SOME APPLICATIONS OF TRISTRAM-LEVINE SIGNATURES AND RELATION TO VASSILIEV INVARIANTS

This is a preprint. I would be grateful for any comments and corrections!

A. Stoimenow

Department of Mathematics,
University of Toronto,
Canada M5S 3G3
e-mail: stoimeno@math.toronto.edu,
WWW: http://www.math.toronto.edu/stoimeno/

Abstract. Tristram and Levine introduced a continuous family of signature invariants for knots. We show that any possible value of such an invariant is realized by a knot with given Vassiliev invariants of bounded degree. We also show that one can make a knot prime preserving Alexander polynomial and Vassiliev invariants of bounded degree. Finally, the Tristram-Levine signatures are applied to obtain a condition on (signed) unknotting number.

Keywords: unknotting number, Vassiliev invariant, signature, Alexander polynomial.

1. Introduction

Consider for a knot K and a complex number ξ of unit norm the Hermitian forms $M_\xi = (1-\xi)A + (1-\bar{\xi})A^T$, where A is a Seifert matrix of K. The signatures of these forms M_ξ are the generalized or so-called Tristram-Levine signatures σ_ξ [40, 23], of which the Murasugi [27] signature $\sigma = \sigma_{-1}$ is a special case. These invariants have a variety of relations and applications, within and outside of knot theory. First, via the Tristram–Murasugi inequality [40, 27], the signatures are related to the 4-genus, and hence unknotting number. More recently, they have been of some interest because of their application to the classification of zero sets of algebraic functions on projective spaces [31]. Tristram-Levine signatures have also close relationship to the zeros of the Alexander polynomial, which have been studied for a while and have importance for several subjects, including monodromy of fibered links [34], divisibility [28] and orderability [32] of knot groups, and statistical mechanical models of the Alexander polynomial [26]. Also some relations of these signatures to (a quantum version of) the Jones polynomial have become apparent [11].

Vassiliev invariants [5] have been introduced more recently, and their relations to more classical invariants have been sought. In this paper, we shall treat the possible relations between Vassiliev invariants and generalized signatures σ_ξ with $\xi \in \mathbb{C}$ and $|\xi| = 1$. We extend the result on the Murasugi signature of [37] (and basically following also from [29]) to them, constructing knots of any given possible value of σ_ξ and Vassiliev invariants of bounded degree.

Theorem 1 For any $n \in \mathbb{N}$, any $\xi \in S^1 \setminus \{1\}$, and any admissible value v of σ_ξ, one has a prime knot with given values of Vassiliev invariants up to degree n, realizing $\sigma_\xi(K) = v$.

For the proof the notion of braiding sequences [39] is used. While for generic ξ (in particular if ξ is transcendental), σ_ξ admits – as the usual (Murasugi) signature – only even values on knots, the remaining cases (we describe them exactly) require an additional argument which we provide using Gousarov's result [14] on the existence of n-inverses. The knots constructed in §3 are composite. In order to find prime examples, some additional work is needed. It will be done in §4, using the construction of n-trivadjacent knots of Askitas and Kalfagianni [4]. It shows also the following result, which may be independent interest.

Theorem 2 If K is a knot and $n > 0$, then there exists a prime knot K' n-similar to K, such that $\Delta_{K'} = \Delta_K$.

We will have to show for this that a certain family of the knots arising in the construction of Askitas–Kalfagianni are non-trivial (a suggestive fact, which, however, does not follow from their results). In doing so, we give a new proof, using the Jones polynomial, that the (pure) braid groups are not nilpotent.

The result of theorem 1 on the σ_ξ should be put in contrast to the fact that via the Alexander/Conway polynomial Vassiliev invariants may well pose conditions to the Seifert pairing A. Also, Ng's result [29] that any concordance class contains a (possibly composite) knot with given Vassiliev invariants of bounded degree does not imply ours (even for composite knots), since by work of Levine [24], the signatures may not be concordance invariants in the zeros of the Alexander polynomial (see remark 4). Namely, there are slice knots with non-zero (singular) signatures.

Albeit Levine writes down only the Seifert forms of such knots, we will be able to find later many concrete examples: any slice knot with zeros of Δ on the unit circle, which has unknotting number one, turns out to be such. This follows from work given in an appendix, where we obtain a condition on signed unknotting number by analyzing the eigenvalues of M_ξ. (This work should be differentiated from the rest of the paper, since it uses mainly the same background on the signatures, but not the arguments in the main part.) In particular, we will also show for an amphicheiral unknotting number one knot, no zeros of Δ_K lie on the complex unit circle (corollary 2).

It is a more challenging question (also pointed out by the referee) whether one can realize not just every individual signature, but rather every signature function, using an n-trivial knot. Although the methods we apply clearly provide a way to approach such a question, they do not put into perspective a rigorous, but elegant and short solution.

2. Preliminaries

We shall briefly introduce the main notions appearing in the sequel (and give a few additional references to those in the introduction for further details).

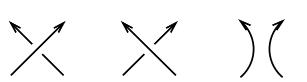
Recall that if A is a Seifert matrix of size $2g \times 2g$ corresponding to a genus g Seifert surface of a knot K, then for any $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\xi \neq 1$ we define

$$M_\xi(K) := (1-\xi)A + (1-\bar{\xi})A^T\,,$$

where bar denotes conjugation and $\cdot^T$ transposition. This is a Hermitian matrix, and all eigenvalues are real. By $\sigma(M_\xi)$ and $n(M_\xi)$ we denote the signature (sum of signs of eigenvalues) and nullity (number of zero eigenvalues) of M_ξ. They turn out to be independent in the surface and Seifert matrix, and are thus invariants of K, denoted by $\sigma_\xi(K)$ and $n_\xi(K)$ respectively. $\sigma_\xi(K)$ is called a generalized or *Tristram-Levine signature*. It satisfies, as the usual signature $\sigma = \sigma_{-1}$, the rules

$$\begin{aligned} \sigma_\xi(L_+) - \sigma_\xi(L_-) &\in \{0,1,2\}\,, \\ \sigma_\xi(L_\pm) - \sigma_\xi(L_0) &\in \{-1,0,1\}\,, \\ \sigma_\xi(!L) &= -\sigma_\xi(L)\,, \\ \sigma_\xi(L\#K) &= \sigma_\xi(L) + \sigma_\xi(K)\,. \end{aligned} \tag{1}$$

(Whether to have $\{0,1,2\}$ or $\{0,-1,-2\}$ in (1) is a matter of convention.) Here L_+, L_0, L_- form a skein triple



L_+ L_- L_0

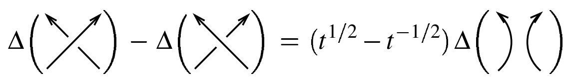
and $!L$ is the mirror image of L. $K_1\#K_2$ denotes the connected sum of K_1 and K_2, and $\#^n K$ denotes the connected sum of n copies of K.

The main difference to the usual signature is that σ_ξ may be odd even on knots, and that nice combinatorial formulas, as for alternating links (see [27, 18, 13]), are lacking.

The (normalized) *Alexander polynomial* [3] can be defined from a Seifert matrix A by

$$\Delta_K(t) = t^{-g}\det(A - tA^T).$$

Δ satisfies the skein relation



(2)

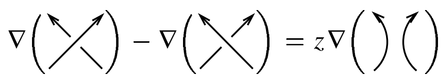
which defines it alternatively (up to a factor, fixed by demanding $\Delta(\bigcirc) = 1$).

We will sometimes modify Δ up to units in $\mathbb{Z}[t, t^{-1}]$, as in the original definition of Alexander.

Let ∇_K denote the *Conway polynomial* [8], given by

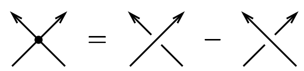
$$\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t).$$

Consequently, ∇ also satisfies a skein relation, namely



(3)

A knot invariant is called *Vassiliev invariant* of degree n, if, when linearly extended to linear combinations of knots, it vanishes of the subspace of $(n+1)$-singular knots, in which each singular knot is mapped to a linear combination of knots by the rule



See [5]. We consider Vassiliev invariants valued in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

'R.h.s' (resp. 'l.h.s') will abbreviate 'right hand-side' (resp. 'left hand-side'). In the sequel the symbol '$\subset$' denotes a not necessarily proper inclusion. Finally, let $\Re e$ and $\Im m$ denote the real and imaginary part, respectively. We will also write $i = \sqrt{-1}$ for the imaginary unit, in situations where no confusion (with the usage as index) arises.

3. Vassiliev invariants and generalized signatures

3.1. Outline of results

Here we consider the generalized signatures (see [23]) and show that they are all as independent from any finite number of Vassiliev invariants as the classical signature is.

First we determine the value range of σ_ξ on knots. This result, albeit possibly known, was never stated explicitly in previous publications. We will give a proof of it, both because it involves some subtleties which are worth remarking, and because it demonstrates some facts used to prove the result on Vassiliev invariants stated below.

Proposition 1 The value range $V_\xi \subset \mathbb{Z}$ of σ_ξ ($|\xi| = 1$, $\xi \neq 1$) on knots is given by

$$V_\xi = \begin{cases} \mathbb{Z} & \text{if } \left(2\Re e\frac{\xi-1}{|\xi-1|}\right)^{-2} \text{ is an algebraic integer} \\ 2\mathbb{Z} & \text{else} \end{cases} . \tag{4}$$

(An algebraic integer is the root of a polynomial in $\mathbb{Z}[x]$ with leading coefficient 1.)

The main result we prove in this section is a weaker version of theorem 1, without the primeness property.

Theorem 3 Any $s \in V_\xi$ is realizable as the value of $\sigma_\xi(K)$ of some (possibly composite) knot K which is n-similar to any fixed knot K' for any fixed n.

Here we call a knot K *n-similar* to K' in Gousarov's [14] sense if the Vassiliev invariants of degree $< n$ of K and K' coincide. We will sometimes write $K \sim_n K'$. For the proof of theorem 3 we will need the construction of [37] recalled below.

3.2. n-trivial rational tangles

Rational tangles were introduced by Conway [8]. The *Conway notation* of a rational tangle is a sequence of integers, to which a canonical diagram of the tangle is associated. (The order of the numbers is a matter of convention, so that some authors use the reverse sequences.) In the tangle notation of Conway, shown on figure 1, this diagram corresponds to the expression

$$a_1 \cdots a_n = ((a_1 a_2) a_3 \ldots) a_n$$

(that is, the 'product', which is not associative, is written as if it is left-associative); see [1, §2.3]. A *rational knot* is the closure of a rational tangle.

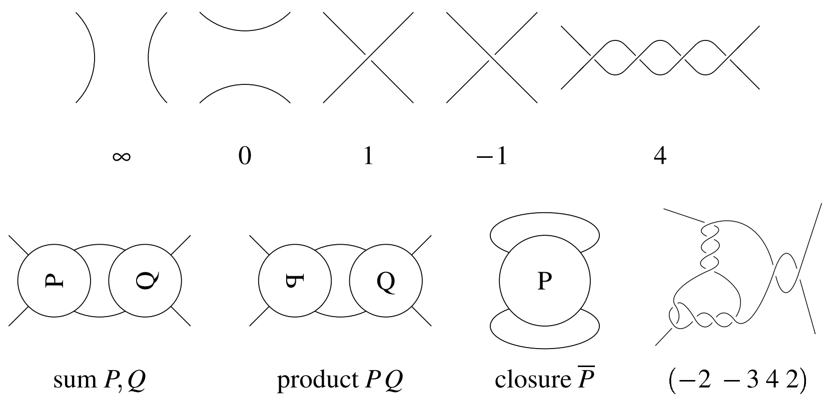


Figure 1: Conway's tangles and operations with them. (The designation 'product' is very unlucky, as this operation is neither commutative, nor associative, nor is it distributive with 'sum'. Also, 'sum' is associative, but not commutative.)

Define the *iterated fraction* (IF) of a sequence of integers $a = (a_1, \ldots, a_n)$ recursively by

$$IF(a_1) := a_1, \quad \ldots \quad IF(a_1, \ldots, a_{n-1}, a_n) := \frac{1}{IF(a_1, \ldots, a_{n-1})} + a_n.$$

It will be helpful to extend the operations '+' and '1/.' to $\mathbb{Q} \cup \{\infty\}$ by $1/0 = \infty$, $1/\infty = 0$, $k + \infty = \infty$ for any $k \in \mathbb{Q}$. The reader may think of ∞ as the fraction $1/0$, to which one applies the usual rules of fraction arithmetics and reducing. In particular reducing tells that $-1/0 = 1/0$, so that for us $-\infty = \infty$. This may appear at first glance strange, but has a natural interpretation in the rational tangle context. A rigorous account on this may be found in Krebes's paper [22].

In this sense, IF is a map ($\forall n \in \mathbb{N}$)

$$IF : \mathbb{Z}^n \longrightarrow \mathbb{Q} \cup \{\infty\}.$$

It is known [1], that diagrams of sequences of integers with equal IF belong to the same tangle (up to isotopy; where isotopy is defined by keeping the endpoints fixed). The correspondence is

$$a_1 \ldots a_n \longleftrightarrow IF(a_1, \ldots, a_n). \tag{5}$$

If $IF(a_1, \ldots, a_n) = p/q$ with $(p,q) = 1$, we call p the *numerator* of the tangle $(a_1 \ldots a_n)$ and q the *denominator*.

Using the correspondence (5), one can convince himself, that a rational tangle T has a representation with all numbers of the same sign, or a different representation with all numbers even (and both signs), if one of numerator or denominator is even.

Define for a finite sequence of integers $a = (a_1 \ldots a_n)$ its reversion $\overline{a} := (a_n \ldots a_1)$ and its negation by $-a := (-a_1 \ldots -a_n)$. For $b = (b_1 \ldots b_m)$ the term ab denotes the concatenation of both sequences $(a_1 \ldots a_n\, b_1 \ldots b_m)$.

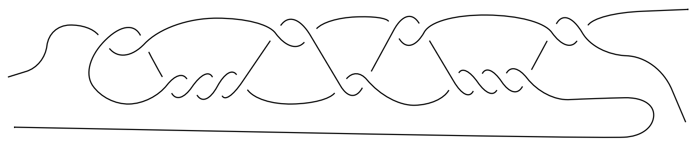
We call a tangle n-trivial, if all its Vassiliev invariants (defined analogously to the knot case) of degree $< n$ are the same as for the 0-tangle. The following construction of such tangles was introduced in [37].

Proposition 2 ([37]) Fix some even $a_1,\dots,a_n \in \mathbb{Z}$ and build inductively the integer sequences $w_k = w_k(a_1,\dots,a_k)$ by

$$w_1 := (a_1), \quad \dots \quad w_k := w_{k-1}(a_k)\overline{-w_{k-1}}\,. \tag{6}$$

Then the rational tangles with Conway notation w_n are n-trivial, and, if all $a_k \neq 0$, non-trivial, i. e., not (isotopic to) the 0-tangle.

Example 1 For $a_1 = 2$, $a_2 = -4$ and $a_3 = 2$ we have $w_1 = (2)$, $w_2 = (2\ -4\ -2)$ and $w_3 = (2\ -4\ -2\ 2\ 2\ 4\ -2)$. The tangle w_3 is shown below:



In other words, the replacement of a 0-tangle by a w_n-tangle in some diagram (possibly) changes the knot type, but preserves the values of Vassiliev (knot) invariants of degree $< n$.

3.3. Proof of theorem 3

We start by a proof of a part of proposition 1. Even although there are more direct arguments (coming from perturbation theory of linear forms [19]), we prefer not to be minimalistic, as we need to set up notations and tools needed in the following.

Lemma 1 If for $\xi \in \mathbb{C}$ with $|\xi| = 1$, $z_0 = \mathring{\xi}^{1/2} - \mathring{\xi}^{-1/2}$ is a *simple* zero of ∇_K along the line between 0 and $2i$, where

$$\mathring{\xi} \ := \ -\left(\frac{1-\bar{\xi}}{|1-\bar{\xi}|}\right)^2 = \frac{\bar{\xi}-1}{1-\xi} = \xi^{-1}\,, \tag{7}$$

then $\sigma_\xi(K)$ is odd.

Proof. Let

$$A_\xi \,=\, (1-\xi)A + (1-\bar{\xi})A^T$$

and $\xi = \xi(t) = e^{2\pi i t}$ (note that $A = \bar{A}^T$ so that all eigenvalues of A_ξ are real). We have

$$\begin{aligned}
\det(A_\xi) \quad &= \quad (1-\xi)^{2g} \det\left(A - \frac{\bar{\xi}-1}{1-\xi}A^T\right) \\
&= \quad (1-\xi)^{2g} \left(\frac{\bar{\xi}-1}{1-\xi}\right)^g \Delta_K\left(\frac{\bar{\xi}-1}{1-\xi}\right) \\
&= \quad (-1)^g |1-\xi|^{2g} \Delta_K\left(\xi^{-1}\right) \\
&= \quad (-1)^g |1-\xi|^{2g} \nabla_K\left(2i\Im m\sqrt{\xi^{-1}}\right) \\
&= \quad (-1)^g |1-\xi|^{2g} \nabla_K\left(2i\Re e\frac{\xi-1}{|\xi-1|}\right)\,.
\end{aligned}$$

As both maps $t \mapsto |1-\xi(t)|$ and

$$t \mapsto z(t) := 2i\Re e\frac{\xi(t)-1}{|\xi(t)-1|}$$

have non-zero derivatives for $t \in (0, 1/2)$ ($\xi = -1$ corresponds to the determinant which is never zero), $t_0 = z^{-1}(z_0)$ is a simple zero of

$$t \mapsto \det A_{\xi(t)}\,.$$

This shows that

$$\frac{\partial}{\partial t}\det A_{\xi(t)}\Big|_{t=t_0} \neq 0\,.$$

Consider $\chi(x,t) = \chi_{A_{\xi(t)}}(x)$ to be the characteristic polynomial of $A_{\xi(t)}$ (whose absolute term is $\det A_{\xi(t)}$). We have

$$\chi(0,t_0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\chi(0,t_0) \neq 0\,.$$

Then, by the Implicit Function Theorem there is an $\varepsilon > 0$ and a function $\hat{t} : [-\varepsilon, \varepsilon] \to \mathbb{R}$ with $\hat{t}(0) = t_0$, such that

$$\chi(x,t) = 0 \iff t = \hat{t}(x)$$

for $x \in [-\varepsilon, \varepsilon]$ and t close to t_0. This means that, for t close to t_0, each eigenvalue $\alpha_i = \alpha_i(\xi(t))$ of $A_{\xi(t)}$ (with $i = 1, \dots, 2g$) in $[-\varepsilon, \varepsilon]$ is attained for only one value of t. Thus there is a *unique* eigenvalue of $A_{\xi(t)}$ in a neighborhood of 0 for t around t_0, and that eigenvalue indeed changes sign as $\hat{t}$ exists for both positive and negative arguments. Finally, the eigenvalue may be multiple, but it must have odd multiplicity because ∇_K changes sign around z_0 (and this multiplicity is locally constant because the dependence of the eigenvalues of $A_{\xi(t)}$ on t is at least continuous). Thus $A_{\xi(t_0)}$ has odd nullity, and hence also odd signature. □

Remark 1 We will later show that in fact the multiplicity of the eigenvalue is indeed 1.

The fact that for a simple zero ζ of Δ on S^1, σ_ξ changes by ± 2 near $\xi = \mathring{\zeta}$ implies that the signature cannot vanish on both sides of $\mathring{\zeta}$. Thus we have

Corollary 1 If K is achiral, then Δ_K has no simple zero on S^1. □

The argument clearly shows the corollary also for slice knots, but in this case it easily follows from the Milnor-Fox condition $\Delta(t) = f(t)f(1/t)$ [9]. It shows that in fact all the zeros of Δ on S^1 are of even order. It is tempting to conjecture that this is also true for achiral knots. It follows from [15] for strongly (positively or negatively) amphicheiral knots. It is also true for all the amphicheiral knots of up to 16 crossings. I have no proof in general, though.

Lemma 2 If $\left(2\Re e\frac{\xi-1}{|\xi-1|}\right)^{-2}$ is an algebraic integer, then V_ξ contains some odd integer.

Proof. If $|\zeta| = 1$, then $\Delta_K(\zeta) = \nabla_K(2i\Im m(\sqrt{\zeta}))$. Now, the polynomial $\tilde{\nabla}_K(x) := \nabla_K(ix) \in 1 + x^2\mathbb{Z}[x^2]$, and, as well-known, any polynomial in this affine ideal is ∇_K, and hence $\tilde{\nabla}_K$, for some knot K. Therefore, x is a zero of $\tilde{\nabla}_K$ for some $K \iff 1/x^2$ is an algebraic integer. (Note, that here $x = 2\Im m\sqrt{\xi^{-1}} = 2\Re e\left(\frac{\xi-1}{|\xi-1|}\right)$ is non-zero.) Moreover, in this case we can choose $\tilde{\nabla}$ so as x to be a simple zero – as extensions of $\mathbb{Q}$ are separable, simply take the minimal polynomial of x. Then apply lemma 1. □

Remark 2 Since for $|\xi| = 1$ we have $\mathring{\xi} = 1/\xi$, which is holomorphic and of non-zero derivative for $\xi \neq 1$, and

$$\mathring{\xi} \mapsto z(\mathring{\xi}) = \mathring{\xi}^{1/2} - \mathring{\xi}^{-1/2}$$

is holomorphic and of non-zero derivative for $\mathring{\xi} \neq -1$, we have for $\xi \neq \pm 1$ by Cauchy-Riemann that

$$\begin{array}{ccc} z(\mathring{\xi}) \text{ is a simple zero of } \nabla_K \text{ along line between 0 and } 2i & \iff & \mathring{\xi} \text{ is a simple zero of } \Delta_K \text{ along } S^1 \quad \iff \\ z(\mathring{\xi}) \text{ is a simple zero of } \nabla_K & \iff & \mathring{\xi} \text{ is a simple zero of } \Delta_K\,. \end{array}$$

Proof of proposition 1. We prove in (4) only the inclusion '$\subset$'. The reverse inclusion will follow from theorem 3, when taking the r.h.s. of (4) as a definition of V_ξ.

The case $\xi = 1$ is trivial, so we need to examine for which other ξ the matrix $(1-\xi)A + (1-\bar{\xi})A^T$ can be made singular (of odd nullity) for a Seifert matrix A of a knot K. This happens only if $\Delta_K\left(\xi^{-1}\right) = 0$ for some K (the zero being of odd order). But from the proof of lemma 2, this is equivalent to $\left(2\Re e\frac{\xi-1}{|\xi-1|}\right)^{-2}$ being an algebraic integer. $\square$

Remark 3 The map $\xi \mapsto \left(2\Re e\dfrac{\xi-1}{|\xi-1|}\right)^{-2}$ for $\xi \in S^1 \setminus \{1\}$ is 2-1, the preimages of an element being conjugate and (up to this conjugacy) the dependence of the one quantity on the other is algebraic (in fact even quadratically radical, and the construction can be performed using ruler and compasses) so that the transcendency of the one is equivalent to the transcendency of the other. Therefore, in particular, as remarked in the introduction, for transcendent ξ, σ_ξ admits only even values on knots. (See [36] for more details on the algebraicity arguments.)

Remark 4 It is known that if $\Delta_{K_1}(\mathring{\xi}) \neq 0 \neq \Delta_{K_2}(\mathring{\xi})$, and K_1 is concordant to K_2, then $\sigma_\xi(K_1) = \sigma_\xi(K_2)$. However, by the examples of [24], there are slice knots with $\sigma_\xi \neq 0$ for $\Delta(\mathring{\xi}) = 0$. Ng showed in [29] that any concordance class of knots contains one n-similar to a given knot K (modulo Arf invariant). This result implies ours for values of ξ, for which $\Delta(\mathring{\xi})$ is never zero (on knots), e.g. for $\xi = -1$ or ξ transcendental, but not for general ξ. Also, since Ng's knots are composite, her result does not imply any of the statements about prime knots we will make below.

Now we prove theorem 3.

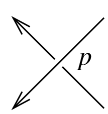
Proof of theorem 3. We split the proof into three lemmas. For simplicity we consider only the cases where $\left(2\Re e\frac{\xi-1}{|\xi-1|}\right)^{-2}$ is an algebraic integer. The other cases are simpler and the argument for them is obtained by omitting irrelevant parts of the argument we describe for the $V_\xi = \mathbb{Z}$ case. (As said, these cases can also be obtained from [29] using the results of [23].) Theorem 3 then follows by taking K to be the connected sum of K' with the knots constructed in the lemmas.

Lemma 3 There is for any n an n-trivial knot K_n of odd σ_ξ.

Proof. As we already saw in the proof of lemma 2, there is a knot K with $\sigma_\xi(K)$ odd. Let K' be an n-inverse of K, that is, a knot such that $K'\#K$ is n-trivial (n-similar to the unknot). The existence of such n-inverses was proved by Gousarov [14]. Note, that a knot n-similar to an n-inverse as also an n-inverse. Thus we would be through (considering $K_n = K\#K'$), if we show that K' can be chosen so that $\Delta_{K'}\left(\xi^{-1}\right) \neq 0$. This follows from the lemma below. $\square$

Lemma 4 For any knot K', any $n > 0$ and any $\xi \in S^1 \setminus \{1\}$ there is a knot K n-similar to K' such that $\Delta_K\left(\xi^{-1}\right) \neq 0$.

Proof. By [30], we can assume up to n-similarity without loss of generality, that K' has unknotting number $u(K') = 1$. Assume that $\Delta_{K'}\left(\xi^{-1}\right) = 0$. Consider an unknotting number one diagram D of K' and a crossing p in D whose switch unknots K'.



(The sign choice of p here is irrelevant; the argument goes through also with the mirrored diagrams.)

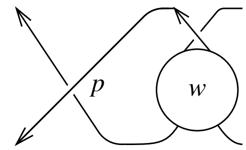
Let D_0 be the diagram obtained from D by smoothing out p, and K_0 the 2 component link represented by D_0. As

$$\Delta_{K'}\left(\xi^{-1}\right) = 0 \neq 1 = \Delta_{\bigcirc}\left(\xi^{-1}\right),$$

the skein relation (2) shows that

$$\Delta_{K_0}\left(\xi^{-1}\right) \neq 0.$$

Then consider the diagram

$D_w =$  (8)

belonging to a knot K'_w, where $w = w_n = w_n(a_1,\dots,a_n)$ is the n-trivial rational tangle considered before, depending on the even integers $a_1,\dots,a_n$. The new knot K'_w is n-equivalent to K'. Thus we need to show that for some choice of the a_i the resulting K'_w has $\Delta_{K'_w}\left(\xi^{-1}\right) \neq 0$.

If we draw the tangle diagram of a rational tangle with all coefficients in the Conway notation even, it is easy to see that the half-twists counted by these coefficients are all reverse. Then the skein relation (2) for Δ shows that, in analogy to the braiding polynomials considered in [39], any value of the Alexander polynomial of a rational knot depends polynomially on the coefficients in the Conway notation in the form with all coefficients even, and the dependence is linear in each single (even) coefficient.

Thus

$$(a_1,\dots,a_n) \;\mapsto\; P(a_1,\dots,a_n) = \Delta_{K'_{w_n(a_1,\dots,a_n)}}\left(\xi^{-1}\right) \in \mathbb{R}[a_1,\dots,a_n]$$

depends polynomially on the a_i's, and the degree of a_i in this polynomial is less than or equal to the number of occurrences of a_i in the Conway notation of w_n, that is, 2^{n-i}. The top degree coefficient of $P(a_1,\dots,a_n)$, that is, the coefficient of the monomial $\prod_{i=1}^{n} a_i^{2^{n-i}}$, is (up to a sign) equal to the product of a power of the (non-zero) number

$$\left(t^{1/2} - t^{-1/2}\right)\Big|_{t=\xi^{-1}}$$

with $\Delta_{\tilde{D}_w}\left(\xi^{-1}\right)$. Here the link diagram $\tilde{D}_w = D_{w(\infty,\dots,\infty)}$ arises from D_w by replacing w_n by w_∞, and w_∞ is obtained in the same way as w_n, but formally setting all $a_i = \infty$, meaning composition with the ∞-tangle in the tangle calculus of figure 1.

Then w_∞ is just the ∞-tangle, and thus $\tilde{D}_w$ depicts K_0, for which, as observed, we have $\Delta_{K_0}\left(\xi^{-1}\right) \neq 0$. Thus the top degree coefficient of $P(a_1,\dots,a_n)$ is non-zero, so that for some choice of the a_i's, $P(a_1,\dots,a_n)$ will be non-zero, as desired. □

Lemma 5 There exists an n-trivial (rational) knot $\hat{K}_n$ of $\sigma_\xi \in \{\pm 2\}$.

Proof. Consider for $s \in 2\mathbb{Z}$ the knots $K_{s,w_n} = \overline{(w_n, s)}$. They unknot only switching crossings in a group of reverse twists (those counted by a_n) and hence have $|\sigma_\xi| \leq 2$ (such a move alters only a single diagonal entry in the Seifert matrix). We need to show that for some s and w_n, $\sigma_\xi(K_{s,w_n}) \neq 0$.

The definitions of σ_ξ and Δ in terms of Seifert matrices show (see the proof of lemma 1), analogously to the classical case $\xi = -1$ that, provided $\Delta_K\left(\xi^{-1}\right) \neq 0$, $\sigma_\xi(K)$ is even, and $4 \mid \sigma_\xi(K)$ exactly if the (real) number $\Delta_K\left(\xi^{-1}\right)$ is positive, when Δ is normalized so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$. Thus we would be through if for some s and w_n we could make $\Delta_{K_{s,w_n}}\left(\xi^{-1}\right) < 0$. Again this value depends polynomially on s and all a_i in w_n, and the top degree coefficient of this polynomial P is obtained, up to some non-zero multiple, by replacing s and all a_i by ∞. But the resulting diagram is an unknot diagram, so that the top degree coefficient of P is non-zero. As the top degree monomial of P, in which the a_i and s occur with the same multiplicities as in the notation (w_n, s), is linear in s and a_n, so is P itself. But any non-trivial polynomial in k variables admitting only non-negative values on the whole $(2\mathbb{Z})^k$ has even degree in *all* variables. This shows the lemma. □

To finish the proof of theorem 3, now consider the families of knots

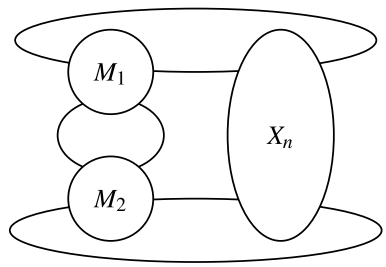
$$\mathcal{F}_1 = \{K'\#K_n\#\#^l\hat{K}_n\}_{l\in\mathbb{N}}, \;\; \mathcal{F}_2 = \{K'\#\#^l\hat{K}_n\}_{l\in\mathbb{N}}, \;\; \mathcal{F}_3 = \{K'\#K_n\#\#^l!\hat{K}_n\}_{l\in\mathbb{N}}, \;\text{ and }\; \mathcal{F}_4 = \{K'\#\#^l!\hat{K}_n\}_{l\in\mathbb{N}}. \quad (9)$$

Here K_n is the knot of lemma 3, and $\hat{K}_n$ the one of lemma 5. Since any $\sigma_\xi \in \mathbb{Z}$ is realized by some of the knots, one of the families contains the knot K we sought. □

4. Constructing prime knots

The knots in the previous construction are composite. That we can modify them into prime knots we prove now. This is related to the proof of theorem 2.

Proof of theorem 2. We proceed as in [20]. Let K be a composite knot with prime factors K_1 and K_2 (more prime factors are dealt with inductively). Then we represent K_1 and K_2 as closures of prime tangles M_1 and M_2. We consider a knot $K' = \overline{(M_1, M_2)X_n}$ for a suitable tangle X_n.



We need the following 3 properties of X_n:

1. X_n is prime,
2. X_n is n-trivial, and
3. replacing X_n for the 0-tangle preserves Δ.

We recall from [20] that a tangle $X \subset B^3$ is called prime, if it has no connected summand (a ball intersecting it in a knotted arc), and has no separating disk, i.e. a properly embedded disk $D \subset B^3$, with both components of $B^3 \setminus D$ containing parts of X. The rational tangles are exactly those with no connected summand, but with a separating disk.

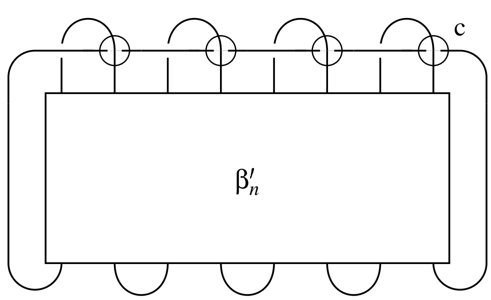
When X_n satisfies the above 3 conditions, then K' is prime from condition 1 by [20], and has the same Vassiliev invariants and Alexander polynomial by conditions 2 and 3.

To find X_n, we turn to a (special case of a) construction of n-trivadjacent knots due to Askitas and Kalfagianni [4].

Let B_n be the n-strand braid group and σ_i the Artin generators. Define a sequence of (pure) braids $\beta_n \in B_n$ by $\beta_2 = \sigma_1^{\pm 2}$ and inductively $\beta_n = [\beta_{n-1}, \sigma_{n-1}^{\pm 2}]$, where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ is the commutator and $\beta_{n-1} \in B_n$ is meant with respect to the canonical inclusion $B_{n-1} \hookrightarrow B_n$. (All the signs '$\pm$' can be chosen independently.) It is easy to see that β_n is "Brunnian", that is, the removal of any strand(s) from β_n gives a trivial braid on the remaining strands.

For $\beta_n \in B_n$ let $\beta'_n \in B_{2n}$ be the doubled braid. (Each letter $\sigma_i^{\pm 1}$ in β_n is turned into $\sigma_{2i}^{\pm 1}\sigma_{2i-1}^{\pm 1}\sigma_{2i+1}^{\pm 1}\sigma_{2i}^{\pm 1}$.) Consider the knot K_n built up from β'_n in the following obvious way (depicted for $n = 4$):

K_n =

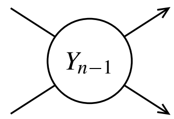


. (10)

(Note, that here n of the strands of β'_n are reversely oriented.)

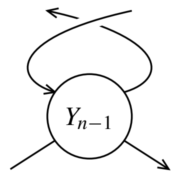
Such knots are constructed in [4]. K_n is n-trivadjacent, i.e. for the set of n encircled crossings in (10) the switch of any non-empty set of these crossings gives an unknot diagram. Thus by [4, theorem 5.2], $\Delta_{K_n} = 1$ for $n \geq 3$.

Let



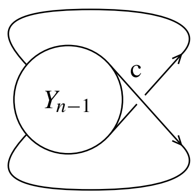
be the complementary tangle to the rightmost encircled crossing c in (10). Let

X_{n-1} =

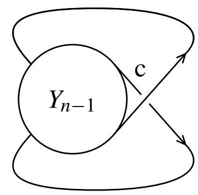


.

Since

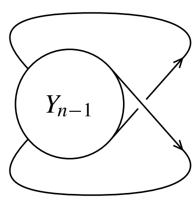


and



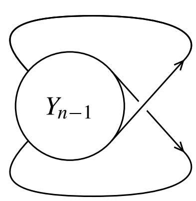
have unit Alexander polynomial, by a simple skein argument (see [6]) one sees that replacing Y_{n-1} for the 0-tangle preserves the Alexander polynomial, and hence the same is true for X_{n-1}. Also, one easily sees that Y_{n-1}, and hence X_{n-1}, are $(n-1)$-trivial. Finally, we must show that X_{n-1}, or equivalently Y_{n-1}, is prime.

For this we must show that Y_{n-1} has no connected summand and is not rational. The first property is clear since



is the unknot. The second property would follow from the fact that

K_n =



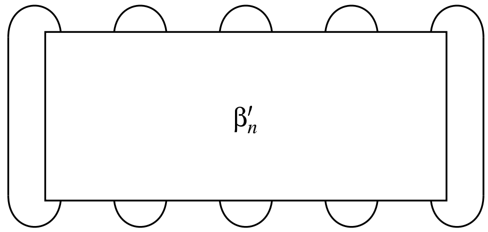
has unit Alexander polynomial, provided one can show that K_n is non-trivial, since no non-trivial rational knot has $\Delta = 1$. Thus the proof is concluded with the below lemma. □

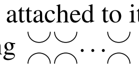
Lemma 6 K_n is non-trivial.

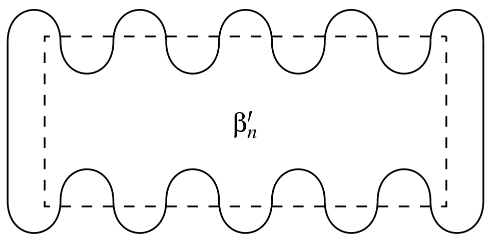
Proof. We prove by contradiction. Assume K_n is trivial.

We know from [35] that if in a skein triple L_+, L_0, L_- the links $L_\pm$ are l-component unlinks, then L_0 is the $(l+1)$-

component unlink. By iterating this argument, we conclude that

$K_n' =$ 

is the $(n+1)$-component unlink. Now, K_n' is the boundary of a disk D with n bands attached to it (corresponding to the doubled strands of β_n). The disk is obtained by removing the bands, i.e. replacing  for β_n':

$U_n =$  .

Let U_n be the unknot bounding this disk. By [10] we know that if a link L is obtained from two (split) components L_1 and L_2 by band connecting, L has a minimal genus Seifert surface containing the band. Since the n bands attached to D can be viewed as connecting the $(n+1)$ components of K_n' to build an unknot, the assumption K_n is the unlink means that one must find a disk bounding U_n containing each band. As this disk D is unique up to isotopy, for each single band, D must be isotopable so as to contain it without intersecting the rest of the link K_n'. In particular, if one shrinks the bands into strands (i.e., ignores the framing), and connects their endpoints by arcs in D, one finds that the usual braid closure $\hat{\beta}_n$ of β_n must be the n-component unlink.

To show that $\hat{\beta}_n$ is non-trivial (and hance to have the contradiction we wish), consider the Jones polynomial value $\left|V\left(e^{\pi i/3}\right)\right|$ (or alternatively the number of torsion numbers of the double branched cover homology group divisible by 3) [25, 12]. Since this is invariant under the insertion of $\sigma_i^{\pm 3}$, the assumption that

$$\hat{\beta}_n = \widehat{\ }\left([\dots[\sigma_1^{\pm 2},\sigma_2^{\pm 2}],\dots,\sigma_{n-1}^{\pm 2}]\right)$$

is trivial means for

$$\tilde{\beta}_n = [\dots[\sigma_1^{\mp 1},\sigma_2^{\mp 1}],\dots,\sigma_{n-1}^{\mp 1}]$$

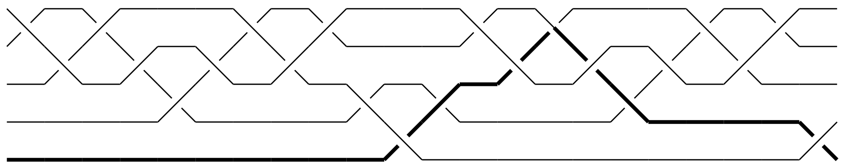
that we have

$$\left|V_{\widehat{\tilde{\beta}_n}}\left(e^{\pi i/3}\right)\right| = \sqrt{3}^{\,n-1}.$$

In particular, by [17, proposition 15.3], $b(\widehat{\tilde{\beta}_n}) = n$, with b denoting the braid index.

Thus it suffices to show that $\tilde{\beta}_n$ is not a minimal strand number representation for $\widehat{\tilde{\beta}_n}$. $\tilde{\beta}_n$ contains as braid word two copies of $\sigma_{n-1}^{\pm 1}$ (of opposite sign). Let w_{n-1} be the subword of $w_n = \tilde{\beta}_n$ between (and not including) these two letters. w_{n-1} in turn has two copies of $\sigma_{n-2}^{\pm 1}$ (again of opposite sign) and so on. By induction one finds subwords w_k of $\tilde{\beta}_n$ for $2 \le k \le n$ containing only letters σ_i with $i < k$, with σ_{k-1} occurring twice (and with opposite sign), unless $k = 2$, in which case $w_2 = \sigma_1^{\pm 1}$ is a single letter. Let c be the crossing in the diagram $\widehat{\tilde{\beta}_n}$ corresponding to this letter.

Numbering generators from left to right, and composing words from bottom to top (i.e. orienting strands upward), consider the segment of the knot strand in the diagram $\widehat{\tilde{\beta}_n}$ between the right outgoing arc of c until the right incoming arc of this same crossing. Here is an example for $n = 5$ (rotated by $-\pi/2$ to save space):



.

All crossings this arc passes correspond to a pair of (oppositely signed) $\sigma_i^{\pm 1}$ for $2 \le i \le n-1$. Thus in $\widehat{\tilde{\beta}_n}$ this arc can be shrunk and eliminated, giving a smaller strand number braid representation. $\square$

Note that the construction of theorem 2 applied on the example in theorem 3 does not immediately prove theorem 1, because σ_ξ may be changed. Also, we cannot control the behaviour of σ_ξ if $\mathring{\xi}$ is a zero of the Alexander polynomial of order > 1. Thus a bit more care is needed.

Proof of theorem 1. We start with the knots occurring in the proof of theorem 3, built for given K'. Now look at which family in (9) contained the knot K found in the proof of theorem 3 (or one of these knots, which we fix in the following). Apply a certain number of times the construction of theorem 2 to make $K'\#K_n$ or K' (dependingly on whether K was in family $\mathcal{F}_1 \cup \mathcal{F}_3$ or $\mathcal{F}_2 \cup \mathcal{F}_4$) into a prime knot P, and by [20] represent P as the closure of a prime tangle P'. Let $T_n = (w_n, s)$ be the rational tangle whose closure is $\hat{K}_n$. Note that we have chosen the a_i in w_n so that $\sigma_\xi(\hat{K}_n) \ne 0$. Also $\Delta_{\hat{K}_n}(\mathring{\xi}) \ne 0$.

Then consider the tangles $S_l = (P', !T_n, \dots, !T_n) \cdot X_n$ (with l copies of $!T_n$) if K is the l-th knot in family $\mathcal{F}_3$ or $\mathcal{F}_4$, or $S_{-l} = (P', T_n, \dots, T_n) \cdot X_n$, if K is in family $\mathcal{F}_1$ or $\mathcal{F}_2$, with X_n being the tangle found in the proof of theorem 2. The tangle S_l is built up as sum of two prime tangles (see [33]). Thus its closure knot $\overline{S_l}$ is prime. It has the same Vassiliev invariants of order $< n$ as K' and K, and the same Alexander polynomial as K.

Assume $\mathring{\xi}$ is a zero of Δ_K of order $n_\xi(K) \bmod 2 \in \{0,1\}$. Then by lemma 1 if $\sigma_\xi(K)$ is even, so is $\sigma_\xi(\overline{S_l})$, and if $\sigma_\xi(K)$ is odd, so is $\sigma_\xi(\overline{S_l})$. Now let $l \in \mathbb{Z}$ vary. Then $\overline{S_l} \sim_n K$. In general $\Delta_{\overline{S_l}} \ne \Delta_K$, but $\Delta_{\hat{K}_n}(\mathring{\xi}) \ne 0$ implies that $\mathring{\xi}$ is zero of all $\Delta_{\overline{S_l}}$ of the same order. Hence still $\sigma_\xi(K) - \sigma_\xi(\overline{S_l})$ is even for all l. Also by the fact that T_n turns by undoing $2a_n$ positive reverse half-twists into the 0-tangle, we have $0 \le \sigma_\xi(\overline{S_{l+1}}) - \sigma_\xi(\overline{S_l}) \le 2$ (see proof of lemma 5). Thus it suffices only to show that by adjusting $l \in \mathbb{Z}$ one can make $\sigma_\xi(\overline{S_l})$ arbitrary large or small. But this is clear since by a crossing change one turns X_n into the 0-tangle, and hence $\overline{S_l}$ into $\#^l !\hat{K}_n \# P$ (resp. $\#^{-l} \hat{K}_n \# P$ for $l < 0$), and $\sigma_\xi(\hat{K}_n) \ne 0$.

What remains is to justify our assumption that $\mathring{\xi}$ is zero of Δ_K of order 0 or 1. As $\Delta_{\hat{K}_n}(\mathring{\xi}) \ne 0$, this is equivalent under replacing Δ_K by Δ_P. That is, we must show that we can modify P up to n-similarity so as $\mathring{\xi}$ to be a zero of order $n_\xi(K) \bmod 2$ of Δ_P. This is proved in the below lemma. The knot found there may be composite, but applying the construction of theorem 2 gives again a prime knot. $\square$

Lemma 7 Let K be a knot, $\xi \in S_1 \setminus \{1\}$ and $n > 0$. Then there is a knot $K' \sim_n K$ with $\Delta_{K'}(\mathring{\xi}) \ne 0$. Also there is a knot $K' \sim_n K$ with $\mathring{\xi}$ being a zero of $\Delta_{K'}$ of order one, provided $\Delta_K(\mathring{\xi}) = 0$.

Proof. That we can achieve $\Delta_{K'}(\mathring{\xi}) \ne 0$ follows from the proof of lemma 4.

Assume now $\Delta_K(\mathring{\xi}) = 0$. We would like to find K' such that $\mathring{\xi}$ is a simple zero of $\Delta_{K'}$ and $K' \sim_n K$.

Let K_1 be a knot such that $\mathring{\xi}$ is a simple zero of Δ_{K_1}. That such K_1 exists follows from the separable extension argument in the proof of lemma 2, and the remark after this proof. Let K_1' be an n-inverse of K_1, chosen by the proof of lemma 4 to have $\Delta_{K_1}(\mathring{\xi}) \ne 0$. Finally, let $\tilde{K}$ be a knot n-similar to K such that $\Delta_{\tilde{K}}(\mathring{\xi}) \ne 0$, also found by lemma 4. Then take $K' = \tilde{K} \# K_1 \# K_1'$. $\square$

Appendix A. Unknotting numbers

We would like to note here an application of Tristram-Levine signatures independent from our previous constructions. It seems useful, and builds on some of the background we introduced, but is possibly too short to deserve a stand-alone exposition.

Let $u(K)$ be the *unknotting number* of a knot K, the minimal number of crossing changes needed to make the unknot out of K. We call $K = K_0 \to K_1 \to \ldots \to K_n = \bigcirc$ ($\bigcirc$ denoting the unknot) an *unknotting sequence* of K, if K_i and K_{i+1} differ by a crossing change. Thus the unknotting number of K is the minimal length n of an unknotting sequence of K.

We say that K has *positive unknotting number n* (denoted by $u_+(K) = n$) if it unknots by switching n, but not less than n, positive crossings to negative. Similarly $u_-(K) = n$ denotes the *negative unknotting number*. These concepts were introduced in [7]. Trivially $u_+(K) = n \iff u_-(!K) = n$, $!K$ being the mirror image of K, and $u(K) = 1 \iff u_+(K) = 1$ or $u_-(K) = 1$. If in the unknotting sequence $K = K_0 \to K_1 \to \ldots \to K_n = \bigcirc$ we have that K_{i+1} differs from K_i by a change of a positive crossing, then we call the unknotting sequence *positive*. In case K has no positive unknotting sequence, we set $u_+(K) = \infty$.

Theorem 4 If there is a $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\sigma_\xi(K) = 0$ and $\Delta_K(\mathring{\xi}) = 0$ (with $\mathring{\xi}$ given as in (7)), then $u_\pm(K) = \infty$. In particular, $u(K) \geq 2$.

Proof. Assume that $u_+(K) = n < \infty$. Since if K_+ and K_- differ by a crossing which is positive resp. negative, we have $A_\xi(K_+) - A_\xi(K_-) = \mathrm{diag}\,(2 - 2\Re e\xi, 0, \ldots, 0)$. Then we know that if we order the eigenvalues of $A_\xi(K_\pm)$ non-increasingly $\alpha_{1,\pm} \geq \alpha_{2,\pm} \geq \ldots \geq \alpha_{2g,\pm}$, we have $\alpha_{i,+} \geq \alpha_{i,-}$. This follows from a theorem attributed to Courant-Fischer and Weyl [42], and is now known as a special case of the complete description of inequalities for the eigenvalues of sums of Hermitian matrices due to Helmke-Rosenthal [16] and Klyachko [21]. Set e.g. $j = 1$ in formula (3) of [2].

Then

$$\sigma_\xi(K_+) = \sum_i \mathrm{sgn}\,(\alpha_{i,+}) \geq \sum_i \mathrm{sgn}\,(\alpha_{i,-}) = \sigma_\xi(K_-)\,.$$

Thus $\sigma_\xi(K_+) = \sigma_\xi(K_-)$ only if $\mathrm{sgn}\,(\alpha_{i,+}) = \mathrm{sgn}\,(\alpha_{i,-})$ for all $i = 1, \ldots, 2g$. If $\Delta_{K_+}(\mathring{\xi}) = 0$, then some of the $\alpha_{i,+}$, and hence $\alpha_{i,-}$ vanishes, so that also $\Delta_{K_-}(\mathring{\xi}) = 0$. If now $K = K_0 \to K_1 \to \ldots \to K_n = \bigcirc$ is a positive unknotting sequence, we have $\sigma_\xi(K_{i+1}) \leq \sigma_\xi(K_i)$ with strict inequality at least once, as the zero $\mathring{\xi}$ of Δ_K must disappear under some of the crossing changes. But then $\sigma_\xi(\bigcirc) < \sigma_\xi(K) = 0$, a contradiction.

The argument for $u_-(K) = \infty$ is analogous. □

Since all Tristram-Levine signatures vanish on achiral knots, we have

Corollary 2 If K is achiral and $\Delta_K(S^1) \ni 0$, then $u(K) > 1$. □

Unfortunately, this is not necessarily true for slice knots, as by the examples of [24], we may have ξ with $\sigma_\xi \neq 0$ and $\Delta(\mathring{\xi}) = 0$. In fact, corollary 2 gives an easy way of finding such examples, without examining the Seifert form. (Note, that in [24], Levine just writes down Seifert forms, and claims they come from knots without giving the knots explicitly, though.)

Example 2 Consider, for example, 8_{20}, which is slice and of unknotting number one, and $\xi = e^{\pi i/3}$. As 8_{20} has the Alexander polynomial of the square (and granny) knot, $\Delta(\mathring{\xi}) = 0$, and one can in fact calculate that $\sigma_\xi(8_{20}) = -1$ and $n_\xi(8_{20}) = 1$.

The signed unknotting number information of the theorem can be combined with other conditions.

Proposition 3 Assume K satisfies the premise of theorem 4. If additionally (a) $V_K\left(e^{\pi i/3}\right) = -3$ or (b) there is some ξ' with $\sigma_{\xi'}(K) = \pm 4$, then $u(K) > 2$.

Proof. If $u(K) = 2$, the additional conditions imply $u_{\pm}(K) = 2$, contradicting theorem 4. (For condition (a) this follows from the inequality $\sigma_{\xi'}(K_+) \leq \sigma_{\xi'}(K_-) + 2$, analogous to Murasugi's inequality for $\xi' = -1$, and for condition (b) from the argument of Traczyk [41].) □

To obtain some interesting examples, we consider twist knots. It is easy to see that for those of even crossing number, the Alexander polynomial has no zeros on the unit circle, while for those of odd crossing number, there in one pair of conjugate zeros, moving towards 1 when the crossing number increases.

Example 3 $u(3_1\#3_1\#!7_2) = 3$. (Here the factor knots are so mirrored so as the knot to have signature 2.)

In particular, we have

Corollary 3 If K is achiral and $0 \in \Delta_K(S^1)$ and $V_K\left(e^{\pi i/3}\right) = -3$, then $u(K) \geq 3$. □

Example 4 $u(3_1\#!3_1\#4_1) = 3$.

This solves one of the undecided numbers in the tables of [38]. No previous method seems to give this result.

Remark 5 As we proved above, $\Delta(\mathring{\xi}) = 0$ and $\sigma_\xi = 0$ imply that $\mathring{\xi}$ is (at least) a double zero of Δ. Already the existence of such a double zero, unfortunately, limits the space of applicable examples, and makes it most likely to obtain new information for composite knots, as we saw above.

The argument in the proof of theorem 4 works in fact assuming a more general condition on the nullity $n_\xi(K)$.

Theorem 5 Assume K is a knot for which there is a $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\xi \neq 1$ with $n_\xi(K) > \sigma_\xi(K)$. Then $u_+(K) = \infty$.

Proof. Assume the contrary. By the argument in the proof of theorem 4, we obtain that the $n_\xi(K)$ zero eigenvalues must become negative in $K_n = \bigcirc$, in order $\Delta_{K_n}(\mathring{\xi}) \neq 0$, but then $0 = \sigma_\xi(K_n) \leq \sigma_\xi(K) - n_\xi(K) < 0$, a contradiction. □

The condition of this theorem is unfortunately seldom satisfied. For example, if $K = 3_1\#3_1\#!3_1$ and $\xi = e^{\pi i/3}$, then $n_\xi = 3 > 1 = \sigma_\xi$. However, for the really interesting example $K = 8_{10}$ (with the same Alexander polynomial), we have only $n_\xi = 1$ (a single eigenvalue changes sign cubically), so that the desired conclusion $u_+(K) > 1$ fails.

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