VASSILIEV INVARIANTS AND RATIONAL KNOTS OF UNKNOTTING NUMBER ONE

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Abstract. Introducing a way to modify knots using n-trivial rational tangles, we show that knots with given values of Vassiliev invariants of bounded degree can have arbitrary unknotting number (extending a recent result of Ohyama, Taniyama and Yamada). The same result is shown for 4-genera and finite reductions of the homology group of the double branched cover. Closer consideration is given to rational knots, where it is shown that the number of n-trivial rational knots of at most k crossings is for any n asymptotically at least $C^{(\ln k)^2}$ for any $C < \sqrt[2\ln 2]{e}$.

1 Introduction

In [S], Stanford introduced a way to modify knots into alternating prime ones using 3 braids [BM], not affecting (i. e., changing the values of) any finite number of Vassiliev invariants [Bi, BL, BN, BN2, BS, Va, Vo]. The 3 braids were chosen to be iterated pure braid commutators and so they are n-trivial in the sense of Gousarov [G], see [St].

In this paper, we give another such construction by means of rational tangles, which we describe in section 2. It can be applied to any diagram of a knot, not only to closed braid diagrams. While Stanford's construction is useful not to augment the braid index (if it is ≥ 3), our construction is useful, when applied in an arborescent diagram, not to spoil arborescency of a knot. Hence, a similar argument to Stanford's allows us to prove an 'arborescent' version of his modification theorem:

Theorem 1.1 Let $v_1,\dots,v_n$ be Vassiliev invariants. Then for any knot K there is some prime alternating knot K' with $v_i(K) = v_i(K')$ for $1 \leq i \leq n$. If K is arborescent, then K' can be chosen to be so as well.

In Gousarov's language two knots K_1 and K_2 having the same Vassiliev invariants of degree up to n are called n-similar. We denote this by $K_1 \sim_n K_2$.

Applied to the 1-crossing-diagram of the unknot, our method produces (infinite) series of n-trivial 2-bridge knots for given $n \in \mathbf{N}$. Hence we have

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Corollary 1.1 For any n there exist infinitely many n-trivial rational knots of genus 2^n. Infinitely many of them have unknotting number one.

The number of such knots will be (asymptotically) estimated more accurately in section 3. The important feature of this estimate is that it is asymptotically independent on the degree of triviality. Such an estimate does not appear to have been known before (see remark 3.2).

Our knots differ in several regards from previous constructions. Lin's iterated Whitehead doubles [L] have genus and unknotting number one, and are non-alternating, Ng's knots [Ng] are slice and of unknotting number at most two but their genus is difficult to control, the same being true for Stanford's alternating braid knots.

Ng's construction offers an analogy to another outcome of our work. She showed that, beside the Arf invariant, Vassiliev invariants give no information on knot cobordism. This helps completing a picture, realized soon after Vassiliev invariants became popular, that all classical knot invariants (that is, those known before the "polynomial fever" [Ro, Preface] broke out with [J]), are not, or stronger (almost) unrelated to, Vassiliev invariants, see [Bi]. Our method exhibits the same picture for the unknotting number.

Theorem 1.2 Let K be some knot and n, u positive integers. Then there exists a prime knot $K_{n,u}$ of unknotting number u having the same Vassiliev invariants of degree up to n as K. Moreover, for fixed K and n, $K_{n,u}$ can be chosen to be alternating (and prime) for almost all u.

We show this result in section 4. It extends the result of Ohyama-Taniyama-Yamada [OTY] (see also [Oh]), which is the claim of the theorem for $u = 1$. Their result is used in the proof, together with an application of our method, given K, how to construct $K_{n,u}$ for any $u \geq u(K)$. The use of the tangle calculus of [KL] allows to ensure primality in most cases, contrarily to Ng's knots, which are composite. Since we will use the signature for the proof of theorem 1.2, the same statement holds via the Murasugi–Tristram inequality also for the 4-genus $g_s > 0$ instead of the unknotting number, thus extending the case $g_s = 0$ studied by Ng.

Theorem 1.2 is a bit surprising, as the picture changes when considering other unknotting operations, or, at least conjecturally, special classes of knots, see §5. Also, the situation differs when considering infinitely many Vassiliev invariants, because for example the Jones polynomial, which by [BL] is equivalent to such a collection, does carry some (albeit modest) unknotting number information, see [LM, St3, Tr].

Beside signatures or 4-ball genera, for the unknotting number results we use the estimate of Wendt [We], the number of torsion coefficients of the homology $H_1(D_K, \mathbf{Z})$ of the double branched cover D_K of S^3 over K. As a by-product, we obtain a similar result to the ones above regarding the homology of D_K over rings of positive characteristic (see theorem 4.2). It would be more interesting (but much more difficult) to examine the situation with the whole $\mathbf{Z}$-module $H_1(D_K, \mathbf{Z})$.

Finally, in section 5, we conclude by summarizing some problems suggested by our results.

2 Rational tangles

In this section we introduce the type of (rational) tangles which will be applied in the subsequent constructions.

Rational tangles were introduced by Conway [Co]. The Conway notation $C(a_1, \dots, a_n)$ of a rational tangle is a sequence of integers, to which a canonical diagram of the tangle is associated, see [Ad, §2.3]. Define the iterated fraction (IF) of a sequence of integers $a = (a_1, \dots, a_n)$ recursively by

$$IF(a_1) := a_1, \dots, IF(a_1, \dots, a_{n-1}, a_n) := \frac{1}{IF(a_1, \dots, a_{n-1})} + a_n.$$

It will be helpful to extend the operations '$+$' and '$1/.$' to $\mathbf{Q}\cup\{\infty\}$ by $1/0=\infty$, $1/\infty=0$, $k+\infty=\infty$ for any $k\in\mathbf{Q}$. The reader may think of ∞ as the fraction $1/0$, to which one applies the usual rules of fraction arithmetics and reducing. In particular reducing tells that $-1/0=1/0$, so that for us $-\infty=\infty$. This may appear at first glance strange, but has a natural interpretation in the rational tangle context. A rigorous account on this may be found in Krebes's paper [Kr].

In this sense, IF is a map ($\forall\, n\in\mathbf{N}$)

$$IF\,:\,\mathbf{Z}^n\longrightarrow\mathbf{Q}\cup\{\infty\}\,.$$

It is known [Ad], that diagrams of sequences of integers with equal IF belong to the same tangle (up to isotopy; where isotopy is defined by keeping the endpoints fixed). The correspondence is

$$C(a_1,\dots,a_n)\longleftrightarrow IF(a_n,\dots,a_1)\,.$$

Using this fact, one can convince himself, that a rational tangle T has a diagram which closes (in the way described in [Ad], see also figure 1) to an alternating reduced prime diagram of a link (the only exception for reducedness being the tangle with notation (1)), which has 1 or 2 components (as in our examples below). This diagram is obtained by taking a representation of $IF(a)=IF(c)$ for a Conway notation a of T, such that all numbers in c are of the same sign (it is easy to see that such a sequence c always exists). In particular, $|c|:=\sum_i|c_i|$ is the crossing number of the closure of T (see [Ka, Mu, Th]), and so T is trivial, i. e., the 0-tangle, iff $IF(a)=0$ (as for a the 0-tangle and $IF(a)\neq 0$ we had $|c|>0$ and $c\neq(1)$, and thereby a contradiction). We also see this way, that rational links are prime (see [Me]; this result independently follows from the additivity of the bridge number proved by Schubert [Ad, p. 67]).

Define for a finite sequence of integers $a=(a_1,\dots,a_n)$ its reversion $\overline{a}:=(a_n,\dots,a_1)$ and its negation by $-a:=(-a_1,\dots,-a_n)$. For $b=(b_1,\dots,b_m)$ the term ab denotes the concatenation of both sequences $(a_1,\dots,a_n,b_1,\dots,b_m)$. We also write $IF(a,a_{n+1})$ for $IF(a_1,\dots,a_n,a_{n+1})$, and analogously $IF(a,a_{n+1},b)$ etc.

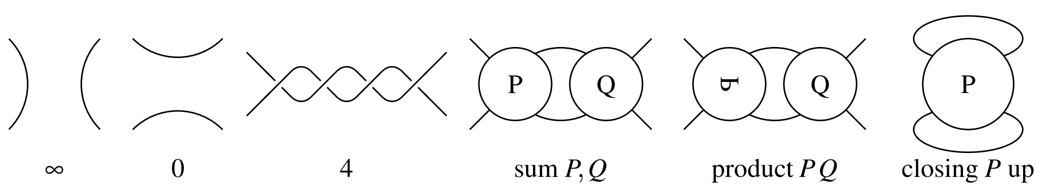


Figure 1: Operations with rational tangles

Proposition 2.1 Fix some even $a_1,\dots,a_n\in\mathbf{Z}$ and build inductively the integer sequences w_n by

$$w_1:=(a_1),\quad\dots\quad w_n:=w_{n-1}(a_n)\overline{-w_{n-1}}\,. \tag{1}$$

Then the rational tangles with Conway notation w_n are n-trivial, and, if all $a_i\neq 0$, non-trivial, i. e., not (isotopic to) the 0-tangle.

Proof. For given n consider the braiding polynomial P [St] of some Vassiliev invariants (which may be assumed to be zero on the 0-tangle), on the w_n-tangle as polynomial in $a_1,\dots,a_n$. By the discussion of Stanford's examples in [St] and the previous remarks, we need to show that $IF(\dots,0,\dots)\equiv 0$, and so $P\big|_{a_i=0}\equiv 0\;\forall i\leq n$, and $IF(a_1,\dots,a_n)\neq 0$, if all $a_i\neq 0$.

Do this by the inductive assumption over n. For $n=1$ the claim is evident. For fixed n by induction assumption $IF\big|_{a_i=0}\equiv 0\;\forall i<n$, as $IF(a,a_n,\overline{-a})$ is independent of a_n if $IF(a)=0$, and $IF(a,0,\overline{-a})=0$ for any integer sequence a. But therefore also $IF\big|_{a_n=0}\equiv 0$.

To see that for $a_1 \neq 0, \dots, a_n \neq 0$ the tangle is non-trivial, use that by induction for $a_1, \dots, a_{n-1} \neq 0$ we have $IF(w_{n-1}) \neq 0$ and that therefore the map

$$a_n \quad \longmapsto \quad IF\left(a_n + \frac{1}{IF(w_{n-1})}, \overline{-w_{n-1}}\right)$$

is a bijection of $\mathbf{Q} \cup \{\infty\}$, so $a_n = 0$ can only be a unique zero. $\square$

Example 2.1 For $a_1 = 2$, $a_2 = -4$ and $a_3 = 2$ we have $w_1 = (2)$, $w_2 = (2, -4, -2)$ and $w_3 = (2, -4, -2, 2, 2, 4, -2)$.

3 Modifying knots

Prepared with the above tangles, we can now describe our modification technique.

Proposition 2.1 already allows to prove the special case of theorem 1.1 given in the introduction as corollary 1.1. We first give this proof, before going to prove theorem 1.1 itself.

Proof of corollary 1.1. Corollary 1.1 follows directly from the proposition 2.1 by replacing the 0-tangle in the unknot diagram $C(0,c)$ for any $c \in \mathbf{Z}$ by some of the tangles in question. To see that indeed infinitely many examples arise this way, take c even and use the well-known fact that the expression of a rational knot with all Conway coefficients even is unique. The number of even entries is known to be equal to twice the genus, hence the genus is as asserted. We obtain the unknotting number property by taking $a_n = \pm 2$. $\square$

We can now prove the arborescent refinement of Stanford's result from our setting.

Proof of theorem 1.1. Given a knot K, take some reduced non-composite diagram of K (which exists even if K is composite) and choose a set S of crossings, which need to be switched to obtain from it an alternating diagram. Then near each such crossing p plug in an n-trivial rational tangle T (in a diagram with alternating closure), so that the right-most crossing of T cancels with p by a Reidemeister 2 move (see fig. 2).

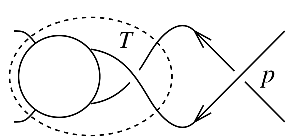


Figure 2: Plugging in T

By applying this modification at all crossings in S, we are done. $\square$

Remark 3.1 More generally, this construction shows that one can preserve the Conway basic polyhedron.

We conclude this section with the announced more specific enumeration result concerning the knots in corollary 1.1.

Corollary 3.1 For any $n_o \in \mathbf{N}$ the number of n_o-trivial rational knots of at most k crossings is asymptotically at least $C^{(\ln k)^2}$ for any constant $C < \sqrt[2\ln 2]{e}$.

Proof. First note, that the freedom to vary C allows us to replace for convenience k by $k/2$, or equivalently to consider at most $2k$ crossing diagrams (instead of at most k crossings).

A diagram of the kind constructed in the proof of proposition 2.1 with $2k$ crossings in the groups of twists except the first one corresponds to writing

$$k = \sum_{i=0}^{n} 2^i |w_i|$$

for some $w_i \in \mathbf{Z}$ and $n \in \mathbf{N}$. For $n \leq n_o$ the number of such representations in polynomially bounded in k, hence, assuming we can show the lower bound for the diagrams including these with $n \leq n_o$, it is possible to neglect them and assume $n \geq n_o$, so that all diagrams are n_o-trivial.

Let

$$D_k := \left\{ (w_0, \dots, w_n) \,:\, k = \sum_{i=0}^{n} 2^i |w_i|,\, w_i \neq 0,\, n > 0 \right\}$$

and $d_k := \# D_k$. Then $d_1 = 2$ and

$$d_k = 2 \sum_{i=1}^{\lfloor k/2 \rfloor} d_i \quad \text{for } k \geq 2.$$

To prove the corollary it suffices to show that

$$c_k := C^{(\ln k)^2} < (2-\varepsilon) \sum_{i=1}^{\lfloor k/2 \rfloor} C^{(\ln i)^2} + D \tag{2}$$

for some $D \in \mathbf{R}$, $\varepsilon > 0$ and sufficiently large k, as then (for possibly larger k) $D < C^{(\ln k)^2} \cdot \frac{\varepsilon}{2}$, so

$$C^{(\ln k)^2} < 2 \sum_{i=1}^{\lfloor k/2 \rfloor} C^{(\ln i)^2},$$

and hence $d_k \geq C' \cdot c_k + C''$ (for some $C', C'' \in \mathbf{R}$, $C' > 0$), but C' and C'' can be eliminated by varying C.

To show (2), first use that $i \mapsto C^{(\ln i)^2}$ is monotonously growing for $i \geq 1$, so

$$\int_1^{(k-1)/2} C^{(\ln t)^2} dt < \sum_{i=1}^{\lfloor k/2 \rfloor} C^{(\ln i)^2}. \tag{3}$$

Now for (2) it suffices to show the inequality for the derivations of the left hand-sides of (2) and (3) for sufficiently large k.

But putting $C = e^p$ with $p < \frac{1}{2\ln 2}$, we have that

$$\frac{d}{dk}\left(e^{p(\ln k)^2}\right) < (2-\varepsilon)\frac{d}{dk}\left(\int_1^{(k-1)/2} e^{p(\ln t)^2} dt\right)$$

is equivalent to

$$e^{p(\ln k)^2} \frac{2\ln k}{k} p < \frac{2-\varepsilon}{2} e^{p(\ln(k-1)-\ln 2)^2},$$

and logarithming we get

$$p(\ln k)^2 + \ln 2 + \ln\ln k - \ln k + \ln p < \ln(2-\varepsilon) - \ln 2 + p(\ln(k-1))^2 - 2p\ln 2\ln(k-1) + p(\ln 2)^2.$$

This is for some $D' \in \mathbf{R}$ the inequality

$$p\left((\ln k)^2 - (\ln(k-1))^2\right) + \ln\ln k < (1 - 2p\ln 2)\ln(k-1) + D'.$$

Now as $k \to \infty$, the first term on the left goes to 0, and then the claim is obvious from the condition on p. $\square$

Remark 3.2 It should be remarked that the asymptotical estimate itself does *not* depend on n. Such an unconditional statement does not seem to have been known before. For example, the number of Lin's iterated n-fold Whitehead doubles for fixed n grows exponentially in k, because of the result of [W] and the uniqueness of the companion, but the base of this exponential heavily depends on n– roughly augmenting n by 1 requires to take the fourth root of the base. On the other hand, the dependence on n in our estimate is present, namely in how quickly the numbers attain their asymptotical behaviour. Thus our result does not imply the existence of knots which are n-trivial for all n. In fact, as our knots are alternating, no one of them can have this property.

4 Unknotting numbers and n-triviality

Here we record some consequences of the preceding results concerning unknotting numbers. The first one is rather easy, and will be later refined to give a proof of theorem 1.2.

Proposition 4.1 Let K be some knot. Then for any $n \in \mathbf{N}$ and $u_0 \geq u(K)$ there exists a knot K_{n,u_0} with $u(K_{n,u_0}) = u_0$ and $v(K_{n,u_0}) = v(K)$ for any Vassiliev invariant v of degree up to n.

Proof. Consider the knots $K_{(i)} := K\#\big(\#^i K'\big)$ with K' being an $n+1$-trivial rational knot of unknotting number one (provided by corollary 1.1). Then the Vassiliev invariants of degree up to n of all $K_{(i)}$ are the same as those of K, and that any $u_0 \geq u(K)$ is the unknotting number of some $K_{(i)}$ follows from the obvious inequality $u(K_{(i+1)}) \leq u(K_{(i)}) + 1$ and the reverse estimate $u(K_{(i)}) \geq d_{K_{(i)}} = d_K + i$, where $d_K = \operatorname{rank} H_1(D_K, \mathbf{Z})$ is the number of torsion coefficients of $H_1(D_K, \mathbf{Z})$ and D_K is the double cover of S^3 branched along K, see [We]. □

Now we indicate how to modify the proof of proposition 4.1 to signatures and 4-genera. (This can also be deduced from Ng's work, but the proof is now brief, so we can give it in passing by.)

Theorem 4.1 Let $n \in \mathbf{N}$ and K be some knot. Then

i) for any $s \in 2\mathbf{Z}$ there is a knot $K_{n,s} \sim_n K$ with $\sigma(K_{n,s}) = s$.

ii) for any integer $g \geq 0$ there is a knot $K_{n,g} \sim_n K$ with $g_s(K_{n,g}) = g$, except if $Arf(K) = 1$, $g = 0$ (and $n > 1$).

Proof sketch. By the result of Ohyama, Taniyama and Yamada, Ng's work for $g_s = 0$ (which we cite, but do not cover with our arguments) and the previous arguments, together with the standard inequalities $|\sigma/2| \leq g_s \leq u$ (see [Mu2, Ts]), we are basically left with showing that the n-trivial rational knot of corollary 1.1 can be chosen to be of signature ± 2. For this we remark that the determinant shows that the signature of a rational knot $S(p,q)$ with $p,q > 0$ (in Schubert's notation [Sh]) is divisible by 4 exactly if $p \equiv 1 \bmod 4$. Violating this property reduces to making the number c in the proof of corollary 1.1 small or large enough in order to adjust the desired sign of $IF(w_n, c)$. □

Next, we state and prove the result for the homology of the double branched cover.

Theorem 4.2 Let $p > 2$ be an odd integer, H be a finite $\mathbf{Z}_p$-module, $n > 0$ be a natural number and K be any knot. Then there is a knot K', which can be chosen to be prime and alternating, with $K' \sim_n K$, such that $H_1(D_{K'}, \mathbf{Z}_p) = H$.

We start the proof by two lemmas, lemma 4.1 and 4.3. Then we prove theorem 4.2 by taking the connected sum of the knots constructed in the lemmas.

Lemma 4.1 Let $p > 1$ be an odd integer. Then for any $k \in \mathbf{Z}_p$ and any n there is an n-trivial rational knot K with $\det(K) \equiv k \bmod p$ and $u(K) = 1$.

Proof. Let w_n be the sequences of integers as in (1) with all $a_i \in \{\pm 2\}$. Then, by the calculation used in [KM, proof of theorem 1, (ii)⇒ (iii)] we find

$$\left| IF(w_n) \right| = \frac{2^{2^n-1}}{\sum_{i=1}^n \pm 2^{2^n-2^i}}$$

for certain signs in the sum depending on the signs of the a_i.

Then for the rational tangle with Conway notation (w_n,s) for a natural (not necessarily even) number s we have

$$\det\big(\overline{(w_n,s)}\big) = \sum_{i=1}^{n} \pm 2^{2^n-2^i} + s\cdot 2^{2^n-1},$$

and the existence of proper choice of s follows from the fact that 2^{2^n-1} and p are relatively prime. □

Lemma 4.2 The (tautological) homomorphism $h_{p,q}: \mathbf{Z}_p^* \to \mathbf{Z}_q^*$ for any $q \mid p$ is onto (where $\mathbf{Z}_p^*$ is the group of units of $\mathbf{Z}_p$ or the relatively prime to p rest classes modulo p).

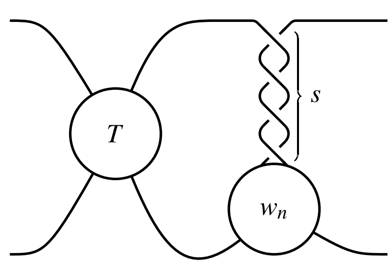
Proof. This surjectivity follows because $|\mathbf{Z}_p^*| = \phi(p)$, $\mathbf{Z}_{ab}^* = \mathbf{Z}_a^* \times \mathbf{Z}_b^*$ for $(a,b)=1$ and because obviously $\left|\ker(h_{p^u,p^{u-1}})\right| = p = \phi(p^u)/\phi(p^{u-1})$ for any prime p and $u>1$. □

Lemma 4.3 Let $p>1$ be an odd integer and K be an unknotting number one knot. Then for any n there is a knot $K' \sim_n K$ with $\det(K')$ relatively prime to p.

Proof. We use the tangle calculus of Krebes (see [Kr] for details). He showed that the pair of determinants of the closures of a tangle T and its flipped version $T0$ (the product of T with the 0 tangle in the notation of figure 1) can be viewed as the numerator and denominator of a certain generalized rational number, denoted here by $R(T)$, lying in $\tilde{\mathbf{Q}} := \mathbf{Z}\times\mathbf{Z}\big/(a,b)\sim(-a,-b)$, which (up to signs) is additive under tangle sum (as in figure 1), and generalizes IF for rational tangles.

The fact that $u(K)=1$ shows that K can be presented as the closure $\overline{T}$ of a tangle T such that the closure $\overline{T,2}$ of the tangle sum of T with the 2-tangle (clasp) is the unknot. Krebes's calculus then shows that $R(T) = (\pm 2k \pm 1)/k \in \tilde{\mathbf{Q}}$ for certain signs and a natural number k.

Then consider the tangle sum of T with the rational tangle $S=(w_n,s,0)$ for a (not necessarily even) integer s.

$T,S =$ 

Then $\overline{T,S} \sim_n K$ and by Krebes's calculus

$$\det(\overline{T,S}) = k2^{2^n-1} + (\pm 2k \pm 1)\Big[\sum_{i=1}^{n} \pm 2^{2^n-2^i} + s\cdot 2^{2^n-1}\Big].$$

Changing s by ± 1 causes the expression to change by $(\pm 2k \pm 1)2^{2^n-1}$. Thus we could finish the proof as in the case of lemma 4.1 unless $2k \pm 1$ and p are not relatively prime. In this case let $l = (2k \pm 1, p)$ be their greatest common divisor. Clearly $(l, k2^{2^n-1}) = 1$ and so $(l, k') = 1$ for

$$k' = k2^{2^n-1} + (\pm 2k \pm 1) \sum_{i=1}^{n} \pm 2^{2^n-2^i}.$$

We would be done if we can find an $s' \in \mathbf{Z}$ with $(k' + l \cdot s', p) = 1$. Then set

$$s := s' \cdot \frac{1}{2^{2^n-1}} \cdot \frac{l}{2k \pm 1}$$

in $\mathbf{Z}_p$. Here the meaning of the second factor is clear, as 2^{2^n-1} is invertible in $\mathbf{Z}_p$. The third factor means some (fixed) preimage under $h_{p,p/l}$ of the (multiplicative) inverse of $(2k \pm 1)/l \in \mathbf{Z}^*_{p/l}$. The existence of this preimage follows from lemma 4.2. In turn, the existence of s' is equivalent to the surjectivity of the homomorphism $h_{p,l}$, which again follows from lemma 4.2. $\square$

Proof of theorem 4.2. We can write

$$H = \bigoplus_{i=1}^{l} \mathbf{Z}_{p_i}$$

with $p_i \mid p$. Let $\hat{K}$ be the knot found to K in lemma 4.3, and K_i be the knots from lemma 4.1 for $k = p/p_i$. Then $H_1(D_{\hat{K}}, \mathbf{Z}_p) = 1$, and $H_1(D_{K_i}, \mathbf{Z}_p) = \mathbf{Z}_{p_i}$, since K_i are rational by construction. Thus

$$K'' = \hat{K} \ \# \ \mathop{\#}_{i=1}^{l} K_i$$

is a knot with the desired values of Vassiliev invariants and homology group. It remains to make K'' into a prime alternating knot K', which will be the knot we sought.

To obtain K' from K'', take a prime diagram of K'', and apply the plugging technique in the proof of theorem 1.1 with a tangle w_n of the form (1) with $a_n = 2p$. Then by the work of Gordon and Litherland [GL] on the Goeritz matrix, this plugging preserves the structure of $H_1(D_{K''}, \mathbf{Z}_p)$, since w_n turns into the 0-tangle by changing some of its Conway coefficients by a multiple of p. $\square$

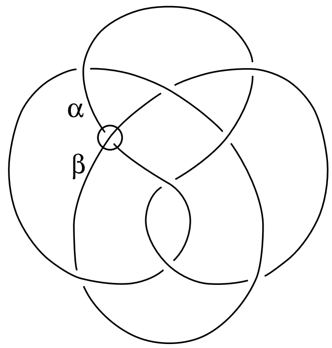
Remark 4.1 It is uninteresting to consider p to be even, because for any *knot* (although not link) K, $H_1(D_K, \mathbf{Z})$ has no 2-torsion, so its reduction modulo $2p$ is equivalent to its reduction modulo p.

Remark 4.2 Instead of making K' in theorem 4.2 alternating and prime, we can also achieve, setting $K' = K''$, that it has $u(K') \leq \operatorname{rank} H_1(D_{K'}, \mathbf{Z}_p) + 2$, as the knot in lemma 4.1 had unknotting number 1, and this constructed in lemma 4.3 has unknotting number 1 or 2.

We conclude this section with the proof of theorem 1.2. For this we use the prime tangle calculus of [KL]. Recall that a tangle is called prime if it contains no properly embedded separating disk, and no one of the strands has a connected summand (i.e. a sphere intersecting it in a knotted arc). First we need a simple lemma.

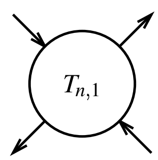
Lemma 4.4 There are prime tangles with unknotted closure.

Proof. Consider the knot 9_{34}, which has unknotting number 1, and the encircled crossing, whose switch unknots it.



Switching the crossing, and cutting the edges α and β we obtain (up to change of the unbounded region) a tangle U with unknotted closure. To show primeness, we need to show first that it has no connected summand. However, this is clear since the closure is unknotted. Then, we need to ensure that it is not rational. For this consider the other closure $\overline{U \cdot 0}$ of U. It has an alternating diagram with Conway polyhedron [Co] 6^*, and hence it is not rational. Thus U is not a rational tangle, and is therefore prime. $\square$

Proof of theorem 1.2. Fix K and n. Let $K_{n,1}$ be the knot constructed in [Oh]. Since $u(K_{n,1}) = 1$, by [Sc, Zh], $K_{n,1}$ is prime, and thus by [KL], $K_{n,1} = \overline{T_{n,1}}$, with $T_{n,1}$ being a prime tangle. We can without loss of generality assume that the orientation of $T_{n,1}$ is like



.

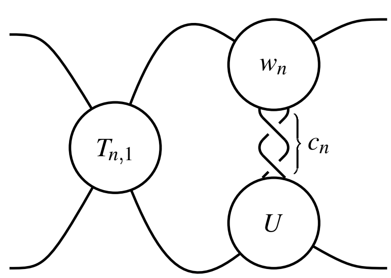
Otherwise, we can replace $T_{n,1}$ by its sum with a one-crossing tangle. This sum is again a prime tangle (see [V]). Let w_{n+1} be a $(n+1)$-trivial rational tangle, and $T_n' = w_{n+1} \cdot c_n$. Let U be a prime tangle with unknotted closure and set $T_n'' = U \cdot T_n'$. Then T_n'' is also prime.

Since smoothing out a crossing in the group of c_n gives the link $\overline{w_{n+1}}$, which has non-zero determinant, as in the proof of theorem 4.1, by choosing c_n large or small enough, we can achieve that $\sigma(\overline{T_n''}) \neq 0$. Also, by choosing $a_{n+1} = \pm 2$, we can achieve that $u(\overline{T_n''}) = 1$.

Now consider

$$T_{n,k} = T_{n,1}, \underbrace{T_n'' \cdot 0, T_n'' \cdot 0, \ldots, T_n'' \cdot 0}_{k-1 \text{ times}}.$$

$T_{n,2} =$



We have that

$$u(\overline{T_{n,k+1}}) \leq u(\overline{T_{n,k}}) + 1\,. \tag{4}$$

Then, because of the above choice of orientation of $T_{n,1}$, the tangle $T_{n,k}$ differs from $T_{n,k},\infty$ by a band connecting (plumbing of a Hopf band). But the closure of $T_{n,k},\infty$ is $\overline{T_{n,1}\cdot 0}\,\#\,(\#^{k-1}\overline{T''_n})$, and since $\sigma(\overline{T''_n}) \neq 0$, we have

$$2u(\overline{T_{n,k}}) \,\geq\, |\,\sigma(\overline{T_{n,k}})\,| \,\geq\, \left|\,\sigma\big(\overline{T_{n,1}\cdot 0}\#\,(\#^{k-1}\overline{T''_n})\big)\,\right| - 1 \,\geq\, (k-1)|\,\sigma(\overline{T''_n})\,| - |\,\sigma(\overline{T_{n,1}\cdot 0})\,| - 1 \,\longrightarrow\, \infty,$$

as $k \to \infty$. This, together with (4) and $u(K_{n,1}) = 1$, shows that each natural number u is realized as the unknotting number of some $\overline{T_{n,k}}$, with $k \geq u$. Since

$$\underbrace{T''_n\cdot 0, T''_n\cdot 0, \dots, T''_n\cdot 0}_{k \text{ times}}$$

is prime for $k \geq 1$ by [V], $\overline{T_{n,k}}$ is a prime knot for $k \geq 2$, and also for $k = 1$ by [Sc].

To show the claim for prime alternating knots, it suffices to replace in the above argument $T_{n,1}$ by an alternating tangle $\hat{T}_{n,1}$, obtained from $T_{n,1}$ by the operation described in the proof of theorem 1.1 (and on figure 2), and to take instead of T''_n the alternating tangles T'_n, mirrored in such a way so as $T_{n,k}$ to remain alternating. □

5 Odds & Ends

There are a lot of questions and problems suggested by the above results. Here we give an extensive summary of what one could think about to improve and push further.

We start by a problem concerning the construction itself.

Question 5.1 Although they easily achieve alternation, both our and Stanford's constructions live at the cost of exponential (in n) crossing number augmentation (at least in the diagrams where n-triviality is achieved). Contrarily, the series of examples of n-trivial knots by Ng [Ng] have crossing number which is linearly bounded in n. There knots are, however, not (*a priori*) alternating or positive, slice (so all have zero signature), and so not to distinguish among each other by such *ad hoc* arguments as [Ka, Mu, Th]. Is it possible to combine the advantages of both series of examples in a new one?

As for the applications of our construction, the results of the section 4 suggest two more problems.

Question 5.2 Does an alternating prime knot K_{n,u_0} exist for any choice of n and u_0 in theorem 4.1?

Theorem 1.2 can be interpreted as saying that any finite number of Vassiliev invariants does not obstruct to any (non-zero) value of the unknotting number. On the other hand, it is remarkable that such obstructions do exist for other unknotting operations, as the Δ move of Murakami and Nakanishi [MN]. Moreover, certain properties of Vassiliev invariants with respect to the ordinary unknotting operation can be suspected in special cases.

Conjecture 5.1 Let (K_i) be a sequence of (pairwise distinct) positive knots of given unknotting number, and $v_2 = -{}^1\!/_6V''(1) = {}^1\!/_2\Delta''(1)$ and $v_3 = -{}^1\!/_{12}V''(1) - {}^1\!/_{36}V'''(1)$ be the (standardly normalized) Vassiliev invariants of degree 2 and 3, where V is the Jones [J] and Δ the Alexander polynomial [Al]. Then the numbers $\log_{v_2(K_i)} v_3(K_i)$ (which are well-defined for $K_i \neq !3_1$) converge to 2 as $i \to \infty$.

This conjecture is related to some results of [St2], but it would take us too far from the spirit of this paper to describe the relation closer here.

Question 5.3 Can any finite number of Vassiliev invariants be realized by a quasipositive (or strongly quasipositive) knot?

Remark 5.1 Rudolph showed that any Seifert pairing can be realized by a quasipositive knot, so there are no constraints to quasipositivity from Vassiliev invariants via the Alexander polynomial.

The consideration of the homology of the double branched cover suggests several questions about further generalizations and modifications, basically coming from the desire to remove the reduction modulo some number. We should remark that n-similarity poses via the Alexander polynomial a congruence condition on the determinant, and ask whether this is the only one.

Question 5.4 Is there for any $n \in \mathbf{N}$ and any knot K a knot K' with $K' \sim_n K$ (or weaker, an n-trivial knot K') with

i) any (finite) abelian group of order $\det(K') \equiv \pm\det(K) \mod 4^{\lfloor (n+1)/2 \rfloor}$ as homology of the double branched cover, or weaker

ii) any odd positive integer $\det(K') \equiv \pm\det(K) \mod 4^{\lfloor (n+1)/2 \rfloor}$ as determinant?

As a weaker version of part i), is any (non-constant) knot invariant depending (only) on the homology of the double branched cover not a Vassiliev invariant?

Remark 5.2 Note, that there is no chance to get K' with some of the above properties in general to be alternating, as for K alternating $\det(K) \geq c(K)$.

Remark 5.3 The weaker statement that any non-constant knot invariant depending on finite reductions of the homology of the double branched cover is not a Vassiliev invariant is, as seen, true, and beside from its stronger versions proved above, originally follows from the results on k-moves in [St].

At least for part ii) the strategy followed in §4 appears promising – use Krebes calculus and construct arborescent tangles by properly inserting n-trivial rational tangles. This leads to a question on the image of Krebes's invariant R on the set of n-trivial arborescent tangles, whose first part is a specialization of ii) of the question above, and whose second part addresses another unrelated by appealing property.

Question 5.5 Let T_n be the set of n-trivial arborescent tangles (of the homotopy type of the 0-tangle).

i) Can for any n and any odd c (or weaker $c = 1$) some $(d,c) \in \tilde{\mathbf{Q}}$ be realized as $R(T)$ for some $T \in T_n$?

ii) Is the image of $R(T_n)$ under the (tautological) homomorphism $\tilde{\mathbf{Q}} \to \mathbf{Q} \cup \{\infty\}$ dense in $\mathbf{R} \cup \{\infty\}$? Is it even the whole $\mathbf{Q} \cup \{\infty\}$?

Remark 5.4 Note, that Krebes in his paper (basically) answers positively both questions in part ii) for $n = 0$.

In view of the desire to consider the detection of orientation (which is a much more relevant problem than just the detection of knottedness), the constructions of n-similar knots suggest one more general and final problem.

Question 5.6 In [St], I gave a generalization of Gousarov's concept of n-triviality, called n-invertibility, which, *inter alia*, led by use of [BM] to an elementary construction of a 14 crossing (closed) 3-braid knot, whose orientation cannot be detected in degree ≤ 11, the argument being provided without any computer calculation. The argument applied there does not seem (at least straightforwardly) to be recoverable from n-triviality alone (in particular, because the knot is not 11-trivial). However, yet, I have no series of examples of arbitrary degree (as those here), where this generalized argumentation shows indeed more powerful, that is, where the failure of Vassiliev invariants of degree $\le n$ to detect orientation can be explained via n-invertibility, but not via n-similarity to some invertible knot. Do such examples exist?

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