THE SECOND COEFFICIENT OF THE JONES POLYNOMIAL

A. Stoimenow

Graduate School of Mathematical Sciences,
University of Tokyo,
3-8-1, Komaba, Tokyo 153-8914, Japan
e-mail: stoimeno@ms.u-tokyo.ac.jp
WWW: http://www.ms.u-tokyo.ac.jp/~stoimeno

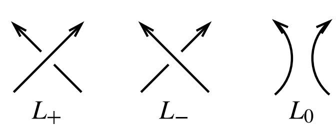
Abstract. This is a research-expository style transcript of my talk at the conference "Intelligence of Low Dimensional Topology 2004" taking place at Osaka City University October 25 - October 27, 2004, for the proceedings of the conference.

Keywords: Jones polynomial, twist number, hyperbolic volume, Mahler measure, adequate link
AMS subject classification: 57M25 (primary)

The *Jones polynomial* V (now commonly used with the convention of [J]) is a Laurent polynomial in one variable t of oriented knots and links, and can be defined by being 1 on the unknot and the *(skein) relation*

$$t^{-1}V(L_+) - tV(L_-) = -(t^{-1/2} - t^{1/2})V(L_0)\,. \tag{1}$$

Herein $L_{\pm,0}$ are three links with diagrams differing only near a crossing.



(2)

When

$$V_K = a_0t^k + V_1t^{k+1} + \ldots + a_dt^{k+d} \tag{3}$$

with $a_0 \neq 0 \neq a_d$ is the Jones polynomial of a knot or link K, we will use throughout the paper the notation $V_i = V_i(K) = a_i$ and $\bar{V}_i = \bar{V}_i(K) = a_{k-i}$ for the the i-th or i-th last coefficient of V, and will write for d the *span* $\operatorname{span} V_K$ of V, for k the *minimal degree* $\min\deg V_K$ and for $k+d$ the *maximal degree* $\max\deg V_K$.

For quite a while one is wondering what topological information the Jones polynomial contains, and in connection with this, one posed the

Question 1 Does there exist a non-trivial knot with trivial Jones polynomial?

While the existence of non-trivial links with trivial polynomial is now settled for links of two or more components by Eliahou-Kauffman-Thistlethwaite [EKT], the (most interesting) knot case remains open. The question remains unanswered, though some classes of knots have been excluded from having trivial Jones polynomial. These results are obtained in [Ka, Mu, Th2] for alternating knots, [LT] for adequate knots, [St2] for positive knots, and also in [Th2] for the Kauffman polynomial of semiadequate knots. Except for these (meanwhile classical) results, and despite considerable (including electronic) efforts [Bi, Ro, DH, St5], even nicely defined general classes of knots on which one can exclude trivial polynomial are scarce. (I came across some work of Yamada who stated that he verified all knots up to 21 or 22 crossings, but I have no reference to it.)

More recently some excitement is caused by the

Conjecture 1 (Volume conjecture [MM]) Some complicated colored Jones polynomial values converge to the Gromov norm of the knot complement (= hyperbolic volume of all hyperbolic parts in the JSJ decomposition = hyperbolic volume for hyperbolic knots).

This conjecture seems, unfortunately, little helpful to determine the volume. One would require a convergence (error term) estimate, but even a proof of the limit seems so far out of reach. On the other hand, at least numerically, we can do easier by computers[1]. However, one still hopes to get some fundamental new insight in what is going on there. For example:

Proposition 1 The Volume conjecture implies that no non-trivial knot has trivial *colored* Jones polynomial.

So it relates to question 1, and also poses

[1] I must confess that by long and bad experience with several quantum topologists (outside Japan) made me lose motivation to work on this subject. While later I had similar, sometimes even worse, experience with other mathematicians (again outside Japan), this did not make me redeem quantum topology.

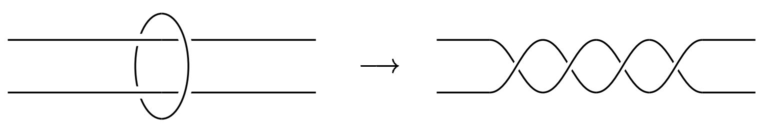
Question 2 What are the relations between volume and V?

In other words, if we sacrifice '$=$' for a '$\leq$', are there more tangible and practical ways to relate volume to V?

What are general ways to estimate hyperbolic volume?

Recent work of Lackenby-Agol-Thurston [La] has opened a way of estimating hyperbolic volume of knots by means of a diagrammatic feature called *twist number*. While twist numbers occur naturally and were considered before, the relation to hyperbolic volume added new interest in them.

Thurston's[1] hyperbolic surgery theorem considers volume under surgery, which for us will be of the following particular type:



D_∞ D_n

By fixing k circles, we have analogously diagrams $D(n_1,\dots,n_k)$.

Theorem 1 (Thurston) $\operatorname{vol}(D(n_1,\dots,n_k)) \underset{\leq}{\longrightarrow} \operatorname{vol}(D_{\infty,\dots,\infty})$ when $\min_{i=1}^{k}|n_i| \to \infty$.

A further remark of Thurston that for any link K, we have $\operatorname{vol}(D) \leq 4v_0c(D)$ (where $v_0 = \operatorname{vol}(4_1)/2$ is the ideal tetrahedral volume and $c(D)$ the crossing number), implies

$$\operatorname{vol}(D) \leq C \cdot t(D)$$

where $t(D)$ is the twist number. The merit of Agol-Thurston in (the appendix of) [La] was to identify the best possible C, and that of Lackenby himself to prove a reverse inequality for *alternating* diagrams:

Theorem 2 (Lackenby [La]) $\operatorname{vol}(D) \geq C' \cdot t(D)$ for alternating knot diagrams D.

Turning back to the Jones polynomial we can ask

Question 3 Are there upper estimates on vol from V?

[1]Here the famous – Bill – Thurston, the father of the previously named – Dylan – Thurston, is meant.

The situation is quite unclear. For example, the only way to show that such estimates do not exist is to find knots (or links) K_i with equal Jones polynomial but vol $\to \infty$.

- Kanenobu [K] found infinitely many knots with the same Jones polynomial, but these knots have bounded twist number. Since bounded twist number implies bounded volume, Kanenobu's series can not be used.
- Similarly fails the construction in [EKT], which yields non-hyperbolic links.

For *lower bounds* on vol from V it makes sense to seek an increasing function of some feature of V.

For example, as candidates for such a feature consider

1) *Mahler measure.* If $f(t)$ is a nonzero complex polynomial with (complex) zeros a_i,
$$f(t) = b \cdot \prod_{i=1}^{n} (t - a_i) \in \mathbb{C}[t],$$
then its *Mahler measure* [Ma] is
$$M(f) = \exp \int_0^1 \log |f(e^{2\pi i\theta})| d\theta = |b| \cdot \prod_{i=1}^{n} \max\{|a_i|, 1\}.$$
So one can ask:

Are there knots K_i with $M(V(K_i))$ bounded (fixed?), but vol$(K_i) \to \infty$?

Are there knots K_i with bounded vol(K_i) but $M(V(K_i)) \to \infty$?

If we take knots as in theorem 1 (bounded twist number), then $M(V)$ is bounded, as we proved in joint work with Dan Silver and Susan Williams [SSW]. (Kofman and Champanerkar also proved convergence [CK] in certain situations.) Twisting more strands will not help either. Are there other constructions that yield bounded volume knots and can one control V under such constructions? (As I'm not a hyperbolic expert, I don't know.)

2) *2-norm.*
$$||f||_2^2 = \int_0^1 |f(e^{2\pi i\theta})|^2 d\theta = \sum_{i=-\infty}^{\infty} ([f]_{t^i})^2,$$
with $[f]_{t^i}$ the coefficient of t^i in f. (1-norm is similar.) To find knots with bounded volume but $||V||_2 \to \infty$ is easy, but are there knots with bounded $||V||_2$ and vol $\to \infty$?

There exist knots with (distinct, i.e. not *à la* Kanenobu) polynomials with bounded $||V||_2$. For example $(3,q)$-torus knots (use Jones' formula for the polynomial of a torus knot [J, proposition 11.19]; maybe (p,q)-torus knots with any $p \geq 3$ fixed will also do, but I did not check it). But all sequences of that sort that come to my mind seem non-hyperbolic.

3) *span*. The span is the difference between minimal and maximal exponent of non-trivial monomials, very famous in [Ka, Mu, Th2].

   Is it at all possible that $\mathrm{span}(V)$ is bounded, but we have infinitely many different Jones polynomials?

   One should note that infinite families of 2-component links were indeed constructed by Traczyk [Tr], for polynomials that differ up to units. But, again, these links are non-hyperbolic. For knots polynomials cannot differ by non-trivial units (because of $V(1) = 1$ and $V'(1) = 0$), and the answer is unknown.

The conclusion is:

For every family of knots or links with peculiar behaviour of the Jones polynomial, the volume is zero or bounded.

A digression on the Alexander polynomial. For the Alexander polynomial most of these questions are settled.

- Knots (even alternating) with $M(\Delta) \to \infty$ but bounded volume are easy to find (see [SSW]).
- There are (even alternating) knots with $M(\Delta) = 1$ but vol $\to \infty$ [SSW].
- For general knots, if one drops the alternation assumption, there are knots with arbitrarily large volume that have the *same* Alexander polynomial. For the trivial polynomial see [Kf]; I have a different construction that applies to any (fixed) Alexander polynomial. However,
- for *alternating* knots, the *Euclidean* Mahler measure $M_e(\Delta)$ bounds volume from below increasingly ([SSW]). Euclidean Mahler measure is the Mahler measure of the polynomial made monic by rescaling.
- For alternating knots, a different volume bound is conjectured by Dunfield, and proved by myself [St4]:

$$\mathrm{vol}\,(K) \leq \log_\gamma |\Delta(-1)| = \log_\gamma |V(-1)|\,,$$

  with $\gamma > 1$ a constant.

For the Jones polynomial of special types of knots, more is known.

In [DL] Dasbach-Lin gave a description of the twist numbers of alternating diagrams by means of the second coefficient of their Jones polynomial. They considered $T_i(K) := |V_i| + |\bar{V}_i|$ and proved

Lemma 1 ([DL]) For an alternating knot diagram D, we have $t(D) = T_1(D)$.

This way they obtained, and then further empirically speculated about, certain relations between coefficients of the Jones polynomial and hyperbolic volume.

In fact, we have a qualitative improvement of the Dasbach-Lin result, stating that

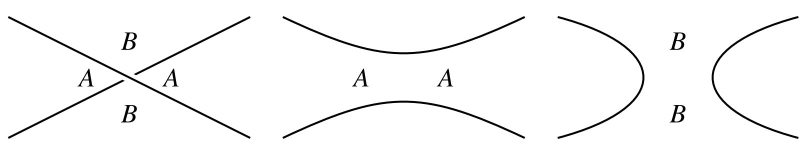
Theorem 3 Every coefficient V_i of the Jones polynomial gives rise to a(n increasing) lower bound for the volume of alternating knots.

The previous occurrence of the second coefficient of the Jones polynomial in a different situation in [St] motivated the quest for understanding V_1, $\bar{V}_1$ in a broader context.

We consider the bracket [Ka] (rather than Tutte, as Dasbach-Lin) polynomial.

The concept of an adequate link was introduced by Lickorish and Thistlethwaite in [LT] to help determining the crossing number of certain links. Adequacy consists of the combination of two weaker properties called jointly semiadequacy. They are defined as follows.

Below are depicted the A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.



(4)

One says a diagram D is *A-adequate* if the number of loops obtained after A-splicing all crossings of D is more than the number of loops obtained after A-splicing all crossings except one. Similarly one defines the property B-adequate. Then we set

$$\begin{aligned} \text{adequate} &= A\text{-adequate } \textit{and } B\text{-adequate}\,, \\ \text{semiadequate} &= A\text{-adequate } \textit{or } B\text{-adequate}\,, \end{aligned}$$

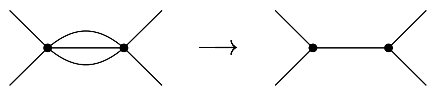
We call a link adequate resp. (A/B/semi)-adequate if it has an adequate resp. (A/B/semi)-adequate diagram.

Note that semiadequate links are a much wider extension of the class of alternating links than adequate links. For example, only 3 non-alternating knots in Rolfsen's tables [Ro2, appendix] are adequate, while all 55 are semiadequate.

An alternative way to understand A-adequacy is to keep the trace of the crossings after each splitting. Then we have each of the traces of the crossings joining two loops, obtained after the splittings. The property A-adequate means that, in the set of loops obtained by A-splitting all crossings, each crossing connects two different loops. We call this set of loops the *A-state* of the diagram.

In the following, we shall explain the second coefficient of the Jones polynomial in semiadequate diagrams. Bae and Morton [BMo] and Manchon [Mn] have done work in a different direction, and studied the extreme coefficients of the bracket (which are ± 1 in semiadequate diagrams) in more general situations.

Let $v(G)$ and $e(G)$ be the number of vertices and edges of a graph G. Let $\overline{G}$ be G with multiple edges removed (so that a simple edge remains).



We call $\overline{G}$ the *reduction* of G. Let $A(D)$ be the *A-graph* of D, a graph with vertices given by loops in the A-state of D, and edges given by crossings of D. (The trace of each crossing connects two loops.)

So a link diagram D is A-adequate, if $A(D)$ has no edges connecting the same vertex. (Anything with B is analogous.)

Theorem 4 ([LT]) If D is A-adequate then $V_0 = \pm 1$. If D is B-adequate then $\bar{V}_0 = \pm 1$. If D is adequate then $V(D) \neq 1$.

Now we have

Theorem 5 If D is A-adequate then $|V_1| = b_1(\overline{A(D)})$ is the first Betti number (number of cells) of the reduced A-graph. Similarly if D is B-adequate then $|\bar{V}_1| = b_1(\overline{B(D)})$.

Key observation: If $b_1(\overline{A(D)}) = 0$, then D admits a positive orientation, i.e., can be oriented so that all crossings become as L_+ in (2).

Corollary 1 No (non-trivial) semiadequate knot has $V = 1$.

Proof. If $V = 1$ then $V_1 = 0$, so the knot must be positive, but no non-trivial positive knot has $V = 1$. $\square$

Actually: There is no non-trivial semiadequate link with trivial Jones polynomial (i.e., polynomial of the same component number unlink), even up to units $\pm t^k$.

Some applications:

Whitehead doubles. Untwisted Whitehead doubles have trivial Alexander polynomial, and are one suggestive class of knots to look for trivial Jones polynomial. (Practical calculations have shown that the coefficients of the Jones polynomial of Whitehead doubles are absolutely very small compared to their crossing number.)

Proposition 2 Let K be a semiadequate non-trivial knot. Then the untwisted Whitehead doubles $Wh_{\pm}(K)$ of K (with either clasp) have non-trivial Jones polynomial.

(Because V and Δ determine the degree-2-Vassiliev invariant v_2 simultaneously, among Whitehead doubles only untwisted ones may have trivial Jones polynomial.)

This generalizes a result for adequate knots in [LT] and positive knots in [St2] and considerably simplifies the quest for trivial polynomial knots among Whitehead doubles. One can combine this condition with the previous ones, the vanishing of the Vassiliev invariants of degree 2 and 3 on K (see [St2, St5]), to extend the verification of [St5] and establish that no non-trivial knot of ≤ 16 crossings has untwisted Whitehead doubles with trivial Jones polynomial.

Montesinos links

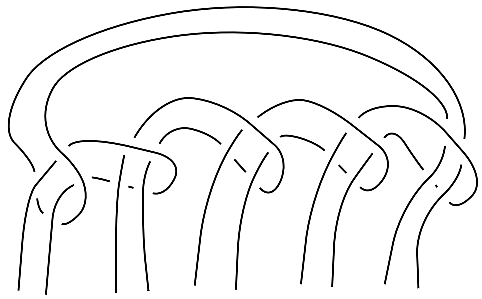
Corollary 2 Montesinos links are semiadequate. So no Montesinos link has trivial Jones polynomial up to units.

Strongly n-trivial knots

The next application concerns strongly n-trivial knots. They were considered first around 1990 by Ohyama, and studied more closely recently [To, HL, AK]. While one can easily verify by calculating the Jones polynomial that a given example is non-trivial, the proof of non-triviality for a family of knots with arbitrarily large n remained open for a while. (For $n > 2$ the Alexander polynomial is trivial [AK].) A proof that partially features the Jones polynomial value $V(e^{\pi i/3})$ was given in [St6], but nothing about the Jones polynomial of the examples directly could be said. How to evaluate the Jones polynomial was also asked by Kalfagianni [Kf]. Now we can deal with a different class of examples, proving the polynomial non-trivial.

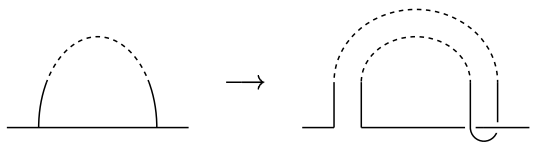
Proposition 3 There exist for any n strongly n-trivial knots with non-trivial Jones polynomial.

Apply the construction $K(G)$ of [AK] on the Suzuki graph G obtained by gluing into a circle the ends of the following family of tangle diagrams (shown for $n = 5$):



(5)

(The signs of the clasps at the arrow hooks are irrelevant.)



3-braids

We call a braid word semiadequate (A-, B-adequate, adequate = etc.) if the closure diagram is semiadequate (etc.). A braid is semiadequate (etc.) if it has a semiadequate (etc.) word.

Thistlethwaite's work [Th] implies that if β is a semiadequate (etc.) braid then its semi-adequate (etc.) words are of minimal (Artin generator) length in the conjugacy class of β (i.e. also for all braids conjugate to β). The interesting feature of 3-braids is that the converse holds for semiadequacy:

Theorem 6 A minimal length word in any 3-braid conjugacy class is semiadequate.

(One can also explicitly describe such words algebraicly.)

Corollary 3 3-braid links are semiadequate, and so have non-trivial Jones polynomial up to units.

It has been speculated for a while that no non-trivial 3-braid knot has trivial Jones polynomial, but a proof was never given (see [B, St3]). It was known that the Burau representation determines the Jones polynomial for 3 and 4-braids [J]. Then, somehow one believes that if the Burau representation is faithful (as known for 3-braids), no knot has trivial polynomial. Bigelow [Bi] is hoping(?) that 4-strand Burau many not be faithful, and is challenging the computers with this idea to find a $V = 1$ knot among closed 4-braids. The closed 4-braid $(\sigma_2\sigma_1\sigma_3\sigma_2)^2\sigma_1^3\sigma_3^{-3}$ (which is among the links given in [EKT]), however, has trivial polynomial up to units. So it cautions about attempts to understand the (possible) non-existence of trivial polynomial knots among 3- or 4-braids in terms of the (possible) faithfulness of the Burau representation. Our proof for 3-braids has indeed little to do with Burau. By the above example, our result also fails for 4-braids.

Combining braid semiadequacy with work in [St3, BM, Xu], we can actually classify all 3-braid links with given Jones polynomial. In particular, we know that

Corollary 4 There are only finitely many closed 3-braids with the same Jones polynomial.

This was known to be true for the skein polynomial [St3]. The links of Traczyk [Tr] show that this is not true for Jones polynomials up to units, and by connected sum for fixed polynomials on 5-braids. (The status of 4-braids here remains unclear.) Also Kanenobu [K2] constructed finite families of 3-braids of any arbitrary size, so that our result is the maximal possible.

The corollary implies the existence of some upper bound on the volume in terms of the Jones polynomial. We can make an estimate more concrete:

Corollary 5 If K is a 3-braid link, which is not a closed positive or negative 3-braid, then $\mathrm{vol}\,(K) \le C' \cdot T_1$ as in Dasbach-Lin.

Also the following is true:

Proposition 4 There exists an upper bound on the volume of a Montesinos link in terms of the Jones polynomial.

For an explicit bound, however, I must involve T_2, and prove a formula similar to theorem 5 for V_2. It depends on more than just $A(D)'$, and even $A(D)$.

Acknowledgement. I would wish to thank to Mikami Hirasawa, Efstratia Kalfagianni and Dan Silver for stimulating me with their remarks into some of the investigations described here.

I would like to thank also Professor Akio Kawauchi and the Osaka City University staff for their invitation to Osaka and support.

References

[AK] N. Askitas and E. Kalfagianni, *On knot adjacency*, Topol. Appl. **126(1-2)** (2002), 63–81.

[BMo] Y. Bae and H. R. Morton, *The spread and extreme terms of Jones polynomials*, J. Knot Theory Ramifications **12(3)** (2003), 359–373.

[Bi] S. Bigelow, *Does the Jones polynomial detect the unknot?*, J. Knot Theory Ramifications **11(4)** (2002) (Knots 2000 Korea, Yongpyong, Vol. **2**), 493–505.

[B] J. S. Birman, *On the Jones polynomial of closed* 3*-braids*, Invent. Math. **81(2)** (1985), 287–294.

[BM] ——— " ——— and W. W. Menasco, *Studying knots via braids III: Classifying knots which are closed 3 braids*, Pacific J. Math. **161**(1993), 25–113.

[CK] A. Champanerkar and I. Kofman, *On the Mahler measure of Jones polynomials under twisting*, preprint `math.GT/0404236`

[DH] O. Dasbach and S. Hougardy, *Does the Jones polynomial detect unknottedness?*, Experiment. Math. **6(1)** (1997), 51–56.

[DL] ——— " ——— and X.-S. Lin, *A volume-ish theorem for the Jones polynomial of alternating knots*, preprint `math.GT/0403448`.

[EKT] S. Eliahou, L. H. Kauffman and M. Thistlethwaite, *Infinite families of links with trivial Jones polynomial*, Topology **42(1)** (2003), 155–169.

[HL] H. Howards and J. Luecke, *Strongly n-trivial Knots*, `math.GT/0004183`, Bull. London Math. Soc. **34(4)** (2002), 431–437.

[J] V. F. R. Jones, *Hecke algebra representations of of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.

[Kf] E. Kalfagianni, *Alexander polynomial, finite type invariants and volume of hyperbolic knots*, preprint, available at `http://www.math.msu.edu/~kalfagia/publications.htm`.

[K] T. Kanenobu, *Infinitely many knots with the same polynomial invariant*, Proc. Amer. Math. Soc. **97(1)** (1986), 158–162.

[K2] ——— " ———, *Examples on polynomial invariants of knots and links II*, Osaka J. Math. **26(3)** (1989), 465–482.

[Ka] L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.

[La] M. Lackenby, *The volume of hyperbolic alternating link complements*, with an appendix by I. Agol and D. Thurston, Proc. London Math. Soc. **88(1)** (2004), 204–224.

[LT] W. B. R. Lickorish and M. B. Thistlethwaite, *Some links with non-trivial polynomials and their crossing numbers*, Comment. Math. Helv. **63** (1988), 527–539.

[Ma] K. Mahler, *On some inequalities for polynomials in several variables*, J. London Math. Soc. **37** (1962), 341–344.

[Mn] P. M. G. Manchon, *Extreme coefficients of Jones polynomials and graph theory*, Jour. Knot Theory and its Ramif. **13 (2)** (2004), 277–295.

[MM] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186(1)** (2001), 85–104.

[Mu] K. Murasugi, *Jones polynomial and classical conjectures in knot theory*, Topology **26** (1987), 187–194.

[Ro] D. Rolfsen, *The quest for a knot with trivial Jones polynomial: diagram surgery and the Temperley-Lieb algebra*, Topics in knot theory (Erzurum, 1992), 195–210, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **399**, Kluwer, Dordrecht, 1993.

[Ro2] ——— " ———, *Knots and links*, Publish or Parish, 1976.

[SSW] D. S. Silver, A. Stoimenow and S. G. Williams, *Euclidean Mahler measure and Twisted Links*, in preparation.

[St] A. Stoimenow, *On polynomials and surfaces of variously positive links*, preprint `math.GT/0202226`.

[St2] ——— " ———, *Positive knots, closed braids and the Jones polynomial*, `math/9805078`, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **2(2)** (2003), 237–285.

[St3] ——— " ———, *The skein polynomial of closed 3-braids*, J. Reine Angew. Math. **564** (2003), 167–180.

[St4] ——— " ———, *Graphs, determinants of knots and hyperbolic volume*, preprint.

[St5] ——— " ———, *On the Polyak-Viro Vassiliev invariant of degree 4*, to appear in Canad. Math. Bull.

[St6] ——— " ———, *Some applications of Tristram-Levine signatures*, to appear in Adv. Math.

[Th] M. B. Thistlethwaite, *On the Kauffman polynomial of an adequate link*, Invent. Math. **93(2)** (1988), 285–296.

[Th2] ——— " ———, *A spanning tree expansion for the Jones polynomial*, Topology **26** (1987), 297–309.

[To] I. Torisu, *On strongly n-trivial 2-bridge knots*, to appear in Math. Proc. Camb. Phil. Soc.

[Tr] P. Traczyk, 3-*braids with proportional Jones polynomials*, Kobe J. Math. **15(2)** (1998), 187–190.

[Xu] Peijun Xu, *The genus of closed* 3-*braids*, J. Knot Theory Ramifications **1(3)** (1992), 303–326.