ON THE POLYAK-VIRO VASSILIEV IN Variant OF DEGREE 4

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Current version: April 4, 2006  First version: August 28, 2001

Abstract. Using the Polyak-Viro Gauß diagram formula for the degree-4-Vassiliev invariant, we extend some previous results on positive knots and the non-triviality of the Jones polynomial of untwisted Whitehead doubles.

1 Introduction

When some new knot invariants are introduced, one is interested what properties of knots they measure. For example, the Alexander polynomial can reflect the property of a knot to be fibered or slice, while the Jones polynomial reflects more strongly other properties like being achiral or alternating. Briefly after Vassiliev invariants were introduced [Va], it was shown that whenever any finite number of such invariants admit given values on some knot, they do so also on alternating [S] or slice [Ng] knots (up to the condition of zero Arf invariant), so that Vassiliev invariants are useless in detecting these properties.

Contrarily, via the Gauß diagram formulas of Fiedler [Fi, Fi2, FS] and Polyak-Viro [PV], Vassiliev invariants turned out to have numerous applications to positive and almost positive knots [St, St2]. In particular, in [St] I showed that the (properly normalized) Vassiliev invariants of degree 2 and 3 are bounded below by a multiple of the crossing number of a positive reduced diagram, implying that a positive knot has only finitely many positive (reduced) diagrams.

In this note we prove a similar type of inequality for the degree-4-Vassiliev invariant, given in [PV]. The method of proof is similar to the one in [St], but technically more difficult, since the formula lacks the configuration of linked pairs of crossings and contains a negative (coefficient) term, whose contribution must be compensated.

An advantage of the Gauß diagram formulas is that they allow to calculate the invariant in polynomial time with respect to the crossing number of the diagram, and thus allow practical computation for very high crossing number, where the calculation of the link polynomials is considerably slower. Based on such a calculation, we then discuss some applications of the Polyak-Viro formula to untwisted Whitehead doubles and the non-triviality of their Jones polynomial. In particular, we will establish this non-triviality for untwisted Whitehead doubles of knots of up to 15 crossings.

*Supported by a DFG postdoc grant.
2 Gauß sums

2.1 Basic definitions

We use the Alexander-Briggs notation and the Rolfsen [Ro] tables to distinguish between a knot $K$ and its obverse (mirror image) $\overline{K}$. “Projection” is the same as “diagram”, and this means a knot or link diagram. Diagrams are always assumed oriented.

We recall briefly the definition of Gauß sum invariants.

**Definition 2.1 ([Fi2])** A Gauß diagram of a knot diagram is an oriented circle with arrows connecting points on it mapped to a crossing and oriented from the preimage of the undercrossing (underpass) to the preimage of the overcrossing (overpass).

We will call the two arrow ends also *hook* and *tail*.

**Example 2.1** As an example, figure 1 shows the knot $6_2$ in its commonly known projection and the corresponding Gauß diagram.

![Figure 1: The standard diagram of the knot 6_2 and its Gauß diagram.](image)

**Definition 2.2** The writhe is a number ($\pm 1$), assigned to any crossing in a link diagram. A crossing as in figure 2(a) has writhe 1 and is called positive. A crossing as in figure 2(b) has writhe $-1$ and is called negative.

![Figure 2](image)

**Definition 2.3** A knot is called positive if it has a positive diagram, i.e. a diagram with all crossings positive. A knot is called almost positive if it is not positive, but has an almost positive diagram, i.e. a diagram with all crossings positive except one.

Positive knots (or certain sub- or superclasses of them) have been studied, beside by their intrinsic knot-theoretical interest, in a variety of contexts, including singularity theory [A, BoW], algebraic
2.2 The main result

curves [Ru, Ru2], dynamical systems [Wi] and (in some vague and yet-to-be understood way) in 4-dimensional QFTs [Kr].

A Gauß sum is evaluated by choosing arrows from the Gauß diagram of a knot diagram matching a given configuration, and summing for each such matching choice of arrows a quantity called weight. Usually (if not specified, by default) the weight is the product of the writhes of the crossings whose arrows match the configuration. We will for convenience identify a configuration with its evaluation on a fixed specified diagram.

The simplest (non-trivial) Vassiliev invariant is the Casson invariant \( v_2 = \nabla_2 \), with \( \nabla_i = |\nabla|_i \) being the coefficient of \( z^i \) in the Conway polynomial, for which Polyak-Viro gave the simple Gauß sum formula

\[
v_2 = \frac{8}{3} \cdot \frac{1}{12} V_2 - \frac{1}{36} V_3, \tag{1}
\]

Here the point on the circle corresponds to a point on the knot diagram, to be placed arbitrarily except on a crossing.

Other formulas Polyak-Viro, and also Fiedler, gave for the degree-3-Vassiliev invariant \( v_3 \). To make precise which variation of the degree-3-Vassiliev invariant we mean, we have

\[
v_3 = -\frac{1}{12} V_2 - \frac{1}{36} V_3,
\]

where \( V \) is the Jones polynomial [J], and \( V_i := V^{(i)}(1) \). (Note that \( v_2 = \nabla_2 = -1/6 V_2 \).) Fiedler’s formula for \( v_3 \) [Fi2, FS] reads

\[
4v_3 = \sum_{(3,3)} w_pw_qw_r + \sum_{(4,2)0} w_pw_qw_r + \frac{1}{2} \sum_{p,q \text{ linked}} (w_p + w_q), \tag{2}
\]

where the configurations are

\[
\begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{config1} & \includegraphics[width=0.2\textwidth]{config2} & \includegraphics[width=0.2\textwidth]{config3} \\
(3,3) & (4,2)0 & p,q \text{ linked}
\end{array}
\]

Here chords depict arrows which may point in both directions and \( w_p \) denotes the writhe of the crossing \( p \). If two chords \( a \) and \( b \) intersect, we call the corresponding crossings linked and write \( a \cap b \) (read “\( a \) intersects \( b \)”). In a linked pair, we call distinguished the crossing, whose over-pass is followed (when passing the diagram in the orientation direction) by the under-pass of the other crossing.

We note, that \( v_3 \) is asymmetric, i. e., \( v_3(\overline{K}) = -v_3(K) \), so that achiral knots have zero invariant.

2.2 The main result

For the invariants of higher degree the formulas are more complicated, and the analysis becomes more involved. For the degree-4-Vassiliev invariant \( v_4 \), the Polyak-Viro formula has the form

\[
v_4 = \includegraphics[width=0.2\textwidth]{config4} + 6 \includegraphics[width=0.2\textwidth]{config5} + 2 \includegraphics[width=0.2\textwidth]{config6} + 3 \includegraphics[width=0.2\textwidth]{config7} + \includegraphics[width=0.2\textwidth]{config8} + 2 \includegraphics[width=0.2\textwidth]{config9} + 2 \includegraphics[width=0.2\textwidth]{config10} + \includegraphics[width=0.2\textwidth]{config11} - \includegraphics[width=0.2\textwidth]{config12} + \includegraphics[width=0.2\textwidth]{config13} + \includegraphics[width=0.2\textwidth]{config14} + \includegraphics[width=0.2\textwidth]{config15} + 2 \includegraphics[width=0.2\textwidth]{config16} + \includegraphics[width=0.2\textwidth]{config17} + \includegraphics[width=0.2\textwidth]{config18} + \includegraphics[width=0.2\textwidth]{config19} . \tag{3}
\]

\(^1\)Note the factor 4 by which (2) differs from the definition in [St]!
Here the weights are taken to be the products of the writhes of the single arrows.

For simplicity denote the diagrams in (3) (without coefficients) by \(1\), \(2\), \(16\) in order of occurrence. Thus \(v_4\) is written as \(1 + 6 \cdot 2 + \ldots + 16\) with \(1\) = \(\begin{array}{c}
\end{array}\) and \(16\) = \(\begin{array}
\end{array}\).

As noted in [PV], \(v_4\) is symmetric (that is, has the same value on mirror images), takes the value 3 on the trefoil(s) and 2 on the figure-eight-knot. In is also primitive. However, the values on 3 non-trivial knots are needed to identify the invariant, since the space of symmetric primitive Vassiliev invariants of degree \(\leq 4\) is 3-dimensional. Using the additional value \(v_4 = 25\) on \(5_1\) (on its usual 5-crossing diagram, all terms in (3) are zero, except \(14 = 5\) and \(12 = 15\) and some calculation, one arrives at the expression

\[
v_4 = \frac{V_4}{144} + 2V_2^2 - \frac{5}{2}V_4 - \frac{V_3}{24} + \frac{V_2}{12}.
\]

Our main aim in this note will be to prove

**Theorem 2.1** If \(D\) is a positive reduced knot diagram of \(c\) crossings, then \(v_4(D) \geq \frac{3c}{4}\).

From the formulas (1) and (2), it is obvious that on a positive diagram \(v_3\) and \(v_5\) are non-negative, since the formulas in this case basically count the number of matching choices of arrows for some of the configurations. By some more detailed arguments, in [St] we showed that this number is bounded below by a positive multiple of the crossing number of the diagram. To prove theorem 2.1, a similar analysis is necessary. It is more difficult, though, than for the Vassiliev invariants of lower degree, because linked pairs are not counted, and the contribution of a negative (coefficient) term occurs. For this reason, and because of the large number of terms, it appears difficult to prove a reasonable estimate of \(v_4\) on almost positive knots. (In a diagram with a negative crossing there are too many configurations of negative contribution to take account for.)

### 2.3 Two examples

There are examples of knots, for which the new inequality is violated, but those for the Vassiliev invariants of degree 2 and 3 shown in [St] are not. Particularly interesting is the knot 16\(1377111\) on figure 3. It has \(v_2 = 7\) and \(v_3 = 7\), so that with \(c = 16\) we have \(v_2 \geq c/4\) and \(v_3 \geq |\frac{c-1}{4}|\). However, \(v_4 = 9 < \frac{3c}{4}\). The knot has the positive Conway polynomial \(V(\sqrt{2}) = |1| 7 13 9 2\) and the (more general) condition of [CM] involving the skein polynomial is also satisfied. The knot also satisfies the equalities

\[
\min \deg V = \min \deg P/2 = \max \deg P/2 = \min \deg F/2 = g, \tag{5}
\]

with \(P\) being the skein and \(F\) the Kauffman polynomial. (The genus \(g = 5\) was determined using [St3].)

The only alternative methods to show non-positivity for this example are the property \(\max \deg V = \max \deg P\) (see [Cr]), the condition of [Yo], \(\min \deg P = \min \deg F\), and the condition of [Th] on the positivity of the “critical line” coefficients \(|F|_{\pm 1}\) with \(m - l\) minimal (following since positive diagrams are \(A\)-adequate). However, these conditions (and also those in (5)) involve invariants which are of exponential complexity, and are therefore hard to apply for more complicated knots. For example, \(v_4\) takes just a few seconds on diagrams of about 65 crossings, while the calculation of the skein polynomial using the skein method may take up to several days! The genus cannot even always be determined in general, except by the work of Haken, which has proved impractically complicated.

One can find other examples, like 12\(2088\), for which the equalities in (5) and the above three conditions violated by 16\(1377111\) are satisfied (in this case \(g = 2\) was verified in [St4]), but ours is not, etc.

Thus our criterion is new, and sometimes more effective.
3 Proof of theorem 2.1

Before we start with the proof, we recall a property and a move of positive diagrams, introduced in [St].

**Lemma 3.1 (extended even valence eevev(c), see [St])** In the Gauß diagram of a knot diagram $D$, any arrow $c$ is intersected by an even number of other arrows. If $D$ is positive, exactly one half of the arrows intersecting $c$ are distinguished in the linked pair with $c$ (and the other half are not).

Alternatively, we also say that exactly one half of the arrows intersecting $c$ intersect it in the one or the other direction.

**Definition 3.1** The loop-move from a knot diagram $A$ to a diagram $B$, consists in choosing a segment of the line in $A$ between the two passings of a crossing $c$, such that it has no self-crossings, removing of this segment by switching some (for $A$ positive exactly one half) of the crossings on it, and elimination of all reducible crossings thereafter.

$$
\begin{array}{c}
\begin{array}{c}
\uparrow c \\
\downarrow c
\end{array}
\rightarrow
\begin{array}{c}
\uparrow c \\
\downarrow c
\end{array}
\rightarrow
\begin{array}{c}
\uparrow c \\
\downarrow c
\end{array}
\rightarrow
\begin{array}{c}
\uparrow c \\
\downarrow c
\end{array}
\end{array}
$$

On the Gauß diagram the loop-move means to choose an arrow $c$, such that one of the half-arcs, into which the ends of $c$ separate the circle, contains no endpoints of arrows $b \parallel c$, then to remove $c$ and all $a \cap c$, and finally to delete all chords $b$ which have become isolated after this removal (these are the chords $b \parallel c$, such that for all $a \cap b$ also $a \cap c$).

Since the loop move transforms a knot diagram into another knot diagram, its Gauß diagram version preserves the realizability of the Gauß diagram (by a knot diagram). Also, the move preserves the positivity of a knot diagram (and its Gauß diagram).

**Proof of theorem 2.1.** We split the proof into three steps, each one estimating the contribution of an appropriate part of the Gauß sum (3), and include some intermediate statements as lemmas.

**Step 1.** We first show that $v_4 \geq 0$. For this we need to account for the negative (coefficient) term $\langle 9 \rangle$. Consider the sum $2 \langle 6 \rangle + 2 \langle 7 \rangle + \langle 8 \rangle - \langle 9 \rangle$ and symmetrize with respect to mirror image, noting that

![Figure 3: Two knots, on which our criterion is rather effective.](image-url)
\[ \frac{1}{2} \left( 2 \begin{array}{c} 2 \\ 2 \end{array} + 2 \begin{array}{c} 3 \\ 3 \end{array} + \begin{array}{c} \circ \circ \\ \circ \circ \end{array} - \begin{array}{c} \circ \circ \\ \circ \circ \end{array} + \right). \] (6)

Here we replaced the double arrow in (8) and its arrow-reversed counterpart by a single one, since the knot diagram is positive (and the weight does not change), and the Gauß diagrams gain no symmetries after the replacement (contrarily to (12), which would gain a cyclic symmetry of order 3, and whose coefficient would have to be multiplied by 3 accordingly).

We define now the upper index \( U(a, b) \) of two linked chords \( a \) and \( b \) as follows.

\[ \begin{array}{c} b \\ \circ \circ \end{array} \]

The arrow \( b \) separates the circle into two half-arcs. Count the arrows \( c \) (with both ends) on the half-arc of \( b \), on which the overpass (arrow hook) of \( a \) lies, which intersect \( a \) in the in the opposite direction to that of \( b \).

\[ \begin{array}{c} b \\ \circ \circ \end{array} \]

In the same way define the lower index \( u(a, b) \), only counting arrows \( c \) on the half-arc of \( b \), on which the underpass (arrow tail) of \( a \) lies.

\[ \begin{array}{c} b \\ \circ \circ \end{array} \]

Now all the configurations in (6) have a distinguished (vertical) arrow, intersecting (as unique arrow) all other arrows in the configuration. Thus the evaluation of (6) on \( D \) can be split into the sum for any fixed vertical arrow.

Let \( a \) be such an arrow, and let \( b_1, \ldots, b_r \), be the arrows intersecting \( a \), such that they are distinguished in the linked pair with \( a \) (that is, look like \( b \) in (7)). Then set \( r' = 2r \), and let \( b_{r+1}, \ldots, b_{r'} \) be the arrows intersecting \( a \), such that \( a \) is distinguished in the pair.

Denote for simplicity by \( \langle (1) \rangle, \ldots, \langle (8) \rangle \) the diagrams in (6). Now consider \( (6)' \), obtained from (6) by reversing the knot orientation. This clearly leaves \( v_4 \) invariant too, but for the Gauß diagrams it means to take the mirror image. Accordingly, for \( k = 1, \ldots, 8 \), set \( \langle (k') \rangle \) to be the mirror image of \( \langle (k) \rangle \). (Arrows that point from left to right in \( \langle (i) \rangle \) point from right to left in \( \langle (i') \rangle \) and vice versa.) Then we have

\[ \langle (1) \rangle + \langle (2) \rangle = \sum_{i=1}^{r} \left( \begin{array}{c} U(a, b_i) \\ 2 \end{array} \right) \quad \langle (3) \rangle = \sum_{i=1}^{r} U(a, b_i) \]

\[ \langle (5) \rangle + \langle (6) \rangle = \sum_{i=1}^{r} \left( u(a, b_i) \right) \quad \langle (7) \rangle = \sum_{i=1}^{r} u(a, b_i) \]

\[ \langle (4') \rangle = \langle (8') \rangle = \sum_{i=1}^{r} u(a, b_i) U(a, b_i). \]

Five analogous identities, with the sum over \( i = r + 1, \ldots, r' \), hold for the mirrored Gauß diagrams (i.e., reversed knot orientation). Abbreviating \( u_i = u(a, b_i) \) and \( U_i = U(a, b_i) \), we obtain then

\[ \frac{1}{2} \langle (6) + (6)' \rangle = \frac{1}{4} \left[ \sum_{i=1}^{r} 2 \left( U_i \right) + 2 \left( u_i \right) + U_i + u_i - 2U_iu_i \right] \]

\[ = \frac{1}{4} \sum_{i=1}^{r} (U_i - u_i)^2 \geq 0. \]
Since the other configurations are non-negative, this shows \( v_4 \geq 0 \).

As we will need it later, denote the r.h.s. in the above estimate

\[
\tilde{w}(a) = \frac{1}{4} \sum_{i=1}^{r'} (U(a, b_i) - u(a, b_i))^2,
\]

the sum being taken over all chords \( b_i \) linked with \( a \), distinguished or not.

Now we give a better description of the crossings with \( \tilde{w}(a) > 0 \).

**Lemma 3.2** \( \tilde{w}(a) > 0 \) \( \iff \exists b, c \cap a \) in opposite directions, \( b \neq c \). Moreover, in this case \( \tilde{w}(a) \geq 1/2 \).

**Proof.** ‘\( \implies \)’. If \( \tilde{w}(a) > 0 \), clearly not all \( u_i \) and \( U_i \) vanish. Thus such \( b, c \) must exist.

‘\( \impliedby \)’. Let such \( b \) and \( c \) exist, and fix \( b \) and \( c \).

\[
\begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
\end{array}
\]

(8)

We construct now a family \( L \) of chords intersecting \( a \) with \( b, c \in L \) by induction as follows.

Set \( b_0 = b \) and \( b_1 = c \) (this indexing is unrelated to the preceding argument). Then assume that for some \( k \geq 0 \) and \( n > 0 \), \( L_{k,n} = \{ b_i : -k \leq i \leq n \} \) is already defined. If \( U(a, b_{-k}) > 0 \), set \( b_{n+1} \) to be one of the chords counted by \( U(a, b_{-k}) \), and if \( u(a, b_n) > 0 \) set \( b_{n+1} \) to be one of the chords counted by \( u(a, b_n) \). Otherwise, if \( U(a, b_{-k}) = u(a, b_n) = 0 \), set \( L = L_{k,n} \).

Then any two arrows in \( L \) do not intersect, and \( b_i \) and \( b_{i+1} \) intersect \( a \) in opposite directions. Then at least two of the \( b_i \), namely \( b_{-k} \) and \( b_n \), have \( U(a, b_i) \neq u(a, b_i) \), and thus \( \tilde{w}(a) \geq 1/2 \). \( \square \)

Denote by \( (\cdot)_s \) the symmetrization with respect to both orientation and mirroring. Then we just showed that

\[
\left( 2 \langle 6 \rangle + 2 \langle 7 \rangle + \langle 8 \rangle - \langle 9 \rangle \right)_s = \sum_a \tilde{w}(a) \geq \left\lvert \left\{ a : \tilde{w}(a) > 0 \right\} \right\rvert / 2 .
\]

(9)

**Step 2.** Now we consider the arrows \( a \) with \( \tilde{w}(a) = 0 \). Clearly all chords \( c \cap a \) with \( c \) distinguished must intersect all chords \( b \cap a \) with \( b \) non-distinguished

\[
\begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
\end{array}
\]

(10)

If we have \( c \) and \( b \) intersecting in such a way that

\[
\begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
\end{array}
\]

(10)

then \( (a, b, c) \in \langle 12 \rangle \) in (3). (Note that in a positive diagram, \( \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
\end{array} = \frac{1}{3} \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
\end{array} \), but it is convenient to retain the distinction of the double arrow as we will soon see.)

Thus consider the case where no \( b \) and \( c \) with (10) occur. Then we have a picture like this:

\[
\begin{array}{c}
  A \\
  B \\
  C \\
  D \\
  E \\
  F \\
  G \\
  H \\
\end{array}
\]

(11)
with
\[ A := \{ b : b \cap a, a \text{ distinguished} \} \quad \text{and} \quad B := \{ b : b \cap a, b \text{ distinguished} \}. \] (12)

Here we drew only \( a \) and all \( b \cap a \). Because of \( \text{eev}(a) \) we have \( |A| = |B| \).

Consider the arrow \( c \) in \( B \) whose arrow hook is closest to the arrow hook of \( a \). Since \( |A| + 1 \) arrows (\( a \) including) intersect \( c \) in one direction, and from the arrows drawn at most \( |B| - 1 \) intersect \( c \) in the other direction, by \( \text{eev}(c) \) there is at least one other chord \( d \) not drawn in (11) (that is, \( d \not\cap a \)), with \( d \cap c \) in the direction opposite to \( a \).

Assume (without loss of generality up to mirroring the diagram and interchanging \( A \) and \( B \) ), that \( d \) is on the left of \( a \) in (11):

```
We have drawn \( d \) to intersect also arrows in \( A \), as we will now justify.

**Lemma 3.3** \( d \) intersects arrows in \( A \) and \( B \).

**Proof.** Assume \( d \) intersects arrows only in \( B \), but not in \( A \). Then all arrows \( d' \not\cap d \) on the opposite side of \( d \) to \( a \) also intersect (possibly arrows in \( B \), but) no arrows in \( A \). By performing loop moves at a suitable choice of such \( d' \) we can achieve that all they disappear, that is, there is no \( d' \not\cap d \) on the opposite side of \( d \) to \( a \). Call this new positive diagram \( D' \). Applying in \( D' \) a loop move on \( d \) gives a positive diagram \( D'' \).

We had \( |A| = |B| \) in \( D \), and the moves from \( D \) to \( D' \) did not affect \( A \), but at least the loop move on \( d \) deleted (at least) one arrow from \( B \). (We take \( A \) and \( B \) in \( D' \) to be defined as in (12), in the same way as for \( D \).) But then \( |B| < |A| \) in \( D' \), and thus \( \text{eev}(a) \) is violated, a contradiction to (12), in the same way for \( A \).

\( \square \)

It follows from the lemma that \( \exists c \in A, \ e \in B \) or \( e \in A, \ c \in B \), such that \( d \cap c, \ e \) and \( d \cap c \) in the opposite direction to \( a \). Then \( (a, c, d, e) \in (4) \cup (5) \).

In summary we showed in step 2 that if \( \tilde{w}(a) = 0 \), then \( a \) participates in at least one of the configurations (12), (4) or (5) of (3). In case \( a \) participates in (12), consider the configuration in which \( a \) is the double arrow. Thus we have assigned to \( a \) configuration, which we call matching configuration for \( a \).

We have that the changes of both orientation (mirroring the Gauß diagram) and mirroring (reversing the arrows) preserve (12), and interchange (4) and (5). Any configuration \( (4) \cup (5) \) is realized as matching configuration at most twice (\( a \) is one of the two arrows that intersect only two of the remaining 3 arrows in the configuration, and not all 3), and after symmetrization appears with coefficient 2. (12) is realized for at most one \( a \) (it must be the double arrow) and appears with coefficient 1.

Denote for simplicity by \( \langle 12 \rangle_{\tilde{w} > 0} \) and \( \langle 12 \rangle_{\tilde{w} = 0} \) the number of configurations of type (12), in which the double arrow \( a \) has \( \tilde{w}(a) > 0 \) resp. \( \tilde{w}(a) = 0 \). Then we have
\[
\left( 3 \langle 4 \rangle + \langle 5 \rangle + \langle 12 \rangle_{\tilde{w} = 0} \right) \geq \left| \{ a : \tilde{w}(a) = 0 \} \right|.
\] (13)

**Step 3.** We come back to the arrows \( a \) with \( \tilde{w}(a) > 0 \), because we can estimate another contribution of theirs to the Gauß sum, different from \( \tilde{w} \).
Lemma 3.4 Any $a$ with $\tilde{w}(a) > 0$ participates in $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$, or $\langle 12 \rangle$.

Proof. Choose chords $c$ and $b$ with (8) from lemma 3.2. If all $d$ with $d \cap b, c$ intersect $b, c$ in the same direction as $a$, analogously to the proof of lemma 3.3, loop moves at $b$, and possibly previously on some chords $b' \cap b$ on the opposite side of $b$ to that of $c$ would create a positive diagram with violated condition $\text{ev}(c)$.

Thus there is a $d$ with $d \cap b, c$ in the opposite direction to $a$. If $d \cap a$, then $(b, d, a)$ or $(c, d, a) \in \langle 12 \rangle$. Otherwise, $(a, b, c, d) \in \langle 1 \rangle \cup \langle 2 \rangle \cup \langle 3 \rangle$.

Define for each $a$ with $\tilde{w}(a) > 0$ as before a matching configuration. If $a \in \langle 12 \rangle$, then set the matching configuration of $a$ to be the one configuration $\langle 12 \rangle$, in which $a$ is the double arrow. Otherwise, set the matching configuration of $a$ to be one of the configurations $\langle 1 \rangle \cup \langle 2 \rangle \cup \langle 3 \rangle$, in which $a$ participates.

Since any configuration in $\langle 1 \rangle \cup \langle 2 \rangle \cup \langle 3 \rangle$ is matching for $\leq 4$ arrows $a$, $\langle 12 \rangle$ is matching for at most one $a$ (the double arrow), and since the symmetrization preserves $\langle 1 \rangle$ and interchanges $\langle 2 \rangle$ and $\langle 3 \rangle$, we have
\[
\left(\langle 1 \rangle + 6 \langle 2 \rangle + 2 \langle 3 \rangle + \langle 12 \rangle_{\tilde{w} > 0}\right) > \left[\{a : \tilde{w}(a) > 0\}\right].
\]
Putting now (13), (9), and (14) together, we obtain, using the symmetry $v_4 = (v_4)_s$,
\[
v_4 \geq \left[\left(\langle 1 \rangle + 6 \langle 2 \rangle + 2 \langle 3 \rangle + \langle 12 \rangle_{\tilde{w} > 0}\right) + (3 \langle 4 \rangle + 5 + \langle 12 \rangle_{\tilde{w} = 0}) + (\langle 2 \rangle + 2 \langle 7 \rangle + \langle 8 \rangle - \langle 9 \rangle)\right]_s \\
\geq \frac{\left[\{a : \tilde{w}(a) > 0\}\right]}{4} + \left[\{a : \tilde{w}(a) = 0\}\right] + \frac{\left[\{a : \tilde{w}(a) > 0\}\right]}{2} \\
\geq \frac{3}{4}c(D).
\]
The positive 4 crossing diagram of the trefoil shows that the constant $\frac{3}{4}$ cannot be improved (at least without additive correction). Also, because of the same type of examples given in [St], one cannot obtain a lower bound for $v_4$ in general positive diagrams $D$ growing faster than $O(c(D))$.

4 On the triviality of the Jones polynomial

We conclude with some applications of the formula (4). From its integrality, evident from (3), it follows that the Jones polynomial determines $\nabla_4$ mod 2. But we have in fact

Proposition 4.1 A knot $K$ with trivial Jones polynomial has $4 \mid \nabla_4$.

Proof. If $\nabla_2 = 0$ and $4 \mid \nabla_4$, then we have for the determinant $\det K = V(-1) = \nabla(2i)$, that $\det K = 1 + 16\nabla_4 - 64(\nabla_6 - 4\nabla_8 + \ldots) \neq 1 \mod 64$, in particular $\det K \neq 1$, a contradiction. \hfill \square

Corollary 4.1 If $K$ has $\nabla \neq 1$ and $V = 1$, then $g(K) \geq 3$.

Proof. We have $\nabla_2 = 0$, because $\nabla_2 = -\frac{1}{6}\nabla_2$, and if $g(K) \leq 2$ and $\nabla_4 \neq 0$, we have $\det K = |16\nabla_4 - 1| \neq 1$, a contradiction. Thus either $g \geq 3$, or $g \leq 2$ and $\nabla_4 = 0$, in which case $V = 1$. \hfill \square

Corollary 4.2 If $V_K = 1$, then $10 \mid v_4(K)$.

Proof. The only term surviving in (4) is $\frac{5}{2}\nabla_4$, and then apply proposition 4.1. \hfill \square

We have also an application to twofold (iterated) untwisted Whitehead doubles. Let $w_\pm$ denote the untwisted double operation of knots with positive (resp. negative) clasp, and $w_{\pm_1, \pm_2} = w_{\pm_2} \circ w_{\pm_1}$. 

\hfill \square
Proposition 4.2 \(v_4(w_{\pm,\pm}(K)) = -8v_2(K), \quad v_4(w_{\pm,\mp}(K)) = 8v_2(K).\)

Proof. We know from [St] that the dualization \(w^*_\pm\) of \(w_\pm\) is a nilpotent endomorphism of the space of Vassiliev invariants preserving its (degree) filtration. A basis for the space of Vassiliev invariants of degree \(\leq 4\) is
\[
1, \, \nabla_2, \, v_3, \, \nabla_2^2, \, \nabla_4, \, v_4. \tag{15}
\]
If \(w^*_\pm(v_4)\) has a non-zero coefficient of \(v_4\) in the basis (15), then \(w^*_\pm\) cannot be nilpotent, since it kills any of the invariants in (15) except \(v_4\) and \(v_3\), but \(v_3\) has smaller degree than \(v_4\). Thus
\[
w^*_\pm(v_4) = a_\pm v_3 + \text{(terms depending on } \nabla \text{ only)}. 
\]
Therefore,
\[
w_{\pm,\pm}^*v_4 = a_{\pm,\pm}w_{\pm,\pm}^*v_3 = \pm 2a_{\pm,1}v_2
\]
by [St]. To determine \(a_\pm\) it is sufficient to calculate \(v_4(w_{\pm,\pm}(K))\) for some \(K\) with \(v_2 \neq 0\). Take the figure-eight-knot \(K = 4_1\) with \(v_2 = -1\).

Then we find by calculation (in a few seconds, despite that the diagrams of \(w_{\pm,\pm}(K)\) have 78 crossings) that
\[
v_4[w_{\pm,\pm}(4_1)] = \pm 8 = 2a_\pm \cdot -1, \tag{16}
\]
whence \(a_\pm = \mp 4\). \(\square\)

Corollary 4.3 If \(K\) is the twofold untwisted Whitehead double of a knot, then \(8 \mid v_4(K)\). \(\square\)

Corollary 4.4 If \(K\) is the twofold untwisted Whitehead double of a knot with \(v_2 \neq 0\), then \(V_K \neq 1\). \(\square\)

In particular, by [St2, St] we obtain

Corollary 4.5 The twofold untwisted Whitehead double of a positive or almost positive knot has non-trivial Jones polynomial. \(\square\)

Remark 4.1 T. Stanford ([S2]) has found that
\[
\frac{\nabla_2(\nabla_2 + 1)}{4} + \frac{\nabla_4}{2} + \frac{v_3}{2}
\]
is always integral. This statement is equivalent to \(\frac{\nabla_2(\nabla_2 + 1)}{2} + \nabla_4 \equiv v_3 \mod 2\), which can be established by checking it on a few knots, since the space of (chord, unitrivalent etc.) diagrams has no 2-torsion in degree \(\leq 4\), and so a degree-4-Vassiliev invariant \(\mod 2\) is the reduction \(\mod 2\) of a degree-4-Vassiliev invariant over \(\mathbb{Z}\), for which a basis is well-known (see (15)).

Using the integrality of \(v_4\), one can obtain from this that
\[
\frac{\nabla_2^2}{4} + \frac{7\nabla_2}{12} - \frac{V_4}{144}
\]
is always integral. Thus any knot with trivial Conway polynomial satisfies \(144 \mid V_4\).

We finish with a simple necessary condition for the untwisted Whitehead double of a knot to have trivial Jones polynomial.

Proposition 4.3 If \(w_+(K)\) or \(w_-(K)\) have trivial Jones polynomial, then \(v_2(K) = v_3(K) = 0\).
Proof. That \( v_2(K) = 0 \) follows from [St], since \( v_3(w_\pm(K)) = \pm 2v_2(K) \). To show is that \( v_3(K) = 0 \). We have from the proof of proposition 4.2 that

\[
\begin{align*}
w_+ v_4 &= a_2^+ v_2 + a_2^+ v_2^2 + a_4^+ v_4 + a_3^+ v_3, \quad \text{and} \\
w_- v_4 &= a_2^- v_2 + a_2^- v_2^2 + a_4^- v_4 + a_3^- v_3.
\end{align*}
\]

Since

\[
w_+ v_4(K) = v_4(w_+(K)) = v_4(!w_+(K)) = v_4(w_-(!K)) = w_- v_4(!K)
\]

\[
= a_2^- v_2(K) + a_2^- v_2^2(K) + a_4^- v_4(K) + a_3^- v_3(K)
\]

by comparing coefficients with (17), we obtain

\[
a_2^+ = a_2^-, \quad a_2^- = a_2^+, \quad a_4^- = a_4^+, \quad \text{and} \quad a_3^- = -a_3^+.
\]

Since we know already that \( a_3^+ \neq 0 \), it follows that

\[
v_4(w_+(K)) \neq v_4(w_-(K)) \iff v_3(K) \neq 0.
\]

But by a simple skein argument at the clasp of the Whitehead double, if one of \( w_+(K) \) or \( w_-(K) \) has trivial Jones polynomial, so have both, and since their (trivial) Conway polynomials also coincide, \( v_4 \) vanishes on both doubles. This shows the assertion.

\[\square\]

Corollary 4.6 If \( K \) is a (prime or composite) knot of \( \leq 15 \) crossings, then \( w_+(K) \) and \( w_-(K) \) have non-trivial Jones polynomial.

Proof. By the simple skein argument it suffices to consider only \( w_+(K) \), and from a pair of mirror images only one knot \( K \), as \( !w_+(K) = w_-(!K) \). Among the 322,033 such knots up to 15 crossings (up to mirroring and orientation), there are only 7,116 with \( v_2 = v_3 = 0 \), and calculating the polynomials of their Whitehead doubles was feasible (even if after some time).

Since \( v_2 \) and \( v_3 \) are fast to calculate (on each of the aforementioned 78 crossing diagrams of the twofold untwisted Whitehead doubles of the figure-eight-knot the calculation took less than a second!), the condition is applicable in practice also to more complicated examples. It appears that in average about 97% of the knots can be excluded this way.

Acknowledgement. I would like to thank to T. Stanford for some helpful remarks and discussions.

References


