjecture holds for L'.*

Proof. It follows from (17) and theorem 4.1. □

Corollary 5.2 *Let L' be an alternating tree plumbing link. Then $\nabla_{L'}(\sqrt{z})$ is an alternating log-concave polynomial. Thus $\Delta_{L'}(t)$ is triangular, with the stronger property that for $M = \lfloor \max\deg\Delta \rfloor$ and $\kappa = \lfloor k \rfloor$ (denoting greatest integer),*

$$\frac{|[\Delta]_{k-1}|}{|[\Delta]_k|} > \frac{M+\kappa}{M-\kappa+1} \tag{18}$$

in all admissible degrees k with $2 \le 2k \le 1-\chi(L)$.

Proof. The property of $\nabla_{L'}$ follows because $\nabla_L(\sqrt{z})$ is known to be a positive polynomial for a positive link L (see [St]). Regarding $\Delta_{L'}$, use the conversion (3), and observe the contribution to $\Delta_{L'}$ of each monomial of $\nabla_{L'}$ is triangular. (Note also that the alternation of $\Delta_{L'}$ follows from that of $\nabla_{L'}$, without needing to invoke Crowell-Murasugi.)

By looking more carefully at the mentioned contributions, we can obtain the stronger stated inequality. The top degree z-term in ∇ gives the lowest ratio, which is $\dfrac{\binom{2M}{M-\kappa+1}}{\binom{2M}{M-\kappa}}$. □

We remind that usual trapezoidality of $\Delta_{L'}(t)$ is known from [Mu2] for a larger class of arborescent alternating links. For ordinary (± 1 full twist) Hopf bands, see also Hironaka and Liechti [HL]. We can now work out a little more. (See also remark 5.3 below.)

Corollary 5.3 *Let L' be an alternating tree plumbing link. Then the polynomial $\Delta_{L'}(t)$ is real-rooted, and log-concave.*

Proof. It follows from corollary 5.1 also for $\Delta_{L'}(t)$ that every root t must have $t+t^{-1}-2$ to be (real) non-negative, whence t is (real) positive. From here one can obtain log-concavity using lemma 3.1. □

Remark 5.1 It is possible that log-concavity of Δ follows from that of ∇ directly, without looking at roots. Modulo lemma 3.2, this is the question: if $P(z)$ is positive and log-concave, is then $Q(t) = P(t^{-1}+2+t)$ also?

Corollary 5.4 *Let L' be an alternating tree plumbing link of n components. Then the signature $\sigma(L')$ satisfies $|\sigma(L')| \le n-1$ (in particular, $\sigma(K') = 0$ if K' is a knot).*

Proof. It follows from the argument in the proof of corollary 5.3 (for an excessive link, as discussed below, also from proposition 5.1) that Δ has no root on $S^1 \setminus \{1\}$. The rest follows from the jump behavior of the signature. (For example see [GL] or the appendix of [L].) □

This corollary and corollary 5.2 conform also to the slight addition to the Trapezoidal conjecture made in [St3, §9] that, for knots K, the number m in (6) satisfies $2m \le |\sigma(K)|$. (This prediction was rediscovered in [HM, Conjecture 5.11]. The triangular property in the case $\sigma(K) = 0$, which occurs here, was independently conjectured by Murasugi in private communication.)

5.2 Excessive alternating trees

For some alternating tree plumbing links one can show a little bit more.

Let T' be an alternatingly labelled tree. If in each vertex v of valence ψ_v, the label $l_v \in \mathbb{Z}$ satisfies $|2l_v| \geq \psi_v$, then call T' *excessive*. (This notion leans on, but is slightly different from, the definition in [Mu2].) Note that in particular this property always holds if T' is a labelled path, i.e., $L' = L(T')$ is a 2-bridge link. (However, keep in mind that a general 2-bridge link arises from an arbitrarily integer labelled path, not just an alternatingly labelled one.)

Proposition 5.1 *If $L' = L(T')$ is an excessive alternating tree plumbing link, then all roots of $\nabla_{L'}(\sqrt{z})$ are in $[0,4]$. Followingly, all roots of $\Delta_{L'}$ are (real and) in $[3-\sqrt{8}, 3+\sqrt{8}]$.*

Proof. One can show that the positive plumbing link L is alternating, by arguing with the Conway notation. This is a standard procedure (it appears, *inter alia*, in some form in [Mu2]), and we do not like to elaborate on the details, beyond the below brief explanation. (Conway notation and changing it for arborescent tangles is well treated, e.g., in Adams' book [Ad].)

Fix a root w of T, which is a leaf (valence $\psi_w = 1$). Order the vertices w.r.t. distance from the root. Then each $v \in V(T) \setminus \{w\}$ has $\phi_v = \psi_v - 1$ edges to a higher distance vertex. This ordering gives a Conway notation of L. Manipulate the tangles from vertices in top-down distance order. Since $|2l_v| > \phi_v$, at least one half-twist of the $|2l_v|$ will always remain after the tangle corresponding to v and all of higher distance that hangs on it is made alternating. At the very last stage we will need $|2l_v| > \psi_v$ for $v = w$, but this is no problem either when $\psi_w = 1$.

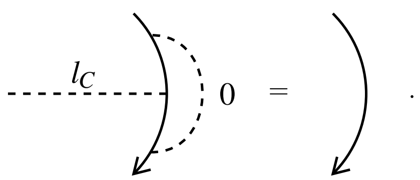
Thus L is alternating. A positive link which is alternating is special alternating, and all roots of Δ_L lie on S^1 (see [St2]). Thus all roots of $\nabla_L(\sqrt{z})$ are on $[-4,0]$. It follows that all roots of $\nabla_{L'}(\sqrt{z})$ are in $[0,4]$, so all roots t of $\Delta_{L'}$ satisfy $t^{-1} - 2 + t \in [0,4]$. □

5.3 Weakly positive/alternating plumbing

Our last piece of work is a generalization of the relationship (17). This leads into a more knot-theoretical area, but it seems appropriate to accomodate here this short digression.

There is no obstacle to define $L(\mathbf{A},\mathbf{B})$ when some $C \in \mathbf{A} \cup \mathbf{B}$ has $l_C = 0$. We avoided such *trivial* bands, because they are exceptional in many ways. (Among others, (9) does no longer hold.) However, we will need zero-labelled chords here, so let us say that if $l_C \geq 0$ for $C \in \mathbf{A} \cup \mathbf{B}$ we call the plumbing link $L = L(\mathbf{A},\mathbf{B})$ *weakly positive* and $L' = L(\mathbf{A},-\mathbf{B})$ *weakly alternating*. For these links, all equalities in (9) still hold, except for '$\max\deg\nabla(L) = v(G)$', which becomes '$\max\deg\nabla(L) \leq v(G)$'. (We will see, though, with remark 5.2 how to express $\max\deg\nabla(L)$ exactly.) Also, the diagrams $D(\mathbf{A},\mathbf{B})$ and $D(\mathbf{A},-\mathbf{B})$ are still positive resp. alternating.

A link diagram is *connected* if it is a connected set in $\mathbb{R}^2$, otherwise it is *disconnected*. A crossing p in a diagram is *nugatory* or *reducible*, if there is a closed curve in $\mathbb{R}^2$ intersecting the diagram only in p. A diagram is reducible if it has a reducible crossing, otherwise it is called *reduced*. Reducible crossings can always be removed, to make a diagram reduced. (These are standard facts well explained in [Ad], for instance.) The simplest pattern of nugatory crossings in $D(\mathbf{A},\mathbf{B})$, and its removal, is



(19)

However, there are more complicated ways in which nugatory crossings can appear in $D(\mathbf{A},\mathbf{B})$. The general rule would be rather to replace the label l_C by 0 and retain the dashed chord.

We will also need to use the *connected sum* $D_1 \# D_2$ of diagrams, and the well-known behavior

$$\nabla(D_1 \# D_2) = \nabla(D_1) \cdot \nabla(D_2)\,. \tag{20}$$

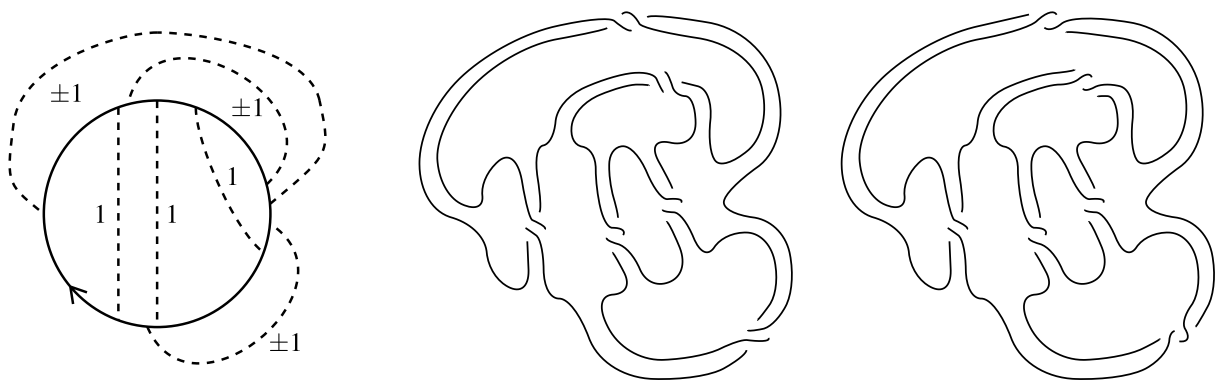
For a treatise of *Murasugi atoms*, see [QW, St7]. We introduced and studied extensively the notion of $\sim$*-equivalence* of crossings; e.g., it was used over the length of almost the entire book [St3]. (We thus prefer not to take space and get into these details again here.)

Theorem 5.1 *For an arbitrary weakly positive bipartite plumbing link $L = L(\mathbf{A},\mathbf{B})$ and its corresponding weakly alternating plumbing link $L' = L(\mathbf{A},-\mathbf{B})$, We assume that the diagram $D(\mathbf{A},\mathbf{B})$ is reduced. We then have*

$$\nabla_{L'}(z) \,=\, \nabla_L(iz)\cdot(-i)^{1-\chi(L)}\cdot(-1)^{m_-}\,, \tag{21}$$

where $\chi(L) = \chi(L')$ is the Euler characteristic, and m_- is the number of negative Murasugi atoms of an even number of components of the diagram $D(\mathbf{A},-\mathbf{B})$.

It is not necessary that $G(\mathbf{A},\mathbf{B})$ is a tree, as illustrates the following example (with $\chi = -5$ and $m_- = 3$):

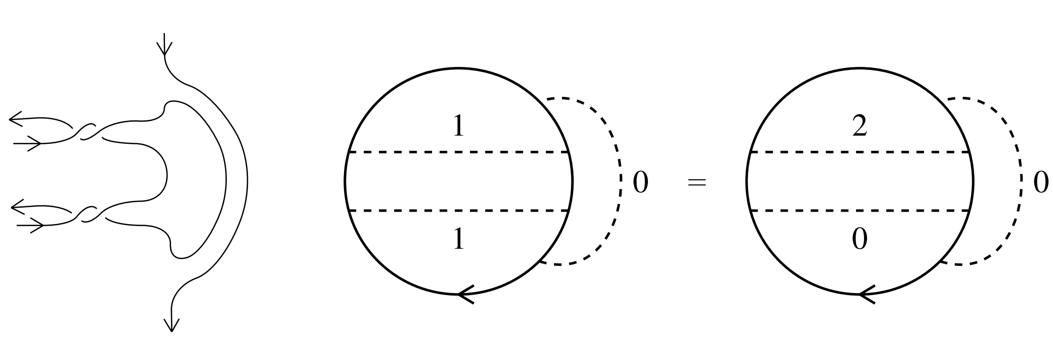


$\nabla(z) = -4z^2 + 7z^4 - z^6$ $\qquad$ $\nabla(z) = 4z^2 + 7z^4 + z^6$

Proof. The proof of lemma 5.1 can be adapted thus. We again do recursive skein resolution of non-trivial bands. Note that, if w is not a leaf, the middle term in (16) leads to a diagram which cannot be obviously simplified to be a positive bipartite plumbing $L(\mathbf{A},\mathbf{B})$ as we defined before. Thus we must work recursively over diagrams $L(\mathbf{A},\mathbf{B})$ in which we allow labels $l_C = 0$, leading to weakly positive or weakly alternating plumbing links L.

This recursion does work well, though, and in fact shows the relationship (21) for this even larger class of diagrams/links, but with the proper modification. Below, we mention several caveats.

In some diagrams in the recursive skein calculation by successively applying (16), twists coming from different Hopf bands will join.



To formalize what crossings 'join', we can use $\sim$-equivalence. It is important to remark that when starting with a positive diagram, all diagrams obtained in the skein resolution will be positive, and similarly with alternating. This ascertains that when $\sim$-equivalent crossings join, all have the same sign, so that none ever cancel.

Thus, use induction over the number of positive $\sim$-equivalence classes. If this number is zero, we have a negative alternating link bounding a planar surface, so its ∇ is a monomial (or $\nabla = 0$, which does not cause any problem, though). Its Murasugi atoms are connected sum factors. Mirroring changes the sign as stated. Then check (it does require some thought) that signs fit in (16).

If $D(\mathbf{A},\mathbf{B})$ is disconnected, then $\nabla = 0$. While this causes no problem in (21), one should see why neither it does in applying (16). Assume that the left diagram in (16) is not disconnected. If the second term on the right of (16) is disconnected, then the crossings are nugatory, thus remove them, and use (16) on some other crossings. If the first term on the right of (16) is disconnected, then $D(\mathbf{A},\mathbf{B})$ has a connected sum factor, which is a (generalized reverse) Hopf link (i.e., $L(\mathbf{A}',\mathbf{B}')$ for $|\mathbf{A}'\cup\mathbf{B}'| = 1$). This case can also be handled directly by (20). □

Thus we see:

Corollary 5.5 *If L' is a weakly alternating bipartite plumbing link, then $\nabla_{L'}$ is alternating and inequality (18) holds.* □

Remark 5.2 The proof of theorem 5.1 also shows a modification of (9) when $l_C = 0$ is allowed. If $D(\mathbf{A},\mathbf{B})$ is connected, then $1-\chi(L)$ is the number of $\sim$-equivalence classes that remain in $D(\mathbf{A},\mathbf{B})$ after all nugatory crossings are removed via (19) or the more general remark below it.

Remark 5.3 A result of Hirasawa-Murasugi [HM, Theorem 6.4] asserts that $\Delta_{L'}$ is real-rooted for a (non-weakly) alternating bipartite plumbing link L'. Of course, we recovered and complemented this result for the restricted class of alternating tree plumbing links. On the opposite side, one can start from Hirasawa-Murasugi's result and use our arguments unrelated to independence polynomials to recover and generalize most results in the last two sections. (For instance, one can then see that $\nabla_{L'}$ in corollary 5.5 is also log-concave – in the non-weak case.) Related work has been done by Liechti in his thesis [L2, Theorem 5.4 and Corollary 5.5]. In such an approach, the connection to graphs remains unclear, though.

This circumstance vaguely raises motivation to seek extensions of Hirasawa-Murasugi's result to weakly alternating bipartite plumbing links. This may be the topic of a separate (longer and, as noted at the end of §4.2, hardly graph theory related) study.

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