ON SOME RESTRICTIONS TO THE VALUES OF THE JONES POLYNOMIAL

A. Stoimenow

Graduate School of Mathematical Sciences,
University of Tokyo,
3-8-1, Komaba, Tokyo 153-8914, Japan
e-mail: stoimeno@ms.u-tokyo.ac.jp
WWW: http://www.ms.u-tokyo.ac.jp/˜stoimeno

Abstract. We prove that Jones polynomials of positive and almost positive knots have positive minimal degree and extend this result to an inequality for k-almost positive knots. As an application, we classify k-almost positive alternating achiral knots for $k \le 4$, and show a finiteness result for general k. Another consequence is a proof that almost positive and fibered positive links (with the obvious exceptions) are non-alternating (latter extending the result for torus knots known from Murasugi, Jones, and Menasco–Thistlethwaite), and that if a positive knot is alternating, then all its alternating diagrams are positive.

1 Introduction

In 1984, Jones [J] discovered, using C^*-algebras, a new polynomial invariant V of oriented knots and links, initiating a variety of far-reaching results in knot theory. This polynomial is a Laurent polynomial in one variable t and can be defined by being 1 on the unknot and the (skein) relation

$$t^{-1}V_+ - tV_- + (t^{-1/2} - t^{1/2})V_{)(} = 0\,, \tag{1}$$

with $V_+, V_-, V_{)(}$ denoting diagrams equal except near one crossing, which is resp. positive, negative and smoothed out.

Although similarly defined, the Jones polynomial strikingly contrasts to its predecessor, the Alexander/Conway polynomial. E. g., it distinguishes (in general) knots from their mirror images, and via cables knots it fails to distinguish themselves [MS]. Although much was observed and proved about and using the Jones polynomial (see e.g. [MT, Mu, MTr, Ka2, Fi, LT]), the 18 years since its discovery have seen little progress towards its conceptual understanding.

The main focus of this paper will be put on studying properties of the values of the polynomial on k-almost positive knots.

The "k-almost" nomenclature was introduced by C. Adams [Ad] for alternating links. Here k is the minimal number of crossing changes necessary to perform in any diagram of the link to obtain an alternating diagram. The (alternating) case $k = 0$ is classic and much is known about it [Ka, Th, MT, Cr, Me]. Unfortunately, for the case $k = 1$ none of the nice properties of knot polynomials carry over. C. Adams proved in [Ad], that a prime satellite link is not (1-)almost alternating, extending the result for alternating links due to Menasco [Me]. However, for $k = 2$ this is also no longer true (Adams quotes the Whitehead double of the trefoil in his book [Ad2] as a counterexample), and so for these cases almost nothing seems to be known. In particular, it seems unclear whether k-almost alternating knots exist for infinitely many $k \in \mathbf{N}$.

Here we consider the analogous notion for positivity. The method applied to prove the degree bounds for the Jones polynomial on positive links is a simple application of the Jones skein and Kauffman bracket relation and we extend it to k-almost positive links. Thus we obtain, in a contrast to the alternating case, a property of the Jones polynomial of a k-almost positive link for any given k, which serves as an obstruction to k-almost positivity. This obstruction appears hard to use in practice for higher k, but it can be applied to certain classes of links. We discuss as one such application the existence of k-almost positive knots for each $k \in \mathbf{N}$ (example 4.4). We also extend the classification of achiral k-almost positive knots for $k \le 2$, in the case the knot is alternating, to $k = 3$ and $k = 4$, and prove a finiteness statement for general k (a consequence of theorem 5.5).

Some of the results of this paper on k-almost positive links for $k \le 3$ have also been announced (without published proofs) by Przytycki and Taniyama [PT][1]. They perform a more detailed analysis of some exceptional cases, while our aim was here to use slightly different arguments, which can be generalized to arbitrary k.

1.1 Definitions, notations, and conventions

In the following $[\text{polynomial}]_{\text{monomial}}$ denotes the coefficient of 'monomial' in 'polynomial'. For a Laurent polynomial $V \in \mathbf{Z}[t, t^{-1}]$ define

$$\min\deg V = \min_{[V]_{t^a} \neq 0} a\,, \quad \max\deg V = \max_{[V]_{t^a} \neq 0} a\,, \quad \operatorname{span} V = \max\deg V - \min\deg V\,,$$

and $\min\operatorname{cf} V = [V]_{t^{\min\deg V}}$.

Diagrams and links are always assumed oriented. Knots are considered links with only one component. A *split component* of a link is an equivalence class of link components modulo the (equivalence) relation, which is the transitive expansion of the relation "inseparable". Two link components are inseparable, if there is no isotopy in $\mathbf{R}^3$ making them separable by a hyperplane (or their projections in a diagram in $\mathbf{R}^2$ separable by a line). A split component may contain several link components. A split component is trivial, if it (has a single link component which) is the unknot.

A split component of a link diagram is a connected component of the plane curve of the diagram. Obviously, a link diagram cannot have more split components than the link it represents.

The term "*projection*" will be used in the same meaning as "*diagram*", and both mean a knot or link diagram. For a link diagram D, $c(D)$ denotes the *crossing number* of D, $w(D)$ its writhe, and $s(D)$ the number of its Seifert circles. The number of components of a link or link diagram will be denoted by n, and by p the number of split components.

We use the notation of [Ro] for knots with up to 10 crossings, renumbering $10_{163} \dots 10_{166}$ by eliminating 10_{162}, the Perko duplication of 10_{161}. We also use the convention of the Rolfsen pictures to distinguish between the knot and its obverse whenever necessary. Knots of > 10 crossings are denoted according to [HT], appending for given crossing number the non-alternating knots after the alternating ones (instead of using superscripts 'a' and 'n').

V denotes the Jones and Δ the Alexander polynomial. σ denotes the signature of a knot, u its unknotting number, g its (Seifert) genus and g_s its 4-ball (slice) genus.

Recall three classic properties of the Jones polynomial. We have

$$V_{!L}(t) = V_L(t^{-1}) \tag{2}$$
$$V_{L\cup L'}(t) = (-t^{1/2} - t^{-1/2})\,V_L(t)\,V_{L'}(t) \tag{3}$$
$$V_{L\#L'}(t) = V_L(t)\,V_{L'}(t)\,, \tag{4}$$

[1] Notice that according to their definition, 'k-almost positive' is what was for us '$\le k$-almost positive'.

where $L \cup L'$ are the split union of the links L and L', $L\#L'$ is (one of) their connected sum(s), and $!L$ is the mirror image (obverse) of L.

The symbol g will be used also for diagrams and for links, with the following meaning (this choice of convention differs from some other papers and is made for technical reasons).

For a diagram D of n components and p split components, define its genus $g(D)$ to be

$$g(D) = \frac{n-\chi(S)}{2} = \frac{c(D)-s(D)+n}{2} = \text{genus}(S)+n-p\,, \tag{5}$$

where S is the Seifert surface obtained by applying the Seifert algorithm to D, and $\chi(S)$ is its Euler characteristic. S will be called the *(canonical) Seifert surface of* D.

For a link L of n components and p split components, define its modified genus $\hat{g}(L)$ to be

$$\hat{g}(L) = \frac{n-\chi(L)}{2} = g(L)+n-p\,, \tag{6}$$

where $\chi(L)$ is the maximal Euler characteristic of a Seifert surface of L. (If L is a knot, i.e. $n=p=1$, then $\hat{g}(L)$ is just the usual genus of L.)

It is known from [Cr2] that for a homogeneous (in particular, alternating or positive) diagram D of a link L, $g(D)=\hat{g}(L)$.

The symbol $\Box$ denotes the end or the absence of a proof. In latter case it is *assumed* to be evident from the preceding discussion/references; else (and anyway) I welcome any feedback.

2 The Jones polynomial on positive and almost positive links

Definition 2.1 A link is called n-almost positive, if the minimal number of negative crossings in all its diagrams is n. A link is called positive, if it is 0-almost positive, and almost positive, if it is 1-almost positive.

Obstructions to positivity exist via the Conway [Bu, Cr2], HOMFLY [Cr2, CM] polynomials and the degree-3-Vassiliev invariant [St2]. Here we give some conditions using the Jones polynomial and extend them to almost positivity.

Theorem 2.2 If L is a link with a positive or almost positive diagram of n components and $c>0$ crossings, then

$$\min\deg V(L) \geq \frac{1-n}{2}$$
$$\max\deg V(L) \leq 2c+\frac{n-5}{2}\,.$$

Moreover, for any $t \in (0,1]$ we have $(-1)^{n-1}V_L(t) \geq 0$.

Proof. Consider the binary tree [Cr2, LM] associated to a positive or almost positive diagram D of L obtained by computing $V(L)$ via the Jones skein relation. In [Cr2] it is shown that it can be chosen so that each crossing is switched at most once. We may even demand that any single initially fixed crossing, call it p, is never switched (see the proof of corollary 2.1 in [Cr2]; p may, however, be eliminated at some stage by separating split components). If D is almost positive choose p to be the negative crossing, otherwise choose p arbitrarily.

Now, the only recurrence relations to apply for a positive or almost positive diagram are of the form

$$V_+ = (t^{3/2}-t^{1/2})V_{)(} + t^2V_-\,. \tag{7}$$

This together with the value of the Jones polynomial on a trivial n-component split link, which is $(-t^{1/2}-t^{-1/2})^{n-1}$, inductively shows that for any link diagram D', appearing in the binary tree,

$$\min\deg V(D') \geq \frac{1-n_{D'}}{2}$$
$$\max\deg V(D') \leq 2\#^+_{D'} + \frac{n_{D'}-1}{2}\,.$$

Here $n_{D'}$ is the number of components and $\#^+_{D'}$ the number of positive crossings of D', with the restriction that p is not counted in $\#^+_{D'}$ (if it has remained in D'). For $D = D'$ this is the first assertion.

The second assertion follows similarly, as for $t \in (0,1]$, the quantities $t^{3/2}-t^{1/2}$ and $-t^{1/2}-t^{-1/2}$ are both negative. □

Remark 2.3 Using the same argument and the inequality

$$\min\deg_v P(D') \geq 1-n_{D'}$$

one can show that the minimal degree of v in the HOMFLY polynomial (in the convention of [Cr2]) of a positive or almost positive link is at least $1-n$, where n is the number of components of L. The other (evaluation positivity) criterion can also be extended to the HOMFLY polynomial. This was done by Morton and Cromwell [CM]. In fact, our result follows from their inequality $P(t^2, t^{-1}-t) \geq 0$ for $t \in (0,1]$, the identity $V(t) = P(-t, -t^{1/2}+t^{-1/2})$ and the fact, that for n components, we have $P(v,z) = (-1)^{n-1}P(-v,z)$.

Corollary 2.4 If L is a positive or almost positive link of n components, then $(-1)^{n-1}\min\mathrm{cf}\,V(L) > 0$.

Proof. Consider the second statement in theorem 2.2 for $t \ll 1$. □

The similarity of the arguments for the positive and almost positive case generalizes all properties of positive knots proved using skein (relation) arguments (such as the theorem of [CM] and all its previously known specializations) to almost positive knots. This, in a way, reveals a manco of the skein approach – using skein arguments it is difficult to distinguish positive from almost positive knots. For example, it is not straightforward to prove that the knot $!10_{145}$ is not positive. It was done by Cromwell [Cr2] using properties of homogeneous links. Another method applicable to this knot are Seifert surfaces [St3].

Thus one should look for alternatives to the skein approach. We discuss one such alternative in the following paragraph.

3 Some applications of Kauffman's state model

A different interpretation of the Jones polynomial than via skein rules has been developed by Kauffman [Ka2, Ka3] (see also [Ad2, §6.2]). The "Kauffman state model" or "Kauffman bracket" is sometimes more useful than the skein approach, as we shall also show in our case, sharpening two of the previous (skein argument) results (theorem 2.2 and corollary 2.4) for positive links.

Recall, that the Kauffman bracket $[D]$ of a diagram D is a Laurent polynomial in a variable A, obtained by summing over all states the terms

$$A^{\#A-\#B}\left(-A^2-A^{-2}\right)^{|S|-1}\,. \tag{8}$$

A state is a choice of splittings of type A or B for any single crossing (see figure 1), $\#A$ and $\#B$ denote the number of type A (resp. type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in a state.

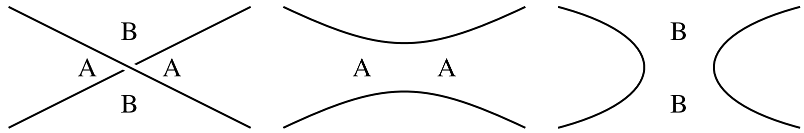


Figure 1: The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.

The Jones polynomial of a link L is related to the Kauffman bracket of some diagram D of L by

$$V_L(t) = \left(-t^{-3/4}\right)^{-w(D)} [D]\Big|_{A=t^{-1/4}}. \tag{9}$$

Theorem 3.1 If L is a positive link of n components, then

$$\min\deg V_L = \hat{g}(L) - \frac{n-1}{2}$$

and $\min\mathrm{cf}\, V_L = (-1)^{n-1}$.

This fact is mentioned for closed positive braids in [Fi] and seems folklore even for positive links, but I rewrite a brief proof. See also [Zu] for a rather different approach.

Proof. In a positive diagram D of L an A splitting of a crossing is the smoothing of the crossing according to the skein rule and the circles obtained in the state of A splittings only are just the Seifert circles of the diagram. We call this state the A-state of D. Then switching an A splitting to a B splitting changes $|S|$ in (8) by ± 1 and reduces $\#A - \#B$ by two. Moreover, at least by the first change from an A splitting to a B splitting $|S|$ is reduced (both fragments of a smoothed crossing belong to distinct Seifert circles and they are connected by the switching). Hence only the A-state gives a contribution to the maximal degree of $[D]$, and it is $(-1)^{s(D)-1}$. The rest follows from (9), the formula (5) for the genus of D, and the minimality of (the genus of) the (canonical) Seifert surface of D (over all Seifert surfaces bounding L), see [Cr2, corollary 4.1]. □

Something more can be said about the minimal degree of V in alternating diagrams.

Theorem 3.2 Let D be an alternating reduced (i.e., with no nugatory crossings) diagram of an n component link L. Let $\#_D^-$ be the number of negative crossings of D. Then

$$\min\deg V_L \le \hat{g}(L) - \frac{n-1}{2} - \#_D^-.$$

Proof. We build the checkerboard shading of D by coloring connected components of its complement in the plane white (resp. black) if they contain just B corners (resp. A corners) of crossings. The number of white regions call W. Then it is known (see [Ad2, exercise 6.10]), that

$$\max\deg[D] = c(D) + 2(W-1).$$

Using the transform (9) of the bracket to V, the formula for the genus of D and its minimality (over the genera of all diagrams of L; see [Cr2]), the inequality we wish to prove is equivalent to

$$-\frac{c(D)}{4} - \frac{W}{2} + \frac{3}{4}w(D) \le \frac{c(D)}{2} - \frac{s(D)}{2} - \#_D^-,$$

which simplifies to

$$-\#_D^- \leq W - s(D)\,. \tag{10}$$

Now any alternating diagram can be transformed by smoothing (according to the skein rule) all negative crossings into a positive alternating diagram, for which both hand sides of (10) are zero. Reversing a smoothing operation augments $\#_D^-$ by 1, whereas it preserves $s(D)$ and changes W by 0 or 1 depending on whether the crossing is nugatory. So the result follows. □

The following corollary summarizes some (partial) answers to questions in [St2, §8].

Corollary 3.3 An alternating diagram of a positive alternating link with no nugatory crossings is positive (or special in the terms of [Cr2]). Therefore, in the case the link is prime, any minimal (crossing number) diagram of it is positive. In the case the alternating link is composite, any prime factor is (alternating and) positive.

This result was shown independently and in a different way by Takuji Nakamura in [N].

Proof. For the first assertion just combine the last two theorems. For the second one use the fact that any minimal crossing number diagram of an alternating prime link is alternating (which follows e.g. from the results in [Ki]). In the case the alternating link is composite, by [Me] so is any alternating diagram of it. So the factors are themselves alternating and they appear as factors in an alternating diagram, and hence are positive by the first assertion. □

There is a well-known inequality $\sigma \geq 2g_s$ for the slice genus g_s, due to Tristram-Murasugi [Mu2, Tr]. (It has noteworthy applications, see [Or] for example.) This inequality is not (in general) sharp for positive knots. In Bennequin's paper [Be] there is a mention of a conjecture of Milnor, that the inequality would be an equality for closed positive braid knots (in fact, Milnor conjectures stronger that $2u = \sigma$ in this case). But even this is false as show some recently obtained unknotting numbers (see [St2, §4]). At least now the inequality reveals to be an equality if the positive knot is alternating.

Corollary 3.4 If a positive knot is alternating, then $2g = 2g_s = \sigma$.

Proof. It follows from the previous corollary, and the observation, that in a positive alternating diagram $\sigma = 2g$ (see [St2]). □

A further consequence (also noticed by Takuji Nakamura) is

Corollary 3.5 Let K be a fibered positive link (in particular, the closure of a positive braid). Then K is non-alternating except if it is the (possibly trivial) connected sum of torus links of type $(2,n)$.

Proof. Assume that a fibered positive link is alternating. Then it has a positive special (or one-block [Cr2, §1]) diagram D, and $\max \mathrm{cf}\,\Delta(D) = 1$. Thus, by [Cr2, proof of theorem 5], the (planar bipartite) Seifert graph $\Gamma(D)$ of D has only cycles of length 2, i.e., its reduced Seifert graph $\hat{\Gamma}(D)$ in the terminology of [MP] is a tree. Then a little thought on such planar graphs shows that D is the connected sum of diagrams D_i with $\Gamma(D_i)$ having only two vertices. (To see this, separate from the graph $\Gamma(D)$ iteratedly the interior of any cell bounded by a 2-cycle.) Such D_i are diagrams of $(2,n_i)$-torus links. (Takuji Nakamura has pointed out that this also follows from proposition 13.25 in [BZ].) □

Remark 3.6 This result follows for the most positive braid links from [Mu3, proposition 7.4]. However, the condition there to exist a positive braid representation of minimal strand number, although often satisfied, is in general difficult to test. Another weaker version of this result for rational knots was shown in [St5]. For torus knots the result is known from Jones [J2, proposition 11.9] and Menasco–Thistlethwaite [MT2].

Remark 3.7 We may also remark that the arguments used in the last proof, together with [Cr] show that for a positive knot K, the following 3 conditions are equivalent (where as before $\hat{\Gamma}(D)$ denotes the reduced Seifert graph of D from [MP]):

1) There is a positive diagram D of K with $\hat{\Gamma}(D)$ a tree.
2) Any positive diagram D of K has $\hat{\Gamma}(D)$ being a tree.
3) K is fibered.

The non-alternation result can also be shown for almost positive links.

Lemma 3.8 If a link L has an almost positive diagram D of n components and genus g, then

$$\min\deg V(L) \geq g-1-\frac{n-1}{2}.$$

In particular for any almost positive knot K, $\min\deg V(K)\geq 0$.

Proof. Apply the skein relation at the negative crossing of D.

$$V_- = -(t^{-1/2}-t^{-3/2})V_{)(} + t^{-2}V_+ \tag{11}$$

The inequality we claim follows from theorem 3.1, noting that in the two terms on the right in (11), a cancelling of the coefficients in the lowest degree occurs. □

Corollary 3.9 An almost positive link L is non-alternating.

Proof. Assume L were alternating and has n components. As a reduced non-positive alternating diagram has at least two negative crossings (because it is homogeneous and a unique negative crossing in a homogeneous diagram must be reducible, see [Cr2, §1]), we have by theorem 3.2

$$\min\deg V_L \leq \hat{g}(L)-2-\frac{n-1}{2}. \tag{12}$$

Now, take an almost positive diagram D of L. Write D_+ for the positive diagram obtained after the switch of the negative crossing and L_+ for the link that D_+ represents. We obtain from the previous lemma

$$\min\deg V(D) \geq \min\deg V(D_+)-1 = \hat{g}(L_+)-1-\frac{n-1}{2}. \tag{13}$$

But L_+ is positive, and hence its modified genus is equal to the genus of D_+, which is the same as for $D=D_-$. Therefore, $\hat{g}(L)\leq\hat{g}(L_+)$ and the r.h.s. of (13) is at least $\hat{g}(L)-1-(n-1)/2$, contradicting (12). □

Example 3.10 The knots $4_1, !6_1$ and $!6_2$ have minimal degree of the Jones polynomial $-2,-2$ and -1 resp., and are hence not almost positive.

4 The Jones polynomial on k-almost positive links

As the bracket model is not of skein origin, one would hope it not to inherit the difficulty in distinguishing between positive and almost positive knots, as in [Cr, corollary 2.2].

Question 4.1 Is there an almost positive knot with $\min\mathrm{cf}\, V\neq 1$ or $\min\deg V\neq g$ (or, if g is not easily computable, $\min\deg V\neq\max\deg\Delta$)?

Unfortunately, the only almost positive knot yet known to me, $!10_{145}$, is not such an example.

Clearly, allowing more and more negative crossings, we lose control on the maximal degree of the bracket. In this regard again the skein approach is more useful, so we will return to it for the more general k-almost positive case.

Proposition 4.2 Let a link L have a k-almost positive genus g link diagram of n components, for $k \geq 1$. Then

$$\min\deg V(L) \geq g - \frac{n-1}{2} + 1 - 2k\,. \tag{14}$$

Proof. Use induction on k. For $k = 1$ this is lemma 3.8, and for the induction step from $k-1$ to k apply the skein relation (11) at a negative crossing of a k-almost positive diagram, and use the induction assumption for the two terms on the right of (11). □

An application of (14) are the following examples.

Example 4.3 The negative trefoil 3_1 and the knot 6_3 have minimal degree of the Jones polynomial -4 and -3 resp., and hence they are not 2-almost positive.

Example 4.4 Let T_k for $k > 1$ odd be the negatively obversed $(2,k)$-torus knot. Then T_k is k-almost positive and the connected sum $T_{k_1}\#T_{k_2}$ is (k_1+k_2)-almost positive. Both facts follow from (14) with the knowledge of

$$\min\deg V(T_k) = -\frac{3k-1}{2}\,.$$

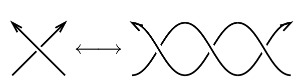
This shows that k-almost positive knots exist for any $k \in \mathbf{N}$, except for possibly $k = 1, 2, 4$. For $k = 1$ the simplest example is $!10_{145}$, see [St]. For $k = 2$ and $k = 4$ examples will be given below.

Proposition 4.2 can be used to generalize the classification of k-almost positive achiral knots for $k \leq 2$ [St, PT] to $k \leq 4$, in case the knot is alternating and – assuming for simplicity – prime. (Although for $k \geq 5$ an exact description is not possible, we will later prove that at least the number of such knots is always finite.)

Proposition 4.5 Let $k \leq 4$. Then the k-almost positive achiral prime alternating knots are exactly those with $2k$ crossings, i.e., 4_1 for $k = 2$, 6_3 for $k = 3$ and 8_3, 8_9, 8_{12}, 8_{17} and 8_{18} for $k = 4$. Consequently, the 10 crossing (alternating) achiral prime knots 10_{17}, 10_{33}, 10_{37}, 10_{43}, 10_{45}, 10_{79}, 10_{81}, 10_{88}, 10_{99}, 10_{109}, 10_{115}, 10_{118} and 10_{123} are all 5-almost positive.

For the proof of this proposition we need to recall some facts on diagrams of given genus (see [St4, St3] for more details).

Theorem 4.6 ([St4]) Knot diagrams of given genus (with no nugatory crossings and modulo crossing changes) decompose into finitely many equivalence classes under flypes [MT] and (reversed) applications of antiparallel twists at a crossing

 (15)

We call such an equivalence class a *series* of diagrams. A knot with an alternating diagram D lying in some series $\mathcal{A}$ is a *generator* of $\mathcal{A}$, if after any possible sequence of flypes D cannot be simplified by the move (15). An alternating diagram of a generator of $\mathcal{A}$ is called a *generating diagram*. A *subseries* $\mathcal{B}$ of a series $\mathcal{A}$ is the series of a diagram obtained by resolving some clasp of an alternating diagram of a generator of $\mathcal{A}$. A series which is not a subseries of another series of diagrams of the same genus is called a *main series*.

The series of diagrams of genus 1 and 2 have been classified in [St4, St3]. Diagrams of genus 1 are generated by the trefoil and the figure–eight knot, and diagrams of genus 2, beside the connected sums of two diagrams of genus 1, decompose into 24 series generated by the following knots: 5_1, 6_2, 6_3, 7_5, 7_6, 7_7, 8_{12}, 8_{14}, 8_{15}, 9_{23}, 9_{25}, 9_{38}, 9_{39}, 9_{41}, 10_{58}, 10_{97}, 10_{101}, 10_{120}, 11_{123}, 11_{148}, 11_{329}, 12_{1097}, 12_{1202}, and 13_{4233}.

The first application of this description we give is:

Proposition 4.7 The only 3-almost positive prime achiral knot of ≤ 16 crossings is 6_3.

Proof. The achiral knots in the tables of [HT] can be identified by checking the reciprocity of their Jones polynomial and then calculating their symmetry group. If such a knot K is 3-almost positive, then from (14) we obtain $\min\deg V(K) \geq -4$. The achiral knots satisfying this property, except 4_1 and 6_3, all have $\min\deg V = -4$. Consider in the following only these knots. It follows from (14) that a 3-almost positive diagram of K must have genus 1, so in particular so has then K. The calculation of the Alexander polynomial shows that $\max\deg\Delta(K) \geq 2$ for all relevant knots except 8_3, which indeed has genus 1. However, from [St4] we can classify genus 1 diagrams of 8_3. They are the rational diagrams with notation $(-5,1,3)$ and $(4,4)$, which are both not 3-almost positive. Thus 8_3 is not 3-almost positive either. □

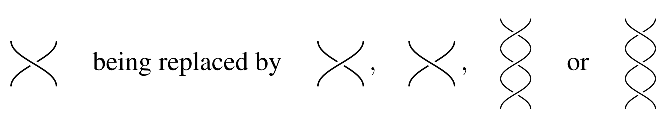
Proof of proposition 4.5. Let D be a k-almost positive diagram of an alternating achiral knot K. Assume K has $c(K) > 2k$ crossings. Since $\operatorname{span} V(K) = c(K)$ and V is reciprocal, we have $\min\deg V(K) < -k$.

If $k = 2$ one readily obtains a contradiction from (14), as $g(D) > 0$.

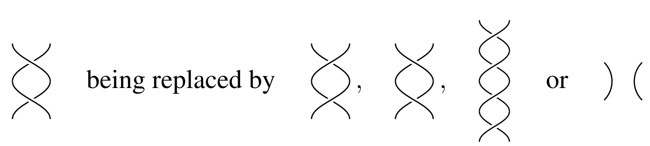
If $k = 3$ then the only option is $g(D) = 1$, which is handled with [St4]. The 3-almost positive genus 1 diagrams, which do not simplify to $\leq$ 2-almost positive ones, are the negative trefoil diagram, which is clearly not of interest, and the pretzel diagrams $P(-3,p,q)$ with $p,q \geq 3$ odd. In latter case one has $\min\deg V \geq 0$. (An easy argument is to observe this for $p = q = 3$, which gives the knot $!9_{46}$, and then to note that the skein relation establishes it for higher p,q.) This deals with $k = 3$.

Thus consider the case $k = 4$. The inequality (14) then shows that either $\min\deg V(K) \in \{-5,-6\}$ and $g(D) = 1$ or $\min\deg V(K) = -5$ and $g(D) = 2$. Using the classification of genus one diagrams, it is straightforward to check that a 4-almost positive genus one diagram has $\min\deg V \geq -4$. This shows the assertion except for the 10 crossing genus two knots, namely 10_{33} and 10_{37}. For them a much less pleasant argument can be applied, namely a rather tedious (electronic) check using the classification of genus two diagrams.

The composite cases are easily dealt with, so consider prime diagrams. By the skein relation for V it suffices to consider the 4-almost positive diagrams with a crossing in the positively switched generating diagram



(where the twists are meant to be antiparallel), or a (reverse) clasp



This also covers all subseries, so we need to consider only the main series 13_{4233}, 12_{1202}, 12_{1097}, 11_{148}, 10_{97}, 9_{41} and 6_3. We need to generate (by computer) the list of (about 18.500) 4-almost positive diagrams and calculate $\min\deg V$. We find that almost always $\min\deg V \geq -4$, except for 6 diagrams of $!6_2$ (where $\min\deg V = -5$), 5 of them in the series of 13_{4233} and one in the series of 10_{97}. But in this case $\min\mathrm{cf}\, V = 1$, while for 10_{33} and 10_{37} we have $\min\mathrm{cf}\, V = -1$. (In fact, this holds for all other achiral 10 crossing knots; it follows more generally from the theorem of [Th], that $\min\mathrm{cf}\, V_K = (-1)^{c(K)/2}$ for any achiral alternating knot K.) This shows that (the V polynomial of) 10_{33} and 10_{37} cannot occur in 4-almost positive genus two diagrams. □

Proposition 4.8 If K is a positive or almost positive knot, then $\min\deg V(K) \geq 1$.

Proof. Using proposition 4.2 and theorem 3.1, the only case remaining to discuss is if K is almost positive and has an almost positive diagram of genus one. However, by the above mentioned result of [St4], knot diagrams of genus one are rational or pretzel knot diagrams, and any almost positive such diagram represents a positive knot. □

5 Applications to alternating and positive braid links

In the previous sections we saw how the Jones skein relation and Kauffman state model can be used to control the minimal degree of the Jones polynomial of some link diagram in terms of the number of its negative crossings.

The aim of this section is to present some applications of the inequalities given in the previous sections.

In the following we use a weaker version of proposition 4.2.

Proposition 5.1 Fix k. Let $n(L)$ be the number of components of a link diagram L. Then the function $L \mapsto \min\deg V(L) + n(L)/2$ is bounded below for all $\leq k$-almost positive link diagrams L. □

This gives an obstruction to k-almost positivity of links for any k. Here is an application of this obstruction.

Theorem 5.2 For any k there are only finitely many braid negative links without trivial split components, which are $\leq k$-almost positive.

This theorem has a plausible explanation. It says that, if one takes for any fixed given k a negative braid with sufficiently many crossings and closes it up, in which way one ever manipulates the resulting link, one can never avoid the existence of k negative crossings in any of its diagrams.

The idea of the proof is to superpose the maximal degree upper bound for the Jones polynomial of braid negative links [St2], which is also a minimal degree upper bound, with our minimal degree lower bound and to show that for sufficiently high number of crossings they contradict.

Proof of theorem 5.2. For a reduced closed negative braid diagram (henceforth called braid negative diagram) D of $c = c(D)$ crossings, $n = n(D)$ components and $p = p(D)$ split components we have by [St2] and (2)

$$\max\deg V(D) \leq -\frac{c}{4} + \frac{p-1}{2}\,. \tag{16}$$

Now by $\leq k$-almost positivity

$$\min\deg V(D) \geq -\frac{n}{2} - C \tag{17}$$

for some constant $C > 0$ (depending on k only, which is fixed). So, given (16) and (17), in order to prove theorem 5.2 it suffices to show that only finitely many reduced braid negative link diagrams D with no trivial split components have

$$\mathrm{span}\, V(D) \leq \frac{n}{2} + C + \frac{p-1}{2} - \frac{c}{4}\,. \tag{18}$$

Equivalently, we show that

$$\rho(D) := 4\operatorname{span} V(D) + c(D) - 2n(D) - 2(p(D) - 1)$$

grows beyond any extent for reduced closed positive/negative braid diagrams D.

Now, ρ is strictly superadditive under split union, so it suffices to show that it is always positive on non-trivial (not equal to the unknot) non-split diagrams D (i.e. $p(D) = 1$), and that it grows unboundedly for any sequence of such diagrams.

As a non-split link diagram of n components has at least $2n-2$ crossings, $\rho(D) > 0$ would follow from $\operatorname{span} V(D) > 1/2$. Clearly $\operatorname{span} V(D)$ is always an integer, so we must argue why $V(D)$ is not a monomial. Assume it were, and use the conditions described in [J2, §12]. The one in (12.7) (and integrality) implies that the coefficient of the monomial is $\pm 3^k$. Then condition (12.1) shows that $k = 0$ and $n = 1$. So we have a knot and wonder whether V can be a unit in $\mathbf{Z}[t^{\pm 1}]$. Now, (12.1) implies that $V = +t^k$, and then (12.2) that $k = V'(1) = 0$, so $V = 1$. However, no non-trivial positive braid knot has trivial polynomial.

To show that $\rho(D)$ grows for non-trivial non-split diagrams D, observe that $\rho(D)$ is also strictly superadditive under connected sum. Then again it suffices to show that $\rho(D)$ grows for prime non-split diagrams (we already ensured positivity). We show that in fact already $c - 2n$ does.

Assume contrarily $D_i = \hat{\beta}_i$ are diagrams, coming from closures of positive braid(word)s β_i, with $c(D_i) - 2n(D_i)$ bounded. Since $c(D_i) \to \infty$, so does $n(D_i)$. Since the number $s(\beta_i)$ of strands of the braids β_i satisfies $s(\beta_i) \geq n(D_i)$, we have that $c(D_i) - 2s(\beta_i)$ is bounded. As we assumed $D_i = \hat{\beta}_i$ reduced, no generator in the braids β_i occurs only once. Therefore, we have that almost all generators σ_j appear exactly twice, say for index $j > j_0$ (where j_0 does not depend on i). Then if for some $j > j_0$ the two pairs of letters of σ_j and σ_{j+1} appear in cyclic order $\sigma_j, \sigma_j, \sigma_{j+1}, \sigma_{j+1}$ in β_i, we have a non-prime diagram D_i. Thus they always occur in cyclic order $\sigma_j, \sigma_{j+1}, \sigma_j, \sigma_{j+1}$. Then one can rearrange by commutativity relations and cyclic permutations the letters so that the generators with indices above j_0 occur in two subwords $\sigma_{j_0+1}\sigma_{j_0+2}\dots\sigma_{s-1}$ in β_i, with $s = s(\beta_i)$. But the induced permutations of such braids have only a bounded number of cycles, in contradiction to the previous conclusion $n(D_i) \to \infty$. The proof is now completed. □

Originally, I tried to continue in (16) using $c \geq 2n - 2p$ to estimate the r.h.s. by $p - (n+1)/2$, then obtaining the inequality $\operatorname{span} V(L) \leq C + p - 1/2$ instead of (18). Since the case $p = 1$ is again the essential one, we have

Conjecture 5.3 Only finitely many closed positive (or negative) braid knots have Jones polynomial of fixed span.

Traczyk [Tz] showed that this is not true (unlike I also expected) for 2-component links on 3-braids. Apart from other generalizations of his examples that would be of interest (for instance, prime links on more strands), the case of knots remains qualitatively different. Namely, unlike links (and in particular as in Traczyk's examples), there are no two *knot* polynomials differing by a non-trivial unit in $\mathbf{Z}[t^{\pm 1}]$. (This follows from the arguments in the above proof.) So we would have to have fixed span polynomials with very large coefficients. The existence of such a sequence of polynomials is not known even among general knots or links.

Another suggestive problem related to theorem 5.2 is that its explanation appears appealing also for the larger class of negative links, but to such links the results on $\max\deg V(L)$ of [St2] do not extend.

Question 5.4 Does theorem 5.2 hold for negative links instead of braid negative?

The next application of proposition 5.1 is less plausible (and so more surprising) and avoids an analogous question (by not restricting itself to alternating braid representations).

Theorem 5.5 For all k only finitely many alternating links without trivial split components are $\leq k$-almost positive and (simultaneously) $\leq k$-almost negative.

Proof. Fix k. Proposition 5.1 imposes an upper bound for the span of the Jones polynomial of an n-component k-almost positive k-almost negative link

$$\operatorname{span} V(L) \leq Q+n \tag{19}$$

for some constant $Q>0$ (depending on k only, which is fixed). Now if L has an alternating diagram D of c crossings, n components and p split components, by (3) and Kauffman [Ka2], Murasugi [Mu] and Thistlethwaite [Th]

$$\operatorname{span} V(L) = c+p-1 \geq n+p-1\,, \tag{20}$$

where $c \geq n$, as L has no trivial split components and so any component must pass through at least two crossings in D. The inequalities (19) and (20) combined give

$$p \leq Q+1\,. \tag{21}$$

Now a non-split link diagram of n components has at least $2n-2$ crossings, and so a link diagram of p split components has at least $c \geq 2n-2p \geq 2n-2Q-2$ crossings. Therefore (20) implies

$$\operatorname{span} V(L) \geq 2n-2Q-2\,.$$

But on the other hand we have (19). Then it follows that n is bounded, implying by (19) that so is $\operatorname{span} V(L)$ and finally by (20) that for D alternating so is c. □

Corollary 5.6 The number of achiral alternating k-almost positive knots is finite for any k. □

This fact came rather surprising, as *a priori* the concepts of alternation and positivity do not have much in common. But even surprised, one might still be unsatisfied.

Question 5.7 Is alternation needed in theorem 5.5 (and its corollary)?

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