THE CANONICAL GENUS OF A CLASSICAL AND VIRTUAL KNOT

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Abstract. A diagram D of a knot defines the corresponding Gauss Diagram G_D. However, not all Gauss diagrams correspond to the ordinary knot diagrams.

From a Gauss diagram G we construct closed surfaces F_G and S_G in two different ways, and we show that if the Gauss diagram corresponds to an ordinary knot diagram D, then their genus is the genus of the canonical Seifert surface associated to D. Using these constructions we introduce the virtual canonical genus invariant of a virtual knot and find estimates on the number of alternating knots of given genus and given crossing number.

1. Introduction

1.1. Knots (smooth embeddings of S^1 into $\mathbb{R}^3$) are usually presented by knot diagrams that are generic immersions of S^1 into the $\mathbb{R}^2$-plane enhanced by information on over-passes and under-passes at the double points. A knot diagram is said to be *alternating* iff as one follows the knot diagram under-crossings and over-crossings alternate. An *alternating knot* is one which has an alternating diagram.

A Seifert surface F_K of a knot K is an oriented surface embedded into $\mathbb{R}^3$ such that $\partial(F_K)$ has one connected component and this connected component equipped with the natural orientation is the knot K. The *genus* $g(K)$ *of a knot* K is the classical knot invariant that is equal to the minimum of genus $g(F_K)$ of Seifert surfaces F_K for K.

For each planar diagram $P(K)$ of a knot K equipped with the information about over and under-passes there is *a canonical Seifert surface* $F_{P(K)}$ associated with this diagram, i.e. the surface obtained by applying Seifert's algorithm on $P(K)$ (see for example [12], p. 120). The minimal genus of all canonical Seifert surfaces for K is called the *canonical genus* $g_c(K)$ *for* K. It was shown by Moriah [8] that the difference between $g(K)$ and $g_c(K)$ can be arbitrarily large. Namely for any $n \in \mathbb{N}$ there exist a knot K such that $g_c(K) - g(K) > n$. (Results of similar spirit were later obtained by Kawauchi [6] and one of the authors [14]. See also Kobayashi and Kobayashi [7].)

But (see [11], [2]), the genus and the canonical genus are coincide for alternating knots. This allows us to use the canonical genus to give estimates on the number of alternating knots of given genus and given crossing number. In this paper we prove

Theorem 1.2. *The number of alternating genus* g *knots of* n *crossings is* $O_n(n^{6g-4})$ *as* $n \to \infty$.

The proof bases on associating to an alternating knot diagram a word in a certain alphabet called Wicks form, and interpreting the (canonical) genus of the alternating knot diagram as genus of its Wicks form. For this we will need to introduce several ways of constructing surfaces out of Wicks forms and show their equivalence. Then the theory for such forms (see [15], [1]) is combined with the structure on the set of alternating knots of given genus studied in [13]. Using the computations of [1], we show that the bound in theorem 1.2 is exact for genus 3 (extending the previous first author's results for genus 1 and 2). Since not every Wicks form gives rise to a knot diagram, we start with introducing an extension of knot diagrams, the *virtual knot diagrams*, introduced in [5], and their associated *Gauß diagrams*, studied in [4], as a combinatorial interpretation of Wicks forms. We would like to remark, that Wicks forms are often called "ideal triangulations" in the literature (see, for example [9]).

2. Virtual knot diagrams and virtual knot genus

2.1. *Diagrams of Knots and Gauss Diagrams.* A Gauss diagram is a circle equipped with some number $n \in \mathbb{N}$ of oriented chords that connect n pairs of points on the circle. (The $2n$ points are all distinct.) The chords of a Gauss diagram are equipped with signs. Gauss diagrams are considered up to an orientation preserving homeomorphism of the underlying circle.

Knots (smooth embeddings of S^1 into $\mathbb{R}^3$) are usually presented by knot diagrams that are generic immersions of S^1 into the $\mathbb{R}^2$-plane enhanced by information on over-passes and under-passes at the double points. To correspond a Gauss diagram to a knot diagram D one connects by a chord the preimages of each double point of the immersion. The orientation of the chords is chosen from the over-passing branch to the under-passing one. The sign of a chord is the sign of the corresponding double point. The obtained Gauss diagram G_D is said to be *the Gauss diagram of the knot diagram D.*

Gauss diagrams that are obtainable as Gauss diagrams of some knot diagrams are said to be *realizable.* A knot diagram corresponding to a realizable Gauss diagram can be recovered only up to a certain ambiguity. However the isotopy type of the corresponding knot is recoverable in a unique way.

A *virtual knot diagram* is a generic immersion of a circle into the $\mathbb{R}^2$-plane with the double points divided into real crossings and virtual crossings. The real crossings are enhanced by information on over- and under-passes (as for the classical knot diagrams). At a virtual crossing the crossing branches are not divided into an over-pass and an under-pass. The *Gauss diagram of a virtual knot diagram* is constructed in the same way as for a classical knot diagram, but all the virtual crossings are disregarded. One can show that every Gauss diagram is a Gauss diagram of a virtual knot diagram.

2.2. It is well-known that when a knot changes by a generic isotopy, its diagram undergoes a sequence of *Reidemeister moves* shown in Figure 1. A virtual knot diagram is allowed to undergo the same Reidemeister moves as well as the moves shown in Figure 2. The additional moves are the *virtual moves*. A *virtual knot* is a class of virtual knot diagrams consisting of diagrams that can be transformed to each other by a sequence of Reidemeister and virtual moves. Such a sequence of moves is called a *virtual isotopy*. (Observe that the moves shown in Figure 3 are prohibited. If one allows these moves then the theory of virtual knots becomes trivial and every virtual knot can be unknotted.)

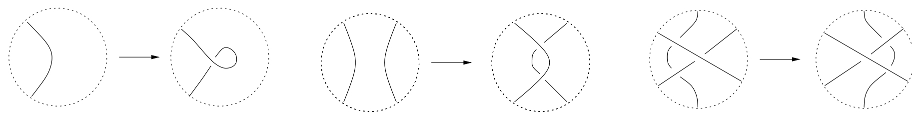


FIGURE 1

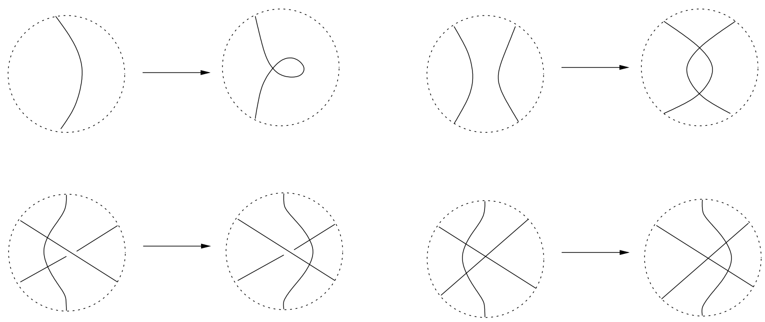


FIGURE 2

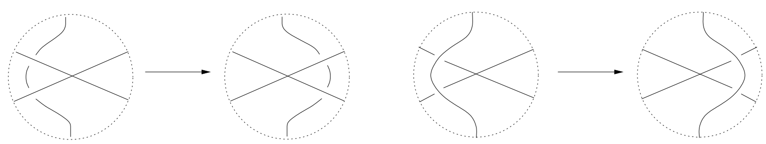


FIGURE 3

Theorem 2.3 (M. Goussarov, M. Polyak and O. Viro [4] and L. H. Kauffman [5]). *A Gauss diagram defines a virtual knot up to virtual moves.*

2.4. *Virtual canonical genus.*

For a Gauss diagram G put F'_G to be the orientable surface with boundary obtained as follows. Take a thin annulus (that we can identify with a thickened circle), and for each chord of G take a thin strip and glue its ends to the boundary of the annulus in the places that correspond to the end points of the chord so that the obtained surface is orientable.

Put F_G to be the closed orientable surface obtained by gluing all the boundary components of F'_G with the 2-disks, and put *the genus* $g(G)$ *of the Gauss diagram* to be the genus $g(F_G)$ of F_G.

Theorem 2.5. *Let* $P(K)$ *be a planar diagram of a knot* K*, let* G *be the Gauss diagram of* $P(K)$*. Then* $g(F_G) = g(F_{P(K)})$*.*

2.6. *Proof of Theorem 2.5.* The Euler characteristic $\chi(F_{P(K)})$ of $F_{P(K)}$ is the number of Seifert circles minus the number of crossings of $P(K)$.

The Euler characteristic $\chi(F_G)$ of F_G is the number of disks that were glued to the boundary components of F'_G minus the number of chords in G. Or equivalently $\chi(F_G)$ is the number of boundary components of F'_G minus the number of crossings of $P(K)$.

Geometrical considerations show that there is a natural bijection between the boundary components of F'_G different from the outer boundary of the annulus and the Seifert circles appearing in the construction of the canonical Seifert surfaces for $P(K)$.

Thus $\chi(F_G) = \chi(F_{P(K)}) - 1$ ($F_{P(K)}$ has one boundary component) and hence $g(F_G) = g(F_{P(K)})$. □

Definition 2.7. For a virtual knot K put *the virtual canonical genus $g_{vc}(K)$ of K* to be the minimum over all the Gauss diagrams G that realize K of $g(F_G)$

The following corollary follows immediately from Theorem 2.5.

Corollary 2.8. *If a virtual knot K admits a planar diagram without virtual crossings that realizes the isotopy class of an ordinary knot $\tilde{K}$, then $g_c(\tilde{K}) \geq g_{vc}(K)$.*

Theorem 2.9. *F_G has minimal genus among all surfaces, on which G can be embedded.*

2.10. *Proof of Theorem 2.9.* If M is a surface of minimal genus, on which G can be embedded, then all components of $M \setminus G$ must be disks. But then, M can be obtained by gluing disks into a band-thickening of G, and hence $\chi(M) = \chi(F_G)$. □

We remark the following combinatorial description of $g(F_G)$. If we denote by $s_{2n} \in S_{2n}$ the permutation of $2n$ elements sending $i \in \{1, \dots, 2n\}$ to $i + 1 \bmod 2n$, then each Gauss diagram G of n chords defines a fix-point free involution $s_G \in S_{2n}$ up to a conjugating with a power of s_{2n}. Then the number of discs to glue in to obtain F_G is equal to the number of cycles $c(s_{2n} \circ s_G)$ of $s_{2n} \circ s_G$, and hence

$$g(F_G) = \frac{n - c(s_{2n} \circ s_G) + 1}{2}.$$

There is yet another way of associating surfaces (and genera) to Gauss diagrams, namely the following:

Definition 2.11. Let G be a Gauss diagram of n chords. We indicate the end points of every chord of G by the same letter in some alphabet, but with the opposite powers 1 and -1; different letters correspond to different chords. So, on the circle we have a cyclic word W in some group alphabet.

Subdivide the circle into $2n$ arcs so that every arc contains exactly one end point of a chord. Let S_G be the unique orientable surface obtained from the disk (whose boundary is the circle of the Gauss diagram) by gluing together the arcs that are connected by a chord.

Again this genus coincides with the previous one(s).

Theorem 2.12. $\chi(S_G) = \chi(F_G)$. *In particular, if G is realizable as a Gauss diagram $G(P(K))$ of a knot diagram $P(K)$ (of a knot K), then $g(S_G) = g(F_G) = g(F_{P(K)})$.*

2.13. *Proof of Theorem 2.12.* The boundary of the disk from definition 2.11 becomes a graph on the surface S_G. The number of edges is the same as the number of chords of G. Geometric considerations show that the number of vertices is equal to the number of boundary components of F'_G in the proof of theorem 2.5, omitting the outer one (or if G is realizable by $P(K)$, the number of $P(K)$'s Seifert cycles). So, we finish the proof by applying Euler's formula. □

3. On the number of alternating knots of given genus

It is well known (see [11], [2]), that the genus and the canonical genus are coincide for alternating knots. This allows us to use the canonical genus to give estimates on the number of alternating knots of given genus and given crossing number.

Definition 3.1. A chord a in a Gauss diagram G is called *strongly isolated*, if its endpoints occur as

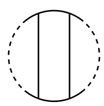
$$\dots aa^{-1}\dots$$

when denoting the endpoints of the chords of the diagram in cyclic order by a letter and its inverse for each chord[1]. More generally, a is called *isolated* if there is no other chord b such that the cyclic notation is of the type

$$\dots a\dots b^{-1}\dots a^{-1}\dots b\dots$$

If G is realizable by a knot diagram D, then the crossing in D corresponding to a is called nugatory. Two chords a and b in a Gauss diagram are called *parallel*, if their endpoints occur as

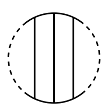
$$\dots ab^{-1}\dots ba^{-1}\dots$$



in this cyclic order notation. Three chords a, b and c are called parallel, if a and b, and b and c are parallel, i.e. we have

$$\dots abc\dots c^{-1}b^{-1}a^{-1}\dots$$

or

 .

The basic notion introduced in [13] was a weaker version of deleting parallel chords, which does not spoil realizability. For its definition, which we rephrase here slightly in more Gauss diagrammatic terms, we need the flype move of [10].

Definition 3.2. A knot diagram is called $\bar{t}_2$ irreducible, if after any sequence of flypes its Gauss diagram does not have a triple of parallel chords. A Gauss diagram is $\bar{t}_2$ irreducible, if it corresponds to a $\bar{t}_2$ irreducible knot diagram.

[1]Here the use of the inverse is made only to conform to standard notation, but the choice for which of the two occurrences to use the letter and for which the inverse is completely irrelevant.

Theorem 3.3. ([13, theorem 3.1]) *The set $\mathcal{D}_g$ of $\bar{t}_2$ irreducible alternating diagrams with no nugatory crossings and genus g is finite. We call elements of $\mathcal{D}_g$ 'generating diagrams' of genus g.*

This result allows to make the following definition.

Definition 3.4. For a knot diagram D, let $c(D)$ be its crossing number. And let $d(D)$ be the number of $\sim$-equivalence classes of crossings under the equivalence relation $a \sim b$ iff a and b can be made to be parallel chords in the Gauss diagram after some sequence of flypes on D.

Set

$$c_g := \max\{\, c(D) \,:\, D \in \mathcal{D}_g \,\}\,,$$

$$d_{g,e} := \max\{\, d(D) \,:\, D \in \mathcal{D}_g,\, c(D) \text{ even} \,\}\,,$$

$$d_{g,o} := \max\{\, d(D) \,:\, D \in \mathcal{D}_g,\, c(D) \text{ odd} \,\}\,,$$

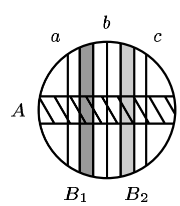
and

$$d_g := \max(d_{g,e}, d_{g,o})\,.$$

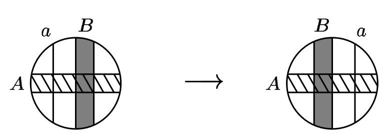
We have that

$$d_g \le c_g \le 2d_g\,.$$

The first inequality in obvious. For the second inequality one needs to note that if for three chords a, b and c, a and b, and b and c can be made parallel after flypes, then a, b and c be made parallel simultaneously, so that the diagram becomes $\bar{t}_2$ reducible. Namely, the property to be transformable into a parallel pair after flypes means that the two chords intersect the same set of other chords. If this applies to a, b and c, then in the Gauss diagram we have a situation like this:



(Here the dashed/shaded regions A and $B_{1,2}$ should symbolize collections of chords, whose ends lie on the segments of the circle touched by the region.) Then by applying flypes, which on the Gauss diagram look like



one obtains three parallel chords.

A consequence of the result of [10] is an asymptotical bound for the number $a_{n,g}$ of alternating knots of n crossings and genus g. For two sequences of positive integers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ we write $a_n = O_n(b_n)$ iff $\overline{\lim_{n\to\infty}}\, \frac{a_n}{b_n} < \infty$, and $a_n \asymp_n b_n$ (a_n is asymptotically proportional to b_n) iff $a_n = O_n(b_n)$ and $b_n = O_n(a_n)$.

Corollary 3.5. *$a_{n,g} = O_n(n^{d_g-1})$. More exactly, $a_{n,g} \asymp_n n^{d_{g,e}-1}$ for n even and $a_{n,g} \asymp_n n^{d_{g,o}-1}$ for n odd.*

Proof. To see that $a_{n,g} = O_n(n^{d_g-1})$, it suffices to prove that the number of diagrams D' of crossing number n in a series generated by a $\bar{t}_2$ irreducible alternating diagram D of genus g is $O_n(n^{d_g-1})$. But if D has d $\sim$-equivalence classes, then each D' corresponds to writing a composition $n = n_1 + \dots n_d$, with the i-th $\sim$-equivalence class in D' having n_i crossings (n_i must be even or odd depending on whether the i-th $\sim$-equivalence class in D has one or two crossings). But the total number of compositions of n into d parts is $\binom{n+d-1}{d-1} \asymp_n n^{d-1}$. Since $d \le d_g$, the claim follows.

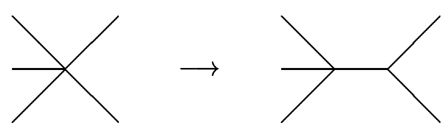
The same argument shows $a_{n,g} = O_n(n^{d_{g,e}-1})$ for n even and $a_{n,g} = O_n(n^{d_{g,o}-1})$ for n odd. To see that also $\liminf a_{n,g}/n^{d_{g,e}-1} > 0$, consider the series of a generating diagram D of genus g with $d_{g,e}$ $\sim$-equivalence classes, and therein diagrams D' of n crossings giving compositions with all n_i's ($1 \le i \le d_{g,e}$) pairwise distinct and increasingly ordered ($n_{i+1} > n_i$). Since flypes descend to $\sim$-equivalence classes, the result of [10] assures that each alternating knot has at most one such diagram. It remains to note that the number of such compositions is asymptotically proportional to $n^{d_{g,e}-1}$. □

The last corollary motivates the quest for some good upper bound on d_g. The bound on d_g given in [13] was obtained by estimating c_g, and this estimate was exponential in g. Here we improve the bound on d_g (and hence c_g) to the conjecturally best possible.

Theorem 3.6. *A $\bar{t}_2$ irreducible Gauss diagram of genus g has at most $6g-3$ classes of $\sim$-equivalent chords.*

For the proof we need a small graph-theoretical definition.

Definition 3.7. An anticontraction is an operation on a graph of the following type



It makes a vertex of valence $v \ge 4$ into two vertices $p_{1,2}$ with valences $v_{1,2} < v$ with $v_1 + v_2 = v + 2$, such that $p_{1,2}$ are connected by an edge.

3.8. *Proof of Theorem 3.6.* Consider the Gauss diagram of a generating diagram D of genus g and remove a chord in each pair of $\sim$-equivalent chords, obtaining a (not necessarily realizable) Gauss diagram G of d_G chords. As D is generating, $d_G \le d_g$. In this Gauss diagram there is no pair of parallel chords. We can also assume that in this Gauss diagram there is no isolated chord.

If you consider a $2d_G$-gon with chords indicating how to glue its edges according to definition 2.11, you obtain by Theorem 2.12 a surface S_G of genus g with a graph $\hat{G}$ on it. As G has no pair of parallel chords and no isolated chord, the graph $\hat{G}$ has no vertices of valence 1 or 2. Moreover, the number of edges of $\hat{G}$ is the number of chords of G.

If now $\hat{G}$ has a vertex of valence $n \ge 4$, then it can be made into $n-2$ vertices of valence 3 by anticontractions, each one of which inserts one edge. Repeat this procedure until you obtain a trivalent graph $\tilde{G}$ on S_G. (Neither $\tilde{G}$ needs to be unique, nor coming from a realizable Gauss diagram, but both points are relevant here.)

It suffices to see that $\tilde{G}$ has $6g-3$ edges. As each edge has valence 3, for the number of edges e and the number of vertices v in $\tilde{G}$ we must have $e = 3k$ and $v = 2k = 2e/3$ for some $k \in \mathbb{N}$. The Euler characteristic of the surface S_G is then $v - e + 1$, because of the one face of the triangulation given by $\tilde{G}$, and on the other hand $2 - 2g$, so that

$$v - e + 1 = -\frac{e}{3} + 1 \;=\; 2 - 2g\,,$$

whence $e = 6g - 3$, as desired. □

A diagram is *prime* iff the intersection graph of its Gauß diagram is connected. (The intersection graph of a Gauß diagram is a graph with vertices corresponding to arrows in the Gauß diagram and edges connecting vertices or intersecting arrows.)

The following corollary, as well Theorem 1.2, is a consequence from the work of [13] described above (note, that g and the number of $\sim$-equivalence classes of crossings are additive under connected sum of diagrams, so that considering prime diagrams suffices), Theorem 2.6 and the proof of corollary 3.5.

Corollary 3.9. $d_g \le 6g-3$ *and* $c_g \le 12g-6$.

The work done in [1] can be used to calculate the explicit values for c_3 and d_3.

Proposition 3.10. $c_3 = 23$ *and* $d_3 = 15$. *In particular, the number of alternating genus 3 knots of* n *crossings grows not faster than* n^{14}.

3.11. *Proof of Proposition 3.10.* Consider the Gauss diagram G of a generating diagram of genus 3, realizing c_3 or d_3, and remove in it from each pair of parallel chords one of them, obtaining a diagram G'. (This preserves the genus, even if it spoils realizability).

We say that a word W' of a free group on a set X' is a *orientable word* if each element of X' that occurs in W' exactly twice, once with exponent $+1$ once with exponent -1.

A quadratic word W is *irredundant* if there is no pair of distinct, noninverse letters x, y which appear in W only in subwords (xy) and $(xy)^{-1}$. A quadratic word W is *cyclically reduced* if there are no subwords of the form xx^{-1} or $x^{-1}x$ for a letter x.

It is known (see, for example [3]), that the maximal length of a genus g cyclically reduced orientable irredundant word is $12g-6$, and every (cyclically reduced orientable irredundant) genus g word can be obtained from a (cyclically reduced orientable irredundant) word of maximal length by substituting some letters (and their inverses) by unit.

In the language of chord diagrams it means that each Gauss diagram of genus g with no pair of parallel chords and no strongly isolated chord (that is, in particular with no isolated chord) can be obtained by deleting chords from the subset of such diagrams that correspond to the words of maximal length.

Such words were explicitly enumerated in [1] and the explicit list $\mathcal{L}_g$ was generated by computer for $g \le 3$. On the level of Gauss diagrams they are the diagrams such that the endpoints of each chord a participate in a cycle of the type

$$\dots ab^{-1} \dots bc^{-1} \dots ca^{-1} \dots \quad \text{or} \quad \dots ba^{-1} \dots cb^{-1} \dots ac^{-1} \dots$$

for (the endpoints of) some other chords b and c, when denoting the endpoints of the chords of the diagram in cyclic order. These diagrams all have $6g-3$ chords.

To calculate c_3 and d_3 we proceeded as follows. First we factored out the elements in $\mathcal{L}_3$ by inversion. Then we considered diagrams obtained by just retaining or doubling chords in diagrams in $\mathcal{L}_3$. We generated by computer all such diagrams with decreasing crossing number (from 30 downward), and tested realizability.

The first crossing number where we found realizable diagrams was 23. The existence of such a diagram already shows that $d_3 = d_{3,o} = 15$. (As in [1] flypes were not considered, some of the diagrams obtained may contain chords at different positions, which can be made parallel after flypes, and so the realizable diagrams had again to be subjected to a test for $\bar{t}_2$ irreducibility.) Then we found such diagrams also for crossing number 22, ensuring also $d_{3,e} = 15$.

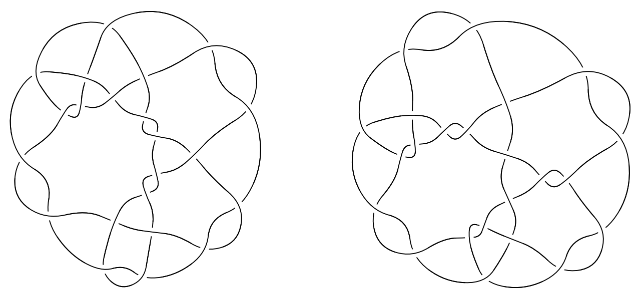


FIGURE 4. Two $\bar{t}_2$ irreducible diagrams of genus 3 of maximal even and odd crossing number.

Then we needed to check that, with the additional option for each chord to be deleted from a diagram in $\mathcal{L}_3$, there are no realizable diagrams of more than 23 crossings. This establishes that indeed $c_3 = 23$. (There were, in fact, some realizable 24 crossing diagrams, but they all were not $\bar{t}_2$ irreducible.) □

4. QUESTIONS

We conclude by several questions. (Here we say that a diagram D realizes c_g resp. d_g if $D \in \mathcal{D}_g$ and $c(D) = c_g$ resp. $d(D) = d_g$.)

Question 4.1. *Is always $d_g = 6g-3$ and $c_g = 10g-7$?*

Question 4.2. *Is for any $g > 1$, d_g attained by both even and odd crossing number diagrams? Do the maximal even and odd crossing numbers of such diagrams differ by ± 1?*

Question 4.3. *Are for any $g > 1$ some (any?) alternating genus g diagrams realizing c_g special (i.e., they are also positive)?*

Question 4.4. *Is some (any?) knot attaining c_g also attaining d_g (which does not happen for $g = 1$)?*

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