REALIZING STRONGLY QUASIPOSITIVE LINKS AND BENNEQUIN SURFACES

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Abstract. Adapting the graph index to be used for braided surfaces, we gain realization results on string numbers of Bennequin surfaces. In particular, all fibered alternating links and pure alternating pretzel links have a minimal string Bennequin surface (as can be proved also for many cables of such knots). We also obtain a visual test for strong quasipositivity, which we call pseudo-positivity, and use it to classify strongly quasipositive pretzel knots and reverse Montesinos links. This test also supports determining the strong quasipositivity status of all so far tabulated (up to 16 crossing prime) knots.

Keywords: braid group, strongly quasipositive link, cable link, alternating link, Bennequin surface, positive link, pretzel link, genus, braid index, graph, alternative link

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1 Introduction, motivation and main results

The notion of a Bennequin surface, so named by Birman-Menasco [BM3], originates from Bennequin's work [Be], and refers to a braided Seifert surface of minimal genus. Rudolph [Ru4] has shown that every Seifert surface can be made into a braided form, so that Bennequin surfaces exist for every link. These surfaces are closely related to (and particularly important for) strongly quasipositive links.

Various notions of positivity of links have been studied also with motivation outside the field of knot theory. If the zero set of a complex polynomial $f:\mathbb{C}^2\to\mathbb{C}$ intersects the unit sphere $S^3=\{\,(u,v)\in\mathbb{C}^2 : |u|^2+|v|^2=1\,\}$ transversely, the intersection forms a link in S^3. By work of Rudolph [Ru] and Boileau-Orevkov [BO] it was proved that these links are precisely the quasipositive links.

A link is called *quasipositive* if it is the closure of a braid β of the form

$$\beta=\prod_{k=1}^{l} w_k\sigma_{i_k}w_k^{-1}$$

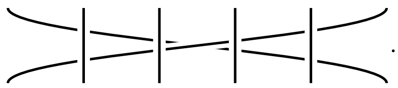
where w_k is any braid word and σ_{i_k} is a (positive) standard Artin generator of the braid group. (In [Ru3] there is some overview of this topic.)

If the words $w_k\sigma_{i_k}w_k^{-1}$ are of the form[1]

$$\sigma_{i,j}=\sigma_i^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}\sigma_{j-2}\dots\sigma_i\,, \tag{1.1}$$

[1] Note that thus the bands are, from our perspective, *behind* the Seifert disks put into the braid strands to obtain a strongly quasipositive Seifert surface. We will maintain this convention throughout, except Remark 4.11.

then they can be regarded as embedded bands



Links which arise this way are called *strongly quasipositive links*, and it was proved that they contain the class of *positive* links, i.e., links with diagrams all of whose crossings are positive (right-hand) [Ru2]. We will re(dis)cover this fact in the context of obtaining explicit (upper) estimates on the string number of Bennequin surfaces of links (see, e.g., [IK]). One such result is the following (for further generalizations, see Proposition 4.6 and Corollary 4.24).

Theorem 1.1 If L is a positive non-trivial non-split link of crossing number $c(L)$, Euler characteristic $\chi(L)$ and braid index $b(L)$, then it has a strongly quasipositive surface on at most

$$\min\left(\frac{c(L)-\chi(L)+1}{2},\ b(L)-\chi(L)\right) \tag{1.2}$$

strands.

Positive links in turn (properly) contain the class of positive braid links, i.e., closures of braids which are positive words in the σ_i [St].

In relation, we obtain that (strong) quasipositivity is preserved under (zero framed) braid-pattern cabling (§3.3.2). The applications to fibered alternating links (§3.3.3) follow as another outcome of our approach (see Proposition 3.18).

Proposition 1.2 Any fibered alternating link L has a minimal string Bennequin surface, i.e., a Bennequin surface on braid index $b(L)$ strings.

There is also an estimate on the string number for a general alternating link similar to Theorem 1.1; see Theorem 4.2. This will result from some work to adapt to braided surfaces first Yamada's braid diagram algorithm (§3.2) and then the Seifert circle reduction process measured by the graph index (§4). As further applications, we gain some estimates for the braid index of a homogeneous link (§4.4.1 and Corollary 4.18), visual tests for strong quasipositivity (Corollary 4.24) and for minimality of canonical surfaces (Corollary 4.44).

In particular, we find an explicit quasipositive realization of a surface in the constructive direction of [FLL, Theorem B]. For an extension of the other direction, see Lemma 4.30. (Several other main results of [FLL] can be extended for multiple components.) One fallout of it is an estimate of the 4-genus of a general link in §4.4.5. Another one is the confirmation of Baker-Motegi's Question 2.2 for canonical surfaces (Corollary 4.33). This is then used to determine the strong quasipositivity of parallel Montesinos links in §5.2.

We show that the strong quasipositivity test gives an easy condition (5.3) for parallel (odd-twist) pretzel knots, which is in fact exact (Proposition 5.1). This suggests that the test is rather efficient. We complete the class of pretzel knots in §5.3. The proof is far less evident than one may expect. The Ozsváth-Szabó τ invariant, smooth 4-genera, plumbing, minimal genus Seifert surfaces and their uniqueness, the Alexander polynomial and skein polynomial, all play their part.

The last section §6 compiles a number of questions in relation to our discussion.

There are two appendices. In the first appendix, we correct some errors from [Cr, St3] about the homogeneity of 10 crossing knots, and add information which ones are alternative. In the second one, we summarize the decision process of strong quasipositivity for all tabulated knots (up to 16 crossings) using the methods previously developed in this paper.

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2 Preliminaries

2.1 Generalities

The symbols $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ denote the integer, natural and complex numbers, respectively.

We write $\lfloor x \rfloor$ for the greatest integer, $\operatorname{sgn}(x) = x/|x|$ for its sign (when $x \neq 0$), and $|S|$ for the cardinality of a set.

The notation $[P]_i = [P]_{l^i} = a_i$ is used for a *coefficient* of a *Laurent* polynomial $P = \sum_i a_i l^i \in \mathbb{Z}[l, l^{-1}]$. The *span* $\operatorname{span} P$ of a polynomial is the difference between its *maximal degree* $\max\deg P$ and its *minimal degree* $\min\deg P$. (For multi-variable polynomials, we will indicate by a subscript the variable with respect to which degrees are taken.)

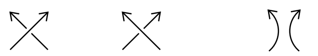
We say that an inequality '$a \leq b$' is *exact* (or sharp) if $a = b$ and *strict* (or unsharp) if $a < b$.

For notation from knot tables, see e.g. [St15, §2.3]; we will not (need to) take care of mirroring issues for table knots. We follow Rolfsen's numbering up to 10 crossings, except for the removal of the Perko duplication.

We use the abbreviations 'w.l.o.g.' for 'without loss of generality' and 'r.h.s.' (resp. 'l.h.s.') for 'right hand-side' (resp. 'left hand-side'). 'W.r.t.' will stand for 'with respect to'. Some further notations will be introduced at an appropriate place in the text.

2.2 Links and diagrams

All link diagrams and links are assumed oriented, even if orientation is not always displayed. Crossings in oriented diagrams are called



positive negative smoothed out (2.1)

Smoothing out is the replacement of a (positive or negative) crossing by a smoothed out crossing. A crossing change is the replacement of a positive crossing to a negative or vice versa.

Here, and in the sequel, for a knot or link K, we write $!K$ for its obverse, or *mirror image*. Similarly $!D$ is the mirror image of a link diagram D, obtained by changing all crossings in D. By $K_1 \# K_2$ we denote the *connected sum* of K_1 and K_2. (The connected sum is uniquely defined for knots, and depends on the choice of components for links.)

A crossing of a link diagram is called *mixed*, if both crossing strands belong to different components, otherwise it is called a *self-crossing*.

The *number of components* of a link L will be written as $n(L)$ (so $n(K) = 1$ if K is a knot). If D is a diagram of L, set $n(D) = n(L)$.

If L is obtained from L' by deleting components, we call L a *sublink* of L' and L' a *superlink* of L.

Here, and in the sequel, for a knot or link K, we write $!K$ for its obverse, or *mirror image*. Similarly $!D$ is the mirror image of a link diagram D. By $K_1 \# K_2$ we denote the *connected sum* of K_1 and K_2.

If a link L is not a connected sum in a non-trivial way, i.e., with one of L_i being an un*knot*, then L is *prime*. A diagram D is *prime*, if there is no simple closed curve transversely intersecting it in two points, with something different from an unknotted (crossing-less) arc in both interior and exterior.

We say a class of link diagrams is *visually prime* if every prime diagram in this class represents a prime knot or link. It was proved by Menasco [Me] that alternating diagrams are visually prime, and by Ozawa [O] that positive diagrams are.

A link diagram D is called *split*, or disconnected, if it can be non-trivially separated by a simple closed curve in the plane. Otherwise we say the diagram is *non-split*, or connected. A split link is a link with a split diagram. Other links are said to be non-split. In the *split union* of diagrams or links, the latters can be separated by a curve or sphere. A non-split sublink L' of L which can be separated by a sphere from $L \setminus L'$ is a *split component* of L. A split component is *trivial* if it is the unknot. (Note that a split component of L may contain several components of L.)

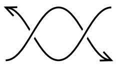
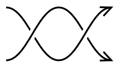
The *trivial split link* or *unlink* of n components is the one with all split components trivial and is written U_n. For simplicity, we set $U_1 = U$ for the unknot and $U_{n(L)} = U_L$ for a link L.

A crossing in a non-split diagram is *reducible*, if its smoothing gives a split diagram. A diagram is reducible if it has a reducible crossing, otherwise it is called *reduced*. To avoid confusion, unless otherwise stated, in the sequel all diagrams are assumed reduced, that is, with no nugatory crossings, and links are non-split.

A diagram which is simultaneously alternating and, up to mirror image, positive is is called *special alternating*, and so is called a link with such a diagram. Every alternating diagram of a special alternating link is positive or negative (so special alternating; see [Na]). But not every positive or negative diagram of a special alternating link is alternating. More generally, a diagram is *special* if no Seifert circle is *separating*, i.e., no Seifert circle contains Seifert circles in both its interior or exterior (compare with §3.2).

Let D be an oriented knot or link diagram. We denote by $c(D)$ the *crossing number* of D. The crossing number $c(L)$ of a knot or link L is the minimal $c(D)$ over all diagrams D of K. It is known, by Kauffman, Murasugi and Thistlethwaite, that $c(D) = c(L)$ if D is a reduced alternating diagram, i.e., that such a diagram is *minimal*.

Let $c_\pm(D)$ be the number of positive, respectively negative crossings of a diagram D, so that $c(D) = c_+(D) + c_-(D)$ and the *writhe* is $w(D) = c_+(D) - c_-(D)$. A diagram D is *positive* if $c_-(D) = 0$ and *negative* if $c_+(D) = 0$; so is called a link with such a diagram.

We call a *reverse clasp* to be  ; and a *parallel clasp*  . We call a clasp *trivial* if both its crossings have opposite sign. Such a clasp can be eliminated by a Reidemeister II move.

The *(Seifert) genus* $g(L)$ resp. *Euler characteristic* $\chi(L)$ of a knot or link L is said to be the minimal genus resp. maximal Euler characteristic of a *Seifert surface* of L. We have

$$2g(L) = 2 - n(L) - \chi(L)\,.$$

Similarly write $\chi_4(L)$ for the *smooth 4-ball* (maximal) Euler characteristic and

$$\chi_c(L) \;=\; \max\{\,\chi(D)\,:\,D \text{ is a diagram of } L$$

for the *canonical Euler characteristic* resp.

$$g_c(L) = (2 - n(L) - \chi_c(L))/2 \tag{2.2}$$

for the *canonical genus* (which will be mostly used for knots L.

We write $s(D)$ for the number of *Seifert circles* of a diagram D, the loops obtained by smoothing out all crossings of D. For a diagram D of L, we define $g(D)$ to be the genus of the *canonical* Seifert surface (obtained by Seifert's algorithm) on D, and $\chi(D)$ its Euler characteristic. We have $\chi(D) = s(D) - c(D)$ and $2g(D) = 2 - n(L) - \chi(D)$. We say that D is of minimal genus, or a *minimal genus diagram*, if

$$g(D) = g(L)\,, \quad \text{or equivalently} \quad \chi(D) = \chi(L)\,. \tag{2.3}$$

Since we will need this property so often, we say that L is *canonically minimal* if it has a diagram D with (2.3). Positive ([St]) and alternating diagrams ([Cw, Mu2]) are of minimal genus, so that the corresponding links are canonically minimal.

For *Murasugi atoms*, see [QW, St11]; these are the minimal pieces under which one can decompose a diagram under (diagrammatic) connected and *Murasugi sum* ($*$-product). Note that all these atoms are special diagrams. In [Cr] their Seifert graphs are called *blocks*. We will write $a(D)$ for the *(Murasugi) atom number* of a diagram D. The inequalities

$$a(D) \le 1 - \chi(D) \tag{2.4}$$

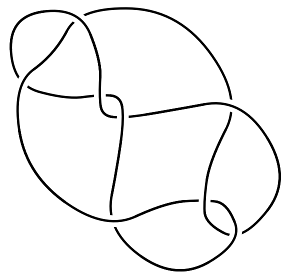
and

$$a(D) \le c(D)/2 \tag{2.5}$$

are easily observed. For later reference, set

$$a'(D) = \min\left(1 - \chi(D), c(D)/2\right). \tag{2.6}$$

A diagram is *alternative* if all crossings in a connected component of the complement of the Seifert circles have the same sign. A diagram is *homogeneous* [Cr] if all Murasugi atoms are special alternating. The difference to 'alternative' is minimal: only if a connected component of the complement of the Seifert circles is a diagrammatic connected sum, its different summands in a homogeneous diagram need not be special alternating of the same sign. The properties are extended to knots and links defined by admitting the corresponding type of diagram.



All properties of homogeneous diagrams (and links) apply to alternative as well. In particular, an alternative diagram of a positive link is positive. One difference is that homogeneity of a diagram is preserved by flypes, while alternativeness in general is not. Above on the side is an example of a homogeneous knot diagram which is not alternative.

Example 2.1 Simplest examples of *knots* which are homogeneous but not alternative are 10_{150}, 10_{151}, 10_{156}, 10_{158}, 10_{160}, and 10_{163} (the third last knot in Rolfsen's table, according to our handling of the Perko pair). They are fibered knots of genus 3 and to exclude alternativeness, it is enough to test alternative diagrams up to 12 crossings (by the same argument as Cromwell's for homogeneous ones).

Kauffman had conjectured that alternative links are equivalent to pseudo-alternating ones, which by [Cr, §1] would imply also equivalence to homogeneous ones. Kauffman's conjecture was disproved in [Si], by finding a pseudo-alternating knot 10_{145} which is non-homogeneous. The above examples show that the inclusion of homogeneous links in alternative ones is also proper.

See also the appendix to this paper. In particular, keep in mind throughout that 'semi-homogeneous' in [St11, St12] is what should be 'homogeneous', while 'homogeneous' there refers to what is properly called 'alternative'.

Let $\Gamma(D)$ be the *Seifert graph* of D. A vertex is made for every Seifert circle of D, and an edge for every crossing connecting two (distinct) Seifert circles. Keep in mind that $\Gamma(D)$ is always bipartite and planar (but not mandatorily simple).

We write $D = D_1 * \cdots * D_l$ for Cromwell blocks D_i, so that $\Gamma(D) = \Gamma(D_1) * \cdots * \Gamma(D_l)$ and $\Gamma(D_i)$ have no cut vertex. Note that, while the planar embedding of $\Gamma(D)$ is not unique, the D_i determine the planar embedding of $\Gamma(D_i)$, which w call *canonical planar embedding*.

An extensive account on related further standard facts (which we would not like to repeat all here) is given in [IS, §2].

For reference and orientation, we include already here the following hierarchy diagram between various classes of links, whose definition (and explanation) will be completed in §4.4.4, 4.4.6.

		canonically minimal	$\supset$	pseudo-homogeneous	$\supset$	homogeneous	$\supset$	alternative	$\supset$	alternating	$\supset$	special alternating
				$\cup$				$\cup$			$\cap$	
quasi-positive	$\supset$	strongly q.p.	$\supset$	pseudo-positive		$\supset$		positive		$\supset$		positive braid

Inclusions are meant up to mirror image. No other inclusions except those given (and their composition) exist, and all inclusions are checked to be proper.

For space reasons some knot diagrams below will not be drawn, but given instead in terms of their *Dowker-Thistlethwaite (DT) notation* [DT]. It is preceded, as in the format in [HT], by the crossing number of the diagram and an integer identifier.

2.3 Polynomials and invariants

Let Δ be the (1-variable) *Alexander polynomial*. We understand it here normalized to $\Delta(U) = 1$ and with the skein relation

$$\Delta(D_+) - \Delta(D_-) = (t^{1/2} - t^{-1/2})\Delta(D_0)\,, \tag{2.7}$$

with $D_\pm$ and D_0 being a (skein) triple of diagrams equal except near one crossing which is as in (2.1), from left to right. When $\Delta(L) \neq 0$, we have

$$2\max\deg\Delta(L) \le 1 - \chi(L)\,. \tag{2.8}$$

For a knot L, this inequality is more commonly written as

$$\max\deg\Delta(L) \le g(L)\,. \tag{2.9}$$

This inequality is exact (i.e., an equality) when L is, among others, a non-split positive, alternating (see [Cr]), almost positive link [St7], or a fibered link, for example. We say L is *almost positive* if it is not positive but has a diagram D with $c_-(D) = 1$.

The *skein (HOMFLY) polynomial* P is understood here via the relation

$$l^{-1}P(D_+) + lP(D_-) \,=\, -mP(D_0)\,, \tag{2.10}$$

and the normalization $P(U) = 1$. The convention in (2.10) is similar to (and uses the same variables as) the one in [LMi] but differs by the interchange of l and l^{-1}. A few basic properties of P we will need are Morton's inequalities [Mo] stating that

$$1 - \chi(D) \,\ge\, \max\deg_m P(L) \tag{2.11}$$

for every diagram D of a link L, and

$$\min\deg_l P(D) \ge 1 - s(D) + w(D)\,. \tag{2.12}$$

as well as

$$\max\deg_l P(D) \le s(D) - 1 + w(D)\,. \tag{2.13}$$

Also

$$\min\deg_l P(D) \le \max\deg_m P(D)\,, \tag{2.14}$$

from [LMi, Proposition 21].

The *Kauffman polynomial* $F = F(a,z)$, and its special case, the *BLMH polynomial* $Q(z) = F(1,z)$, will only be tenuously required, in relation to Fact 4.52 and Theorem 4.53, and is used with the convention of [St5, St20].

We will also need the *Ozsváth-Szabó* τ *invariant*, defined for knots (the Rasmussen invariant instead would do as well). We require only few of its properties. It satisfies under crossing change

$$\tau(D_+) - 1 \,\le\, \tau(D_-) \,\le\, \tau(D_+)\,, \tag{2.15}$$

(with the convention below (2.7)).

An obstruction of Livingston[2] (see [He, Theorem 1.1] and the explanation below it) is that when K is strongly quasipositive, then

$$\tau(K) = g_4(K) = g(K)\,, \tag{2.16}$$

with $g_4(K)$ being the smooth 4-genus. The connection of τ to the *signature* σ for an alternating knot,

$$2\tau(K) = \sigma(K) \tag{2.17}$$

implies that

$$\tau(K) \le g(K)\,,$$

and equality holds if and only if K is (positively) special alternating.

Also τ satisfies the inequality, analogous to (2.12),

$$2\tau(K) \,\ge\, w(D) - s(D) + 1 \tag{2.18}$$

for any diagram D of K. (Under (2.15) this is more or less equivalent to (2.16).)

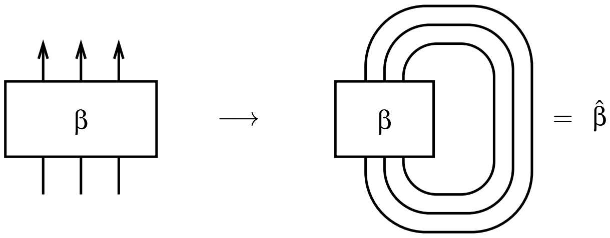
[2] found in various versions by others as well, e.g., Rasmussen has observed it for his invariant and so on...

2.4 Braids and braided surfaces

We write B_n for the braid group on *n strands* or *strings*. The relations between the *Artin generators* σ_i, $i=1,\dots,n-1$ are given by $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1\le i\le n-2$ and by $\sigma_i\sigma_j=\sigma_j\sigma_i$ for $1\le i<j-1\le n-2$. The trivial braid in B_n will be written Id_n.

Let us first fix a convention. In diagrams drawn we number braid strands by 1,2,3,... from left to right and compose words in $\sigma_i^{\pm1}$ from bottom to top. Consequently, the orientation of braid strands is assumed to be upward, even if usually we don't explicitly display arrows. See (2.23) for an example.

Braid *closure* $\hat\beta$ is defined in planar diagrams with



(2.19)

A braid β whose closure $\hat\beta$ is a given link L is called a *braid representative* of L. The minimal number of strands for a braid representative of a link L is called the *braid index* of the link, and will be denoted by $b(L)$. (See, e.g., [DHL, Mo, Mu].) A braid representative $\beta\in B_{b(L)}$ with $\hat\beta=L$, i.e., with the fewest strands, is called a *minimal braid*.

With the *Morton-Franks-Williams bound*

$$\mathrm{MFW}(L)=1+\frac{1}{2}\,\mathrm{span}_l P(L)\,, \tag{2.20}$$

the Morton-Franks-Williams inequality [Mo, FW] states that

$$b(L)\ge \mathrm{MFW}(L)\,. \tag{2.21}$$

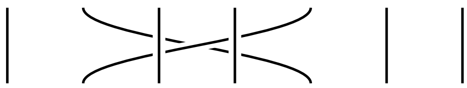
Alternatively to the standard Artin generators, one considers also a presentation of the braid groups by means of an extended set of generators (1.1) and their inverses

$$\sigma_{i,j}^{\pm1}=\sigma_i^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{\pm1}\sigma_{j-2}\dots\sigma_i \tag{2.22}$$

for $1\le i<j\le n$. Note that

$$\sigma_i=\sigma_{i,i+1}\,.$$

We will call $\sigma_{i,j}$ *positive bands* and $\sigma_{i,j}^{-1}$ *negative bands*. For example, $\sigma_{2,5}\in B_7$ can be drawn (by using braid relations for cosmetic's sake) thus:



(2.23)

A representation of a braid β, and its closure link $L=\hat\beta$, as word in $\sigma_{i,j}^{\pm1}$ is called a *band representation* [BKL]. A band representation of β spans naturally a Seifert surface S of the link L: one glues disks into the strands, and connects them by half-twisted bands along the $\sigma_{i,j}$. The resulting surface is called *braided Seifert surface* of L. (See, e.g., [St2].) In fact, a result of Rudolph [Ru4] (later rediscovered independently by M. Hirasawa) says that any Seifert surface is of this form.

A minimal genus braided Seifert surface is called a *Bennequin surface* (see [HS, St15]). Bennequin's work [Be] implies that any *strongly quasipositive Seifert surface*, a braided Seifert surface with only positive bands, is a Bennequin surface; thus if a link L has a strongly quasipositive Seifert surface S on s strands with l bands, then $\chi(L)=\chi(S)=s-l$. The converse is an open question we will return to. (Our discussion will lead to a more general version of this problem, given in Question 6.6.)

Question 2.2 (Baker and Motegi; see [Kp]) Is every minimal genus surface of a strongly quasipositive link strongly quasipositive?

Our contribution towards this question is its resolution for (minimal) canonical surfaces in Corollary 4.33. For an application, see §5.2.

It is important to remark here that the answer is affirmative if a minimal genus surface is unique, in particular for a fibered link. This will enable us to use plumbing and Murasugi sum operations to discern about strong quasipositivity. We will extensively (and implicitly) invoke Rudolph's characterization [Ru5] that for two Seifert surfaces $S_{1,2}$, a Murasugi sum $S_1 * S_2$ is strongly quasipositive if and only if both $S_{1,2}$ are strongly quasipositive. As explained there, Gabai's work showed this relationship for 'strongly quasipositive' replaced by 'minimal genus'.

One can define the *Bennequin braid index* by

$$b_b(L) := \min\{\, n : L \text{ has a Bennequin surface on } n \text{ strands}\,\}. \tag{2.24}$$

If β gives a Bennequin surface S for $L = \hat{\beta}$ and β is a minimal braid for L, so S is a Bennequin surface on $b(L)$ strings, we call S a *minimal string Bennequin surface*. Then

$$b_b(L) = b(L). \tag{2.25}$$

It is known, for instance, that (2.25) holds when $b(L) \le 3$. This is a result Birman-Menasco attribute to Bennequin, which becomes very explicit in Xu's 3-braid normal form [Xu].

Obviously in general $b(L) \le b_b(L)$, but sometimes this inequality is strict (see below Question 6.2). We will make substantial effort to either establish (2.25) or at least obtain good *upper* estimates on $b_b(L)$.

The case of strongly quasipositive links was considered previously by Rudolph. If S is a strongly quasipositive surface of a (strongly quasipositive) link L, define $b(S)$ to be the minimal number of strings on which S is spun *as a strongly quasipositive surface*. By definition, $b(S) < \infty$. Set

$$b_q(L) := \min\{\, b(S) : S \text{ is a strongly quasipositive surface of } L\,\}.$$

Then, since a positive band representation gives a Bennequin surface, we have

$$b(L) \le b_b(L) \le b_q(L). \tag{2.26}$$

Question 2.3 (Rudolph; see [St15, Remark 8.3.3]) If L is strongly quasipositive, does L have a strongly quasipositive braid representative on $b(L)$ strands, i.e., is $b_q(L) = b(L)$?

Obviously, because of (2.26), $b_q(L) = b(L)$ implies $b_b(L) = b(L)$. But the converse is also true, as we will soon remark. However, it does not seem to have been considered whether $b_q(L) = b_b(L)$ holds in general, i.e., that if a strongly quasipositive surface of L is realized as a Bennequin surface, then one is realized on a positive band representation with no more strings. This does not seem entirely trivial, as long as Question 2.3 is not settled.

Many geometric methods for strong quasipositivity have an inherent deficiency of telling little about string numbers of a strongly quasipositive surface. Indeed, as Example B.2 cautions, $b(S)$ does not appear to behave in an easy manner.

For a way to address Rudolph's question, see Corollary 4.35. Further evidence is discussed in the appendix §B.1; certainly no knot up to 16 crossings can answer the question negatively (Proposition B.1). The method in [IS2] mentioned below Example 4.25 will also behave economically w.r.t. string numbers.

To account for the situation $b_b(L) > b(L)$, and in accordance with Question 2.3, in [IK, Stronger form of Conjecture 1] a related bound was conjectured. For a link L, define the *Thurston-Bennequin number* $TB(L)$ by

$$TB(L) = \max\{\, w(D) - s(D) : D \text{ a diagram of } L\,\}, \tag{2.27}$$

and the *Bennequin defect* of L by

$$\delta(L) = \frac{-\chi(L)}{2} - \frac{TB(L)}{2}.$$

Then *Bennequin's inequality* states that

$$\delta(L) \geq 0\,, \tag{2.28}$$

and also δ is an integer. We call L *Bennequin-sharp* if $\delta(L) = 0$. The *slice Bennequin inequality* states that

$$TB(L) \leq -\chi_4(L)\,. \tag{2.29}$$

For an improvement of this inequality, as an estimate of the 4-genus (which subsumes Rudolph's improvement [Ru]), see §4.4.5.

Note the inequalities

$$2\delta(L) \geq 1 - \chi(L) - \min\deg_l P(L) \tag{2.30}$$

and for knots K,

$$\delta(K) \geq g(K) - \tau(K)\,, \tag{2.31}$$

that follow from (2.12) and (2.18).

Also note that

$$TB(L_1 \# L_2) \geq TB(L_1) + TB(L_2) + 1\,, \tag{2.32}$$

whence $\delta(L_1 \# L_2) \leq \delta(L_1) + \delta(L_2)$. (It would interesting to know if always equality occurs.) In particular, the connected sum of Bennequin-sharp links is Bennequin-sharp.

Obviously, for a strongly quasipositive link K, we have $\delta(K) = 0$. It has been asked whether the converse is true (see introduction of [FLL]):

Conjecture 2.4 Every Bennequin-sharp link is strongly quasipositive.

For our contribution to this problem, see Corollary 4.31. For a very general construction of knots with large δ, see Corollary 4.51.

Ito-Kawamuro made the following (rather optimistic) conjecture.

Conjecture 2.5 (Ito-Kawamuro) $b_b(K) \leq b(K) + \delta(K)$.

The conjecture, for $\delta(K) = 0$, is equivalent to a positive answer to Question 2.3 plus the resolution of Conjecture 2.4.

We remind that because of the proof of the Jones-Kawamuro conjecture (see Theorem 2.6 below) every minimal string (band representation of a) Bennequin surface of a strongly quasipositive link L must be strongly quasipositive, i.e., that $b_b(L) = b(L)$ implies $b_q(L) = b(L)$.

We will come back to the Ito-Kawamuro conjectured inequality after Theorem 4.2.

When β is considered as a word in σ_i, then $\hat{\beta}$ gives a braid diagram of the closure link. (It will be more useful here to disregard braid relations in braid representatives.) Note that the number of Seifert circles of $\hat{\beta}$ is the number of strands of β. The number of crossings $c(\hat{\beta})$ of $\hat{\beta}$ is the *word length* of β in the $\sigma_i^{\pm 1}$, and the writhe $w(\hat{\beta})$ is the *exponent sum* of β, which we will also write as $w(\beta)$.

We will use a few times the notion of minimal writhe from the proof of the (strong version of the) Jones-Kawamuro conjecture [DP, LM]. Let us say that a (s,w)-*representative* is a braid representative of L on s strands of exponent sum (writhe) w.

Theorem 2.6 ([DP, LM]) For every link L there is a number $w(L)$ such that every $(b(L),w)$-representative of L satisfies $w = w(L)$. (weak version)
Every (s,w)-representative of L satisfies $|w - w(L)| \leq s - b(L)$. (strong version)

Using Yamada-Vogel (see §3.1) we can straightforwardly extend the theorem from (s,w)-representatives to (s,w)-*diagrams*, latter meaning diagrams of L having s Seifert circles and writhe w. We can in particular use the weak version to define the *minimal writhe* $w_m(L)$ of a link L by $w_m(L) = w(D)$ whenever we have diagram D of L with $s(D) = b(L)$.

Theorem 2.6, together with Bennequin's inequality (2.28), implies that when L is strongly quasipositive, then

$$w_m(L) = b(L) - \chi(L)\,, \tag{2.33}$$

and then more generally

$$w(D) \geq 2b(L) - s(D) - \chi(L)\,, \tag{2.34}$$

for any diagram D of a strongly quasipositive link L.

We need the half-twist (braid) element

$$\delta_n = \prod_{j=2}^{n} \prod_{l=1}^{j-1} \sigma_l\,. \tag{2.35}$$

Let us also set notation for the index shift map

$$[\sigma_i]_k = \sigma_{i+k}\,, \tag{2.36}$$

whenever the right hand-side has admissible index. We will also below in (2.37) and (2.38) assume notation extended to inverses by taking the inverse of the r.h.s. and then extend to braid words under multiplication of letters. (These operations give homomorphisms of braid groups, but this aspect will be less relevant here.)

We call $\beta \in B_k$ *non-split* if for each $i = 1,\dots,k-1$, some letter σ_i or σ_i^{-1} occurs in β. (It is not assumed that more than one letter of such type occurs, i.e., we do not require that $\hat{\beta}$ is a reduced diagram.)

We call $\beta \in B_k$ *homogeneous* if for each $i = 1,\dots,k-1$, only one type of letters σ_i or σ_i^{-1} occurs in β. If for each i only σ_i occurs, we call β *positive*; if only σ_i^{-1} occurs for all i, we call β *negative*. If σ_i occurs for even i and σ_i^{-1} occurs for odd i, or vice versa, then we call β *alternating*. For any of these four types, β has such type if and only if the closed braid diagram $\hat{\beta}$ has (see [Cr]).

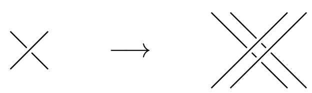
2.5 Cabling

Let $k \in \mathbb{N}$, $\beta \in B_k$ and K be a knot. Define the β-k-cable K_β of K to be the satellite link with companion K and pattern given by the k-string braid β in the complement of its axis, with 0 framing. (See [Sn] or below Example 3.13.) In particular, if β is a torus braid, i.e., a power of $\sigma_1 \cdots \sigma_{k-1}$, we obtain classical cables. When $\beta = Id_k$, the cable $K_{Id_k} = K_k$ is the 0-framed disconnected parallel.

The notion of cabling extends in a natural way to braids: the *k-cable* of $\beta \in B_n$, denoted by $\{\beta\}_k$, is a braid on $n \cdot k$ strings, obtained by replacing σ_i in β by

$$\{\sigma_i\}_k \;=\; \prod_{j=1}^{2k-1} \;\prod_{l=0}^{\min(j-1,2k-1-j)} \sigma_{ki-\min(j-1,2k-1-j)+2l}\,. \tag{2.37}$$

Then if $\hat{\beta} = K$ is a knot, we have $\widehat{\{\beta\}_k} = K_{\delta_k^{2w(\beta)}}$, where δ_k is the half-twist (2.35). See the crossing replacement for the sample case $k = 2$:



To correct for the framing, one has to full-twist the left or right k strings (using the notation (2.36))[3]:

$$(\sigma_i)_k^l = [\delta_k^2]_{(i-1)k}\{\sigma_i\}_k = \{\sigma_i\}_k[\delta_k^2]_{ik} \qquad \text{or} \qquad (\sigma_i)_k^r = [\delta_k^2]_{ik}\{\sigma_i\}_k = \{\sigma_i\}_k[\delta_k^2]_{(i-1)k}\,. \tag{2.38}$$

[3]The superscripts 'l' and 'r' for 'left' and 'right' are used only symbolically as literals and not as exponents.

These braids are not equal, but conjugate by $[\delta_{2k}]_{(i-1)k}$ (from (2.35)). Then $\widehat{(\beta)^l_k} = \widehat{(\beta)^r_k} = K_k$. In fact, one can choose a superscript 'l' or 'r' for each letter of β separately and multiply the resulting braids. All these possible product braids are conjugate braid representatives of K_k.

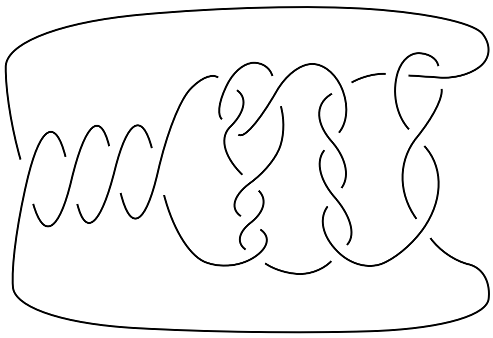
For obtaining a braid whose closure gives an unframed cabling of a *link* $\hat\beta$, one has to use $(\sigma_i)^l_k$ or $(\sigma_i)^r_k$ for σ_i corresponding to self-crossings of $\hat\beta$, and $\{\sigma_i\}_k$ in mixed crossings.

2.6 Montesinos and pretzel links

For the following, we must clarify a few basic facts about Montesinos links. A Montesinos link has the presentation

$$L = M(e, p_1/q_1, \dots, p_n/q_n)\,. \tag{2.39}$$

We use (without repeating much of) the (standard) explanation, e.g., in [St5, §2.7]. The convention should be explained with the below diagram. (For the moment ignore strand orientation.)

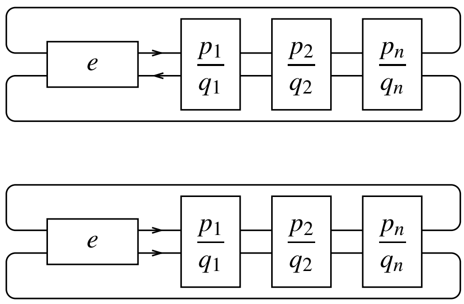


$M(4, 3/11, -1/4, 2/5)$

This notation determines the diagram *as an unoriented diagram*, up to mirroring.

We call e *integer part*, and p_i/q_i *fractional parts*. Their number n is the *length*. If $n \le 2$, the Montesinos link is rational, so we assume (5.1).

Here orientation issues near the integer part become essential:



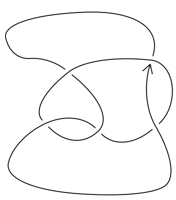
reverse link

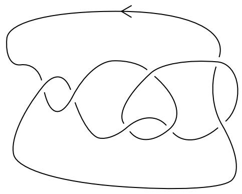
(2.40)

parallel link

Note that for a parallel link q_i are necessarily odd. For this to be a knot, $\sum p_i + e$ is odd. Otherwise, it is a 2-component link.

When an orientation is given, we fix the mirroring in (2.39) so that $e > 0$ corresponds to integer twists which are positive crossings. For $e = 0$, take in the previous mirroring convention $e = 2$ (for a reverse link in (2.40) one needs even e to keep orientation coherent) and then resolve the clasp of these two crossings. For example

$M(2/3, -1/2) =$  , (2.41) even although the crossings in the $2/3$-tangle are negative and those in the $-1/2$ positive, because when adding two integer crossings to alternate with the $2/3$-tangle,

$M(2,2/3,-1/2) =$  ,

these two new crossings orient to be positive.

Note, though, that this convention still leaves an ambiguity of orientation of components not involved in the integer twists. This will not create major problems, in particular for knot diagrams.

When all $p_i = \pm 1$, we have a *pretzel link* diagram

$$D = L(a_1,\dots,a_n) \tag{2.42}$$

(*cf.* [KL, Fig. 1]) with

$$a_i = \pm q_i$$

and

$$L = L(\pm q_1,\dots,\pm q_n) = M\left(\pm\frac{1}{q_1},\dots,\pm\frac{1}{q_n}\right). \tag{2.43}$$

We will call the group of crossings counted by q_i a *twist*. The twist for $q_i > 1$ can be distinguished between *parallel* or *reverse* depending on the type of clasps of its crossings.

Some important remarks are necessary regarding how we will handle the signs in the conversion (2.43). The sign of q_i on the left will generally *not* match the one of $\frac{1}{q_i}$ on the right. For pretzel links, we will constantly assume all components to be oriented, and we will *sign a_i according to the sign of the crossings it its twist*. See (5.35) and above it for comparison. Thus if all $a_i > 0$ in (2.43), then the diagram (2.42) is positive, but not necessarily alternating.

There again remains an ambiguity, in that for various a_i one would have to specify whether the twist is parallel or reverse. We will have to carry this caveat along. However, we will mostly consider the case of a knot diagram or of an alternating diagram, and under any of these assumptions the ambiguity will disappear.

Furthermore, when D is special, in particular always when the pretzel link is parallel, the discrepancy of signs in (2.43) also disappears.

3 General link diagrams and banded diagrams

3.1 Braid diagram algorithms

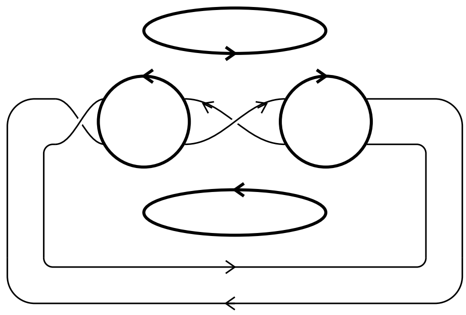
For the constructions described below, it is necessary to review the (now likely most common) algorithms that transform a general link diagram into a (closed) braid diagram.

The Vogel move [Vo] is a Reidemeister II move of a special type. It affects two edges in the boundary of a region R, which have the *same orientation* as seen from inside R, and belong to *distinct Seifert circles*. 'Same orientation' should be henceforth understood so that *none or both* of them coincides with the induced orientation on the boundary of R. Let us call their Seifert circles *Vogel incoherent*. The Vogel move then creates a reverse trivial clasp inside R (while preserving the number of Seifert circles).

Vogel explains that D is a closed braid diagram if and only if D has no pair of Vogel incoherent Seifert circles. He proves that a sequence of such moves always transforms any link diagram into a closed braid diagram (of the same number of Seifert circles).

Another algorithm previously found by Yamada [Y] will be introduced and discussed below. For its treatise it is useful to define a related condition. Let us call two *distinct* Seifert circles of a diagram D to be *incoherent*, if they bound a common region R *in the complement of the union of the Seifert circles of* D and if their orientation is the same w.r.t. R.

There is a subtlety here: Vogel incoherent Seifert circles are incoherent, but if two Seifert circles are incoherent, they may not be Vogel incoherent, since they may not bound a common region of D. An example is:

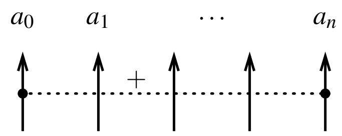


But it is very easy to see (the argument is similar to the proof of the below Lemma 3.5) that the *existence* of either type of 'incoherent' pairs of Seifert circles is equivalent.

3.2 Braiding banded diagrams

Consider a collection of oriented Seifert circles in the plane.

Definition 3.1 We call a *band* to be a dashed line with a $+/-$ label. The dashed line should start and end on a Seifert circle, do not intersect itself, intersect Seifert circles transversely, and with the orientation of Seifert circles being

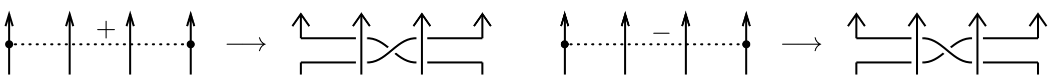


Note that in particular because of the orientation, no band can intersect or connect the same Seifert circle twice.

We say two Seifert circles a, b are *coherent*, if they bound a common region R and are oriented oppositely w.r.t. R's boundary. (Region is again understood in the complement of the Seifert circles.)

Then a band is a dashed line going from Seifert circle a_0, over *intermediate Seifert circles* a_i to Seifert circle a_n, so that a_i, a_{i+1}, for all $0 \leq i < n$, are coherent.

A band should be understood for standing for a fragment like



(3.1)

Of course, an ordinary crossing is a band in this sense (which does not pass through intermediate Seifert circles).

Definition 3.2 A *banded diagram* is a collection of Seifert circles in the plane with a collection of bands, none two of which intersect.

We will implicitly understand a banded diagram to stand for a link diagram in which all replacements (3.1) have been made. Usual link diagrams can obviously be regarded as banded diagrams, with each crossing turned into a band. If a banded diagram is a (closed) braid diagram, it will give a band representation of a closed braid, and the bands become the $\sigma_{i,j}^{\pm 1}$ in (2.22).

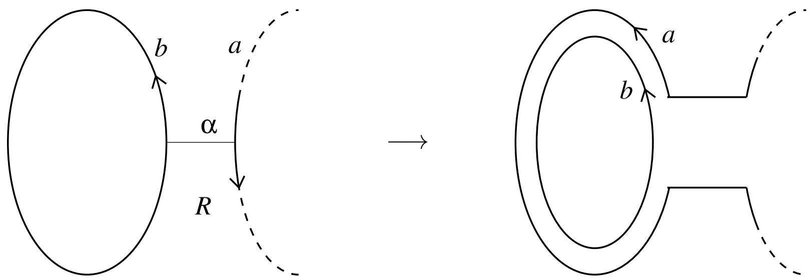
Most of what follows will center around the following lemma.

Lemma 3.3 Let L have a banded diagram D with s Seifert circles and l bands. Then L has a braid representative on s strands with l bands (and the same number of positive and negative ones as D).

Definition 3.4 We say that a pair of Seifert circles in a banded diagram is *admissible* if it is incoherent, and some two points on the Seifert circles can be connected by an arc α not intersecting other Seifert circles and disjoint from bands. We will call α also an *admissible arc*.

Note that when all bands are crossings, this is exactly what we explained above as 'Vogel incoherent'.

Let (a,b) be an admissible pair, with an admissible arc α in a region R of the Seifert circle picture. We now use a special instance of the Yamada move [Y], after threading inside R a small piece of a along α to become close to b:



(3.2)

We call this an *admissible Yamada move* or shortly a *Y move*. Note that this move depends on the arc α chosen between a and b. It also depends, unlike the notion of being an incoherent pair, on the order of these two Seifert circles. While the arc will not be so relevant, except for its existence, we must distinguish the order of (a,b), and will write $Y(a,b)$ for the Yamada move.

It is obvious that the Y move does not change the number of Seifert circles. (In fact, it does not change any Seifert circle except the Seifert circle a.) This was fundamentally used by Yamada in his paper. However, it does not either change the number of bands under the equivalence (3.1), once we meet the crucial convention that from now on *the newly created arc of a should lie on top of all crossings it passes*.

To prove Lemma 3.3, our goal is thus to prove that there is a sequence of Y moves turning any diagram D into a braid diagram.

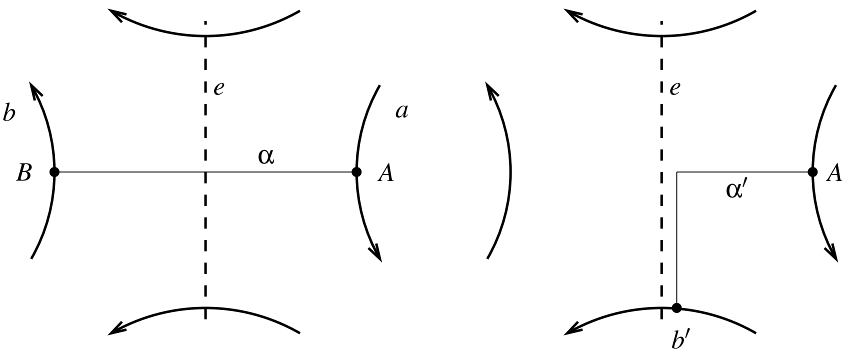
It will be helpful from now on to fix the infinite region of D. Accordingly, every Seifert circle has an *interior* and *exterior*. We say that b,b' *lie on the same side of* a, if both b,b' lie in the interior of a or both b,b' lie in the exterior of a.

Lemma 3.5 Let (a,b) be incoherent. Then there is a Seifert circle b' on the same side of a as b, so that (a,b') is admissible.

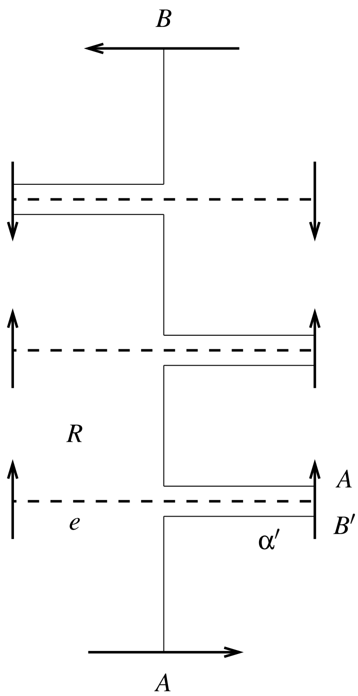
Proof. Let α be an arc in a region R of the complement of the Seifert circles connecting inside R with no self-intersection a point $A \in a$ and $B \in b$.

If α meets no band, then α is coherent, and we can take $b' = b$. Otherwise follow α starting from A until before its intersection with a band e. Then follow the band on either side until it intersects or ends on a Seifert circle b' at a point B'. In exactly one of the two directions one meets a Seifert circle b' incoherent with a. One obtains an arc α'

in R with no self-intersection and disjoint from bands.



If $b' \neq a$, then α' is an admissible arc. Otherwise start off at the other side of e at $b' = a$, and continue as in the below figure.



Assuming all previous arcs connect a with itself, each next arc starts at a. Since we assumed $b \neq a$ (incoherent Seifert circles are by definition distinct), one of the arcs α' cannot connect a with itself, and so it is an admissible arc between a and some Seifert circle $b' \neq a$. Obviously b' bounds R as a and b do, so b, b' are on the same side of a. □

Definition 3.6 1. When b lies in the interior of a, we say *a contains b*. When b lies in the exterior of a, we say b is *outside a*.

2. We say that a Seifert circle is *empty* or *innermost* if it contains no other Seifert circle.

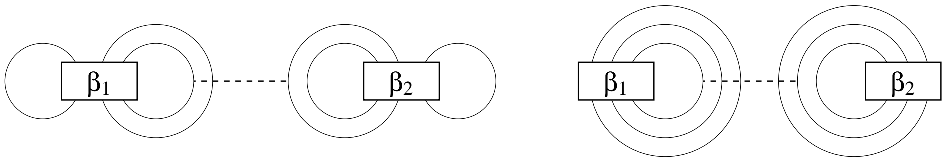
3. We say it is *outermost* no other Seifert circle contains it.

4. We say that *a immediately contains b* if a contains b and there is no Seifert circle c such that a contains c and c contains b.

5. We say that a is *braid-like* if the following is true. Let b be contained in a or $b = a$. Then either b is empty, or b immediately contains exactly one Seifert circle c, and this Seifert circle c is coherent with b.

6. We say that a is *maximal braid-like* if it is braid-like and it is not contained in a braid-like Seifert circle.

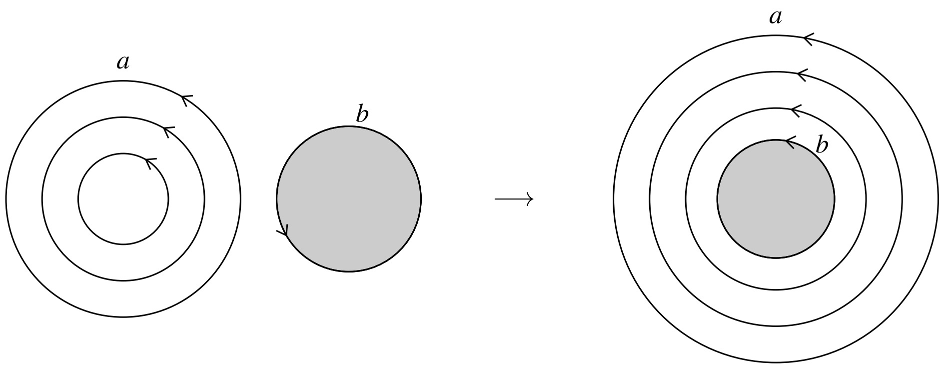
For example the outermost Seifert circles a in the diagrams on both hand sides of (3.3) are maximal braid-like. Note that a diagram is a (closed) braid diagram if and only if all Seifert circles are braid-like, and there is one outermost (or maximal braid-like) Seifert circle, or two outermost Seifert circles, which are coherent. Compare with the diagrams below: only the second (banded) diagram is a closed braid (banded) diagram.



Lemma 3.7 Let a be braid-like and a immediately contain a'. Let b be outside a so that (a,b) is admissible. Then after a $Y(a,b)$ move, (a',b) is admissible.

Proof. This is essentially because an admissible arc between a point $A \in a$ and $B \in b$ can be joined by an admissible arc α' between A and some $A' \in a'$. The arc α' exists, because no two bands intersect, and no band can intersect a twice. □

This means that one can successively move b through all Seifert circles contained in a. We will call this process that a *swallows* b.



(3.3)

If b is braid-like, then after this process a, with b and its interior sitting inside, will remain braid-like.

The proof of Lemma 3.3 is now accomplished by the following two lemmas.

Lemma 3.8 One can perform Y moves on D so that all Seifert circles become braid-like.

Proof. We proceed by induction on the number of non-braid-like Seifert circles. Take a non-braid-like Seifert circle a which is innermost, i.e., not containing another non-braid-like Seifert circle.

1. If a (immediately) contains an incoherent Seifert circle (with a), by Lemma 3.5 a must be admissible with one such Seifert circle b. The move $Y(a,b)$ then moves b out of a (and b remains braid-like). Repeat this process until a immediately contains no incoherent Seifert circle.

2. Now if a immediately contains at least 2 Seifert circles a'_i, which must be coherent with a, they are all incoherent among themselves in pairs. By Lemma 3.5 there is an admissible pair (a'_i, a'_j) (since (a'_i, a) is coherent for every i). Now we assumed that a'_i are braid-like. Then letting one of a'_i and a'_j swallow the other, we have still that all Seifert circles immediately contained in a are braid-like, but there is one fewer such now. Repeat this process until a immediately contains only one, coherent with a, Seifert circle. Then a becomes braid-like. □

Lemma 3.9 Assume in D all Seifert circles are braid-like. One can perform Y moves on D so that it becomes a braid diagram.

Proof. Modify the part 2 of the previous proof. As long as there is a pair of incoherent outermost Seifert circles, there is an admissible pair, and swallowing one Seifert circle in such a pair by the other decreases the number of pairs of incoherent outermost Seifert circles (and all they remain braid-like).

In the end, one remains with outermost Seifert circles all which are coherent. But if a is coherent with b and c, then (b,c) are incoherent, thus there are at most two outermost Seifert circles. Then D is a braid diagram, as in the remark below Definition 3.6. □

Remark 3.10 Since the Y moves only alter the shape of the Seifert circles but do not affect the bands, one can see that the isotopy in lemma 3.3 in fact gives an isotopy of the banded Seifert surface itself (not only of its boundary).

3.3 Some other applications

3.3.1 Positive links

For example, if we start with a positive diagram D, regarding its positive crossings as positive bands, the algorithm in the proof of Lemma 3.3 will give a band representation with positive bands. The below is thus a direct consequence of Lemma 3.3 and recovers, as announced, Rudolph's result [Ru].

Corollary 3.11 A link is strongly quasipositive if and only if it has a positive banded diagram. □

One can easily see a little bit more: by estimating

$$s(D) \le c(L)$$

from [St5, Corollary 3.2] (see the proof of Theorem 1.1, specifically (4.3) without the '-1' at the end), one can get a strongly quasipositive representative on at most $c(L)$ strings. However, comparing this estimate with Ohyama's [Oh]

$$b(L) \le \frac{c(L)}{2} + 1\,, \tag{3.4}$$

for a general braid representative, clearly reveals this bound to be somewhat crude. This provides motivation to us to make a non-trivial step forward toward an improvement of Lemma 3.3 when one starts with a regular (and not banded) link diagram. The result, formulated in Theorem 4.1, requires some independent toolkit to open up, and is moved to §4. Ultimately, the obvious question arises: if L is positive, then does it always have a strongly quasipositive braid on minimum ($= b(L)$) strings? It seems even very legitimate to ask this for strongly quasipositive L as well. See Question 2.3. Several of Ito's papers have studied the (more general or related) version of this question in general open books [IK, HIK].

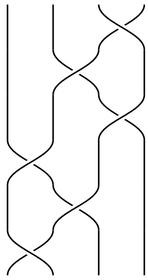
3.3.2 Preservation of strong quasipositivity under cabling

Another small application is as follows. We resume the notation and conventions of §2.5.

Proposition 3.12 If K is a strongly quasipositive knot and $\beta \in B_k$ is strongly quasipositive, then the β-k-cable K_β of K is strongly quasipositive, and

$$\chi(K_\beta) = k\chi(K) - w(\beta)\,. \tag{3.5}$$

Proof. Let first $k=2$ and K be positive. When $\beta = Id_2$, one notes that K_β is obtained from K when replacing each (positive) crossing of a (positive) diagram of K by



(3.6)

which gives a band diagram of K_0 with 2 positive bands for each crossing of D. (Note that for a knot K we can move half-twists between the left and right pair of strands, and we do need this here.) By induction on k generalize (3.6) to give the (positive) band representation

$$(\sigma_1)_k^l = \sigma_{1,k+1}\sigma_{2,k+2}\cdots\sigma_{k,2k} \in B_{2k}\,. \tag{3.7}$$

If β is non trivial (and strongly quasipositive), add its extra bands.

The positive diagram D of K and the positive band surface of K_β is of minimal genus (see §2.4). Thus the Euler characteristic is obtained by counting Seifert circles and bands. Notice that when D has s Seifert circles, then any k-parallel diagram of D will have ks. (From here one also sees estimates on the number of strings, which are k times those for L in Theorem 1.1.)

Now to prove this when K is strongly quasipositive, the only difference is that one has to start with a diagram with positive bands rather than positive crossings and generalize (3.7).

$$(\sigma_{1,m})_k^l = \sigma_{1,(m-1)k+1}\sigma_{2,(m-1)k+2}\cdots\sigma_{k,mk} \in B_{mk}\,. \tag{3.8}$$

The rest of the argument then works the same. □

Example 3.13 Since we start with 0-framing, for a positive braid β and positive knot K, for many of these links K_β one can easily prove that they are not positive. For example, when $k=2$, $\beta=\sigma_1$ and K is the positive (right-hand) trefoil, the above proposition[4] shows that the knot K_β, with the 4-braid representative

$$(\sigma_2\sigma_1\sigma_3\sigma_2)^3\sigma_1^{-5}\,,$$

has genus 2. But by calculation of the (Alexander variable) degree $\max\deg_m P$ in the skein polynomial of K_β, the inequality of Morton (2.11) shows that all its canonical Seifert surfaces have genus at least 3, i.e., for every diagram D of K_β

$$g(D) \geq 3 > 2 = g(K_\beta)\,, \tag{3.9}$$

in particular K_β cannot have a positive diagram D.

This same example has occurred in [Ba]. Based on its discussion there, a different argument here would be to use the fact (which is an exercise[5]) that the fiber surface of every positive link is a Hopf plumbing. See, though, Question 6.5. Yet another proof of (3.9) follows from the result of [St10].

Remark 3.14 Similarly, if L is a link of components $1,\dots,n$, there is a satellite link $L_{\bar\beta}$ for a braid tuple $\bar\beta = (\beta_1,\dots,\beta_n) \in B_k^n$. In a generalization of Proposition 3.12, we have for a strongly quasipositive link L and strongly quasipositive braids β_i (for all $1,\dots,n$) the version of (3.5)

$$\chi(L_{\bar\beta}) = k\chi(L) - 2k(k-1)lk(L) - \sum_{i=1}^{n} w(\beta_i)\,.$$

Here $lk(L)$ is the total linking number of L (sum of all linking numbers between pairs of components of L). The term arises because in the generalized proof the reduction to two bands in (3.6) occurs only in self-crossings of a positive diagram of L.

[4] although, of course, if K is a positive braid knot, like in this instance, the capacity of Lemma 3.3 is not really needed

[5] I have no exact reference, but the best related account I know for is [St8].

Remark 3.15 Proposition 3.12 also holds for 'strongly quasipositive' replaced by 'quasipositive' with a very similar proof, but in (3.5) one would have to use instead of χ the smooth slice Euler characteristic. The converses are more interesting (see Question 6.1).

Another topic is whether something can be said when strands in satellite links have reverse orientation; for this see Example 4.27 and Theorem 3.16.

Similarly, one could ask if the observations of §3.3.2 can be extended when strand orientation in satellite links is incoherent (i.e., algebraic and geometric intersection number are not the same). Whitehead doubles would be the archetype of such links, and Bennequin's example of the trefoil Whitehead double (see Example 4.27) suggests that some result may be possible in this direction.

We are dealing with this question in work in progress with G.-T. Jin. We quote the following outcome, where TB is the Thurston-Bennequin number (2.27). (Compare also with [LN].)

Theorem 3.16 For every knot K, the Whitehead doubles of K with negative clasp are not strongly quasipositive. But the k-twisted Whitehead double of K with positive clasp is strongly quasipositive if and only if $k \geq -TB(K)$.

Thus there emerges a (strange) dichotomy between incoherent and cable patterns. For instance, one can easily check that no 2-cable of the figure-8-knot is strongly quasipositive. For cable patterns, some condition on the companion is thus necessary for the satellite to be strongly quasipositive. We expect (Question 6.1) that this condition is strong quasipositivity. But for incoherent patters of every type (i.e., fixed unequal algebraic and geometric intersection number), for *every* companion knot one can find a pattern making the satellite link strongly quasipositive. We have to move out further discussion to a separate paper.

3.3.3 Stably alternating links

Before we apply Lemma 3.3 further to strong quasipositivity, let us make the first outlined application to alternating links.

Start with a diagram D of a link L whose canonical Seifert surface is of minimal genus (we regard the crossings as bands, as we noticed above) and whose number of Seifert circles gives the braid index of L. Then the proof of the lemma gives a braided Seifert surface with the same number of strands and bands, which is of the same genus.

There are a few such diagrams D. Let us restrict ourselves to alternating diagram D. It is known, by Crowell-Murasugi that the canonical Seifert surface of such alternating diagram is of minimal genus, i.e., (2.3) holds. In some cases D also has the minimal number of Seifert circles $s(D) = b(L)$. In fact, these cases have been recently classified [DHL].

Let us call[6] a link to be *stably alternating*, if it has an alternating diagram D, in which no crossing connects as single crossing its two Seifert circles. (That is, for any pair of Seifert circles, there is either no crossing connecting them, or at least two.) We call D also stably alternating. It is very well known that any fibered alternating link is stably alternating.

Combining (2.21) and (4.16), let us say (as in [St12]) that a diagram D of a link L is ind*-optimal* if the inequalities become exact:

$$\mathrm{MFW}\,(L) = b(L) = s(D) - \mathrm{ind}\,(D)\,. \tag{3.10}$$

The following is a way to state the result of Diao et al., incl. how it is proved.

Theorem 3.17 ([DHL]) For an alternating diagram D of a link L the following are equivalent:

1. $s(D) = b(L)$, i.e., D has the minimal number of Seifert circles,

2. D is stably alternating,

[6] After choosing this terminology, I became aware of [It], where the term 'DHL link' is used.

3. D is ind-optimal.

The partial case of

$$\text{'stably alternating} \Longrightarrow \text{ind-optimal } (\Longrightarrow s(D) = b(L))\text{'}$$

for fibered alternating links was long before proved by Murasugi [Mu]. The implication 'not stably alternating$\Longrightarrow$ $b(L) < s(D)$' (which is not relevant here) is a consequence of the Murasugi-Przytycki move (see §4).

From the theorem and the preceding observations we immediately obtain

Proposition 3.18 Any stably alternating link L has a minimal string Bennequin surface. $\square$

This is among several classes of links for which we have (2.25). See the remarks below Question 6.2 for a summary. Whether any alternating link has a minimal string Bennequin surface remains an open problem (see Question 6.2).

As one last further notice in this subsection, we have the following. Let us call $\beta \in B_k$ to be Δ-*sharp* if β can be written as a product on c bands and

$$2 \max\deg \Delta(\hat{\beta}) = 1 - k + c\,.$$

This means, these are exactly the braids giving a Bennequin surface of a link making (2.8) exact. Examples of such braids are non-split homogeneous braids (in particular non-split positive, negative, or alternating braids), for which the bands are just crossings, or any strongly quasipositive braid β whose closure is a non-split positive or almost positive link (see below (2.9)), or which is fibered. (See [Ba] for some construction of fibered strongly quasipositive links, and [FLL] for the almost positive ones; *cf.* the end of §4.4.6.)

Proposition 3.19 Let K be a stably alternating knot and take $\beta \in B_k$ which is Δ-sharp. Then K_β has a minimal string Bennequin surface and $\chi(K_\beta) = k\chi(K) - c(\beta)$.

Proof. With the idea explained, only some hints are given. Combine the proofs of the Propositions 3.12 and 3.18 by finding a decomposition into bands of the mirror image of (3.6) (and its generalization to $k > 2$) when the framing half-twists are moved to the other cabled strand, i.e., use $(\sigma_i^{-1})_k^r$ from (2.38) instead of $(\sigma_i^{-1})_k^l$, when cabling the negative letters in β. (*Cf*. the remark below (2.38).)

The braid index $b(K_s) = kb(K)$ follows[7] by Corollary 3 of [BM]. The minimality of the braided surface follows from (2.8) and the cabling formula for the Alexander polynomial

$$\Delta(K_\beta)(t) = \Delta(K)(t^k) \cdot \Delta(\hat{\beta})\,.$$

Since β was assumed so, $\Delta(\hat{\beta}) \neq 0$ and (2.8) is exact for $L = \hat{\beta}$, as well as for $L = K$. $\square$

This result then includes again all classical cables of K, *except* the 0-framed one.

3.4 More complexity estimates for strongly quasipositive composition links

This application is only outlined, since it is related to the construction of [St17], and details are given there.

Using a separate construction we achieved that any two given links can be combined to give a strongly quasipositive link. This in particular implies that any link is a sublink of a strongly quasipositive link. We will say that a link L is a *composition* of links L_1 and L_2, and write $L = L_1 \times L_2$, if both L_1 and L_2 are sublinks of L, and if deleting from L the components of L_1 gives L_2 and vice versa. Of course, the operation '$\times$' is highly ambiguous, and we showed that we can always make the result to be strongly quasipositive. We also discussed some complexity issues of the strongly quasipositive link constructed and proved the following. (Complexity can be further reduced if one allows for restrictions on what links to combine, but details are left to [St17].)

Conforming to [St17], let in the sequel $D_i' = \hat{\beta}_i$ be (closed) braid diagrams of L_i and for each letter $d \in \{c, s, w, n\}$ set $d_i = d(D_i')$ and $d = d_1 + d_2$. We will also continuously stipulate $n_1 \geq n_2$, *and that D_1' is not a split diagram.*

[7] for discussion of another, incorrect, proof see [St6]

Proposition 3.20 ([St17]) Let L_i have a braid representatives β_i on s_i strands with word length $c(\beta_i) = c_i$. Then there is a link $L = L_1 \times L_2$ with a positive band representation on $2s$ strands with at most $4c - n_1 + 3n_2$ bands.

To relax ourselves from using braids, we developed Lemma 3.3 to give some complexity estimates of strongly quasipositive links $L_1 \times L_2$ constructed in terms of *an arbitrary* diagram D_i' of L_i. This gives then the generalization of Proposition 3.20:

Proposition 3.21 If L_i have diagrams D_i' with s_i Seifert circles and c_i crossings, then there is a positive band representation of a link $L_1 \times L_2$ of $2s$ strands and at most $4c - n_1 + 3n_2$ bands.

This follows first by applying Yamada's algorithm on D_i' and then a generalization of the construction of $L_1 \times L_2$ from braids. Every Yamada move (when applied carefully, as we developed in §3.2) adds no new bands, but makes crossings into non-crossing bands. Details were given in [St17].

4 Reducing banded Seifert circles

4.1 Outline

We return to the problem that transpires below Corollary 3.11: If one starts in Lemma 3.3 with an ordinary link diagram D (rather than a banded diagram), can one find a braided surface S with $\chi(S) = \chi(D)$ on how many fewer strands than $s(D)$? Specifically, for a minimal genus diagram, with (2.3), one can obtain upper estimates on $b_b(L)$ defined in (2.24).

Ohyama's quoted bound (3.4), for an arbitrary braid representative, relies on Murasugi-Przytycki's move, and there are ways to make this move work for bands, discussed in [St15], but they carry many subtle caveats. The current investigation means to do away with many of these technical problems. The following outcome will already bear its price of proof and explanation. It is not necessary to restrict oneself to positive diagrams, also in order to go beyond Question 2.3.

Theorem 4.1 If L has a connected link diagram D with c crossings and s Seifert circles, then it has a braided Seifert surface S with $\chi(S) = \chi(D) = s - c$ on s' strands with (4.1), with at most $c_+(D)$ positive and at most $c_-(D)$ negative bands.

In fact, we have the estimate (which we owe proof of)

$$s' \leq s(D) - \operatorname{ind}(D)\,. \tag{4.1}$$

The term $\operatorname{ind}(D)$ will be explained in §4.2. Ohyama obtains (3.4) by proving

$$s(D) - \operatorname{ind}(D) \leq \frac{c(D)}{2} + 1\,, \tag{4.2}$$

and using (4.16) discussed below.

What this gives for positive links was presented in the introduction.

Proof of Theorem 1.1. Take a positive reduced diagram D. Use that by [St5, Corollary 3.2] (unless L is alternating, where one could take an alternating diagram D with $c(D) = c(L)$) we have

$$c(D) \leq c(L) - \chi(L) - 1\,. \tag{4.3}$$

Substitute this into the combination of (4.1) and (4.2). This gives the first alternative in the minimum (1.2).

For the second alternative, see the proof of Proposition 4.6 and its statement. The second alternative in (4.13) does readily adapt to (1.2) for a positive link, and because of the use of Theorem 4.1, 'Bennequin surface' can be replaced by 'strongly quasipositive surface'. The increase by 1 in (1.2) is due to the possible failure of the argument that (4.37) is strict. (The r.h.s. of (4.43) goes up by 2.) □

It should be warned that the quantity $c(L)$, even for a positive link L, is less tangible than one may expect: not only are (obviously) positive diagrams not always minimal, but some positive links have no minimal positive diagrams at all (see [St4]). Similar caution applies to 'homogeneous' in Proposition 4.6. (Recall the remark in §2.2 that a homogeneous diagram of a positive link is positive.)

Moreover, (4.3) has its roots in Thistlethwaite's work on the Kauffman polynomial, which is very different from the approach in [St12] drawn upon in the proof of Lemma 4.12 and Proposition 4.6. This Kauffman polynomial estimate does somewhat better than the analogue $c(L)$ of the first choice in (4.13) for a positive (or negative) link. (See the proof of Corollary 4.15.) For strongly quasipositive links, it would be interesting to have $c(L)$ serving any geometric purpose whatsoever.

Theorem 4.2 If L is a non-special alternating, resp. special alternating, non-trivial non-split link, then it has a Bennequin, resp. strongly quasipositive/-negative, surface on at most

$$\min\left(\frac{1}{2}c(L)+1, \quad 2b(L)-3, \quad b(L)-\frac{1}{2}-\frac{\chi(L)}{2}\right), \tag{4.4}$$

resp.

$$\min\left(\frac{1}{2}c(L)+1, \quad 2b(L)-2, \quad b(L)-\frac{\chi(L)}{2}\right) \tag{4.5}$$

strands.

For an alternating link, the third estimate may improve the second one in (4.4) and (4.5) (which relates to the Jones polynomial), and with the factor in front of $b(L)$ gone, it looks more natural and closer, certainly for small $-\chi$.

Proof. One could take an alternating diagram D with $c(D)=c(L)$, on the inequality of (4.2).

For the second alternative of the minimum, use [St2, Theorem 6.1]:

$$2b(L)-2 \geq 2\,\mathrm{MFW}\,(L)-2 \geq s(D)\,. \tag{4.6}$$

The minor improvement for non-special alternating links comes from the extra unsharpness property stated in the theorem.

For the third alternative, use (4.41) in Lemma 4.17 on the second alternative.

And for a special alternating link L, the diagram D would be (up to mirroring) positive, thus so will be the band representation. □

Remark 4.3 For a special alternating link L, [St15, Conjecture 7.5.1] predicts an improvement of (3.4),

$$b(L) \leq \frac{c(L)-g(L)}{2}+1 = \frac{c(L)+1}{2}+\frac{n(L)+\chi(L)}{4}\,.$$

This was verified, using (4.16), for a (special) alternating arborescent link[8], in particular for a (special) alternating Montesinos link. Thus we have an improvement of the first alternative in (4.5) in these cases.

We can relate now Theorem 4.2 to Conjecture 2.5, beyond the numerous links (to be) established with (2.25) (as summarized below Question 6.2). Many meaningful classes of links do satisfy lower bounds on the Bennequin defect making them applicable to our method. One can also use the subsequent estimates for some (more crude) statements about homogeneous links, etc. As an example, see Corollary 4.19.

Proposition 4.4 Conjecture 2.5 holds for alternating links L which satisfy one of the following conditions:

[8] in the sense that its arborescent diagram is alternating; we did not consider the links with Menasco's hidden Conway spheres

1) of minimal writhe $w_m(L) \le 7$ (in particular achiral knots),

2) with $\min\deg_l P(L) \le 3$ (in particular negatively special alternating links),

3) *knots* K with $\tau(K) \le 1$ (among others further, slice knots, knots of unknotting number one, etc.), and with $\tau(K) = 2$ when $c(K)$ is odd.

Proof. We assume throughout that D is a reduced alternating diagram of L. Set $g'(L) = (1-\chi(L))/2$.

1) When $w_m(L) \le 7$, then
$$\delta(L) \ge g'(L) - \frac{7}{2} + \frac{b(L)-1}{2} \;= g'(L) + \frac{b(L)}{2} - 4\,. \tag{4.7}$$
By the last alternative of (4.4) in Theorem 4.2, we have when L is not special alternating,
$$b_b(L) - b(L) \;\le\; g'(L) - 1\,. \tag{4.8}$$
When $b(L) \le 4$, then it is known from [St2] that (2.25) holds. (See Remark 4.5.) Setting $b(L) \ge 5$ in (4.7), we have
$$\delta(L) \ge b_b(L) - b(L) - \frac{1}{2}\,,$$
which is enough because of integrality.

There remains the case that L is special alternating. If L is positive special alternating, then from (4.7)
$$0 = \delta(L) \ge g'(L) + \frac{b(L)}{2} - 4\,, \tag{4.9}$$
which leads to a few low-crossing links that can be checked directly for (2.25). Since $b(L) \le 4$ are known from [St2], there remain up to connected sums,

- for $g'(L) = 1/2$ only reverse $(2,n)$-torus links, and
- for $g'(L) = 1$ the pretzel links $L(p,q,r)$ with $p,q,r > 0$ all even or all odd. They can be handled as described below (5.35).
- For $g'(L) = 3/2$, we need $b(L) = 5$ from (4.9). Assuming $s' > b(L)$ in (4.1), we have that
$$\frac{c(D)}{2} + 1 \ge s(D) - \operatorname{ind}(D) \ge 6\,,$$
so that $c(D) \ge 10$. Using $\chi(D) = -2$ and $b(L) = 5$, from (4.6) we have that $c(D) = 2 + s(D) \le 10$, so there remains $c(L) = c(D) = 10$. A few explicit checks (which are extended below) can finish the argument.

If L is negatively special alternating, then from (2.30),
$$\delta(L) \ge 2g'(L)\,.$$
(Compare with (4.44).) Then the claim follows from the modification of (4.8),
$$b_b(L) - b(L) \;\le\; g'(L) - \frac{1}{2}\,.$$
according to (4.5).

2) If $\min\deg_l P(L) \le 2$, the claim follows like point 3), just using (2.12) instead of (2.18) and $g'(L)$ instead of $g(K)$. If L has even number of components, use that $\min\deg_l P(L) \le 1$ by parity. (This would reinstate the constant '$1/2$' in the generalization of (4.12).)

It is worth focusing a little more when $\min\deg_l P(L) = 3$. Point 2) can be proved by an extra (parity-integrality) argument.

We need to prove the first inequality in

$$b_b(L) - b(L) \le g'(L) - \frac{3}{2} \le \delta(L),$$

keeping in mind that $g'(L)$ is a half-integer (and $n(L)$ is even). It is enough to confirm the second inequality in

$$b_b(L) \le \frac{c(L)}{2} + 1 \le b(L) + g'(L) - \frac{3}{2}, \tag{4.10}$$

which is equivalent to $\frac{s(D)}{2} \le b(L) - 2$, or $s(D) \le 2b(L) - 4$.

Assume $c(L)$ is even. Now if L is not positively special alternating, then the strict (4.6) gives

$$s(D) \le 2b(L) - 3, \tag{4.11}$$

and now use that when $n(L)$ and $c(L) = c(D)$ are even, so is $s(D)$. (Negatively special alternating links have $\max\deg_l P(L) = \chi(L) - 1 \le 0$.)

If $c(L) = c(D)$ is odd, then by integrality (4.10) can be changed to

$$b_b(L) \le \frac{c(L)+1}{2} \le b(L) + g'(L) - \frac{3}{2},$$

leading directly to (4.11). This shows point 2) if $\min\deg_l P(L) = 3$ when L is not (positively) special alternating.

Assume finally L is special alternating. Then $\min\deg_l P(L) = 2g'(L) = 3$. We have to compile the series of such (special alternating) diagrams D and ascertain that they are ind-optimal (3.10). This is a (minorly tedious but very) manageable hand check with the tools explained for Proposition 5.11, and the description of MFW$(L) = 3$ (special) alternating links in [St16]. Composite diagrams are handled like in point 1).

3) From (2.31), for $\tau(K) \le 1$ we have $\delta(K) \ge g(K) - 1$. So it is enough to prove by integrality

$$b_b(K) \le b(K) + g(K) - \frac{1}{2}. \tag{4.12}$$

Use the last alternative in (4.5).

For $\tau(K) = 2$ and $c(K)$ odd use the first. We have to see the second inequality in

$$b_b(K) - b(K) \le \frac{c(D)+1}{2} - b(K) \le g(D) - 2 \le \delta(K).$$

This second inequality can be reorganized as

$$b(K) \ge \frac{s(D)}{2} + 2,$$

or $s(D) \le 2b(K) - 4$, which follows for K not special alternating from the strict version of the right inequality in (4.6) using that $s(D)$ is even (since K is a knot and $c(K) = c(D)$ is assumed odd). If K is special alternating, then $\tau(K) = g(K) = 2$, which have been checked in [St3] for (2.25). □

Remark 4.5 a) In the statement of point 1), $w_m(L)$ (is not always equal to but) can be replaced by

$$1/2(\min\deg_l P(L) + \max\deg_l P(L)),$$

even if (2.21) is not exact. We can apply the result in [St2] that MFW$(L) \le 4$ for an alternating link implies (3.10) and thus (2.25). Based on it, either the argument can be modified by using that $b(L)$ can be replaced by MFW(L) in (4.4) and (4.5), or use point 2).

b) Of course point 3) can be formulated in terms of the signature σ, because of (2.17) when K is alternating. But the property (2.18) underlying the proof is lacking for σ in general. We do not know for example if a proof is possible if $\sigma(L) \leq 2$ for an alternating *link L*. On the other hand one can, for knots, replace τ also by the Rasmussen invariant.

Proposition 4.6 If L is a homogeneous non-trivial non-split link, which is not positive or negative, then it has a Bennequin surface on at most

$$\min\left(c(L)-1\,,\ b(L)-1-\chi(L)\right) \tag{4.13}$$

strands.

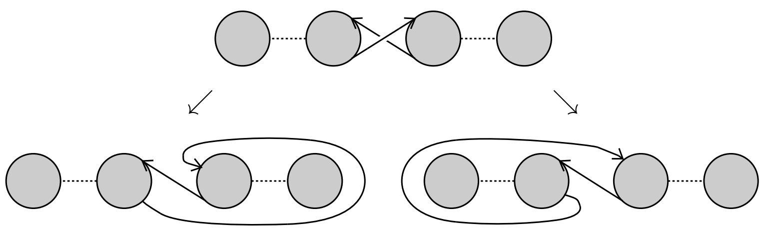
This proof does require a bit of extra work, and thus it will be better to postpone it to §4.4.2, after we familiarized ourselves more with the graph index machinery. The case $\chi = 0$ gives only reverse $(2,n)$-torus links (n even), and they have minimal string Bennequin surfaces by rather obvious arguments (or see for example [Hi]). They are automatically excluded since they are positive or negative.

The rest of this section will mainly deal with the proof of Theorem 4.1. Basically, what we will accomplish is to show that one can apply a maximum of Murasugi-Przytycki moves, while still keeping bands intact, so that the banded diagram of the canonical surface of a diagram can be modified into a braided surface. In particular we will be justified using Ohyama's bound (4.2) then. This requires some detailed technical explanation in the following subsection, where we have to roll out various variants of the graph index. This dust will settle later with Theorem 4.9, but will need to get to that point first.

4.2 Graph index relations

Proof of Theorem 4.1. The following discussion will center around a move (first) found by Murasugi-Przytycki, which serves the reduction of the number of Seifert circles of a link diagram.

Call, in analogy to the terminology from §3.3.3, a crossing *p unstable* if it is the unique crossing connecting its two Seifert circles. Let these Seifert circles be a and b. The Murasugi-Przytycki move then relays one of the crossing strands around Seifert circles adjacent to one of a or b.



(4.14)

The move was rediscovered by several people, e.g., [Ch]. It is important to note (while we will not repeat this detail later) that both the overpass (as in (4.14)) and the underpass can be laid around either of these Seifert circles. For similar reasons as in the case of Yamada's move (see below (3.2)) *we will always choose the overpass*.

Obviously when starting with a link diagram D of $s(D)$ Seifert circles and can successively apply n Murasugi-Przytycki moves, we have

$$b(L) \leq s(D) - n\,. \tag{4.15}$$

Here D is a diagram of a link L with braid index $b(L)$. One is thus interested in the maximal possible n.

Murasugi-Przytycki's definition of graph index bases on a transformation of graphs, which is to model their diagram move on the level of Seifert graphs.

Recall the Seifert graph $\Gamma(D)$ of D from §2.2. When D' arises from D by a Murasugi-Przytycki move, there is a corresponding transformation of $\Gamma(D')$ in terms of $\Gamma(D)$. Then $\operatorname{ind}(D) = \operatorname{ind}(\Gamma(D))$ measures the maximal possible number of such transformations. We will recall the definition of ind in its exact form in §4.3 (or see [St15, §7]), but its role is now important in understanding what follows.

Then we have the inequality

$$b(L) \leq s(D) - \mathrm{ind}\,(D)\,, \tag{4.16}$$

which has some important applications, notably [Oh]. (There is now an improvement of (4.16) in general diagrams; see (4.46).) In relation, Murasugi-Przytycki made the following optimality conjecture for alternating diagrams.

Conjecture 4.7 (Murasugi-Przytycki conjecture) For an alternating diagram D we have equality in (4.16).

Unfortunately, there was a flaw in Murasugi-Przytycki's work: the move Murasugi-Przytycki introduced on $\Gamma(D)$ did not correspond exactly to the effect of their move on D. The problem was treated in [St15, §7] and later a bit more detailedly in [St9]. Roughly, Murasugi-Przytycki's graph move contracts more edges than it should.

Correcting this error results in a new, but far more technical, definition of index, called ind_0, explained in the aforementioned references. (This definition requires bipartacy.) Thus (4.16) must be corrected to

$$b(L) \leq s(D) - \mathrm{ind}_0(D)\,. \tag{4.17}$$

However, it is easy to prove by induction (see Lemma 4.1 in [St9]) that contractions do not augment the index, so that

$$\mathrm{ind}_0(D) \geq \mathrm{ind}\,(D)\,, \tag{4.18}$$

and thus (4.17) implies (4.16). Later Traczyk [Tr] proved the fact (which I suspected based on extensive computations) that actually

$$\mathrm{ind}_0(D) = \mathrm{ind}\,(D)\,, \tag{4.19}$$

so that the Murasugi-Przytycki index, although wrongly defined, gives the right value. Then, indeed, working with ind instead of ind_0 is better, since it is less technical. But one loses the correspondence between graphs and diagrams, and proofs using ind must live in their own (graph-theoretic) setting.

In [St15, §8.2] then the question was tackled how to use Murasugi-Przytycki's move with (braided or Bennequin) surfaces. There a problem arises. With the above convention relaying a strand along a Seifert circle by the move will preserve the structure of bands. But once an unstable crossing is turned into a band, Murasugi-Przytycki moves at this band will destroy it and hence can no longer be allowed.

This then requires the definition of a more restrictive version of index, the *band index* ind_b, with

$$\mathrm{ind}_b(D) \leq \mathrm{ind}_0(D)\,. \tag{4.20}$$

Let the Bennequin braid index be as in (2.24). Then we have

$$b_b(L) \leq s(D) - \mathrm{ind}_b(D)\,. \tag{4.21}$$

Now if we prove

$$s(D) - \mathrm{ind}_b(D) = \mathrm{MFW}\,(L)\,,$$

then the chain of inequalities

$$\mathrm{MFW}\,(L) \leq b(L) \leq b_b(L) \leq s(D) - \mathrm{ind}_b(D)$$

all become exact. (This then also implies the exactness of (2.21), a significant problem on its own, and with (4.19) and (4.20) confirms Murasugi-Przytycki's Conjecture 4.7.) It was proved using such an approach, for example, in [St15, §8.2] that any alternating knot of genus at most 4 has a minimal string Bennequin surface. These extensive tests then also raised the conjectured possibility that always

$$\mathrm{ind}_b(D) = \mathrm{ind}_0(D)\,. \tag{4.22}$$

We will soon return to this equality.

Before we continue, a remark is in place. The Murasugi-Przytycki moves, even in the restricted band form, leave us with what we described above as a banded diagram. To turn it into a braided surface, one has to use Lemma 3.3. Now, the clear concept of a banded diagram and Lemma 3.3 were missing in [St15]. But the proof of Lemma 3.3

shows that more detail is required than what one may originally expect. The work in §3.2 here thus in fact also fills this gap, which is needed to complete the proofs in [St15, §8].

Returning to the treatise of indices, we designed ind_b so that it of course incorporates the fix of Murasugi-Przytycki's error. Unfortunately, this fix carries its corresponding complication. For computer calculations, as in [St15], one would have to pay only with time (although, as was observed, it was not an insignificant factor in comparison between ind_0 and ind), but for a theoretical account, it is useful to seek simplifications.

In this spirit, we will work below with a modified version of ind_b, which we call ind_*, the *false band index*. Essentially, we will take a benefit from the discovery of the error of Murasugi-Przytycki, and intentionally repeat this error, in the context of banded diagrams.

The justification of making this false definition is that again

$$\operatorname{ind}_*(D) \leq \operatorname{ind}_b(D) \tag{4.23}$$

so that (4.21) implies

$$b_b(L) \leq s(D) - \operatorname{ind}_*(D), \tag{4.24}$$

which is what we will use. The proof of (4.23) is analogous to that of (4.18). Again ind_* marks and contracts more edges than ind_b does, and one has to repeat Lemma 4.1 in [St9] in this context. Similarly one proves that when G' is G with an edge marked, then $\operatorname{ind}_*(G') \leq \operatorname{ind}_*(G)$.

With this justification, we – finally – give in the next subsection the definition of the index ind_* we will actually work with. We should also note that, while the Murasugi-Przytycki moves counted by ind_b, and estimated by ind_*, turn crossings into and preserve bands, each removes one crossing (and one Seifert circle) of these banded diagrams, so that the number of positive and negative bands cannot increase in comparison with D. This was part of the statement of Theorem 4.1. To prove (4.1), we will relate it to (4.24).

In an attempt at clarification, we summarize the types of index (of *bipartite* graphs or diagrams) that we explained above (compare [St15, (8.4)], with the same notation):

$$\begin{array}{ccccc} & & \operatorname{ind}_b(D) & \geq & \operatorname{ind}_*(D) \\ & & \mathrel{\rotatebox{90}{$\leq$}} & & \mathrel{\rotatebox{90}{$\leq$}} \\ s(D)-b(L) & \geq & \operatorname{ind}_0(D) & = & \operatorname{ind}(D)\,. \end{array} \tag{4.25}$$

The indices of the first, respectively second, line take, respectively do not take, into account keeping the bands. Thus, these indices can, respectively cannot (yet), be used to estimate Bennequin surface string numbers. The indices of the left, respectively right, side take, respectively do not take, into account the fix of Murasugi-Przytycki's error. The added inequality between $\operatorname{ind}_*(D)$ and $\operatorname{ind}(D)$ follows from the (above discussed) others.

It is possible, like in (4.22), that all inequalities between indices are in fact always equalities. We will prove this below (Theorem 4.9).

This long explanation was inevitable for clarifying the correctness of the following arguments.

4.3 The ordinary and the false band index

The graph terminology we use is fairly standard; see [St15, §7] and [St9]. We write $E(G)$ for the set of edges of G.

We assume for the rest of the paper that *G is bipartite*, which avoids some technical difficulties. In particular G has no loop edges. But it may have multiple (parallel) edges (between the same vertices).

We define a distance function d between edges of G by the following procedure. For every edge e set $d(e,e)=0$. Then if for two edges e,e', the distance $d(e,e')$ is not yet defined, but there is an edge e'' sharing a common vertex with e, such that $d(e'',e)$ is defined, set $d(e',e)=d(e,e')=d(e'',e)+1$. (In this way, for example, distance between parallel edges is 1, although we will not need this special case.)

Similarly, define a distance $d(v,e)$ between a vertex and an edge, by saying $d(v,e) = 0$ if e is incident from v, and if $d(v,e)$ is not yet defined, but there is a vertex v' adjacent to v with $d(v',e)$ defined, set $d(v,e) = d(v',e)+1$. For a vertex v, set

$$\operatorname{star} v = \{ e \in E(G) : d(v,e) = 0 \}.$$

In the Murasugi-Przytycki operation, we choose a *simple* edge e between vertices v and v' and one of the two vertices v of e. (Note that a simple edge in $\Gamma(D)$ corresponds to an unstable crossing in D, as defined above.) Then the transformation they use is

$$G' = G/v := G/\operatorname{star} v. \tag{4.26}$$

Note that this decreases the number of vertices (to correspond to Seifert circles) by more than one. But in fact all vertices adjacent to v different from v' become stumps, and we can tacitly remove them.

Now we use graphs in which we mark edges. Each edge has a $\mathbb{Z}_2$-grading: it is either marked or not. The *marking* of an edge is its marking status. In below diagrams we indicate marked edges by an asterisk. In the initial graph $G = \Gamma(D)$, all edges are unmarked. A marked edge is to be understood as one that cannot be chosen as an edge e. It corresponds to a multiple edge. I point out again that the marking of edges is thus completely unrelated to the $+/-$ marking of the sign of crossings/bands. What happens to the signs was explained above, so these signs will be henceforth completely ignored.

We describe now our version of the Murasugi-Przytycki graph operation, which we call $*$-*operation*, and we write analogously to (4.26)

$$G' = G/_{*}\, e. \tag{4.27}$$

We choose a *non-marked simple* edge e and a vertex v of e. While (4.26) really depends only on the vertex v (as long as there is a simple edge incident from it), now our operation will depend on the edge. Also, it will of course depend as well on which v of the two vertices of e is chosen. Instead of carrying the pair (e,v) along, we will accommodate for it by saying that *e is a directed edge* (towards v). Let v' again be the other vertex of e. Since we need this designation a few times, let us write

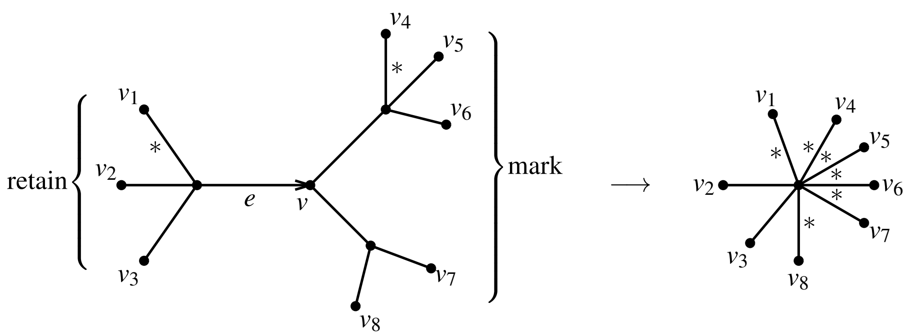
$$v' = o(e,v)$$

for the vertex of e which is not (i.e., which is opposite to) v.

In (4.27), as a graph, $G' = G/\operatorname{star} v$, but we need to explain the markings on G'. Let $e' \in E(G')$. Obviously there is an edge $e' \in E(G) \setminus \operatorname{star} v$ corresponding to it, written with the same letter, and the below cases are given assuming $d(v,e') > 0$ in G.

Case 1. If $d(v,e') \geq 2$, or $d(v,e') = 1$ and $d(v',e') = 0$, then in G', the edge e' retains its marking.

Case 2. If $d(v,e') = 1$ and $d(v',e') > 0$, then e' is marked in G'.



Note that both cases are disjoint, because G has no cycles of length 3.

We say that a sequence of directed edges $e_1,\dots,e_n$ (towards their vertices $v_1,\dots,v_n$) in G is $*$-*independent* if for $G = G_0$, the $*$-operations $G_i = G_{i-1}/_{*}\, e_i$ are all admissible; in particular in G,

$$e_i \notin \bigcup_{j=1}^{i-1} \operatorname{star} v_j\,,$$

and when G_{i-1} is built, the edge e_i therein has not received a marking.

Obviously in a $*$-independent sequence orientation and order of edges matters, but no edge (with whatever orientation) is included twice. We say that a *set* of edges $\{e_1,\dots,e_n\}$ in G is $*$-*independent* if it admits an ordering and each edge admits an orientation so that $e_1,\dots,e_n$ is a $*$-independent sequence. Then we define $\mathrm{ind}_*(G)$ to be the maximal size $|E|$ of a $*$-independent set E of G, and for a link diagram again set $\mathrm{ind}_*(D)=\mathrm{ind}_*(\Gamma(D))$.

With our definition of $\mathrm{ind}_*(D)$, we explained why (4.24) and (4.25) holds. We should also emphasize from what we said about (4.23) that a $*$-independent set is then admissible in terms of sets whose maximal size is counted by $\mathrm{ind}_b(D)$. This means that a $*$-independent set indeed gives a set of crossings of D at which successively Murasugi-Przytycki moves can be applied, in the proper order and way, so as to preserve bands.

The original Murasugi-Przytycki operation has only the difference that markings are ignored, and leads to the definition of *independent set/sequence* of edges and of the index $\mathrm{ind}(D)$. Again, the property to be independent assumes the tacit designation of a vertex for each edge. We will need to recall Traczyk's alternative characterization of an independent set of edges which, though, turns out not to depend on the choice of these vertices.

Theorem 4.8 (Traczyk [Tr2]) A set E of edges of G is independent if and only if each cycle C of length $2n$ of G contains $|C\cap E|\le n-1$ edges in E.

We remind that G is a bipartite graph, so all cycles are even. Also, we can and do regard two copies of a multiple edge as a cycle of length 2. Traczyk's condition for $n=1$ then just says that all edges in E are simple.

The rest of the work here is devoted to the following fact.

Theorem 4.9 For every bipartite planar graph G, we have $\mathrm{ind}_*(G)=\mathrm{ind}(G)$. So all inequalities between indices in (4.25) are in fact always equalities.

Combining this with (4.24) gives (4.1) and completes the argument for Theorem 4.1. □

Proof of Theorem 4.9. Fix a graph G. We will prove, by contraposition, that every independent *set* E of edges in G is $*$-independent; the converse is trivial. (The statement that every independent *sequence* of edges is $*$-independent is easily seen to be false.) This is essentially accomplished by the following lemma.

Lemma 4.10 Let E in G be independent. Then there is an edge $e\in E$ and a vertex v of e, so that no edge e' with

$$d(v,e')\le 1 \text{ and } d(o(e,v),e')>0 \tag{4.28}$$

is contained in E.

Another way of saying the specified condition (4.28) on e' is that $e'\ne e$ and e' is either contracted (if $d(v,e')=0$) or marked (if $d(v,e')=1$) during the $*$-operation turning G into $G/_*e$.

Assume this lemma is proved. Notice that, while the markings are relevant in choosing what is a $*$-independent sequence of edges, once a sequence is chosen, its property to be $*$-independent does not depend on the markings of the edges outside this sequence. Thus the same holds for a $*$-independent set.

Now work by induction on $|E|$. By applying the lemma, consider $G'=G/_*e$. The edge set $E\setminus\{e\}$ survives in G', and by Traczyk's condition is independent there. Thus it is $*$-independent by induction assumption. Now, by the lemma an edge in $E\setminus\{e\}$ was not contracted, and did not receive a marking $G'=G/_*e$. Then one can first perform a $*$-operation on (e,v) in G, and then on the edges $E\setminus\{e\}$ as from their $*$-independence in G'. This gives an ordering as a $*$-independent sequence of the edges in E, and thus proves that E is $*$-independent in G as well. □

It remains now to prove the lemma.

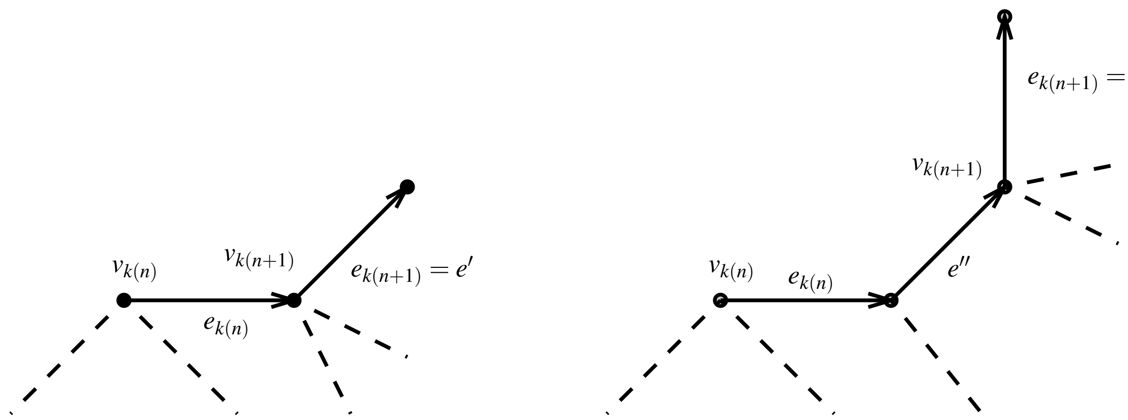
Proof of Lemma 4.10. We will assume that for every edge $e\in E$ and vertex v of e, there is an $e'\in E$ with (4.28). We will use this property to prove that in fact E is not independent, giving a contradiction.

Now we construct for every $n\in\mathbb{N}$ a sequence $e_1,\dots,e_{k(n)}$ of directed edges in G with the following properties:

- $k(1) = 1$ and
$$k(n+1) - k(n) \in \{1,2\}, \tag{4.29}$$
- $e_{k(n)} \in E$ and $e_{k(n)} \neq e_{k(n-1)}$,
- For $i > 1$, the edges e_i and e_{i-1} share a common vertex v_i, and e_{i-1} is oriented towards v_i, while e_i is oriented away from v_i, towards $o(e_i, v_i)$.

We proceed by induction on n. For $n = 1$ take $k(1) = 1$ and any edge $e_1 \in E$ with any orientation, towards a vertex $v = v_2$. Set $v_1 = o(e_1, v)$. (Clearly $E = \varnothing$ is no problem.)

Now assume $e_1, \dots, e_{k(n)}$ is constructed. Since $e_{k(n)} \in E$, by assumption, some edge e' in $E \setminus \{e_{k(n)}\}$ was, in creating $G/_* e_{k(n)}$, either contracted or marked.



Case 1 Case 2

Case 1. e' is contracted. This means $e' \in \operatorname{star} o(e_{k(n)}, v_{k(n)})$. Set $k(n+1) = k(n) + 1$, $v_{k(n)+1} = o(e_{k(n)}, v_{k(n)})$, and $e_{k(n+1)} = e'$, and direct e' away from $v_{k(n)+1}$, towards $o(e', v_{k(n)+1})$.

Case 2. e' is marked. This means that there is an edge
$$e'' \in \operatorname{star} o(e_{k(n)}, v_{k(n)}) \setminus \{e_{k(n)}\}$$
with $d(e'', e') = 1$. Then set

- $k(n+1) = k(n) + 2$,
- $v_{k(n)+1} = o(e_{k(n)}, v_{k(n)})$,
- $e_{k(n)+1} = e''$,
- $v_{k(n)+2} = o(e'', v_{k(n)+1})$,
- $e_{k(n)+2} = e'$,

and direct e_i away from v_i towards $o(e_i, v_i)$ for $i \in \{k(n)+1, k(n)+2\}$.

This completes the construction of the sequence (e_i). Now this sequence is by assumption extendable in(de)finitely, but E has only finitely many edges.

Thus $e_{k(m)} = e_{k(n)}$ for some $m \neq n$. This means that $C = \{e_{k(m)+1}, \dots, e_{k(n)}\}$ forms a cycle in G. But because of (4.29), at least half of the edges in C belong to E, violating Traczyk's condition of Theorem 4.8. Thus E is not independent, giving the desired contradiction. $\square$

Remark 4.11 It is useful to notice that the canonical surface of D in Theorem 4.1 can itself be deformed into a braided surface. See Remark 3.10 for the Y move. The Murasugi-Przytycki move's affect on surfaces will be discussed in more detail in [IS2]. To illustrate the argument, I reproduce here (with my coauthor's consent) in Figure 1 a figure from that paper (although there the mirrored convention for the bands is used).

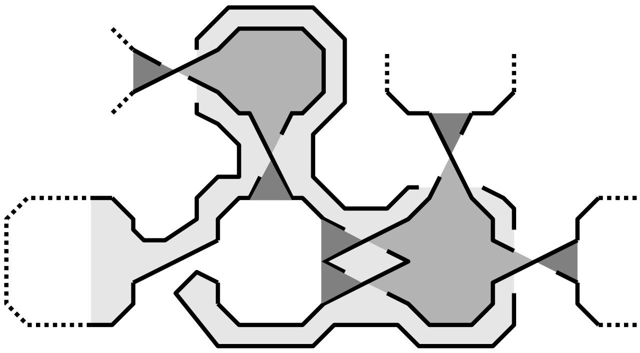


Figure 1: Murasugi-Przytycki's move viewed as an isotopy of the canonical Seifert surface; the Seifert surface inherits a disk-and-twisted-band decomposition.

4.4 Applications and relations

4.4.1 The braid index of a homogeneous link

As a preparation for proving Proposition 4.6, we need to discuss some estimates on the braid index of a homogeneous link. Due to the general difficulties to give good lower bounds on the braid index of links, this discussion has a merit in its own right, completely apart from Bennequin surfaces. Starting with some diagram D of a link L, obviously some regularity is needed to have a chance at a meaningful estimate. We will assume D is homogeneous (and reduced). We will refer to the bound

$$b(L) \geq \frac{c(D)+1}{2} + 2\chi(D)\,, \tag{4.30}$$

which was stated in [St12, Theorem 1.1].

For $\sim$*-equivalence*, $\bar{t}'_2$*-moves*, *generators* and *series*, see [St15] or [St12, §3], for example. We will write $t_1(D)$ for the number of odd $\sim$-equivalence classes of D.

We will often need below the maximal m-term of P in Morton's sense. We restrict ourselves to diagrams, and using brackets to denote a coefficient, we define

$$P_{\max}(D) = [P(D)]_{m^{1-\chi(D)}}\,. \tag{4.31}$$

This is a Laurent polynomial in l. One should pay attention to let $P_{\max}$ depend on D, since we must keep track of $\chi(D)$. An important result of Murasugi-Przytycki [MP] is that if a diagram $D = D_1 * D_2$ decomposes along a separating Seifert circle as a *diagrammatic* Murasugi sum of diagrams D_1 and D_2, then

$$P_{\max}(D_1 * D_2) = P_{\max}(D_1) \cdot P_{\max}(D_2)\,. \tag{4.32}$$

At first, we improve [St12, Theorem 5.6].

Lemma 4.12 Let D be prime reduced homogeneous diagram of a link L. Then with (2.6)

$$\mathrm{span}_l P_{\max}(D) \geq s(D) - 1 - a'(D) \tag{4.33}$$

and

$$\mathrm{span}_l P(D) \geq s(D) + 1 - a'(D)\,. \tag{4.34}$$

In particular, because of (2.21),

$$b(L) \geq \mathrm{MFW}(L) \geq \frac{s(D)+3}{2} - \frac{a'(D)}{2}\,. \tag{4.35}$$

Also, (4.34) (and hence the right inequality of (4.35)) is exact only if D is positive or negative.

By using (2.4) in (4.35), it is very easy to see that (4.30) holds. Thus the outcome is non-trivial, even although the recursive proof of (4.36) may well go over estimates which are negative.

Also, compare (4.34) with (4.6). They are very similar, and (4.34) is slightly weaker, but if for a special alternating prime diagram we replace $a'(D)$ with $a(D) = 1$ in (4.34), we obtain exactly (4.6), as far as $\mathrm{MFW}(L)$ is concerned. Nevertheless, (4.6) emerges in [St2] from a totally different perspective. This also accords with the fact that the replacement a' to a is not possible in (4.33), as shows Example 4.13.

Proof. When substituting the minimum alternatives of (2.6) into the right of (4.33), the one result is exactly twice the other. We will stick for the double, because the single is only better when it is negative, but then the inequality is trivially true. So we have to show

$$\mathrm{span}_l P_{\max}(D) \geq 2s(D) - c(D) - 2\,. \tag{4.36}$$

We work by induction on $c(D)$. If D is a Hopf link diagram, then the claim is true. Otherwise, we use the skein polynomial.

Because of (4.32), for (4.33) it is enough to assume D is prime. It is enough also to assume D special and has no nugatory crossings.

It is better, for this proof, to switch to *Morton's variables* of the skein polynomial $P(v,z)$, used in [Cr], where (2.10) becomes

$$v^{-1}P(D_+) - vP(D_-) = zP(D_0)\,.$$

During the skein calculation it is important to avoid that crossings become nugatory. Whenever we choose a crossing p in D, we let D' be D with p switched, and D_0 be D with p smoothed out. It should be kept in mind that $\min\deg_v P_{\max}(D) = 1 - \chi(D)$ when D is positive.

If there are two crossings connecting the same two Seifert circles, by a flype, they can be made to form a parallel clasp (as D is special). Let p be one of them. If D_0 has a nugatory crossing, then D has a Hopf link factor, and we could proceed with D by induction. Thus D_0 has no nugatory crossing, and then $P_{\max}(D) = v^2 P_{\max}(D_0)$, and $s(D) = s(D_0)$, so the result follows by induction.

So now assume in D there are no two crossings connecting the same two Seifert circles, i.e., the Seifert graph $\Gamma(D)$ is simple. If $\Gamma(D)$ has a valence-2 vertex, then D has a reverse clasp. Then we can proceed as in the proof of [St12, Theorem 5.6]. Note that in a special diagram D, no reverse clasp is bad, unless D has a Hopf link factor. Then D' depicts a positive link L' with $\chi(L') = \chi(D')$, so $P_{\max}(D')$ will enter with a factor v^2 in $P_{\max}(D)$. Also L' has a special reduced diagram with $c(D) - 2$ crossings and $s(D) - 2$ Seifert circles. The claim the follows by induction. (Note that L' may be composite despite that D is prime.)

There remains the case that no vertex of the Seifert graph has valence 2. By a counting argument, there must be one of valence 3. Choose p to be one of the crossings adjacent to this Seifert circle. Under exclusion of reverse clasps in D, we have that D_0 has no nugatory crossings.

By [Cr, Corollary 2.2] an almost positive diagram D' has a positive based diagram resolution (as in [Cr, Corollary 2.1]). Then, by a remark of Traczyk [Cr, Corollary 4.3], $P_{\max}(D')(z)$ is a positive polynomial in z. We must in general allow this polynomial to be zero, but this situation creates no harm. In fact in our case the polynomial is not zero; see the remark on the case $n = 1$ below Corollary 4.44. But in any case its contribution to $P_{\max}(D)$ is not cancelled in any v-degree with the one of $P_{\max}(D_0)$, and hence it is enough study $P_{\max}(D_0)$ only.

But the Seifert graph of D_0 has a valence 2 Seifert circle, and the claim follows by induction. This establishes (4.36), and hence (4.33).

For (4.34) (and (4.35)) it remains to see why

$$\operatorname{span}_l P(D) \geq \operatorname{span}_l P_{\max}(D) + 2\,, \tag{4.37}$$

i.e., $\operatorname{span}_l P(D) > \operatorname{span}_l P_{\max}(D)$. Assume that both were equal. Then under the trivializing substitution from [LMi, Proposition 21], the contribution of both the leading and trailing l-term of $P_{\max}(D)$ will not be cancelled, and at least one will contribute in non-zero l-degree, unless $1 - \chi(D) = 0$. Also, both will contribute in non-zero l-degree, unless $\pm \min\deg_l P(D) = 1 - \chi(D)$, which is the case of positive or negative D (see [Cr, Theorem 4]). Thus if D is neither positive nor negative, then (4.37) is strict. $\square$

Example 4.13 The estimate (4.33) does not hold when $a'(D)$ is replaced by $a(D)$. An example is the alternating diagram of 12_{1097} (known from [St3]). We understood this blunder after noting, in the valence-2 Seifert circle argument above, that L' may be composite despite that D is prime.

On the positive side, the recursion in the proof is sufficiently explicit to allow, or a least suggest, more successful modifications. For example, term $t_1(D)/3$ may be added on the right of (4.36). Unfortunately, I was not able to further profit from this improvement.

Another quantity that may be put through the induction for special alternating diagrams D is

$$\operatorname{span}_l P_{\max}(D) \geq c(D) - t_1(D) + 2 - 2n(D)\,, \tag{4.38}$$

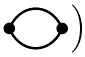
for the number of components $n(D)$ of D (and L). This works well except that in the parallel clasp scenario, one must look at D_p, not D_0, and except for the $*$-product (which is not a connected sum, thus restricting to special alternating diagrams). It then gives, using (4.37),

$$b(L) \geq \mathrm{MFW}\,(L) \geq \frac{c(D) - t_1(D)}{2} + 3 - n(D)\,. \tag{4.39}$$

However, it is interesting to find that in fact (4.39) (but, of course, not (4.38)) is subsumed by the partial case of (4.6) for a special alternating link. This is a matter of some exercise, which is too long to discuss here, and is left to an interested reader.

In some attempt to more formalize and generalize the proof of Lemma 4.12 to facilitate the search for other estimates, we include the following formulation in terms of the Seifert graph $\Gamma(D)$ (defined below (4.15)). Denote by $*$ the *(cut-)vertex join* of graphs, so that $\Gamma(D_1 * D_2) = \Gamma(D_1) * \Gamma(D_2)$.

Proposition 4.14 Let $\mathcal{G}$ be the class of planar bipartite connected graphs with no valence-1 vertices. Let $f : \mathcal{G} \to \mathbb{Z}$ be a function with the following properties.

- $f\Big($  $\Big) \leq 0$
- $f(G_1 * G_2) \leq f(G_1) + f(G_2)$.
- If G has no cut vertex, and e is an edge of G, then $f(G \setminus e) \geq f(G)$. (It is enough that e has a parallel edge or e is incident from a valence-3 vertex.)
- If G has no cut vertex, and v is a vertex of valence 2, then $f(G/\operatorname{star} v) \geq f(G) - 2$.

Then for a homogeneous diagram D of a link L,

$$b(L) \geq \frac{1}{2} f(\Gamma(D)) + 2\,.$$

And '+2' can be replaced by '+3' if L (or D) is not positive or negative. $\square$

As a consequence, we can also improve upon [St12, Corollary 5.14].

Corollary 4.15 Let L be a non-trivial homogeneous link, and not positive or negative, and D a reduced homogeneous diagram of L. Then
$$c(D) \le 2c(L) - 4\,. \tag{4.40}$$

Proof. There is a well-known inequality, which is mostly the combination of (2.11) and (2.21),
$$c(L) \ge \max\deg_m P(L) + \frac{1}{2}\,\mathrm{span}_l P(L)\,,$$
and using D with (4.36) and the unsharpness of (4.37) gives
$$c(L) \ge 1 - \chi(D) + \frac{2s(D) - c(D) + 2}{2}\,.$$
This is equivalent to (4.40). (Obviously also $c(D) \le 2c(L) - 2$ for L positive or negative, but this is subsumed by (4.3).) □

Another easy consequence of the proof of Lemma 4.12 is useful for Conjecture 2.5.

Corollary 4.16 Let L be a non-trivial homogeneous link, and D a non-positive homogeneous reduced diagram of L. Then $\delta(L) \ge c_-(D)/2 + 1$.

Proof. Let D_1 be a positive Murasugi atom of D and D_2 a negative one. Then
$$\min\deg_l P_{\max}(D_1) \,=\, c(D_1) - s(D_1) + 1$$
and using (4.36) and mirroring
$$\min\deg_l P_{\max}(D_2) \,\le\, 1 - s(D_2)\,.$$
By iterating (4.32), we obtain
$$\min\deg_l P_{\max}(D) \,\le\, c_+(D) + 1 - s(D)\,,$$
and since D is not positive,
$$\min\deg_l P(D) \le \min\deg_l P_{\max}(D) - 2 \,\le\, c_+(D) - 1 - s(D)\,,$$
whence the claim follows easily from (2.12). □

4.4.2 Another graph index estimate

Lemma 4.17 Let D be a connected diagram. Then
$$s(D) - \mathrm{ind}(D) \le \frac{s(D)}{2} - \frac{\chi(D)}{2} + 1\,. \tag{4.41}$$

Proof. Rewrite Ohyama's inequality (4.2) by
$$\frac{c(D)}{2} = \frac{s(D)}{2} - \frac{\chi(D)}{2}\,.$$
□

Proof of Proposition 4.6. Use (4.40) for a homogeneous diagram D. This, together with (4.2), gives the first alternative in the minimum (4.13). For the second one, use (4.34) and (2.4), and the argument that for non-positive, and non-negative D, (4.37) is strict, giving
$$2(b(L) - 1) \ge \mathrm{span}_l P(D) \ge s(D) + 3 - a'(D) \ge s(D) + \chi(D) + 2\,, \tag{4.42}$$
so
$$s(D) \le 2b(L) - 4 - \chi(L)\,. \tag{4.43}$$
Using (4.43) in (4.41) gives the assertion. □

Corollary 4.18 The defect of the Morton-Franks-Williams inequality (2.21) is linearly bounded in the Euler characteristic: we have

$$b(L) - \mathrm{MFW}(L) \leq -\mu\chi(L),$$

with $\mu = 1$ for L homogeneous and $\mu = 1/2$ for L alternating.

Proof. Observe that in all estimates involving $b(L)$, in particular on the left of (4.42), it can actually be replaced by $\mathrm{MFW}(L)$ from (2.20). The same applies to the proof of Theorem 4.2; note that (4.6) is in fact stated in this form. □

The existence of (even alternating) examples with strict (2.21) together with iterated connected sums shows that a better than linear bound is not possible. Also, there is no estimate in general: a result of Birman-Menasco [BM2] implies that for the *Kanenobu series* [K] $b(L) - \mathrm{MFW}(L) \to \infty$ while $\chi(L)$ is bounded.

As a sequel to Proposition 4.4 and to emphasize that neither knots nor alternation are needed, we have the following special case of the Ito-Kawamuro conjecture.

Corollary 4.19 Conjecture 2.5 holds for any homogeneous link L

1) with $\min\deg_l P(L) \leq \chi(L) + 3$, in particular for a negative link L,

2) with a reduced homogeneous diagram D which is not positive, and has $c_+(D) \leq 2b(L)$ or $c_-(D) \geq -4 - 2\chi(L)$.

Proof.

1) Let L not be negative. Thus from (2.30),

$$\delta(L) \geq \frac{(1-\chi(L)) - (3+\chi(L))}{2} = -1 - \chi(L).$$

Now use (4.13).

Let L be negative. Then $\min\deg_l P(L) \leq \max\deg_l P(L) = \chi(L) - 1$. Thus from (2.30),

$$\delta(L) \geq 1 - \chi(L). \tag{4.44}$$

Then the right alternative in (1.2) will apply.

2) It follows rather easily from Corollary 4.16,

$$b_b(L) \leq \frac{c(D)}{2} + 1,$$

and the second alternative in (4.13). (Note that the conclusion also holds when D is positive and $c_+(D) = c(D) \leq 2b(L) - 2$, i.e., (4.2) and (4.16) are exact.) □

The following are some easy consequences not of Proposition 4.6, but of Theorem 4.1 directly.

Corollary 4.20 If D is a diagram for which (4.16) is exact and (2.3) holds (in particular an alternating diagram for which Conjecture 4.7 is true), then its link L has a minimal string Bennequin surface. □

This happens among others in these situations.

- Theorem 3.17 discussed above covers precisely the special case of $\mathrm{ind}(D) = 0$ in this conjecture; thus Proposition 3.18.
- By [St12, Lemma 5.19] ([St15, Corollary 7.4.5]) it follows for every alternating link in the *series* of a special stably alternating generator. This is enough to cover, as an application, all alternating pure pretzel links (2.42); see Proposition 5.11.
- It can be also verified with not too much effort that an alternating diagram D of a 2-bridge link L is ind-optimal (see (3.10)), leading to another derivation of the result in [Hi].

4.4.3 A digression on the Jones-Kawamuro conjecture

A slight (and not very practical, but at least worth noticing here) improvement of (4.16) follows from the proof of the (strong version of the) Jones-Kawamuro conjecture (Theorem 2.6). Another way of writing the theorem is this:

Proposition 4.21 If L admits (s_i,w_i)-representatives (or -diagrams), $i=1,2$, then L also admits a

$$\left(\frac{-w_1+w_2+s_1+s_2}{2},\ \frac{w_1+w_2+s_2-s_1}{2}\right)-\text{representative}. \tag{4.45}$$

Proof. By Theorem 2.6, L admits (s_1-l,w_1+l)-representatives for every $l\in\mathbb{Z}$ and (s_2-m,w_2-m)-representatives for every $m\in\mathbb{Z}$ such that $s_2-m=s_1-l$ and $w_2-m\ge w_1+l$. The statement is obtained by setting

$$(s_2-m,w_2-m)=(s_1-l,w_1+l)\,.$$

(Note that if the number $s_2-m=s_1-l$ greater than $b(L)$, then the weak version of the conjecture is contradicted.) □

Define the *superindex* of an oriented diagram D as

$$\text{s-ind}\,(D)\;=\;\text{ind}_+(D)+\text{ind}_-(D)\,,$$

where signed indices count maximal independent sets of positive resp. negative crossings/edges only. It is obvious that

$$\text{s-ind}\,(D)\ge\text{ind}\,(D)\,,$$

and equality holds if D is homogeneous (because of the observation below Corollary 4.44). Also, unlike its predecessor, the superindex does depend on how edges in $\Gamma(D)$ are signed, i.e., how crossings in D are switched.

Then (4.16) generally improves as follows. (The first part (4.46) was independently observed by Ito in [It, Theorem 5].)

Proposition 4.22

$$b(L)\le s(D)-\text{s-ind}\,(D)\,. \tag{4.46}$$

If L is strongly quasipositive and (2.3) holds, then we would have even

$$b(L)\le s(D)-\text{ind}_+(D)-c_-(D)\,. \tag{4.47}$$

Proof. Use proposition 4.21 with

$$\begin{array}{ll} s_2=s(D)-\text{ind}_+(D)\,, & s_1=s(D)-\text{ind}_-(D)\,,\\ w_2=w(D)-\text{ind}_+(D)\,, & w_1=s(D)+\text{ind}_-(D)\,. \end{array} \tag{4.48}$$

The first component of the pair (4.45) gives (4.46).

The improvement (4.47) follows from (2.34) applied on some diagram D' with $w(D')=w(D)-\text{ind}_+(D)$ and $s(D')=s(D)-\text{ind}_+(D)$, and using $\chi(L)=s(D)-c(D)$. Alternatively, modify s_1,w_1 in (4.48) by (leaving s_1 arbitrary and) setting $w_1=s_1+c(D)-s(D)$. □

This can be used in some instances (see Example 5.4) for sharp upper braid index estimates without actually obtaining braid representatives. But the below example shows limitations of this method beyond its non-constructiveness. They seem more common than I expected, and thus we shift attention away from this approach.

Example 4.23 The knot $L=9_{49}$ is the first one where neither (4.16) nor (4.46) becomes exact on a minimal crossing diagram D. Let us write

$$\iota(D)=s(D)-\text{ind}\,(D)-b(L)$$

for the 'defect'. There are two minimal crossing diagrams of 9_{49}, which are positive, so $\text{ind}\,(D)=\text{s-ind}\,(D)$ and also (4.47) does the same as (4.46). For both diagrams $\text{ind}\,(D)=1$ and $s(D)=6$, but $b(L)=4$, so $\iota(D)=1$. The mutants 12_{1376} and 12_{1421} have only positive minimal diagrams, all with $\iota(D)=2$.

4.4.4 Pseudo-positive and strongly quasipositive links

Let us call a set of crossings in a diagram D independent, if their corresponding edges in $\Gamma(D)$ are independent.

Corollary 4.24 (and definition) Let L have a diagram D in which all negative crossings are independent. We call D and L *pseudo-positive*. Then L is strongly quasipositive on (at most) $b_q(L) \leq s(D) - c_-(D)$ strands, and (2.3) holds. □

Example 4.25 An almost positive diagram is pseudo-positive (exactly) when it is what is called in [FLL] of type I. The corollary thus recovers the result in their title in this case.

In [IS2] we will give a different construction that covers (and extends) the main result of [FLL] for type II, but still relies on the work here. In combination, we have the following, which refines Feller-Lewark-Lobb's answer to [St8, Question 3].

Theorem 4.26 ([IS2]) Let a link L have an almost positive diagram D of s Seifert circles. Then L has a positive band representation on (at most)

$$b_q(L) \leq \left\{ \begin{array}{cl} s-1 & \text{if } D \text{ is of type I} \\ s & \text{if } D \text{ is of type II} \end{array} \right\}$$

strands.

Example 4.27 A strongly quasipositive link need not be pseudo-positive, and the converse of Corollary 4.24 is not true. A few examples are listed below:

- The counterexamples to Morton's conjecture

$$\min \deg_l P \leq 2g \tag{4.49}$$

  in [St4, §5.1] (see below (5.4)) are strongly quasipositive, and using (2.14) we see that they satisfy

$$\max \deg_m P(L) > 1 - \chi(L)\,. \tag{4.50}$$

  Thus because of (2.11) they have no diagram D with (2.3).

- Two of the examples where (2.3) was disproved using the method in [St3] for genus-2 knots are strongly quasipositive. (See §B.2.)
- Another example without such D is a Whitehead double of the trefoil, given in [Be]. (This example should be combined with the explanation in [Mo], or see the main result in [St10].) We will generalize this example in work in progress with G.-T. Jin as announced in Theorem 3.16.
- Example 3.13 provides another instance.

Remark 4.28 Note that in Corollary 4.24, $c(D) - l$ can be replaced by $c(D) - l'$, where l' is the maximal size of an independent set of edges of $\Gamma(D)$ containing all negative ones. This is not (further) useful for exhibiting L as strongly quasipositive, but can be useful for string estimates of a strongly quasipositive surface of L. See for this Corollary 4.35.

The following result was originally proved by Przytycki-Taniyama for positive knots.

Corollary 4.29 A pseudo-positive unknotting number $u(K) = 1$ knot K is a positive twist knot.

Proof. Since K is strongly quasipositive, we need $0 < g(K) = g_4(K) \le u(K)$, so that $g(K) = 1$. But we also have bt Corollary 4.24 that the canonical genus $g_c(K) = 1$. Now $u(K) = g(K) = 1$ implies that K is a doubled knot [Ko], and $g_c(K) = 1$ that the companion must be trivial [St10], so that K is a twist knot. The twist knots are alternating, so to be strongly quasipositive, they need to be positive. □

Corollary 4.24 provides a rapid test for strong quasipositivity. There has been some interest [IK, FLL] in determining which low-crossing knots are strongly quasipositive. Since, as mentioned in [FLL], many knots are canonically minimal, it templed us to extend the compilation of lists of strongly quasipositive knots. For this see Appendix B.

Under Theorem 4.8, Corollary 4.24 turns out to be one direction of [FLL, Theorem B], except for the explicit realization of the strongly quasipositive Seifert surface, and the string number estimate. The authors of [FLL] prove a converse; in particular they know that ‘pseudo-positive’ is equivalent to ‘strongly quasipositive and canonically minimal’ for knots, or if Question 2.2 has an affirmative answer.

The below Lemma 4.30 will extend these conclusions and remove the need of their restrictions (Corollary 4.32), by allowing us to look at the link, not at its surface. This argument has then a few other new ramifications.

In the setting of [FLL] (but with our notation), let C be a cycle in $\Gamma(D)$. We emphasize here that we will continuously regard two *parallel edges*, such between the same two vertices in $\Gamma(D)$, to form a cycle of length 2. Since $\Gamma(D)$ is bipartite, cycles have always even length. We write λ_C for the *sum of the sign of the edges* in C (this is en even number). We call this the *weight* of C. Then the condition of the negative crossings in D being independent is equivalent to

$$\lambda_C > 0\,, \tag{4.51}$$

for every cycle C. We will occasionally refer to (4.51) as *Traczyk's cycle condition*.

It should be noted that (4.51) cannot be easily restricted to check on a subset of the cycles C. For example, one can easily find diagrams D where (in some rather obvious planar embedding) all face cycles C of $\Gamma(D)$ satisfy (4.51), but other cycles do not.

We say (following [IS], in particular Section 2) that S_D is the canonical Seifert surface of D and that D is strongly quasipositive, if S_D is so. We say that D is χ_4-*maximal* if $\chi_4(D) = \chi(D)$, and $\chi_4(D) = \chi_4(L)$ when D represents the link L. We remark that the argument behind Corollary 4.24 can be easily modified to show that D is strongly quasipositive, as explained in Remark 4.11. For Bennequin-sharpness recall below Question 2.3.

Lemma 4.30 Assume L is a Bennequin-sharp link and D is a minimal genus diagram of L, i.e., $\chi(D) = \chi(L)$. Then all the negative crossings in D are independent.

Proof. We assume $\chi(D) = \chi(L)$ and L is strongly quasipositive, so $\chi_4(D) := \chi_4(L) = \chi(L) = \chi(D)$. Thus D is χ_4-maximal. We assume further (by Traczyk's condition) that there is a cycle C in $\Gamma(D)$ whose edge sign sum λ_C is $\lambda_C \le 0$. We will derive a contradiction.

Let first $\lambda_C = 0$. When D_0 is obtained from D by smoothing one crossing (positive or negative), then by a standard fact $\chi_4(D_0) \le \chi_4(D) + 1$. It follows that when D is a minimal genus diagram of a Bennequin-sharp link, and D' is obtained from D by smoothing out some crossings, then D' is χ_4-maximal. But one can smooth out crossings in D (and remove nugatory crossings) so that $\Gamma(D') = C$. Then $S_{D'}$ is a(n untwisted) annulus bounding the 2-component unlink. So D' is not χ_4-maximal, a contradiction.

To adapt the case $\lambda_C < 0$, use connected sum. Consider the connected sum $L' = L\#T_{2,-\lambda_C}$ for the reverse $(2,-\lambda_C)$-torus link (which is Bennequin-sharp). For the usual minimal crossing diagram $D_{2,-\lambda_C}$ of $T_{2,-\lambda_C}$, the diagram $D' = D\#D_{2,-\lambda_C}$ is a minimal genus diagram of a Bennequin-sharp link L'. By modifying the procedure for $\lambda_C = 0$, we can obtain from D' by crossing smoothings a diagram $D'_2 = D'_1\#D_{2,-\lambda_C}$, where $\chi(D'_1) = 0$ and D'_1 depicts T_{2,λ_C}. Then $\chi(D'_2) = -1$ while it depicts $L'_2 = T_{2,\lambda_C}\#T_{2,-\lambda_C}$. That (3-component) link L'_2 bounds a ribbon surface of an annulus and a disk, so $\chi_4(L'_2) \ge 1$. Thus D'_2 is not χ_4-maximal. Then D' is not χ_4-maximal either, which is again a contradiction. □

Corollary 4.31 Every canonically minimal Bennequin-sharp link is strongly quasipositive. □

Corollary 4.32 Every link L is pseudo-positive if and only if L is canonically minimal and strongly quasipositive (and if and only if it is canonically minimal and Bennequin-sharp). □

Also, the answer to Baker-Motegi's Question 2.2 is affirmative for minimal *canonical* surfaces (of strongly quasi-positive links).

Corollary 4.33 If a link L is strongly quasipositive, they every canonical surface S of L with $\chi(S) = \chi(L)$ is strongly quasipositive. □

For a demonstration of how to use this, see §5.2. (It will be also very evident there that the link extension in Corollary 4.33 is essential.)

Also notice (from the $\lambda_C = 0$ case argument in the preceding proof) that for D being χ_4-maximal, $\lambda_C \neq 0$ for all C, but of course $\lambda_C < 0$ is possible.

The above corollaries allow us to lift the restriction to knots in all of the main results of [FLL], excluding those (bound to knots because) dealing with the τ invariant. Our proposed argument behind Lemma 4.30 cannot be readily used for τ (unless some proper link extension of τ is invoked). But it turns out to work for the signature, and more generally Tristram-Levine signatures σ_ω. (Since we do not treat them extensively here, see [IS, §2] for definition and conventions.)

Corollary 4.34 If

$$\sigma_{(\omega)}(D) = 1 - \chi(D) \text{ (for some } |\omega| = 1) \tag{4.52}$$

and D is connected, then all negative crossings are independent. In particular, D is strongly quasipositive.

Proof. We repeat and slightly modify the argument for the case $\lambda_C = 0$ in the proof of Lemma 4.30. We would need for D' with $\Gamma(D') = C$ that $\sigma_{(\omega)}(D') = 1$, which would imply that $\lambda_C > 0$. □

Of course (4.52) is somewhat too strong an assumption for a converse to hold in general. Also, the knot 14_{45575} from [IS, Remark 2.21] (an example admitting systematic extension using Theorem 3.16) shows that the restriction to canonical surfaces is necessary. But many knots are canonically minimal. For instance, the corollary implies that every homogeneous knot K with $\sigma(K) = 2g(K)$ is strongly quasipositive, and hence (as observed in [It]) positive.

In the opposite direction, if a link is strongly quasipositive, we can use our tools to address Question 2.3.

Corollary 4.35 Assume L is a strongly quasipositive link and has a diagram D with $\chi(L) = \chi(D)$ and $b(L) = s(D) - \mathrm{ind}(D)$. Then L is strongly quasipositive on the minimal $b_q(L) = b(L)$ number of strands.

Proof. The construction of Theorem 4.1 would yield from D a minimal string Bennequin surface of L. By the remark following Conjecture 4.19, the surface (and its band representation) must be strongly quasipositive. □

Let us, for the sake of some brief discussion, call the diagrams D in Corollary 4.35 *optimal* (pseudo-positive) diagrams. (That is, *only the right* equality in (3.10) is exact.)

For instance, since Conjecture 4.7 is known for many special alternating knot diagrams (including those of at most 18 crossings or genus at most 4; see remarks following Question 6.2), they are all optimal, and the corollary immediately settles Rudolph's question for their knots.

But the $(-3,5,5)$- and $(-3,5,7)$-pretzel knots show that for non-alternating (strongly quasipositive canonically minimal) knots, the converse of the corollary is not true (i.e., minimal string strongly quasipositive surfaces may exist beyond the provision by such a diagram D). See Example 5.4.

Such examples exist in higher genus as well, and can be determined. The following lemma is a simple counting exercise but helpful in restricting possible diagrams.

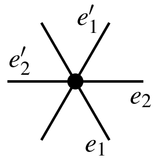
Lemma 4.36 Every (reduced) pseudo-positive diagram of a knot of genus g has writhe $w(D) \geq 2g + 1$. If D is a (reduced) optimal diagram of a genus g braid index b knot, then D can have at most $2(g + b) - 3$ crossings. □

As a result of our test (using such insights), we found that nearly 30% of the non-alternating strongly quasipositive prime knots to 16 crossings have optimal diagrams. (This figure does not discount for the non-canonically minimal knots.) Again, six 16 crossing knots caution that knots may have such diagrams only of non-minimal crossing number.

4.4.5 Estimates for general links

Now let us move away from the restraint to strongly quasipositive. For a general link L, there is no hope of obtaining $\chi_4(L)$ from some combinatorial data of a diagram D of L. However, we can extend the preceding arguments to give estimates in terms of quantities we introduced.

We write $D = D_1 * \cdots * D_l$ for Cromwell blocks D_i, so that $\Gamma(D) = \Gamma(D_1) * \cdots * \Gamma(D_l)$ and $\Gamma(D_i)$ have no cut vertex. We recall that D_i determine the planar embedding of $\Gamma(D_i)$ (see §2.2).



We call two cycles C_1, C_2 in $\Gamma(D)$ *linked* if their edges belong to the same Cromwell block $\Gamma(D_i)$, and either C_1, C_2 share an edge, or they share a vertex v so that the cyclic order at v (with other edges omitted) of the edges of e_i, e_i' of C_i incident from v, $i = 1,2$, is e_1, e_2, e_1', e_2'. (Obviously, there must be then be an even number of such vertices v.) Note again that the cyclic orientation of incident edges at v comes from the planar embedding of $\Gamma(D_i)$ which is canonically determined by D_i. In other words, linked means that cycles border each other, or that their interiors/exteriors (which are determined by the planar embedding) overlap without being contained in one another.

Let $\xi(D)$ be the maximal number of *vertex-disjoint* cycles C in $\Gamma(D)$ of non-positive weight $\lambda_C \leq 0$, let $\xi_e(D)$ be the maximal number of *edge-disjoint* cycles of non-positive weight, and $\xi_u(D)$ be the maximal number of *pairwise unlinked* cycles of non-positive weight. Then

$$\xi(D) \leq \xi_u(D) \leq \xi_e(D).$$

Write $\kappa(D) = c_-(D) - \mathrm{ind}_-(D)$ for the minimal number of negative edges in $\Gamma(D)$ outside of an independent set of negative edges.

Theorem 4.37 For every diagram D of a link L,

$$\chi(D) + 2\kappa(D) \geq -TB(L) \geq \chi(D) + 2\xi(D). \tag{4.53}$$

Also, $\xi_u(D)$ can replace $\xi(D)$ in (4.53) for special diagrams D. Furthermore, if

$$\lambda_C \geq 0 \ \text{ for every cycle } C \text{ of } \Gamma(D), \tag{4.54}$$

then (4.53) extends to

$$\chi(D) + 2\kappa(D) \geq -TB(L) \geq \chi_4(L) \geq \chi(D) + 2\xi(D), \tag{4.55}$$

resp. with $\xi_u(D)$ for special diagrams D.

Proof. The first inequality in (4.53) emerges from how Theorem 4.1 was proved, but does not really require the braided surface, and thus is a consequence of (the fix of) Murasugi-Przytycki's method.

The right inequality in (4.53) is essentially the proof of Lemma 4.30 multiplied under split union. If (4.54), then the connected sum trick in the proof of the lemma is not needed, and the second inequality in (4.55) is (2.29).

If (4.54) does not hold, we use (2.32), which has no analogue for χ_4 and thus disables this part. (The 4-crossing diagram of the figure-8-knot immediately shows that (4.55) fails if (4.54) does not hold.)

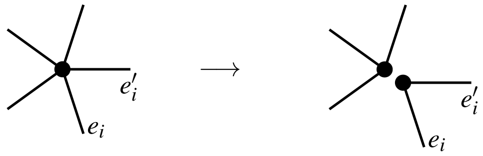
The proof of Lemma 4.30 also shows that for special diagrams D, vertex-disjointness can be relaxed. Here a short argument is needed to explain why the edge-disjoint case can be recurred to the vertex-disjoint (disjoint union of cycles) case, under smoothing out of crossings.

We assume D is special, so that (recall §2.2) $\Gamma(D)$ determines D. Also, $\Gamma(D)$ has a canonical planar embedding (up to spherical moves) determined by D. Thus at every vertex v of $\Gamma(D)$, there is a cyclic orientation of incident edges.

Let C_i be pairwise unlinked cycles in $\Gamma(D)$ of $\lambda_{C_i} \leq 0$. We can obviously assume that no cycle C_i passes a vertex more than once; otherwise it is a vertex join of smaller cycles, and at least one would have non-positive weight. This means that we can assume that every vertex has at most two incident edges e_i, e_i' that belong to a fixed cycle C_i.

Next, since D is special, we can w.l.o.g. assume the interiors of Seifert circles are empty. Assume a cycle C_i has edges e_i, e'_i at vertex v consecutive in the cyclic orientation at v. The assumption that C_i are unlinked ascertained that or every v there is such a cycle C_i.

By adding a trivial clasp inside the Seifert circle of v (which does not change $\chi(D)$), and smoothing out one crossing in the clasp, we can "detach" C_i at v:



By iterating this procedure we can make all C_i vertex-disjoint. $\square$

For an explicit formula for χ_4, we can describe a large part of the cases where (4.53) gets exact.

Corollary 4.38 Assume in $\Gamma(D)$,

1) (4.54) holds, and
2) all C with $\lambda_C = 0$ are vertex-disjoint, or D is special and they are edge-disjoint.

Let $l = \xi_e(D)$ be the number of C with $\lambda_C = 0$. Then

$$TB(L) = -\chi_4(L) = -\chi(D) - 2l\,. \tag{4.56}$$

Proof. Let $E_-(\Gamma(D))$ be the set of negative edges of $\Gamma(D)$ and say a set $S \subset E_-(\Gamma(D))$ is *complementary independent* if $E_-(\Gamma(D)) \setminus S$ is independent. A way of understanding a complementary independent set E is to say that switching sign (from negative to positive) of every edge of E makes all cycles positive-weight. When $\lambda_C \geq 0$ for all C, this is the same as saying that every weight-zero cycle C contains some edge in E.

Also, when $\lambda(C) \geq 0$ for all C, and C_i are edge-disjoint, we can assume w.l.o.g. they are mutually unlinked: by rearranging arcs of C_i, we can always achieve that at no vertex the edges e_i, e'_i of C_i and e_j, e'_j of C_j come in cyclic orientation e_i, e_j, e'_i, e'_j.

Under the assumptions of the corollary, if $\mathcal{Z}$ is a collection of (edge-disjoint) weight-zero cycles in $\Gamma(D)$, then every complementary independent set S contains one edge in some element in $\mathcal{Z}$, with distinct edges for distinct elements. Thus $|E| \geq |\mathcal{Z}|$. This also clarifies, in our special case $\lambda_C \geq 0$, why

$$\xi(D) = \max_{\mathcal{Z}} |\mathcal{Z}| \leq \min_{E} |E| = \kappa(D)\,, \tag{4.57}$$

which obviously must hold from (4.53).

Now take $\mathcal{Z}$ to be the set of all weight-zero cycles of $\Gamma(D)$ and E consist of one edge in each cycle in $\mathcal{Z}$. Then E is complementary independent and $|E| = |\mathcal{Z}|$. Thus equality holds in (the middle of) (4.57). (Resp., replace with $\xi^*(D)$ when D is special.) $\square$

The proof also reveals a quite weaker (but slightly more technical) sufficient condition. It is enough if there is a collection $\mathcal{Z} = \{C_1, \dots, C_l\}$ of vertex- (resp. edge-)disjoint weight-zero cycles and of edges $e_i \in C_i$, such that every weight-zero cycle C contains some edge in $E = \{e_1, \dots, e_l\}$. This will yield $\min|E| \leq \max|\mathcal{Z}|$, which obviously enables the argument. The following two examples should illustrate this situation.

Example 4.39 An example for $l = 1$ is the pretzel (2.42) diagram $L(-k, k, a_3, \dots, a_n)$ (for a parallel link (2.40) if n is even) for $a_i \geq k > 0$, $i > 2$, and k, a_i odd.

Example 4.40 Here is how to construct examples for arbitrary large l. Take any diagram D', and fix edges e_i in $\Gamma(D')$ so that no e_i has a parallel edge, and one needs to pass at least 4 edges to go from a vertex of e_i to a vertex of any e_j, $j \neq i$. Then make all the edges $e \neq e_i$ positive, all e_i negative, and apply a $\bar{t}'_2$-twist on the crossings of e_i. Then for the resulting diagram D, (4.56) will hold with l being the number of e_i featured in a length-4-cycle of $\Gamma(D')$.

In [Ru], Rudolph gives an improvement of (2.29) by

$$-\chi_4(L) \ge -\chi(D) - 2c_-(D) + 2s_-(D)\,, \tag{4.58}$$

where $s_-(D)$ is the number of *negative Seifert circles*, those to which only negative crossings are attached. This improvement is not captured in the left bound in (4.53). However, there is a unified argument that amalgamates both estimates.

Set

$$\mathrm{ind}^*_-(D) \,=\, \max\left\{ \mathrm{ind}_-(\Gamma(D)') - l \,:\, \begin{array}{c} \Gamma(D)' \text{ is obtained from } \Gamma(D) \text{ after some} \\ \text{negative edges and } l \text{ positive edges are removed} \end{array} \right\}$$

Then define $\kappa^*(D) = c_-(D) - \mathrm{ind}^*_-(D)$.

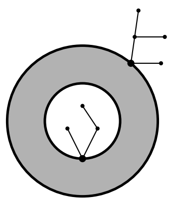
Proposition 4.41 We have

$$-\chi_4(L) \ge -\chi(D) - 2\kappa^*(D)\,. \tag{4.59}$$

Proof. Because $\chi_4(L)$ changes by at most ± 1 under crossing smoothing, if the claim holds for D, it will also hold for a diagram D' so that $\Gamma(D)$ is obtained from $\Gamma(D')$ by the deletion of a negative or positive edge.

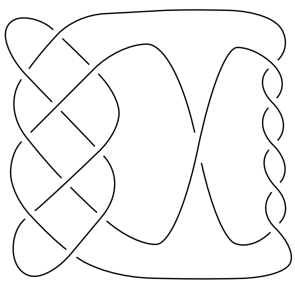
This allows one to recur the statement to prove to the combination of the left inequalitiy in (4.53) and (2.29). (Note, however, that this is *no longer* an estimate on $-TB$, and there is no braided surface constructed.) □

Corollary 4.42 Let D be a connected reduced diagram with at least one crossing. Then the estimate (4.58) holds for D.



Proof. It is very easy to see that $\mathrm{ind}^*_-(D) \ge s_-(D)$. By deleting negative edges only, one can always leave one negative edge connecting the vertex of a negative Seifert circle to the rest of the graph. This can be done by induction in such a way that all these negative edges will form a forest, with each tree attached only at one root vertex to the rest of the graph.

Thus these negative edges will become independent (when star-contracted w.r.t. the vertex away from the root and in order of decreasing distance from the root). The number of these edges is $s_-(D)$, unless D is negative (when it is $s_-(D) - 1$). But for a negative diagram D, by mirroring $\chi_4(L) = \chi(D)$, while (4.58) reads $-\chi_4(L) \ge \chi(D)$. This is then true unless D is the trivial (unknot) diagram. □



Example 4.43 As an illustration of (4.59), consider the diagram on the top with its special orientation (i.e., orientation with which it becomes a special diagram). We have $\chi = -4$, $c_- = 4$, $\mathrm{ind}_- = 1$, $\mathrm{ind}^*_- = 3$ and $s_- = 0$. Thus neither (4.58) nor (4.53) give any more than the trivial bound $-\chi_4 \ge -2$. But (4.59) gives $-\chi_4 \ge 2$.

4.4.6 Pseudo-homogeneous links

The following widens Cromwell's notion of a homogeneous diagram, giving a more general (but still fully combinatorial) condition under which the canonical surface is minimal genus. (See the end of this subsection for a slight further extension.)

For Murasugi atoms, see §2.2. *Dealternators* are the crossings needed to switch in a diagram to obtain an alternating diagram. The (complete) *dealternator set* is determined up to complement; in a special diagram it is the set of (all) crossings of one of either sign.

Corollary 4.44 (and definition) Let L have a diagram D in which in every Murasugi atom D_i one can choose an independent dealternator set E_i. We call D and L *pseudo-homogeneous*. Then (2.3) holds and L has a Bennequin surface on at most $s(D)-n$ strands, with $n=\sum|E_i|$ being the total number of dealternators in all D_i.

Note that $\Gamma(D)$ is a join of $\Gamma(D_i)$, and so every cycle in $\Gamma(D)$ splits into cycles in $\Gamma(D_i)$. Thus a set $E\subset E(\Gamma(D))$ is independent if and only if $E\cap E(\Gamma(D_i))$ is independent in $\Gamma(D_i)$ for all i. But the union of dealternator sets in all D_i is not necessarily a dealternator set in D. The corollary offers thus a greater freedom in choosing a proper set of crossings in D: for k atoms D_i, one among 2^k different possibilities should do. (But also obviously not more than one can.)

Proof. If one looks at a D_i separately, the condition on dealternators implies that the canonical surface of D_i is strongly quasipositive or strongly quasinegative, and so (2.3) holds for D_i. Note that D_i are special, so the dealternators are precisely the positive crossings in D_i or precisely the negative crossings in D_i. Since they are independent, they can be removed by Murasugi-Przytycki moves. Minimal genus is preserved under connected sum and Murasugi sum, so that (2.3) holds. Once we have this property, one can apply the Murasugi-Przytycki moves on D to remove all dealternators in D_i as well (while making the other crossings into bands), and obtains a Bennequin surface. $\square$

For $n=0$ in Corollary 4.44 one has precisely the homogeneous diagrams D in [Cr] (for which the property (2.3) was observed by Cromwell based on (4.32)). For $n=1$, the pseudo-homogeneity condition simply says that the dealternator is unstable. In that case it is known that this condition stands in equivalence to (2.3). Unfortunately, for $n>1$ it is less easy to find a combinatorial condition on dealternators equivalent to (2.3). Note also that in general pseudo-homogeneity may apply even if (2.8) fails to establish (2.3). (This failure does not occur if $n=0$, or $n=1$; see[9] [St8, corollary 5].) Every pseudo-homogeneous diagram in which the $(-3,5,7)$-pretzel diagram is a Murasugi atom is an example. (So for $n=3$ we do have a new tool as compared to (2.8).)

Example 4.45 The knot 10_{145} is pseudo-homogeneous (and pseudo-positive) but not homogeneous (see Appendix A). The knot 9_{48} is an example of a pseudo-homogeneous knot which is not pseudo-positive or homogeneous. (It, though, only has a 10 crossing pseudo-homogeneous diagram.)

Corollary 4.46 If D is a connected pseudo-homogeneous diagram, then for $P_{\max}(D)$ defined in (4.31), we have

$$P_{\max}(D)\neq 0\,. \tag{4.60}$$

In particular, a pseudo-homogeneous non-split link L satisfies $\max\deg_m P(L)=1-\chi_c(L)=1-\chi(L)$.

Proof. If a link L has a diagram D that makes the chain (5.5) exact, then we need for $P_{\max}(D)$, that

$$\min\deg_l P(L)=\min\deg_l P_{\max}(D)=1-\chi(D) \tag{4.61}$$

because of the sequel to (2.14). In particular, $P_{\max}(D)\neq 0$ when D is pseudo-positive. Using (4.32), we see that $P_{\max}(D)\neq 0$ when D is pseudo-homogeneous. Morton's inequality (2.11) then implies that $\chi(D)=\chi_c(L)$. $\square$

Example 4.47 The examples of knots L in [St3] with $2g_c(L)>\max\deg_m P(L)=4$ are not pseudo-homogeneous. These examples turn out to be all genus 2, so that $g_c(L)>g(L)$. But there also exist knots L [St4, Fig. 14] with $g(L)=g_c(L)$, that is, where a diagram D with (2.3) exists, but for which (and then every other such diagram) $P_{\max}(D)=0$.

Thus canonically minimal does not imply pseudo-homogeneous, which refutes a possible (partial) converse of Corollary 4.44. See Question 6.6 for a more difficult geometric version. For another class of links whose genus is detected by the HOMFLY polynomial, but not necessarily by the Alexander polynomial, see [St16].

Remark 4.48 Since the genus of a *diagram* D and the HOMFLY polynomial are mutation invariant, Corollary 4.46 also gives a simple argument to see that $\operatorname{ind}(D)$ is *diagram* mutation invariant. (Gabai's work has long shown that many mutant *knots* have different genus; see e.g. Example 5.9.)

[9] but beware that '$\min\deg\Delta$' is a misprint of '$\max\deg\Delta$' there, and that the normalization is different

The following example briefly returns to the visual primeness account in §2.2.

Example 4.49 Given in DT notation (see end of §2.2), the diagram `9 74 6 10 -14 18 4 16 2 8 12` shows that even special pseudo-positive (and almost positive), and hence pseudo-homogeneous, diagrams are not visually prime. That is, a prime such diagram may represent a composite knot or link. Another example is given in [IS] for what is defined there as a good s.a.p. diagram. (Such a diagram is always pseudo-positive, but usually not special.) Compare also with [St3, Figure 15] for minimal genus diagrams, and with Question 6.9 below.

The argument in the above corollary has a few other implications, which are worth recording.

Proposition 4.50 When D_i are reduced pseudo-homogeneous diagrams of fixed $\chi = \chi(D_i)$, then, up to simultaneously mirroring all D_i,

$$\max \deg_l P(D_i) \to \infty, \tag{4.62}$$

or all D_i have only finitely many HOMFLY polynomials.

Proof. First, let us assume the number of components $n(D_i)$ of D_i is bounded. An unbounded number of components for fixed χ, will require the addition of an unbounded number of unknotted or annular connected components of D_i, which will lead to (4.62).

When $n(D_i)$ and $\chi(D_i)$ is fixed, the number of split components of D_i is bounded, thus we may w.l.o.g. assume that D_i are connected.

Then (4.62) is an easy consequence of the fact that when $\chi(D)$ is fixed, every coefficient $D \mapsto [P(D)]_{l^s m^p}$ of P (in previously fixed degree s, p in l, m) can take only finitely many values on such D, as proved in [St20]. □

'Pseudo-homogeneous' can also be replaced by 'pseudo-positive', in which case the mirroring is unnecessary.

One can also extend the result of [St19]. For example, we have the following. (For notation recall §2.2 and (2.27).)

Corollary 4.51 For any sequence of pseudo-positive knots K_i with distinct HOMFLY polynomials, we have

$$TB(!K_i) \to -\infty.$$

For any sequence of pseudo-homogeneous knots K_i with distinct HOMFLY polynomials, we have

$$\max(\delta(K_i), \delta(!K_i)) \to \infty.$$ □

We cannot exclude the possiblily even among pseudo-positive knots, that infinitely many have the same HOMFLY polynomial, i.e., that they contain a Kanenobu series (see above Corollary 4.19). As pretzel knots (§5.1) readily caution, this property definitely occurs for the Alexander polynomial (already for pseudo-positive diagrams).

Also, control over pseudo-homogeneous diagrams of a knot is far more difficult than over homogeneous ones. Using (4.32), and a $\bar{t}'_2$-twisting argument in positive diagrams, it is rather straightforward that finitely many homogeneous diagrams have the same P (in fact, $P_{\max}$). This is used in Appendix A (with a few added explanations and also, for positive knots, in [St3]).

When we talk about finitely many (reduced) diagrams, there is one obvious (and easy to remove) redundancy: we can add $\sim$-equivalent crossings of opposite sign by Reidemeister II moves, and then disperse them by possible flypes. Thus we must exclude these operations – which also preserve pseudo-homogeneity.

Let us say a diagram is *$\sim$-reduced* if it is reduced and no $\sim$-equivalent crossings have opposite sign. This property is often implicit in many related arguments, like the notion of a *twist vector*, but was not always clearly emphasized.

Fact 4.52 A pseudo-homogeneous knot of genus up to 2 has only finitely many $\sim$-reduced pseudo-homogeneous diagrams.

Proof outline. We were able to bring this into knowledge using a variant of pattern stabilization [St10]. The calculation revealed that, when we regard a $\sim$-equivalence class of 1,2 crossings as fixed and for 3,4 crossings as $\bar{t}'_2$-twistable, pseudo-homogeneous diagrams occur in about every fourth pattern. Let us call such patterns also *pseudo-homogeneous*. To identify them, we had to switch from the Murasugi-Przytycki vertex star-contraction method to Traczyk's cycle condition (4.51). It is much better suited for twisting, but as noted in the context of (4.51), it does require determining all cycles in the Seifert graph $\Gamma(\hat{D})$ of a generator $\hat{D}$.

Pseudo-homogeneous diagrams thus appear nearly as general as minimal genus diagrams (2.3). This explains why beyond (4.61) we have failed to say much about the HOMFLY polynomial. We succeeded in stabilizing P only for about a quarter of the pseudo-homogeneous patterns.

Rather, to ensure stability we shifted attention to the Kauffman polynomial F. Its special case, the BLMH polynomial Q, worked much faster and left over less than 2%, but it, too, seriously failed on some of the remaining patterns. Even for F, a few patterns had to be broken up into subpatterns. The method (described in quite a detail in [St10], up to some minor adaptations needed here) is highly involved, and it is not reasonable to attempt it on genus 3. $\square$

We asked in [St3, Question 14.2] a finiteness property more generally for minimal genus diagrams. This is one place where $\sim$-reducedness was bourne in mind, for otherwise the question would not make much sense. However, investigating genus 2 diagrams (recall (2.2)), we now found out that more subtle cancellation phenomena occur than $\sim$-reduction. Thus Fact 4.52 is quite far from the truth for canonical genus 2 knots. We were able, though, to extend the F-finiteness result, which can be stated thus.

Theorem 4.53 At most $O\big((\max\deg_z F)^8\big)$ knots K with $g_c(K)=2$ (and in particular pseudo-homogeneous knots of genus 2) have the same Kauffman polynomial z-degree $\max\deg_z F$.

Proof outline. This was established by making stabilization work for (most of) the remaining patterns (slightly enhanced by using some result of Gabai), and then identifying completely the cancellation in cases of failure. (The Q polynomial leaves over about 7% of the patterns here.) A more precise property, coming out of stabilization, is that there is a constant d so that for every knot K with $g_c(K)=2$,

$$\max\deg_z F(K) \geq \min\{c(D) : g(D)=2\,, D \text{ is a diagram of } K\} - d\,. \qquad \square$$

An account (incl. of how d is obtained) may be given separately, also possibly further details regarding Fact 4.52. Note that Kanenobu's knots [K] have canonical genus 2, and he also asked ([Ki, Problem 1.91(3)]) if there are infinitely many knots with the same Kauffman polynomial, a question which is still open. It appears now tempting to refine his question: are there only finitely many with given $\max\deg_z F$?

The complexity of genus 2 leaves little hope to control minimal genus diagrams of a knot. Whether pseudo-homogeneity restricts these ways of cancellation to $\sim$-reducedness (beyond genus 2) is a potential direction of further investigation (Question 6.8). For whatever reasons, we failed (so far, even partially checking genus 3 generators) to observe these more complicated cancellation phenomena in pseudo-homogeneous diagrams.

One can introduce an even slightly larger class of diagrams with (4.60). Say D is *weakly pseudo-homogeneous*, if in the Seifert graph $\Gamma(D)$ no parallel edges have opposite sign, and in the *reduced* Seifert graph $\Gamma'(D)$ (obtained from $\Gamma(D)$ by reducing multiple edges to single ones) each Cromwell block has independent positive or independent negative edges. Then Corollary 4.46 and Proposition 4.50 hold even for this more general class. However, the string number of the Bennequin surface in Corollary 4.44 changes.

Example 4.54 The simplest knot diagram which is weakly pseudo-homogeneous but not pseudo-homogeneous is the (reduced) 5-crossing diagram of the figure-8-knot. The diagram $M(1/3,-2/3,-2/3)$ of 9_{48} (see §2.6) is another example. But we saw 9_{48} is pseudo-homogeneous. For the reasons related to the explanation on Fact 4.52, we do not know what is a weakly pseudo-homogeneous knot that is not pseudo-homogeneous (and verifying such an example is beyond feasability).

5 Classifying strongly quasipositive pretzel knots

5.1 Strong quasipositivity of the parallel pretzel links

As an instance of how to gain a quick visual test for strong quasipositivity from Corollary 4.24, together with Traczyk's cycle condition of Theorem 4.8, consider the link L with a pretzel diagram (2.42). To avoid trivial (or easy) cases, we assume no $a_i = 0$, and

$$n > 2\,. \tag{5.1}$$

Sometimes one regards pretzel links only to have all $|a_i| \geq 2$, in which case we call them *pure* (as referred to in §4.4.6 below Corollary 4.20), but here *we will allow for* $|a_i| = 1$.

With $a_i \neq 0$, we sign a_i according to the sign of the twist crossings. We say $a_i \neq \pm 1$ are parallel or reverse if the twist crossings form parallel or reverse clasps (§2.2). We assume there is an $|a_i| > 1$, and all such are reverse, and that D is special (so $\chi(D) = 2 - n$); in particular all a_i are even or all are odd. Also, in the case of odd a_i, which in accordance with (2.40) we will call the *parallel* case, we can obviously assume that

$$\{-1, +1\} \not\subset \{\, a_i \,:\, i = 1, \dots, n \,\}\,. \tag{5.2}$$

Then one observes easily from Corollary 4.24 that L is strongly quasipositive (with $\chi(L) = 2 - n$) if

$$\text{at most one } a_i \text{ is negative, say } a_1 < 0, \text{ and for every other } a_i \text{ we have } a_i > -a_1. \tag{5.3}$$

For the parallel case (all a_i odd), the condition (5.3) is in fact exact. Even if the proof is slightly non-trivial, I decided to afford its inclusion below, as a testimony to the efficiency of our test. See also Example 4.27 above.

Let $L(a_1, \dots, a_n)$ be a knot. If some a_i is even, then all other a_i are odd. If all a_i are odd, then n is also odd, and this is the case we consider here. We will continuously assume (5.1) and (5.2).

The below statement was discussed also, for $|a_i| > 1$ and n odd, in [FLL, Corollary F], and it was mentioned that the case $n = 3$ was previously treated by Rudolph.

Proposition 5.1 The parallel pretzel link $L(a_1, \dots, a_n)$ with all a_i odd and $n > 2$ is strongly quasipositive if and only if (5.3) holds.

We have also an algorithm for parallel Montesinos links, given in §5.2.

To prepare the argument, we need to review a few properties of the skein polynomial.

If a link is strongly quasipositive, then (2.12) shows

$$\min \deg_l P(L) \geq 1 - \chi(L)\,. \tag{5.4}$$

This inequality can be strict (whose revelation refuted a long-standing problem of Morton; see [St4]).

However, if a link L is canonically minimal, then

$$\min \deg_l P(L) \leq \max \deg_m P(D) \leq 1 - \chi(D) = 1 - \chi(L)\,, \tag{5.5}$$

so that if L is also strongly quasipositive, then we have equality in (5.4).

We will often need the maximal m-term of P in Morton's sense. We restrict ourselves to diagrams, and using brackets to denote a coefficient, we recall the definition (4.31) of $P_{\max}(D)$ and its property (4.32). For a positive special alternating diagram it follows (essentially) from (2.14) that

$$\min \deg_l P_{\max}(D) = 1 - \chi(D)\,, \tag{5.6}$$

so that for a negative diagram

$$\min \deg_l P_{\max}(D) < 1 - \chi(D)\,. \tag{5.7}$$

It then follows from (4.32) that

$$\text{an alternating diagram is positive if and only if (5.6) holds.} \tag{5.8}$$

(This was observed in [Cr].) In particular an alternating link is strongly quasipositive iff it is positive (and its alternating diagrams are positive).

Now we argue that the pretzel surface has minimal genus; this will enable us to compare with the 4-genus.

Lemma 5.2 We assume that D is in (2.42), and (5.2) holds. Then we have (2.3). Thus $\chi(L) = 2-n$. Also, if

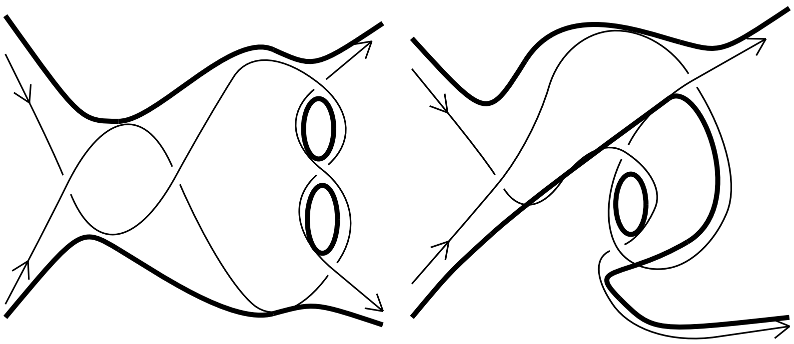
$$\{-1,+1\} \cap \{\, a_i : i = 1,\dots,n\,\} \neq \varnothing\,,$$

then

$$\max\deg_m P(D) = 1 - \chi(D)\,. \tag{5.9}$$

Proof. If no a_1 is ± 1, then it follows from the work of Gabai (see [KL]).

Otherwise, let w.l.o.g. $a_1 = -1$, so that no $a_i = 1$. Assume that some $a_i > 1$. Flype the negative integer half-twist close to a_i and consider the Seifert circle picture of the tangle isotopy



(5.10)

The right diagram has one crossing and one Seifert circle less, and the a_i twist becomes a diagrammatic Murasugi sum (plumbing) with a reverse $(2, a_i - 1)$ torus link. Deplumbing it, one is left with the original diagram in which the a_i twist is gone. In this way the $a_1 = -1$ can successively "gobble up" all $a_i > 0$, leaving us either with a one-crossing unknot diagram, if $a_1 = -1$ is the only negative a_i, or with a negative special alternating pretzel diagram otherwise. We will refer to the iterated desuming procedure also after this proof.

In [Cr], homogeneous diagrams D were used so that one can successively Murasugi desum all Murasugi atoms D_i in the same diagram, and with them being special alternating, we have

$$2\max\deg\Delta(D_i) = \max\deg_m P(D_i) = 1 - \chi(D_i)\,.$$

With (4.32), the corresponding result for Δ (which follows from (4.32) by substitution, but was long before known from Murasugi's work for general geometric Murasugi sums), and (2.8), one has (2.3) and (5.9). Here after every diagrammatic desum one has to apply an isotopy like (5.10), but since *it does not change* $\chi(D)$, we have $P_{\max}$ remaining the same, and the same argument still works. $\square$

It will be useful to set

$$\tilde a_i = |a_i| - 1\,, \tag{5.11}$$

since it will be constantly needed later.

Proof of Proposition 5.1. We argue that the condition is necessary, so assume L is from now on strongly quasipositive. We will assume at least one a_i is negative, say $a_1 < 0$, and at least one is positive; otherwise it is trivial.

Case 1. If some $a_i = 1$, say $a_2 = 1$, then $a_1 < -1$. Using $a_2 = +1$, by the mirrored version of the iterated diagrammatic deplumbing in the proof of Lemma 5.2, deplumb all reverse $(2, \tilde a_i)$ torus links for $a_i < 0$. At least one such occurs, for $a_1 < 0$. Then by Cromwell's argument for (5.8), we have (5.7), so L is not strongly quasipositive.

Case 2. If some $a_i = -1$, we have to prove that for strongly quasipositive L, this is the only negative a_i. If some other $a_i < 0$, say $a_2 < 0$, then after the iterated diagrammatic deplumbing in the proof of Lemma 5.2, remove all $(2, a_i - 1)$ torus links for $a_i > 0$. It remains a non-trivial negative link, so (5.7) holds and L is not strongly quasipositive.

We will need the following

Lemma 5.3 Let

$$D = L(\underbrace{-1, -1, \dots, -1}_{k}, a_{k+1}, \dots, a_n) \qquad a_i > 1, \ i > k.$$

Then

$$\max \deg_m P(D) = 1 - \chi(D) = 2 - n \quad \text{and} \quad \min \deg_l P_{\max}(D) = 3 - \chi(D) - 2k. \tag{5.12}$$

Proof. The first part of (5.12) is (5.9). The second part follows from (4.32). After the iterated deplumbing argument of positive Hopf bands, one remains with the negative $(2, -k)$ parallel torus link diagram D; its $\min \deg_l P_{\max}(D)$ is found by direct calculation. □

Case 3. Assume from now on no $a_i = \pm 1$. We will prove it two steps.

- that if more than one a_i is negative, then L is not strongly quasipositive.
- that if there are $a_i < 0$, $a_j > 0$ with $|a_i| \ge |a_j|$, then L is not strongly quasipositive.

step 3.1. To prove only one a_i is negative, consider

$$D = L(\underbrace{a_1, a_2, \dots, a_k}_{<0}, \underbrace{a_{k+1}, \dots, a_n}_{>0}) \qquad |a_i| < |a_j|, \quad 1 \le i \le k < j \le n.$$

Note that exchanging a_i accounts for mutations which do not change the below formula. Then for $k > 1$ the non-strong quasipositivity of L follows from (5.6) and proving that, with (5.11),

$$\min \deg_l P_{\max}(D) = \begin{cases} 1 - \chi(D) & k = 1, \\ 3 - \chi(D) - 2k - \sum_{i=1}^{k} \tilde{a}_i & k > 1 \end{cases} \tag{5.13}$$

The case $k = 1$ in (5.13) follows from (2.3) of Lemma 5.2 and (5.6) because L is strongly quasipositive. For $k \ge 2$, use induction. From (2.10), we have

$$P_{\max}(D_-) = -l^{-2} P_{\max}(D_+) - l^{-1} P_{\max}(D_0)$$

and applying this to pretzel diagrams, we get identities like

$$P_{\max}(L(a_1 - 2, a_2, \dots)) = -l^{-2} P_{\max}(L(a_1, \dots)) - l^{-1} P_{\max}(L(a_2, \dots)),$$

and similarly can be done with any a_i. The induction should go over a_k and consists in showing that the minimal l-degree term of $P_{\max}$ comes from the first term on the right. For $k = 2$ use induction on a_2 for $a_1 = -1$ first, and then induction on a_1.

step 3.2. So finally it remains to prove that (2.42) is not strongly quasipositive if a negative a_1 satisfies $|a_1| \ge |a_i|$ for some $i > 1$.

If $a_i = -a_1$ for some $i > 1$, then the ribbon genus of L is less than its genus. The diagram exhibits a Seifert ribbon of smaller genus than the pretzel surface. So L is not strongly quasipositive.

If some $0 < a_i < -a_1$, use the argument in the proof of Lemma 4.30, as follows.

Assume L is strongly quasipositive. Then $L \# T(2, -a_1 - a_i)$ is also, and the diagram $D' = L(a_1, \dots, a_n) \# T_{2, -a_1 - a_i}$ is χ_4-maximal; compare above Lemma 4.30. Then by crossing smoothings so is the diagram

$$D_2 = L(a_1, a_i) \# T_{2, -a_1 - a_i} = T_{2, a_1 + a_i} \# T_{2, -a_1 - a_i},$$

which is not the case. □

Example 5.4 If one takes the $(-3,5,7)$-pretzel knot $K = 15_{199038}$ and its $(-3,5,7)$-pretzel diagram D (having $s(D) = 14$), one easily sees from (4.46), together with (2.21), that K has braid index 6. But $\text{s-ind}(D) = 8$ does not measure a set of Murasugi-Przytycki moves actually performable, so from (4.16) (with $\text{ind}(D) = 6$) it is not obvious how to gain a braid representative of K on less than 8 strands. (One can get from (4.16) at least the estimate $b(K) \le 7$ from other minimal crossing diagrams of K.) Also, since the largest independent set of D containing all negative crossings has size 4, the method behind Theorem 4.1 cannot yield a strongly quasipositive surface on less than 10 strands.

5.2 Strong quasipositivity of the parallel Montesinos links

Below follows an algorithm to decide if a parallel Montesinos link is strongly quasipositive, each step with a few explanations what happens and why. We must point out in advance that if we just use the behavior of 'strongly quasipositive' under Murasugi sum and work with a minimal genus surface, this algorithm depends on the positive answer to Question 2.2. To circumvent this problem, we emphasize that we work with canonical surfaces and Corollary 4.33, and use (4.32) to meet the obstruction (5.7).

We refer to §2.6 for explanation about Montesinos links we use below.

step 1. Bring the Montesinos link to minimal crossing diagram D. If the diagram is alternating, then L is strongly quasipositive iff D is positive.

step 2. So now we assume L is not alternating. Thus there are no integer twists, $e = 0$. Assume by convention $q_i > 0$ (and odd) and p_i are signed. Arrange that there are no fractional parts

$$\frac{1}{2} < \left|\frac{p_i}{q_i}\right| < 1$$

of opposite sign, using transformations like in the example

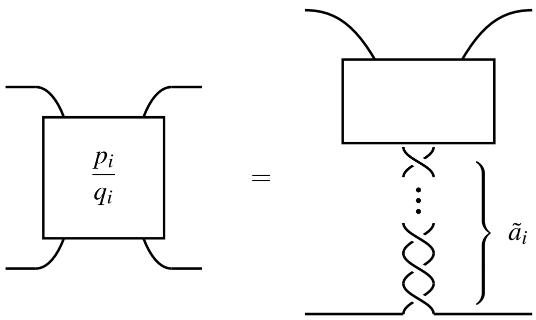
$$M\big(1/3, 5/7, -11/19, 6/11\big) = M\big(1/3, -2/7, 8/19, 6/11\big)\,. \tag{5.14}$$

This diagram is minimal genus (see [KL] or below in step 4).

step 3. Now we will need the numbers

$$\tilde a_i = \operatorname{sgn}(p_i)\cdot\left\lfloor\left|\frac{q_i}{p_i}\right|\right\rfloor,$$

with $\lfloor\,.\,\rfloor$ being integer part. These numbers manifest themselves as the *integer twists*



in the alternating form of the p_i/q_i tangle, and in the box on the right we accommodate the *non-integer twists*.

If some non-integer twist of a p_i/q_i tangle is negative, then L is not strongly quasipositive.

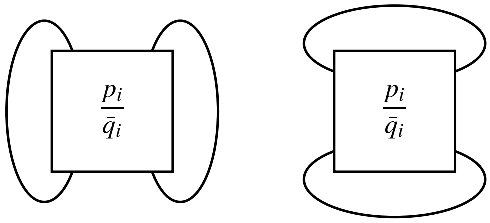
To see this, find the plumbing form of p_i/q_i. Expand

$$\frac{q_i}{p_i} - \operatorname{sgn}(p_i) = \frac{\bar q_i}{p_i} \quad \text{for} \quad \bar q_i = q_i - |p_i|\,,$$

as a continued even fraction
$$\frac{\bar{q}_i}{p_i} = b_{i,1} + \cfrac{1}{b_{i,2} + \cfrac{1}{b_{i,3} + \cfrac{1}{\dots}}}$$
with $b_{i,j} \in \mathbb{Z}$ even. Also $b_{i,j} \neq 0$ unless $j = 1$ and $|q_i| < 2|p_i|$. Since q_i is odd, at least one of $\bar{q}_i$ and p_i is even, so there is such an even continued fraction. The correction $-\mathrm{sgn}(p_i)$ above is needed to get strand orientations right when closing up the tangle, because we need the strand orientation in the $b_{i,j}$ twists to be reverse.

The even continued fraction will give then within the $\bar{q}_i/p_i$-tangle (with the right strand orientation) some plumbing picture of generalized Hopf bands, i.e., reverse $(2, \pm b_{i,j})$-torus links ($b_{i,j}$ even). Note that the sign of these torus links alternates for positive $b_{i,j}$, so it does not coincide with the one of $b_{i,j}$ in general; this is why '$\pm$' was written. Also $b_{i,j} \neq 0$; the case $j = 1$ and $|q_i| < 2|p_i|$ experiences extra handling below (5.15).

Let first $q_i > 2|p_i|$. If some non-integer twist in the alternating form is negative, then not all generalized Hopf bands in the even continued fraction form will be positive, because the left tangle closure of



(5.15)

has a non-positive reduced alternating (2-bridge) link diagram.

The argument needs a little modification when $q_i < 2|p_i|$, because then $b_{i,1} = 0$, and in the left closure of (5.15) there is an integer twist which becomes nugatory crossings. But one can easily argue with the right closure. Thus this extra case makes no harm.

step 4. After deplumbing all (positive) generalized Hopf bands, one is left with a pretzel diagram (2.42) with
$$a_i = \mathrm{sgn}(p_i) + b_{i,1} = \begin{cases} \tilde{a}_i + \mathrm{sgn}(\tilde{a}_i) & \tilde{a}_i \text{ even} \\ \tilde{a}_i & \tilde{a}_i \text{ odd} \end{cases} \tag{5.16}$$
Note that because of step 2, we have (5.2), which implies the minimal genus property. (Thus we also see minimal genus for the diagram in step 2 after (5.14) by Gabai's work.)

Then L is strongly quasipositive if and only if this pretzel surface is strongly quasipositive, which is decided as in Proposition 5.1. This completes the decision algorithm.

5.3 Strong quasipositivity of the reverse pretzel knots

We will complete at least the classification of strongly quasipositive pretzel knots, by studying the reverse ones. As for the link case, in its completeness, (it will transpire very soon why) we have no capacity to discuss it here. (There may be different component orientations and arbitrarily many components.) However, some of the arguments below can be adapted to some types of reverse links (in particular see Corollary 5.8), and we only make brief hints.

'Reverse' should refer to the orientation of strands at the integer twist when regarding a pretzel knot as a Montesinos knot. In that case exactly one a_i is even, and we write (2.42) as
$$L(a_1, \dots, a_{n-1}, a_0) \tag{5.17}$$
with a_0 even. Note that cyclically permuting the a_i is an isotopy, exchanging an $a_i = \pm 1$ with its neighbor is a flype, and otherwise permuting the a_i is a mutation. We will, of course, continue assuming (5.1) and (5.2).

Theorem 5.5 Let L be a reverse pretzel knot. Then the following are equivalent.

1) L is strongly quasipositive.

2) L is positive.

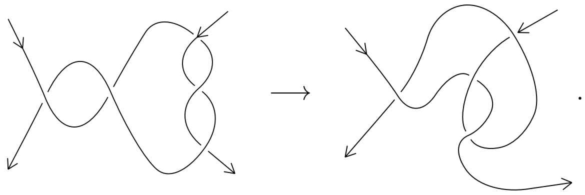
3) In (2.42),

$$\text{all negative } a_i \text{ are } -1 \text{ and there are at most } \left\lfloor \frac{n}{2} \right\rfloor \text{ of them.} \tag{5.18}$$

Proof. 3)$\Longrightarrow$2) Bring by flypes $a_i = -1$ before some odd $a_i = q_i > 0$ (for $i \neq 0$) and use repeatedly the moves of Montesinos diagrams

$$M(\dots, -1, 1/q_i, \dots) \to M(\dots, \frac{1-q_i}{q_i}, \dots),$$

which with orientation look like


(5.19)

The negative crossings of the -1 twists disappear.

1)$\Longrightarrow$3) This will occupy the rest of the proof.

Case 1. n is odd. The a_0 twist is reverse and all odd a_i, $i > 0$, are parallel.

Keep in mind that our signing convention for all a_i, incl. a_0, is that sign is given by the sign of crossings in the twist. In the case here, thus now (5.17) for $a_0, a_i > 0$ will be a non-alternating diagram. Also under the transfer (2.43), a_0 should be turned into $-1/a_0$, not $1/a_0$. For example, $L(1,1,1,1,-2) = M(4,1/2)$ is the 6-crossing diagram of 6_1 with writhe 4, while $L(1,1,1,1,2)$ is a diagram of 5_2.

Let

$$n_+ = |\{\, i : i > 0 < a_i \,\}| \qquad n_- = |\{\, i : i > 0 > a_i \,\}|. \tag{5.20}$$

Case 1.1. If $n_+ \neq n_-$, then the pretzel surface is minimal genus, as one can inspect by deplumbing. After plumbing Hopf bands by smoothing out crossings in parallel clasps, one is left with a plumbing of a reverse $(2, n_+ - n_-)$ and $(2, a_0)$ torus links.

Assume L is a strongly quasipositive knot. Then by Livingston, and (2.15),

Obstruction 5.6 every sequence of crossing changes of length $g(L) - \nu$ will give a knot with $\tau \geq \nu$, and if it gives a knot with $\tau = \nu$, then the sequence must involve only positive crossing changes.

Our understanding of such sequence will be to remove parallel clasps decreasing the genus of the diagram (by 1), so the length of the sequence can be inferred by looking at the genus of the final diagram.

If there is an $a_i \leq -3$, then use $g(D) - 1$ crossing changes to make L into

$$L(\operatorname{sgn}(a_1), \dots, \operatorname{sgn}(a_n), a_0) \tag{5.21}$$

which is a genus-one (alternating) 2-bridge knot, with $\tau \leq 1$. But not all crossing changes were positive, so by Obstruction 5.6 for $\nu = 1$ we are done.

If $a_0 < 0$, then (5.21) is a non-positive genus-one 2-bridge knot, so $\tau \leq 0$, and again by Obstruction 5.6 for $\nu = 0$ we are done.

Now it remains to see that

$$n_- \leq \frac{n-1}{2}.$$

If $n_- > n_+$, then again (5.21) is a non-positive genus-one 2-bridge knot, and the rest is clear. Thus we have

$$n_+ \geq n_-\,, \quad a_i < 0 \Longrightarrow a_i = -1\,, \quad a_0 > 0\,,$$

which conforms with (5.18).

Case 1.2. Let now

$$n_+ = n_-\,. \tag{5.22}$$

By [KL], we get a minimal genus surface by using (n_+ of the) Montesinos link transformations

$$M\left(\dots,\frac{1}{q_1},\dots,\frac{-1}{q_2},\dots\right) = M\left(\dots,\frac{1-q_1}{q_1},\dots,\frac{q_2-1}{q_2},\dots\right) \tag{5.23}$$

for $q_1 = a_i > 0$ and $q_2 = -a_j > 0$ and $i, j > 0$. These changes are done n_+ times to remove all $p_i = \pm 1$; the a_0 twist is not affected (compare [KL, Figure 6]).

Case 1.2.1. Assume first no $a_i = \pm 1$. Then after resolving crossings in parallel clasps, one plumbs down up to mutations to a

$$L(\overbrace{-2,\dots,-2}^{k},\overbrace{2,\dots,2}^{k},a_0) \tag{5.24}$$

pretzel surface, with all clasps reverse. (Since this surface is planar, one can prove minimality by showing $\Delta \neq 0$ using Hoste-Hosokawa's formula for the minimal coefficient of the Conway polynomial [Ho, Hs], which now is a monomial. This then shows that the canonical surface of the right of (5.23) is minimal genus, in accordance with [KL].)

The diagram on the right of (5.23) unknots by removing the parallel clasps in the tangles (excluding $-1/a_0$), which gives a diagram of genus 0. (Keep in mind that $|q_i| - 1$ is even. By resolving $(|q_i| - 1)/2$ parallel clasps, in each tangle on the right of (5.23) one is left with one nugatory crossing.) If one of these parallel twists is negative (i.e., some $a_i < -1$), then not all crossing changes were positive. The Obstruction 5.6 applies for $\nu = 0$.

It remains to see why $a_0 > 0$. Assume $a_0 < 0$. As $n_+ = n_- = (n-1)/2 > 0$, there is some positive tangle on the right of (5.23). By $g(D) - 1$ crossing changes, one leaves one parallel clasp in one positive tangle, and can get $M(-2/3, -1/a_0)$. For $a_0 = -2$ it gives an unknot (this is the mirror image of (2.41)), so $\tau = 0$, and then $\tau \leq 0$ for all $a_0 < 0$ by (2.15). The Obstruction 5.6 applies for $\nu = 1$, so we are done.

Case 1.2.2. Now if there is an $a_i = +1$ or -1 (but not both), a -2 is missing in (5.24) for each $a_i = -1$ and a 2 is missing in (5.24) for each $a_i = +1$.

One can again show that the pretzel surface is minimal genus by showing $\Delta \neq 0$ by Hoste-Hosokawa's formula, unless we have for $a_0 = +2$

$$L(\overbrace{-2,\dots,-2}^{k},\overbrace{2,\dots,2}^{k-1},2)\,,$$

or for $a_0 = -2$ its mirror image

$$L(\overbrace{2,\dots,2}^{k},\overbrace{-2,\dots,-2}^{k-1},-2)\,.$$

The first originates from an L where we have $L(,\dots,1,2)$, which simplifies to $L(\dots,2)$ with a parallel clasp. Then we go to case 2. (The condition (5.18) behaves well under this transition.) Similarly is done with $L(\dots,-1,-2) \to L(\dots,-2)$.

So assume now the diagram on the right of (5.23) is minimal genus. Resolving parallel clasps brings it to a genus 0 diagram, so again all clasps must be positive (by Obstruction 5.6 for $\nu = 0$). Thus all the fractional parts on the right of (5.23) must be positive, and we have all $a_i < 0$ are -1 and we have at most $n_- = (n-1)/2$ of them because of (5.22). It remains to see why $a_0 > 0$.

Assume $a_0 < 0$. As $n_+ = n_- = (n-1)/2 > 0$, there is some negative a_i, which is -1. As some $a_i = -1$, there is a positive tangle on the right of (5.23). Repeat the argument at the end of case 1.2.1 for a sequence of length $g(D) - 1$ and Obstruction 5.6 for $\nu = 1$.

This finishes the case n odd.

Case 2. n is even. Then the a_0 twist is parallel (and sign discrepancies disappear). We modify (5.20) to

$$n_+ = |\{ i : i \ge 0 < a_i \}| \qquad n_- = |\{ i : i \ge 0 > a_i \}|.$$

Case 2.1. $n_+ \ne n_-$

This case is easy. The canonical Seifert surface is minimal, and it is a plumbing of Hopf bands into a twisted annulus. By a result of Kobayashi [Ko], this minimal Seifert surface is unique, so we can apply the remark below Question 2.2, and then by Rudolph it is enough to inspect the positivity of the Hopf bands and annulus. This leads to (5.18) quite easily.

Case 2.2. We assume now

$$n_+ = n_- . \tag{5.25}$$

This is the most difficult case.

Here working with the genus turns out extremely complicated (see Example 5.9 below), thus we will resort to (2.8) and a calculation of the Alexander polynomial.

We say a pretzel link is *twist-parallel* if all a_i twists are parallel. Note that then n is even and the pretzel link is *reverse*.

If Δ_a denotes the polynomial of a diagram with a positive twist of a parallel crossings ($a-1$ parallel clasps), then

$$\Delta_a = \frac{\sqrt{t}^a - (-\sqrt{t})^{-a}}{\sqrt{t}+\sqrt{t}^{-1}}\Delta_1 + \frac{\sqrt{t}^{a-1} - (-\sqrt{t})^{1-a}}{\sqrt{t}+\sqrt{t}^{-1}}\Delta_0 ,$$

which is easy to prove by induction using the skein relation (2.7). Similarly under mirroring,

$$\Delta_{-a} = \left(\frac{\sqrt{t}^a - (-\sqrt{t})^{-a}}{\sqrt{t}+\sqrt{t}^{-1}}\Delta_{-1} - \frac{\sqrt{t}^{a-1} - (-\sqrt{t})^{1-a}}{\sqrt{t}+\sqrt{t}^{-1}}\Delta_0 \right) \cdot (-1)^{a-1}$$

This can be iterated to gain a formula of (2.42) for a twist-parallel link in terms of the polynomials of the links $L(a_1',\dots,a_n')$ with $a_i' \in \{0, \mathrm{sgn}(a_i)\}$. Now note that under the assumption (5.25), we have for $L(a_1',\dots,a_n')$

$$\Delta(L(a_1',\dots,a_n')) = \begin{cases} 1 & |\{ i : a_i = 0 \}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

This gives, using (5.11),

$$\Delta(L(a_1,\dots,a_n)) = (-1)^{\sigma} \prod_{i=1}^{n} \frac{\sqrt{t}^{|a_i|} - (-\sqrt{t})^{-|a_i|}}{\sqrt{t}+\sqrt{t}^{-1}} \sum_{i=1}^{n} \mathrm{sgn}(a_i) \frac{\sqrt{t}^{|a_i|-1} - (-\sqrt{t})^{1-|a_i|}}{\sqrt{t}^{|a_i|} - (-\sqrt{t})^{-|a_i|}} , \qquad \sigma = \sum_{a_i<0} \tilde{a}_i . \tag{5.26}$$

The list of coefficients, from the top degree downward, of

$$\frac{\sqrt{t}^a - (-\sqrt{t})^{-a}}{\sqrt{t}+\sqrt{t}^{-1}}$$

is $1\ -1\ 1\ -1$ etc. The maximal degree of the polynomials (with the common factor) summed in (5.26) is

$$\sum_{i=1}^{n} \frac{\tilde{a}_i}{2} - \frac{1}{2} = g(D) - 1 . \tag{5.27}$$

One can infer from the formula (5.26) about $\max\deg\Delta$ as follows.

If some $|a_i| = 1$, then the corresponding summand in (5.26) vanishes, and since no both $a_i = \pm 1$ occur,

$$\sum_{|a_i|>1} \mathrm{sgn}(a_i) \ne 0 ,$$

so by (5.27),

$$\max\deg\Delta = g(D) - 1\,.$$

Now assume $|a_i| > 1$ for all i. The span of

$$W_{|a_i|} = \frac{\sqrt{t}^{\tilde a_i} - (-\sqrt{t})^{-\tilde a_i}}{\sqrt{t}+\sqrt{t}^{-1}}$$

is $|a_i|-2$. In (5.26) all top coefficients will cancel in the first span $W_{|a_i|}+1$ degrees for all $a=|a_i|$ satisfying

$$|\{\,i\,:\,a_i=a\,\}|\,=\,|\{\,i\,:\,a_i=-a\,\}|\,. \tag{5.28}$$

Thus

$$\max\deg\Delta = g(D) - 1 - \mathrm{span}W_{|a_{i_0}|} - 1 \,=\, g(D) - |a_{i_0}|\,, \tag{5.29}$$

so that $|a_{i_0}|$ is the smallest $a=|a_i|$ for which (5.28) fails:

$$|\{\,i\,:\,a_i=a\,\}|\,\neq\,|\{\,i\,:\,a_i=-a\,\}|\,. \tag{5.30}$$

To summarize this argument, write

$$\max\deg\Delta(D) = g(D) - \alpha\,, \tag{5.31}$$

where

$$\alpha=\alpha(a_1,\dots,a_n)=\left\{\begin{array}{cc} 1 & \text{some } a_i=\pm1 \\ \min\{a>0\,:\,(5.30)\text{ holds}\} & \text{otherwise}\end{array}\right\}. \tag{5.32}$$

For example,

$$\alpha(3,-3,3,5,-7,-3)=5\,,\qquad \alpha(2,-3,3,5,-7,2)=2\,. \tag{5.33}$$

The formula (5.31), of course, does not restrict to knots, except $g(D)$ must be changed to $(1-\chi(D))/2$ for links, and if (5.28) holds for all a, then $\Delta=0$.

Now focus on knots, so exactly one a_i is even, say $a_n=a_0$. We modify the Obstruction 5.6 as follows. It follows from (2.8) together with (2.15) and Livingston's result below it that if a knot L is strongly quasipositive,

Obstruction 5.7 any unknotting or more generally slicing sequence (i.e., sequence of crossing changes giving a smoothly slice knot) has at least $\max\deg\Delta(L)$ crossing changes. Moreover, if it has length $\max\deg\Delta(L)$, then all crossing changes are positive (to negative).

We will again only use crossing changes through removing parallel clasps in D, so it is enough to reach a slice knot diagram D' with $g(D')>\alpha$, or $g(D')=\alpha$ but not all crossing changes are positive. Also, to keep notation at bounds, we will afford permutations of the entries. Mutations will not change an unknot, and the sliceness of the particular knots we arrive at. We add at a few places concrete examples with permuted entries to illustrate the crossing changes[10].

Case 2.2.1. Some $a_i=1$. In particular all negative $a_i\le -2$.

Case 2.2.1.1. All positive a_i are 1. Then the mirrored changes (5.19) will give a negative link, which is not strongly quasipositive.

Case 2.2.1.2. Not all positive a_i are 1, but all are 1 or 2. By $g(D)-2$ crossing changes, one gets the (genus 2) diagram

$$L(\overbrace{1,1,\dots,1}^{k},-3,\overbrace{-1,\dots,-1}^{k},2)\,,$$

which is an unknot. Obstruction 5.7 applies for $\alpha=1$ in (5.32).

Case 2.2.1.3. Some positive a_i is at least 3.

[10] It is likely that one can slightly simplify this argument by using the part of [FLL, Theorem B] we discussed in Lemma 4.30. But we leave this only as a hint, since the simplification is not significant.

Case 2.2.1.3.1. $a_0 > 0$. By $g(D) - 3$ crossing changes, one gets the (genus 3) diagram

$$L(\overbrace{1,1,\dots,1}^{k},3,-3,\overbrace{-1,\dots,-1}^{k+1},2) = L(3,-3,-1,2)\,, \tag{5.34}$$

which is the slice knot 8_{20}. Obstruction 5.7 applies again.

Case 2.2.1.3.2. $a_0 < 0$. Get similarly to (5.34)

$$L(\overbrace{1,1,\dots,1}^{k+1},3,-3,\overbrace{-1,\dots,-1}^{k},-2) = L(3,-3,1,-2)\,,$$

and mirror the argument.

Case 2.2.2. Some $a_i = -1$.

Case 2.2.2.1. All negative a_i are -1. Since there are $n_- = n/2$ of them, we have condition (5.18), as claimed.

Case 2.2.2.2. Not all negative a_i are -1, but all are -1 or -2. Mirror the crossing changes in case 2.2.1.2.

Case 2.2.2.3. Some negative a_i is at most -3. Mirror the crossing changes in case 2.2.1.3.

Case 2.2.3. No $a_i = \pm 1$. Fix some i_0 with $|a_{i_0}| = \alpha$. Then note the following two properties, which we will use below.

- There is always an $|a_{i'_0}| > \alpha$ and $\mathrm{sgn}(a_{i'_0}) = -\mathrm{sgn}(a_{i_0})$. Let us assume up to permutations that $i'_0 = n-1$ or $i'_0 = 0$.
- All a_i with $|a_i| < \alpha$ pair up $(-a,a)$ (*pairing property*).

Case 2.2.3.1. α is even, so $i_0 = 0$. Change crossings to

$$D'' = L(-3,3,1,\dots,1,-1,\dots,-1,-a_0 - \mathrm{sgn}(a_0),a_0)$$

with equally many 1 and -1, with $g(D'') = \alpha + 2$. Then change two crossings to

$$D' = L(-1,1,1,\dots,1,-1,\dots,-1,-a_0 - \mathrm{sgn}(a_0),a_0)\,,$$

with $g(D') = \alpha$, which is an unknot, but not all crossing changes were positive. As an example,

$$D = L(7,-5,5,-2) \to D'' = L(3,-3,3,-2) \to D' = L(1,-1,3,-2)\,, \quad i'_0 = 3\,.$$

Case 2.2.3.2. α is odd. Since only one a_i is even, namely for $i = 0$, because of the pairing property we have $|a_0| > \alpha$. So let $i'_0 = 0$.

Case 2.2.3.2.1. If $\mathrm{sgn}(a_0) = -\mathrm{sgn}(a_{i_0})$, then change crossings to

$$D'' = L(-3,3,1,\dots,1,-1,\dots,-1,a_{i_0},-a_{i_0} - \mathrm{sgn}(a_{i_0}))$$

with equally many 1 and -1, with $g(D'') = \alpha + 2$. Then change two crossings to

$$D' = L(-1,1,1,\dots,1,-1,\dots,-1,a_{i_0},-a_{i_0} - \mathrm{sgn}(a_{i_0}))\,,$$

with $g(D') = \alpha$, which is an unknot, but not all crossing changes were positive. For example,

$$L(3,5,-5,-6) \to L(3,3,-3,-4) \to L(3,1,-1,-4), \quad i_0 = 1\,.$$

Case 2.2.3.2.2. If $\mathrm{sgn}(a_0) = \mathrm{sgn}(a_{i_0})$, then there is an odd $|a_{i''_0}| > \alpha$ and

$$\mathrm{sgn}(a_{i''_0}) = -\mathrm{sgn}(a_{i_0}) = -\mathrm{sgn}(a_0)$$

(and $i_0'' \neq 0$). Examples are

$$L(3,-9,-7,6), \quad L(3,-9,-7,10), \quad i_0 = 1, i_0' = 0, i_0'' = 2.$$

Case 2.2.3.2.2.1. $|a_{i_0''}| > |a_0|$. Proceed as in case 2.2.3.1. For example,

$$D = L(3,-9,-7,6) \to L(3,-7,-3,6) \to L(1,-7,-1,6) = D', \quad i_0 = 1,\ i_0' = 0,\ i_0'' = 2,$$

and here even $g(D') > \alpha$.

Case 2.2.3.2.2.2. $|a_{i_0''}| < |a_0|$. Proceed as in case 2.2.3.2.1, with i_0 replaced by i_0''. For example,

$$D = L(3,-9,-7,10) \to L(3,-3,-7,8) \to L(1,-1,-7,8) = D', \quad i_0 = 1,\ i_0' = 0,\ i_0'' = 3,$$

and again $g(D') > \alpha$. □

5.4 Further properties and examples

We remark a few useful consequences that can be inferred from formula (5.26) and its discussed implication (5.31) on the degree of the Alexander polynomial of twist-parallel links (as defined below (5.25)).

Corollary 5.8

1) Let L be a twist-parallel pretzel knot (5.17) with (5.2). Then (2.8) is an equality if $\max\deg\Delta < n/2$. (In particular the Alexander polynomial detects the unknot among such knots.)

2) Let L be a twist-parallel pretzel link. Then $\Delta = 0$ if and only if all the a_i can be paired up $(k_i, -k_i)$.

Proof.

1) The case $n_+ \neq n_-$ is easy to settle extra, and leads to (5.31) with $\alpha = 0$; it is also treated in [KL].

   Now let $n_+ = n_-$. Assume n and exactly one a_i is even. (See the notes below (5.25) resp. (5.33).) For suggestive reasons, further assume (5.2) and that $n \geq 4$.

   Order a_i so that $|a_i| \leq |a_{i+1}|$ and $|a_{i_0+1}| > |a_{i_0}| = \alpha$, with i_0 set as below (5.29). By looking at the place of the (only) even a_i, one sees that $i_0 < n$. Now use (5.27) and (5.31), setting (5.11), in the form

$$\max\deg\Delta = \frac{\tilde{a}_1 + \tilde{a}_2 + \ldots + \tilde{a}_{i_0-1} - \alpha + \tilde{a}_{i_0+1} + \ldots + \tilde{a}_n}{2}$$

   All $+/-$ terms in the numerator are non-decreasing, and there is at least one after $-\alpha$.

   If $\alpha > 1$, then all terms before $-\alpha$ are at least 2. Thus $\max\deg\Delta \geq n-2 \geq n/2$.

   If $\alpha = 1$, then $i_0 \leq n/2$ because of (5.2), and if $i_0 = n/2$, then L is alternating. Thus assume that $i_0 < n/2$, and then $\max\deg\Delta \geq n/2$.

2) This was noted below (5.33). The links specified in the second part are up to mutations amphicheiral even-component links, which is another explanation why the Alexander polynomial vanishes. □

The below example shows that the first part is not true if $\max\deg\Delta = n/2$.

Example 5.9 The knot $L(-3,3,3,-4) = 13_{7279}$ (for notation, see [St5, §2]) has genus 4, according to the tabulation [St13], but its mutant, the knot $L(3,-3,3,-4) = 13_{7278}$, has genus 2. We have $\max\deg_m P = 8$, so by (2.11) both knots have canonical genus 4. Thus even among pretzel knots there are not canonically minimal ones and genus may be mutation variant.

Example 5.9 hints to major problems in using (minimal genus) surfaces in any attempts to extend to the case of reverse Montesinos links. Similarly, explicit calculation of the Alexander polynomial is hardly possible, which explains why we expect no adequately long proof (at the least, even if a result is adequate) in this case.

Corollary 5.10 Every strongly quasipositive pretzel knot is pseudo-positive.

Proof. Our results on pretzel knots show what the only pretzel diagrams (2.42) where the condition there fails to correctly detect strong quasipositivity are the reverse pretzel diagrams with negative a_i being only $a_i = -1$ and exactly (but not less than) $\lfloor n/2 \rfloor$ of them (compare in (5.18)). But we also saw that these examples are in fact positive, so that going over to a different diagram, we can make the pseudo-positivity condition apply again. $\square$

Thus knots of the type in Example 4.27 do not occur among pretzel knots, at least.

Another (promised) statement, again without restriction to knots, is this. (Compare above Theorem 3.17 and the list below Corollary 4.20.)

Proposition 5.11 Let D be an alternating parallel or pure pretzel link diagram. Then it is ind-optimal. In particular, its link has a minimal string Bennequin surface.

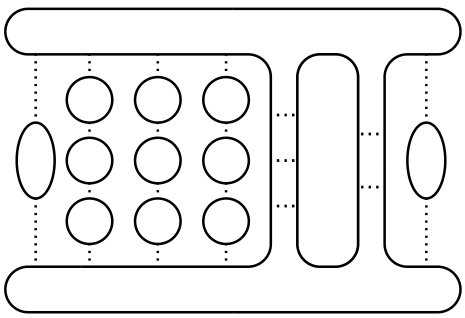
It should be stressed again that at least the property to be pure does not really depend on component orientation, while genus and braid index of course do. The statement is then to be understood for any possible orientations.

Proof. We use the generators and series, as in the proof of Proposition 4.6.

Case 1. Assume first D is parallel. Then it lies in the series of a $(2,n)$-torus link generator. The generator is ind-optimal by Theorem 3.17 (or direct inspection), and its maximal independent set is empty. Thus by [St12, Lemma 5.19], all diagrams in the series are ind-optimal.

Case 2. Next assume D is reverse and pure.

Then as D is alternating, the sign of a_i is determined by whether the twist is parallel or reverse. Let us mirror the diagram w.l.o.g. so that parallel a_i are negative and the reverse are positive. (Note that positive a_i are thus even, as is *the number of* negative a_i.) As a reference, below is the Seifert circle picture (with crossings marked as dashed lines) of $L(2,4,4,4,-3,-2,2)$ in this convention.



(5.35)

Case 2.1. Assume first there are negative a_i (i.e., parallel twists). When each positive a_i is either set to 2 or removed (by smoothing out), the diagram D is stably alternating. (We assumed that the pretzel diagram is pure, thus all negative $a_i \leq -2$, which are not changed.) Since by Theorem 3.17, D is ind-optimal, one can determine $\min\deg_l P(D)$ and $\max\deg_l P(D)$, since with (2.21), also (2.12) and (2.13) become exact. Then show that on the right of

$$P(D_+) = -l^2 P(D_-) - mlP(D_0) \tag{5.36}$$

(which follows from (2.10)), minimal and maximal l-degree terms do not cancel. Then one works by induction over the positive (even) a_i.

Case 2.2. There remains one case, where all $a_i > 0$ (and even). This is again a series of a special alternating generator, the $(2,2,\dots,2)$-pretzel diagram (with all clasps reverse). This is a link, but the argument behind [St12, Lemma 5.19] does not require to restrict oneself to knots. To apply the lemma, one is only left to check that $D = L(2,\dots,2)$ is ind-optimal (with $\text{ind}(D) = 1$). This is again a matter of a simple recursive skein calculation using (5.36). $\square$

If the reverse link is non-pure, the needed verification is more technical (and lengthy), and we prefer not to get into it here. Note, though, that the argument does readily work for (at most) two negative (i.e., parallel) a_i, even if some is -1.

6 Problems

In conclusion, we briefly review several questions upon which we came touching in the discussion. We return to Question 2.3.

The following lists information about this problem known (to me).

- Conditions on minimality of *positive* braids $\beta \in B_n$, i.e., when $n = b(L)$ for $L = \hat{\beta}$, have been studied (see [Na2] and Nakamura's theorem [St2, Theorem 5.2]). It is known for positive braids β with a full twist, i.e., $\beta = \delta_n^2\alpha$ with α positive (likely first observed in [FW]). T. Ito pointed out that this is generalized in [IK, Corollary 1.13]. Another example[11] are words
$$\beta = \prod_j \sigma_{i_j}^{k_j} \tag{6.1}$$
with all $k_j > 1$. There are further constructions possible; see, though, the last point in this list.

- It is true for the positive links among the classes for which Question 6.2 is resolved (see below). In particular it is true for (positively) special alternating links among the 2-bridge links, parallel or pure pretzel links (see below Corollary 4.20 and Proposition 5.11), or knots of genus at most 4.

- The additivity of genus and braid index under connected sum implies that if the property holds for links L_1, L_2, it holds also for $L_1\#L_2$ (regardless of which components the connected sum is performed at).

- It follows from the proofs of Propositions 3.12 and 3.19 that if K is a strongly quasipositive knot on a minimal braid, and β is strongly quasipositive, then K_β is strongly quasipositive on a minimal braid.

- The result is known for 'strongly quasipositive' replaced by 'quasipositive' [Ha], but it is false for 'positive (braids)' [St4, Example 7].

Next we return to the problem outlined in Remark 3.15. (The answer is trivial if $K = U$ is the unknot.) For the related discussion of connected sums, see [Or]. Contrast also with Theorem 3.16.

Question 6.1 If K_β is strongly quasipositive (resp. quasipositive), with $K \neq U$, are both K and β strongly quasipositive (resp. quasipositive)? At least one of both?

An analogous class of problems arose for the more general Bennequin surfaces.

Question 6.2 (see [St15, Question 8.3.4]) If L is an alternating link, does L have a minimal string Bennequin surface? What other type of link? Perhaps even a canonically minimal link (i.e., one which has a diagram D with (2.3))?

[11] These examples are disjoint; in fact it follows from Thistlethwaite's work on adequacy that none of (6.1) is Markov equivalent to a positive word in B_n containing even a $[\delta_3]_l$ for some $0 \leq l < n-1$.

It was proved for example in [St15] that any alternating knot of genus at most 4 has. In the proof of Proposition 4.4 we did as well the special alternating links L with $\chi(L) \geq -2$, and in Proposition 5.11 alternating pure pretzel links. In Proposition 3.18 we discussed stably alternating links (including the fibered ones). It was also checked for all alternating knots of at most 18 crossings (using the method rolled out again in §4). This is further true for an alternating link of braid index at most 4, every (also non-alternating) 3-braid link (see [St2]), and every 2-bridge link [Hi]. On the other hand, some other (non-alternating) knots (failing (2.3)) have no minimal string Bennequin surface [HS]. By Corollary 4.20, Conjecture 4.7 implies the positive answer for an alternating link. This topological outlook may lend new motivation to the study of the graph index.

As mentioned below Conjecture 2.5, it follows from Theorem 2.6, that a minimal string Bennequin surface of a strongly quasipositive link is strongly quasipositive. (This is very closely related to the reason for (2.33) and (4.47).) Thus Question 6.2 for strongly quasipositive links is equivalent to Question 2.3. Example 5.4 and its likes leave Question 2.3 open for pretzel knots even after the description achieved in §5.

Question 6.3 If $L_1\#L_2$ has a minimal string Bennequin surface, do L_1, L_2 have?

The converse is true as per the remark on connected sums below Question 2.3.

Proposition 3.19 prompts also the following version of Question 6.1. (Exclusion of the case $K = U$ is now imperative, by means of the aforementioned examples in [HS].)

Question 6.4 Is the following true (or at least some direction of it, and at least in what special cases)? The cable link K_β (for $K \neq U$) has a minimal string Bennequin surface if and only if K has a minimal string Bennequin surface and β can be written in bands so as to give a Bennequin surface of $\hat{\beta}$.

At least the 'if' direction is very much suggested by the proof of Proposition 3.19. This implication is true if (2.8) is exact for both K and $\hat{\beta}$, and the Alexander polynomial is in general a very efficient algebraic means in estimating genus. The 'if' part is also true for strongly quasipositive K and β, as we noted above.

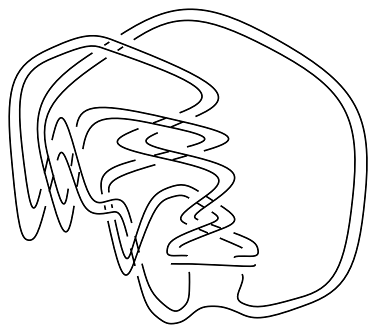
Another question pops up in relation to Example 3.13: the Hopf plumbing argument there can certainly exclude a positive diagram; the same argument would work for a homogeneous diagram, etc., but can it recover an inequality like (3.9) for any diagram D? Extensive computer experiments [St14] have shown that fiber surfaces coming from Seifert's algorithm, already for knots of genus 2, may be very difficult to describe. Nevertheless, even in the below formulated most simple form of the problem, counterexamples do not seem to be known.

Question 6.5 Is every canonical fiber surface a Hopf plumbing?

After Example 4.47 falsified the diagrammatic style of converse to Corollary 4.44, the proof of the corollary suggests a geometric alternative question. The appended remarks clarify why large χ do not fit well, but the essence of the question is to understand better what is the effect of a Murasugi sum.

Question 6.6 Is at least one minimal genus surface for $\chi < 0$ of a general link L a Murasugi sum of strongly quasipositive and strongly quasinegative surfaces? Every minimal genus surface when χ is small? At least a stable Murasugi sum?

It is obviously true for 'one minimal genus surface' for pseudo-homogeneous links. For 'one minimal genus surface' it is also true for a strongly quasipositive link. By Rudolph's work [Ru5], the question for 'every minimal genus surface' of a strongly quasipositive link is equivalent to Question 2.2 (as strongly quasinegative surfaces will not occur as Murasugi summands). It the case $\chi = 0$, no Murasugi sum is possible, and the answer is negative by arguments closely related to Theorem 3.16. (This needs to be explained separately elsewhere.) This provides some intuition against the case $\chi < 0$ as well. However, there is little control over what a Murasugi sum can yield, in particular if we consider small χ.



Example 6.7 The following example for $\chi = -1$ shows that the answer is negative for 'every'. This is a Brittenham [Br] free genus 1 knot, its HOMFLY polynomial shows that it is not strongly quasipositive or -negative. Now assume its free genus 1 surface is a Murasugi sum. A free Murasugi sum must have free summands, and since the Alexander polynomial is monic, by observation the surface must be a Murasugi sum of Hopf bands, thus a fiber surface. But the knot is not a trefoil or a figure-8 knot.

We add here the question which was motivated and discussed in relation to Fact 4.52. (For $\sim$-reduced recall the definition there.)

Question 6.8 Are there only infinitely many pseudo-homogeneous (or pseudo-positive) links with the same P (or Jones) polynomial? Does a link have only finitely many $\sim$-reduced pseudo-homogeneous diagrams?

A final problem is highlighted due to its apparent lack of previous record.

Question 6.9 Are homogeneous diagrams visually prime?

This fits well with the special cases of alternating and positive diagrams (proved by Menasco and Ozawa, resp., as discussed in §2.2). It is interesting that Cromwell [Cr], who initiated the homogeneity concept, asks the question not for homogeneous diagrams, but for minimal genus ones. (Perhaps a reason was, as outlined at the start of §4.4.6, that he designed homogeneity rather as a vehicle to show minimal genus.) But in this generality the question was answered negatively in [St3]. In Example 4.49 above (and in another similar example in [IS]), it is possible to improve the "quality" of such counterexample in the hierarchy of §2.2, but these same checks have insistently confirmed all homogeneous diagrams encountered.

A Appendix: table of alternative and homogeneous prime knots

Cromwell [Cr, Appendix] lists a table of homogeneity of the (non-alternating) Rolfsen knots, which is easily found to contradict Example 2.1. But there is additional confusion.

Without attempts to check the table, some effort to resolve (negatively) the open cases therein was made in [St3]. But there I mistook the definition of homogeneous for alternative, as e.g. correctly reflected in [Si, Theorem 3.6]. (This mixup also led to the relabeling of the true property 'homogeneous' as 'semi-homogeneous' in [St11, St12].) The exclusion of Cromwell's knots in [St3] is thus, as recorded, only from being alternative.

Below we aim to settle all this disarray. We have not investigated what exactly went wrong where in Cromwell's tests. To be on the reliable side, as in Example 2.1, we used only two tools. First, if D is homogeneous, and D' obtained from D by crossing changes making it alternating, then the Alexander polynomials $\Delta(D)$ and $\Delta(D')$ have the same degree with the same leading coefficient up to sign. Then we applied the generator-twisting method that was largely deployed in [St3] (and [St12], etc.)

For verification reference, it is best to give the table in terms of a list of alternative diagrams, written for compactness in the Dowker-Thistlethwaite notation (see end of §2.2).

8	19	4	8	-12	2	-14	-16	-6	-10		
9	43	4	8	10	14	2	-16	6	-18	-12	
9	47	6	-16	-14	12	-4	-2	18	-10	-8	
9	49	6	-10	-14	12	-16	-2	18	-4	-8	
10	124	4	8	-14	2	-16	-18	-20	-6	-10	-12
10	128	4	10	-14	-18	2	-16	-20	-8	-12	-6
10	134	4	10	12	-16	2	8	18	20	-6	14
10	138	4	8	10	16	2	-14	-18	6	-20	-12
10	139	4	10	-14	-16	2	-18	-20	-6	-8	-12
10	142	4	10	-14	-16	2	-18	-20	-8	-6	-12
10	152	6	8	12	2	-16	4	-18	-20	-10	-14
10	154	4	8	12	2	-16	6	-18	-10	-20	-14
10	161	4	12	-16	14	-18	2	8	-20	-10	-6

Below follow the homogeneous (non-alternative) ones for the knots in Example 2.1; keep in mind our disposal over (and of) the Perko duplication.

10	150	4	12	-16	14	18	2	8	-20	10	-6
10	151	4	12	16	14	-18	2	8	20	-10	6
10	156	4	12	16	-14	18	2	-8	20	10	6
10	158	6	-10	-16	20	-14	-2	-18	-4	-8	-12
10	160	4	12	-16	-14	-18	2	-8	-20	-10	-6
10	163	6	-10	14	16	-4	18	2	20	12	8

The remaining knots are non-homogeneous (and non-alternative).

The Cromwell table thus reports 8 and 9 crossing non-alternating prime knots correctly; all those homogeneous knots are also alternative. The entries for 10_{150} and 10_{163} (his 10_{164}) must be positivized. For the five question-marked Cromwell knots, none is alternative (as in the corrected statement of [St3, Corollary 4.1 and Proposition 13.1]), but 10_{151}, 10_{158} and 10_{160} are homogeneous (with its correct definition).

The remaining two knots 10_{144} and 10_{165} (Cromwell's 10_{166}) are indeed non-homogeneous. More generally, [Si, Theorem 3.6] still holds for 'homogeneous', as stated in [St3, Theorem 4.1]. The proof only needs to add the check that the only genus 2 generators with at least 3 Murasugi atoms are 7_6, 7_7, 8_{12}, and 10_{58}, they all admit only type B flypes (see [St3, Figure 2]), and any homogeneous diagram in their series can be flyped to an alternative one. (This upgrade can then also reduce to non-positivity the non-homogeneity proof for 20 more non-alternating genus 2 Rolfsen knots, but was *not* used in the above check.)

As known from [St4], in general knots may realize the properties alternative and homogeneous only in non-minimal diagrams. (Both questions in [Cr, §5] are answered negatively.)

B Appendix: Strongly quasipositive prime knots up to 16 crossings

B.1 Band representations

The following determines which (so far availably) tabulated knots are strongly quasipositive. We have not considered alternating knots, for reasons explained above (and recalled in §B.2).

Since Morton's conjecture (4.49) was known to be true up to 16 crossings [St18, §2], for compiling candidates we tested

$$2\tau = 2g = \min\deg_l P\,.$$

This rendered a list of 22,009 knots. All knots are Bennequin sharp, thus this is the complete list of such knots up to 16 crossings.

The following division into genera has some impact on how knots were (further) handled. The $(-3,5,5)$- and $(-3,5,7)$-pretzel knots are the (only) examples of genus 1.

genus	1	2	3	4	5	6	7
# knots	2	282	4149	10440	6143	957	36

We checked a systematic list of band representations, which confirmed all knots to be strongly quasipositive (in accordance with Conjecture 2.4). However, for many of the knots, we knew strong quasipositivity from diagrams far sooner, and more efficiently, than finding a band representation. The larger the genus and braid index, the more words it is necessary to check, but the more likely it is to find a diagram showing strong quasipositivity by the methods of §B.2. They worked flawlessly for genus 5 or higher.

We can also resolve, for all these knots, Question 2.3.

Proposition B.1 For all prime strongly quasipositive knots up to 16 crossings, there is a minimal string positive band representation.

Proof. This was estiablished by explicitly compiling positive band representations for the non-alternating knots, and then determining that they have minimal strand number.

The braid index of some of the knots is extraordinarily difficult to determine. The hardest cases could not avoid a multiple-week calculation per entry. Among them by far stands out $16_{1057125}$, which required its separate technological upgrade. Further information is moved out to [St13]. □

For the related discussion of composite knots, see [Or].

The list of band representations allowed us also to easily obtain a series of the following type of examples (which seem more common than expected).

Example B.2 Consider the two knots with positive 4-band representations (using the definitions in §2.4):

$$\begin{array}{ll} 12_{1393} & \sigma_3\sigma_{1,4}\sigma_{1,3}\sigma_{2,4}\sigma_1\sigma_2\sigma_1^5 \\ 14_{39299} & \sigma_3\sigma_{1,4}\sigma_{1,3}\sigma_{2,4}\sigma_1^2\sigma_2\sigma_1^6 \end{array}$$

Both knots are fibered, and the (strongly quasipositive genus 5 fiber) surface S of $K = 14_{39299}$ is a (repeated) positive Hopf plumbing of the (strongly quasipositive genus 4 fiber) surface S' of $K' = 12_{1393}$. But while K' has braid index 4, K has an almost positive 3-braid representative. Thus $(b(K) =)b_q(K) = b(S) = 3$ while $(b(K') =)b_q(K') = b(S') = 4$. This means that b_q can decrease under (positive) Hopf plumbing of strongly quasipositive surfaces, even for fiber (and in particular unique minimal) surfaces. Another such (fibered) pair of genus 5 and 6 are $K' = 13_{6169}$, $K = 14_{46970}$.

B.2 Diagrammatic tests

The methods of [FLL] and §4.4.4 here are sufficient (and efficient) to show strong quasipositivity without the need to explicitly find a positive band representation, for all but less than 100 of the knots.

Obviously, (special) alternating diagrams are pseudo-positive, so Corollary 4.24 is infallible for alternating knots. And they were excluded from any further computation.

For non-alternating knots, it turned out that $> 90\%$ of prime strongly quasipositive nom-alternating knots up to 16 crossings are pseudo-positive or almost-positive in minimal crossing diagrams.

Among knots without minimal crossing almost positive diagrams, extensive test of minimal genus diagrams (of higher crossing number) left at most 65 (only genus 3,4) undecided for pseudo-positivity. (We did not test for almost positivity of non-minimal crossing diagrams; this test seems not very restrictive.)

Further 19 knots (all of genus 2) are ruled out from being pseudo-positive (17 by Morton's inequality (2.11) and two more in the list of genus 2 knots where (2.11) was proved unsharp in [St3]). All these mentioned knots are 15,16 crossing, thus in particular until 14 crossings, all strongly quasipositive knots are pseudo-positive or almost positive. (Until 13 crossings, it was mentioned in the introduction of [FLL] that all are pseudo-positive, by reference to previous calculations of ours.)

Beware again that knots may have pseudo-positive diagrams only of far higher crossing number. An example is 14_{38473}, where only a 19 crossing (genus 3) diagram occurs. It is included here for illustrative purposes:

```
19 38473 6 12 20 38 28 4 -18 34 26 2 -10 32 14 22 8 -36 16 24 -30
```

As usual, a small number of such "difficult" cases remain, where complicated diagrams have to be searched and come out rather slowly. This should illustrate that, while all diagrammatic strong quasipositivity tests were ran on a modest laptop within 3-4 days, proper management is crucial for making this a tractable task.

Yet, the pseudo-positivity approach turned out far more efficient for large genus and/or braid index, where the number of band representations to process grows disproportionally. While we knew from Corollary 4.24 much sooner that the knots are strongly quasipositive, we still tried to test positive band representations, also to address Question 2.3. But this was a very time consuming process, and not feasible for some portions of the list. Corollary 4.35 remained the only viable help.

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