SUBLINKS OF STRONGLY QUASIPOSITIVE LINKS

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Abstract. We prove that any two given links can be combined to give a strongly quasipositive link. This in particular implies that any link is a sublink of a strongly quasipositive link. We discuss also some complexity issues of the strongly quasipositive link constructed.

Keywords: braid group, strongly quasipositive link, alternating link, Bennequin surface, positive link.

2020 AMS subject classification: 57K10, 57M50 (primary), 20F36 (secondary)

1 Introduction, motivation and main result

The notion of a Bennequin surface, so named by Birman-Menasco [BM], originates from Bennequin's work [Be], and refers to a braided Seifet surface of minimal genus. Rudolph [Ru4] has shown that every Seifert surface can be made into a braided form, so that Bennequin surfaces exist for every link. These surfaces are closely related to (and particularly important for) strongly quasipositive links.

Various notions of positivity of links have been studied also with motivation outside the field of knot theory. If the zero set of a complex polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ intersects the unit sphere $S^3 = \{ (u,v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1 \}$ transversely, the intersection forms a link in S^3. By work of Rudolph [Ru] and Boileau-Orevkov [BO] it was proved that these links are precisely the quasipositive links.

A link is called *quasipositive* if it is the closure of a braid β of the form

$$\beta = \prod_{j=1}^{n} w_j \sigma_{i_j} w_j^{-1}$$

where w_j is any braid word and σ_{i_j} is a (positive) standard Artin generator of the braid group. (In [Ru3] there is some overview of this topic.)

If the words $w_j \sigma_{i_j} w_j^{-1}$ are of the form[1]

$$\sigma_{i,j} = \sigma_i^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1} \sigma_{j-2} \dots \sigma_i \tag{1}$$

they can be regarded as embedded bands (see the l.h.s. of (19)). Links which arise this way are called *strongly quasipositive links*, and it was proved that they contain the class of *positive* links, i.e., links with diagrams all of whose crossings are positive (right-hand) [Ru2].

[1] Note that thus the bands are, from our perspective, *behind* the Seifert disks put into the braid strands to obtain a strongly quasipositive Seifert surface. We will maintain this convention throughout.

Positive links in turn (properly) contain the class of positive braid links, i.e., closures of braids which are positive words in the σ_i [St].

In some recent papers, Bode and Dennis [B, BD] gave explicit constructions of a complex polynomial realizing a quasipositive link, and generalized this construction to show that every link is a zero set of a semiholomorphic polynomial (in $u, v, \overline{v}$). They also notice that a zero set of a semiholomorphic polynomial can always be extended to a zero set of a holomorphic polynomial. Their construction thus proves that every link L is the sublink of a quasipositive link L'.

Among others, we are going to give here the most general possible result in terms of knot theory, even for strongly quasipositive links. We will say that a link L is a *composition* of links L_1 and L_2, and write $L = L_1 \times L_2$, if both L_1 and L_2 are sublinks of L, and if deleting from L the components of L_1 gives L_2 and vice versa. Of course, the operation '$\times$' is highly ambiguous, and we will show that we can always make the result to be strongly quasipositive.

Theorem 1.1 For any two links L_1 and L_2, there is a composition link $L_1 \times L_2$ which is strongly quasipositive.

This is equivalent to a version of Bode and Dennis' result for strongly quasipositive links, with the restriction that the components added (to obtain L' from L) can form themselves any other given link. Nevertheless, it will be technically easier to deal with the sublink property first (Theorem 3.1). We also give some complexity estimates of L'. Generally, the complexity of the result L' is linear in terms of the complexity of the input L, and can be reduced if one allows for restrictions on what kind of components are added to L to obtain L'. (See Propositions 3.6 and 4.5.)

The theorem gives a stark contrast to many special and related cases. Positive links, positive braid links, k-almost positive links, the successively k-almost positive links of [It2, IMT], all are closed under taking sublinks for obvious reasons.

In §4.1 we introduce the concept of [St4] of banded diagrams, and use it to extend the construction of Theorem 1.1 by starting from arbitrary diagrams of L_i. This gives a more general economic estimate of the quasipositive surface of $L_1 \times L_2$ constructed. This paper was originally contained in [St4], and I am grateful to a referee for convincing me that it pursues a separate idea to better stand on its own. Both concepts ultimately meet in §4.2, but I hope that the connection is sufficiently clear yet smoothed out.

Acknowledgement. I had some helpful related discussions with and references from B. Bode, T. Ito, and S. Orevkov.

I thank the organizers of the Third Pan Pacific International Conference on Topology and Applications (PPICTA), Sichuan University, Chengdu, China, 2019 and the conference "Computing in topological structures: Foundations and implementations", Sirius Mathematics Center, Sochi, Russia, 2022 for the opportunity to participate and give a talk.

This work was partly supported by the National Research Foundation of Korea (grant NRF-2017R1E1A1A03071032) and the International Research & Development Program of the Korean Ministry of Science and ICT (grant NRF-2016K1A3A7A03950702).

2 Preliminaries

2.1 Generalities

The symbols $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ denote the integer, natural and complex numbers, respectively.

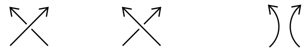
The notation $[P]_i = [P]_{l^i} = a_i$ is used for a *coefficient* of a *Laurent* polynomial $P = \sum_i a_i l^i \in \mathbb{Z}[l, l^{-1}]$.

We say an inequality '$a \le b$' is *exact* (or sharp) if $a = b$ and *strict* (or unsharp) if $a < b$.

We use the abbreviations 'w.l.o.g.' for 'without loss of generality' and 'r.h.s.' (resp. 'l.h.s.') for 'right hand-side' (resp. 'left hand-side'). 'W.r.t.' will stand for 'with respect to'. Some further notations will be introduced at an appropriate place in the text.

2.2 Links and diagrams

All link diagrams and links are assumed oriented, even if orientation is not always displayed. Crossings in oriented diagrams are called



positive negative smoothed out (2)

Smoothing out is the replacement of a (positive or negative) crossing by a smoothed out crossing. A crossing change is the replacement of a positive crossing to a negative or vice versa.

Here, and in the sequel, for a knot or link K, we write $!K$ for its obverse, or *mirror image*. Similarly $!D$ is the mirror image of a link diagram D, obtained by changing all crossings in D. By $K_1\#K_2$ we denote the *connected sum* of K_1 and K_2.

A crossing of a link diagram is called *mixed*, if both crossing strands belong to different components, otherwise it is called a *self-crossing*.

The *number of components* of a link L will be written as $n(L)$ (so $n(K) = 1$ if K is a knot). If D is a diagram of L, set $n(D) = n(L)$.

If L is obtained from L' be deleting components, we call L a *sublink* of L' and L' a *superlink* of L.

A link diagram D is called *split*, or disconnected, if it can be non-trivially separated by a simple closed curve in the plane. Otherwise we say the diagram is *non-split*, or connected. A split link is a link with a split diagram. Other links are said to be non-split. In the *split union* of diagrams or links, the latters can be separated by a curve or sphere. A non-split sublink L' of L which can be separated by a sphere from $L \setminus L'$ is a *split component* of L. A split component is *trivial* if it is the unknot. (Note that a split component of L may contain several components of L.)

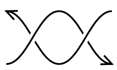
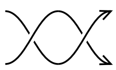
The *trivial split link* or *unlink* of n components is the one with all split components trivial and is written U_n. For simplicity, we set $U_1 = U$ for the unknot and $U_{n(L)} = U_L$ for a link L.

A crossing in a non-split diagram is *reducible*, if its smoothing gives a split diagram. A diagram is reducible if it has a reducible crossing, otherwise it is called *reduced*. To avoid confusion, unless otherwise stated, in the sequel all diagrams are assumed reduced, that is, with no nugatory crossings, and links are non-split.

A diagram which is simultaneously alternating and, up to mirror image, positive is is called *special alternating*, and so is called a link with such a diagram. Every alternating diagram of a special alternating link is positive or negative (so special alternating; see [Na]). But not every positive or negative diagram of a special alternating link is alternating. More generally, a diagram is *special* if no Seifert circle is *separating*, i.e., no Seifert circle contains Seifert circles in both its interior or exterior (compare with [Cr]).

Let D be an oriented knot or link diagram. We denote by $c(D)$ the *crossing number* of D. The crossing number $c(L)$ of a knot or link L is the minimal $c(D)$ over all diagrams D of K. It is known, by Kauffman, Murasugi and Thistlethwaite, that $c(D) = c(L)$ if D is a reduced alternating diagram, i.e., that such a diagram is *minimal*.

Let $c_\pm(D)$ be the number of positive, respectively negative crossings of a diagram D, so that $c(D) = c_+(D) + c_-(D)$ and the *writhe* is $w(D) = c_+(D) - c_-(D)$. A diagram D is *positive* if $c_-(D) = 0$ and *negative* if $c_+(D) = 0$; so is called a link with such a diagram.

We call a *reverse clasp* to be  ; and a *parallel clasp*  . We call a clasp *trivial* if both its crossings have opposite sign. Such a clasp can be eliminated by a Reidemeister II move.

We write $s(D)$ for the number of *Seifert circles* of a diagram D, the loops obtained by smoothing out all crossings of D.

2.3 Polynomial invariants

Let Δ be the (1-variable) *Alexander polynomial*. We understand it here normalized to $\Delta(U) = 1$ and with the skein relation

$$\Delta(D_+) - \Delta(D_-) = (t^{1/2} - t^{-1/2})\Delta(D_0)\,, \tag{3}$$

with $D_{\pm}$ and D_0 being a (skein) triple of diagrams equal except near one crossing which is as in (2), from left to right.

The *skein polynomial P* is understood here via the relation

$$l^{-1}P(D_+) + lP(D_-) = -mP(D_0)\,, \tag{4}$$

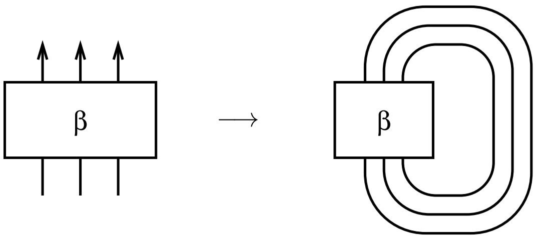
and the normalization $P(U) = 1$. The convention in (4) is similar to (and uses the same variables as) the one in [LMi] but differs by the interchange of l and l^{-1}.

2.4 Braids and braided surfaces

We need some preparations. We write B_n for the braid group on *n strands* or *strings*. The relations between the *Artin generators* σ_i, $i = 1, \dots, n-1$ are given by $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2$ and by $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \le i < j-1 \le n-2$. The trivial braid in B_n will be written Id_n.

Let us first fix a convention. In diagrams drawn we number braid strands by 1,2,3,... from left to right and compose words in $\sigma_i^{\pm 1}$ from bottom to top. Consequently, the orientation of braid strands is assumed to be upward, even if usually we don't explicitly display arrows. See (7) for an example.

Braid *closure* $\hat{\beta}$ is defined in planar diagrams with



$= \hat{\beta}$ (5)

A braid β whose closure $\hat{\beta}$ is a given link L is called a *braid representative* of L. The minimal number of strands for a braid representative of a link L is called the *braid index* of the link, and will be denoted by $b(L)$. (See, e.g., [DHL, Mo, Mu].) A braid representative $\beta \in B_{b(L)}$ with $\hat{\beta} = L$, i.e., with the fewest strands, is called a *minimal braid*.

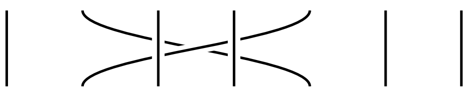
Alternatively to the standard Artin generators, one considers also a presentation of the braid groups by means of an extended set of generators (1) and their inverses

$$\sigma_{i,j}^{\pm 1} = \sigma_i^{-1} \dots \sigma_{j-2}^{-1}\sigma_{j-1}^{\pm 1}\sigma_{j-2} \dots \sigma_i \tag{6}$$

for $1 \le i < j \le n$. Note that

$$\sigma_i = \sigma_{i,i+1}\,.$$

We will call $\sigma_{i,j}$ *positive bands* and $\sigma_{i,j}^{-1}$ *negative bands*. For example, $\sigma_{2,5} \in B_7$ can be drawn (by using braid relations for cosmetic's sake) thus:



(7)

A representation of a braid β, and its closure link $L = \hat{\beta}$, as word in $\sigma_{i,j}^{\pm 1}$ is called a *band representation* [BKL]. A band representation of β spans naturally a Seifert surface S of the link L: one glues disks into the strands, and connects them by half-twisted bands along the $\sigma_{i,j}$. The resulting surface is called *braided Seifert surface* of L. (See, e.g., [St2].) In fact, a result of Rudolph [Ru4] (later rediscovered independently by M. Hirasawa) says that any Seifert surface is of this form.

A minimal genus braided Seifert surface is called a *Bennequin surface* (see [HS, St3]). Bennequin's work [Be] implies that any *strongly quasipositive Seifert surface*, a braided Seifert surface with only positive bands, is a

Bennequin surface; thus if a link L has a strongly quasipositive Seifert surface S on s strands with l bands, then $\chi(L) = \chi(S) = s - l$.

We need the half-twist (braid) element

$$\delta_n = \prod_{j=2}^{n} \prod_{l=1}^{j-1} \sigma_l \,. \tag{8}$$

Let us also set notation for the index shift map

$$[\sigma_i]_k = \sigma_{i+k}\,, \tag{9}$$

whenever the right hand-side has admissible index. We will also below in (10) and (11) assume notation extended to inverses by taking the inverse of the r.h.s. and then extend to braid words under multiplication of letters. (These operations give homomorphisms of braid groups, but this aspect will be less relevant here.)

We call $\beta \in B_k$ *non-split* if for each $i = 1, \dots, k-1$, some letter σ_i or σ_i^{-1} occurs in β. (It is not assumed that more than one letter of such type occurs, i.e., we do not require that $\hat\beta$ is a reduced diagram.)

We call $\beta \in B_k$ *homogeneous* if for each $i = 1, \dots, k-1$, only one type of letters σ_i or σ_i^{-1} occurs in β. If for each i only σ_i occurs, we call β *positive*; if only σ_i^{-1} occurs for all i, we call β *negative*. If σ_i occurs for even i and σ_i^{-1} occurs for odd i, or vice versa, then we call β *alternating*. For any of these four types, β has such type if and only if the closed braid diagram $\hat\beta$ has (see [Cr]).

2.5 Cabling

Let $k \in \mathbb{N}$, $\beta \in B_k$ and K be a knot. Define the β-k-cable K_β of K to be the satellite link with companion K and pattern given by the k-string braid β in the complement of its axis, with 0 framing. (See [Sn] for example.) In particular, if β is a torus braid, i.e., a power of $\sigma_1 \cdots \sigma_{k-1}$, we obtain classical cables. When $\beta = Id_k$, the cable $K_{Id_k} = K_k$ is the 0-framed disconnected parallel.

The notion of cabling extends in a natural way to braids: the *k-cable* of $\beta \in B_n$, denoted by $\{\beta\}_k$, is a braid on $n \cdot k$ strings, obtained by replacing σ_i in β by

$$\{\sigma_i\}_k = \prod_{j=1}^{2k-1} \prod_{l=0}^{\min(j-1,2k-1-j)} \sigma_{ki-\min(j-1,2k-1-j)+2l}\,. \tag{10}$$

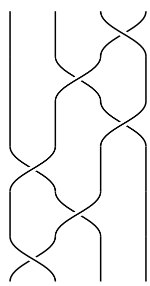
Then if $\hat\beta = K$ is a knot, we have $\widehat{\{\beta\}_k} = K_{\delta_k^{2w(\beta)}}$, where δ_k is the half-twist (8). (See below (15) for the sample case $k = 2$.)

To correct for the framing, one has to full-twist the left or right k strings (using the notation (9))[2]:

$$(\sigma_i)^l_k = [\delta_k^2]_{(i-1)k}\{\sigma_i\}_k = \{\sigma_i\}_k[\delta_k^2]_{ik} \qquad \text{or} \qquad (\sigma_i)^r_k = [\delta_k^2]_{ik}\{\sigma_i\}_k = \{\sigma_i\}_k[\delta_k^2]_{(i-1)k}\,. \tag{11}$$

These braids are not equal, but conjugate by $[\delta_{2k}]_{(i-1)k}$ (from (8)). Then $\widehat{(\beta)^l_k} = \widehat{(\beta)^r_k} = K_k$. In fact, one can choose a superscript 'l' or 'r' for each letter of β separately and multiply the resulting braids. All these possible product braids are conjugate braid representatives of K_k.

Let as an example $k = 2$ and K be positive. When $\beta = Id$, one notes that K_β is obtained from K when replacing each (positive) crossing of a (positive) diagram of K by



(12)

[2]The superscripts 'l' and ''r' for 'left' and 'right' are used only symbolically as literals and not as exponents.

By induction on k one extends (12) to give for $\sigma_1 \in B_2$ the (positive) band representation

$$(\sigma_1)^l_k = \sigma_{1,k+1}\sigma_{2,k+2}\cdots\sigma_{k,2k} \in B_{2k}\,, \tag{13}$$

and then more generally for $\sigma_{1,m} \in B_m$,

$$(\sigma_{1,m})^l_k = \sigma_{1,(m-1)k+1}\sigma_{2,(m-1)k+2}\cdots\sigma_{k,mk} \in B_{mk}\,. \tag{14}$$

3 Realization of strongly quasipositive composition links

3.1 Construction from braid diagrams

We start first with the special case of realizing a link as a sublink of a strongly quasipositive link. We will refine later this construction, and become more specific about complexity.

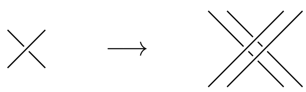
Theorem 3.1 Let L have a braid representative β on s strands with word length $c(\beta) = l$ and exponent sum $w(\beta) = w$.

a) Let L_1 have a braid representative β_1 differing from β by exchange of $\sigma_i^{\pm 1}$ to $\sigma_i^{\mp 1}$ (for *some* letters $\sigma_i^{\pm 1}$ in β and possibly different i). Then there is a link $L' = L \times L_1$ with a positive band representative on $2s$ strands with $4l$ bands. (In particular one can choose $L_1 = U_L$ to be an unlink[3] or $L_1 = !L$.)

b) Let L_1 have a braid representative β_1 differing from β by exchange of some σ_i^{-1} (in β) to σ_i^{+1} (in β_1). Then there is a link $L' = L \times L_1$ with a positive band representative on $2s$ strands with $3l - w$ bands. (In particular one can choose $L_1 = L$ or L_1 to be a positive braid link.)

Proof.

a) We consider the *(blackboard framed) 2-parallel* D_2 of the diagram $D = \hat{\beta}$. It is obtained by doubling every strand and replacing

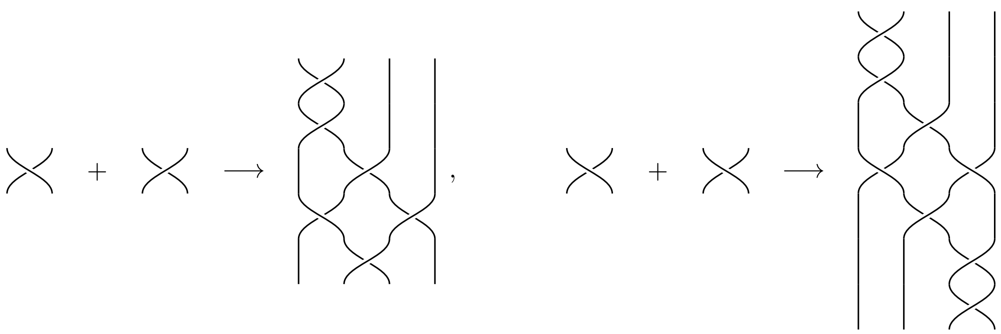


(15)

On the level of braids $D_2 = \hat{\beta}_2$ with $\beta_2 = \{\beta\}_2 \in B_{2n}$ obtained from β by replacing $\sigma_i^{\pm 1}$ by $(\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i})^{\pm 1}$ (see (10)). Note that the subbraid of β_2 on even strands gives β, and so does the one on odd strands.

Now our goal is to find a braid $\beta' \in B_{2n}$ whose subbraid on even strands gives β_1 (and on odd strands β). This is done by locally modifying β'. Obviously, when a crossing in positive in β and β_1, the right of (15) gives a group of positive crossings, which are (four) positive bands, so in that case no modification is needed. But we must take care of the cases then in β and β_1 some letter is negative.

In the below scheme, and later as well, we adopt the notation '$A + B \to C$' to mean that for a crossing A in β and a corresponding crossing B in β_1, the right of (15) has to be modified to C.



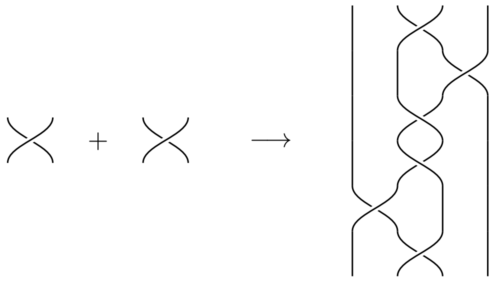
(16)

[3]However, for the unlink see better Proposition 3.6 to follow.

To see that these modifications are fine, consider the subbraids made of strings 1 and 3, and of strings 2 and 4. They must give the two crossings on the left. (The case  +  can be obtained by 180° rotation from the left modification, and reversing strand orientation.)

After all these modifications are done, we obtain from β_2 the desired braid β'. Note that in (16), as in (15), all braids 'C' on the right decompose into 4 (positive) bands, which proves the statement.

b) Here every positive crossing in β remains in β', and then one can modify the right of (15) to


(17)

This economizes two bands for each positive crossing in β, so for $D = \hat{\beta}$, we have

$$4c(D) - 2c_+(D) = 4c_-(D) + 2c_+(D) = 3c(D) - w(D)$$

bands in β_1. □

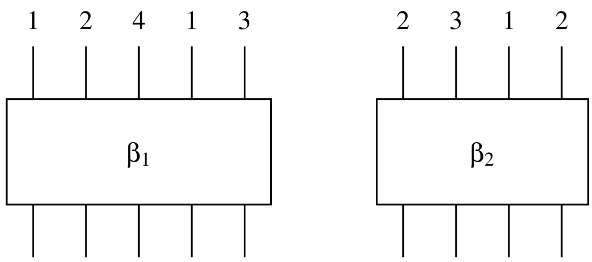
Remark 3.2 In the constructions of Bode and Dennis [BD], the quasipositive link L' is a satellite of the Hopf link. It also appears, but is not yet confirmed, that the components added (in $L' \setminus L$) are unknotted.

To prove the full version of Theorem 1.1, we must include the case that $n(L_1) \neq n(L_2)$, so that crossing changes are not sufficient. We thus now prepare the following lemma, which uses smoothings.

Lemma 3.3 Let $n(L_1) \geq n(L_2)$. Then there is a diagram D_1 of L_1 which gives a diagram D_2 of L_2 under crossing changes and smoothings.

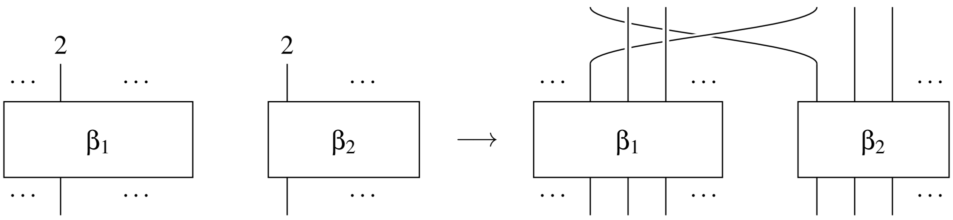
Proof. We will construct a link diagram D such that D_1 is obtained from D by crossing changes, and D_2 is obtained from D by crossing changes and smoothings.

Take the split union of diagrams $\hat{\beta}_1$ of L_1 and $\hat{\beta}_2$ of L_2 as a split braid.



Assume further w.l.o.g. that β_1 is not split (i.e., for each $1 \leq i < s_1$, at least one of σ_i or σ_i^{-1} occurs in β_1).

Now mark on the outgoing strands of β_1 which component of L_1 they correspond to and similarly for β_2. Setting $n_i = n(L_i)$ and with $n_2 \leq n_1$, choose for each $j = 1, \ldots, n_2$ a pair of one braid strand on top of β_1 corresponding to component j in L_1 and a braid strand on top of β_2 corresponding to component j in L_2. Add a *positive* band between these two braid strands,



and stack the resulting bands from bottom to top above β_i. These bands will be called *extra bands*, and will be designated for special treatment later. There are exactly n_2 extra bands.

We claim now that the (braid) diagram D thus obtained satisfies the claim of the lemma.

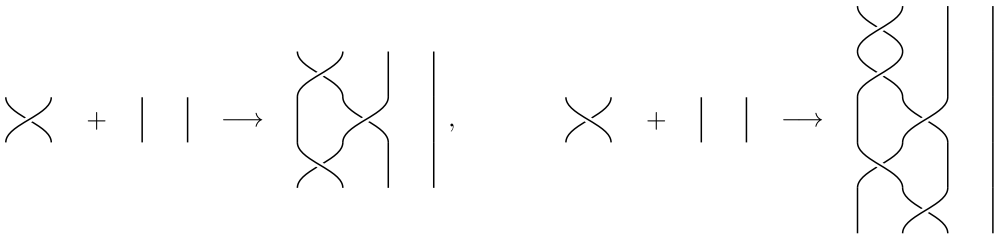
To see that one can obtain a diagram D_1 of L_1 from D, switch crossings in $\hat{\beta}_2$ so that all self-crossings are switched so that the components are unknots and a mixed crossing between components $j_1 < j_2$ of $\hat{\beta}_2$ is switched so that the overpass belongs to component j_1. Then the extra bands connect split unknots, and can be removed together with these unknots, giving the diagram $\hat{\beta}_1$ of L_1.

To obtain a diagram D_2 of L_2 from D, first observe that since $\hat{\beta}_1$ is connected, one can smooth $n_1 - n_2$ crossings in $\hat{\beta}_1$ to obtain a diagram in which components $1, \dots, n_2$ remain.

Namely, consider a graph G made up of vertices labeled $1, \dots, n_1$ and an edge connecting vertices j_1, j_2 if components j_1, j_2 have a mixed crossing. This graph G is connected. As long as there are edges not labeled $1, \dots, n_2$, smooth out a crossing between component $j_1 \le n_2$ and $j_2 > n_2$. This corresponds to contracting an edge in G. Label the new vertex j_1. By repeating this process, one sees the above property.

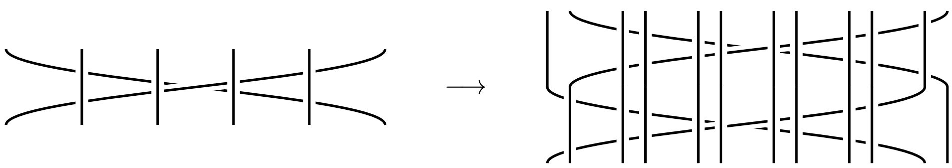
After these smoothings, crossing changes in the smoothed $\hat{\beta}_1$ will give a diagram of D_2 as above. □

Proof of Theorem 1.1. Assume again w.l.o.g. $n(L_1) \ge n(L_2)$. Take the diagram D_1 from the lemma. Modify the proof of Theorem 3.1 by taking the doubled closed braids β_1, β_2 and complementing the rules in (16) by



(18)

This should already suffice to prove the theorem, but let us, for economy's sake, note that there is a better way to treat the extra bands in D_1, which can be doubled



(19)

instead of dealing with all their crossings one by one as in the proof of Theorem 3.1. None of the switches or smoothings transforming D_1 into D_2 affects these extra bands. (We will make use of this economization of bands in the below complexity estimates.) □

The *Y-polynomial* is sometimes defined as

$$Y(L) = [P(L)]_{m^{1-n(L)}} .$$

It is a Laurent polynomial in l. Rudolph has proved that every Alexander polynomial is realized by a strongly quasipositive link (see [Ru3, 88 Corollary]). The below corollary gives a similar result for the Y polynomial, which is one consequence of the theorem proved above. The corollary directly follows from the property expressing the Y-polynomial of a link by the one of its components and linking numbers. (For the skein polynomial P, there are conditions like that all powers of l are positive, see [St4], so the unit $(-l^2)^k$ below is needed.)

Corollary 3.4 Let $n = n(L) > 1$ and there be a link L with $Y(L) = Y$. Then there is a strongly quasipositive link L' with $n(L') = n$ and $Y(L') = (-l^2)^k Y$ for some $k \in \mathbb{Z}$. □

3.2 Complexity

To save notation, let in the sequel $D_i' = \hat{\beta}_i$ be diagrams of braid representatives β_i of L_i and for each letter $d \in \{c,s,w,n\}$ set $d_i = d(D_i')$ and $d = d_1 + d_2$. (We will use D_i to refer to the diagrams from Lemma 3.3.) We will also below continuously stipulate

$$n_1 \geq n_2\,, \tag{20}$$

and that D_1' is not a split diagram.

Proposition 3.5 Let L_i have a braid representatives β_i on s_i strands with word length $c(\beta_i) = c_i$ and $c = c_1 + c_2$. Then there is a link $L = L_1 \times L_2$ with a positive band representation on $2s$ strands with at most

$$4c - n_1 + 3n_2 \tag{21}$$

bands.

Proof. We found a diagram $\hat{\beta}$ of $s = s_1 + s_2$ strands and n_1 components, so that L_1 is obtained from $\hat{\beta}$ by switching crossings and L_2 is obtained from $\hat{\beta}$ by switching crossings and $n_1 - n_2$ smoothings. The diagram $\hat{\beta}$ has $c_1 + c_2$ crossings and n_2 extra bands, which are positive. By doubling to obtain a banded diagram D of $L_1 \times L_2$, every crossing in β_1' or β_2' not smoothed out gives at most 4 bands, but a crossing smoothed to obtain D_2 each economizing one band, so totally economizing $n_1 - n_2$ bands.

So the number of bands coming from β_1' or β_2' is $4c - n_1 + n_2$. We treated the extra bands in (18) as in double (19), adding $2n_2$ doubled extra bands. □

For the more special sublink result, one can save installing extra bands, and the estimates can be simplified. The following is an amplification of Theorem 3.1.

Proposition 3.6 Let L have a braid representative β as a word with s strands and c crossings and writhe (exponent sum) w. Then

(a) there is a strongly quasipositive link $L' = L \times M$ with a positive band representation with $2s$ strands and at most $4c$ bands. M can be chosen to be any link with a diagram differing from $\hat{\beta}$ by crossing changes, in particular, $M = !L$.

(b) If $M = U_L$, then at most
$$\frac{7c - w}{2}$$
bands suffice.

(c) One can choose $M = L$ or M a positive braid link, when using at most $3c - w$ bands.

(d) Also assuming that β is not a split braid word, and $m < n = n(L)$, one can realize $L \times U_m$ with $2s$ strands and at most
$$\min\left(\frac{7c - w}{2}, 4c + m - n\right)$$
(positive) bands.

Proof.

(a) See Theorem 3.1.

(b) Switch at most half of the positive crossings in $D = \hat{\beta}$ to obtain an unlink. (An unlink is amphicheiral, so one can choose the complementary set of crossings.) This economizes at least $c_+(\hat{\beta})$ bands by (17). So we have at most

$$4c(\beta) - c_+(\beta) = \frac{7c - w}{2}$$

bands.

(c) See Theorem 3.1.

(d) One needs $n - m$ smoothings to obtain an m-component link diagram from $\hat{\beta}$, before unlinking by crossing changes. This will economize one band for each smoothed crossing, giving at most $4c + m - n$ bands.

Now there are at least $c_+ + m - n$ positive crossings. Again by changing to the complementary set of crossings, at most half of them need to be switched. So we can economize at least

$$2 \cdot \frac{c_+ + m - n}{2}$$

bands (and if this number is negative at least 0). So we have at most

$$4c + m - n - 2 \cdot \frac{c_+ + m - n}{2} = \frac{7c - w}{2}$$

bands. □

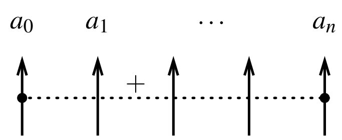
4 General link diagrams and banded diagrams

4.1 Banded diagrams

To extend the construction to general link diagrams, it is necessary to use the (now likely most common) algorithms that transform a general link diagram into a (closed) braid diagram. This is quoted from (and will be considered in much more detail in) [St4].

Consider a collection of oriented Seifert circles in the plane.

Definition 4.1 We call a *band* to ba a dashed line with a $+/-$ label. The dashed line should start and end on a Seifert circle, do not intersect itself, intersect Seifert circles transversely, and with the orientation of Seifert circles being

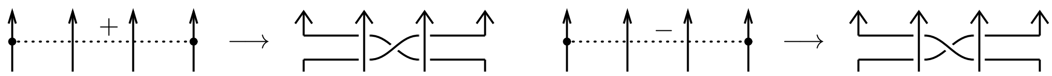


Note that in particular because of the orientation, no band can intersect or connect the same Seifert circle twice.

We say two Seifert circles a, b are *coherent*, if they bound a common region R and are oriented oppositely w.r.t. R's boundary. (Region is again understood in the complement of the Seifert circles.)

Then a band is a dashed line going from Seifert circle a_0, over *intermediate Seifert circles* a_i to Seifert circle a_n, so that a_i, a_{i+1}, for all $0 \le i < n$, are coherent.

A band should be understood for standing for a fragment like


(22)

Of course, an ordinary crossing is a band in this sense (which does not pass through intermediate Seifert circles).

Definition 4.2 A *banded diagram* is a collection of Seifert circles in the plane with a collection of bands, none two of which intersect.

We will implicitly understand a banded diagram to stand for a link diagram in which all replacements (22) have been made. Usual link diagrams can obviously be regarded as band diagrams, with each crossing turned into a band. If a banded diagram is a (closed) braid diagram, it will give a band representation of a closed braid, and the bands become the $\sigma_{i,j}^{\pm 1}$ in (6).

We will need the following lemma, proved in [St4] using a careful analysis of the Yamada [Y] algorithm.

Lemma 4.3 Let L have a banded diagram D with s Seifert circles and l bands. Then L has a braid representative on s strands with l bands (and the same number of positive and negative ones as D).

4.2 More complexity estimates for strongly quasipositive composition links

To turn back to strongly quasipositive links, we developed Lemma 4.3 to give some complexity estimates of strongly quasipositive links $L_1 \times L_2$ constructed in terms of an arbitrary diagram of L_i.

This gives then the generalization of Proposition 3.5 to arbitrary diagrams. Again, let in the sequel D_i' be diagrams of L_i and for each letter $d \in \{c,s,w,n\}$ set $d_i = d(D_i')$ and $d = d_1 + d_2$. We will use D_i to refer to the diagrams from the proof of Lemma 3.3. We will also continuously stipulate (20) *and that D_1' is not a split diagram.* The number $4c - n_1 + 3n_2$ repeats (21).

Proposition 4.4 If L_i have diagrams D_i' with s_i Seifert circles and c_i crossings, then there is a positive band representation of a link $L_1 \times L_2$ of $2s$ strands and at most $4c - n_1 + 3n_2$ bands.

Proof. This follows first by applying Yamada's algorithm on D_i' and then a generalization of the construction of §3. Every Yamada move adds no new bands, but makes crossings into non-crossing bands.

Note that Lemma 3.3 can be applied to ordinary diagrams D_i' as well. (The braid shape of D_i' was not needed in the proof.)

Let us thus proceed as follows, to be precise.

(a) Double (with blackboard framing) the diagrams D_i'.

(b) Apply the doubled Yamada moves giving braid diagrams with doubled bands.

(c) One needs to generalize the modifications (16) and (17) to doubled bands. If one has a doubled band $\sigma_{1,m}$, instead of a doubled crossing $\sigma_1 = \sigma_{1,2}$, one adds $2(m-2)$ strands to the 4-braids on the right of (16) and (17) and conjugates with

$$\sigma_4^{-1}\sigma_5^{-1}\dots\sigma_{2m-1}^{-1}\sigma_3^{-1}\sigma_4^{-1}\dots\sigma_{2m-2}^{-1},$$

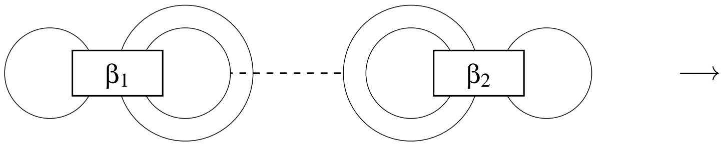
which still gives the same number of bands, where $\sigma_{i,j}$ becomes $\sigma_{i,2m-4+j}$. For the right of (17), the result of this modification is given, with $k=2$, in (14) (and in (13) for $m=2$). This gives banded braid diagrams D_i''.

Each crossing in D_i' gives at most 4 bands in D_i''. But in D_1' there are $n_1 - n_2$ smoothings, each economizing one band. (We can decide before installing extra bands which crossings in D_i' we switch or smooth after installing bands in D.)

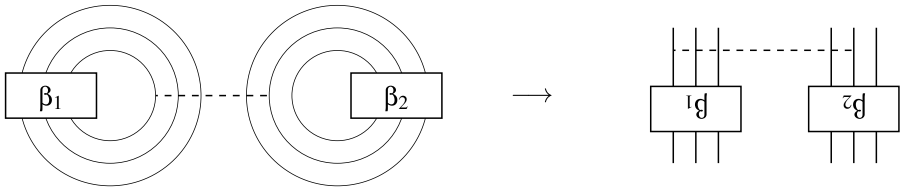
(d) If a Seifert circle bounding the infinite region of D''_1 is incoherent with a Seifert circle bounding the infinite region of D''_2, flip one of the two D''_i by starting with the flipped diagram D'_i.

(e) Add doubled extra bands, as on the right of (19), obtaining the final banded braid diagram D. (Use that we made the Seifert circles of D''_i coherent.)

(f) Use regular isotopy to make the innermost Seifert circle of D''_1 contain D''_2 and D''_1 lie in the infinite region of D''_2.



(23)



(For the last diagram, we stipulate the braid closure as in (5).) This can be done without creating new crossings, *except* at the extra bands, but their band structure is not spoiled when we fix that the added strands pass on top of the bands. So now the union of D''_1 and D''_2, with the extra bands added, is a closed braid diagram. □

The simplification for estimating complexity of a strongly quasipositive link L' with a given sublink L in terms of an arbitrary diagram of L follows along the same lines, and is paraphrased below so as not to repeat the whole wording of Proposition 3.6.

Proposition 4.5 Let a link L have a diagram D with s Seifert circles, c crossings and writhe w. Then all the assertions of Proposition 3.6 hold for L, when replacing

- in part (a) $\hat{\beta}$ by D,
- in part (c) 'positive braid link' by 'positive link' and
- in part (d) demanding that D instead of β is non-split. □

Remark 4.6 The construction can be modified to show that for a link L, the composition $L \times L$ can be made an alternating link. In the doubled diagram D of L change the crossings on the right of (15) not belonging to the two copies of D to have an alternating tangle, and for every non-alternating edge of D (between two under- or overpasses) add a half-twist (with proper sign) into the doubled strands. This will give from a (braid) diagram D of L of c crossings an alternating (braid) diagram of $L \times L$ of at most $6c$ crossings.

There are further constructions available to realize composition links as an alternating link, alternating braid link, etc., but they require slightly less straightforward modifications, and we may present them at a separate place if they become very relevant.

As a final remark in this context, although complexity results like Proposition 3.6 are linear, one could still consider improvements. For example, every $(2,n)$-torus link is a sublink of a closed strongly quasipositive braid on 3 (and not only 4) strands. Perhaps generalizing such an observation to a meaningful extent could have worthwhile proofs.

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