Chord and Gauß diagrams in the Theory of Knots

Research Summary and Plan

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1 Short Characteristics

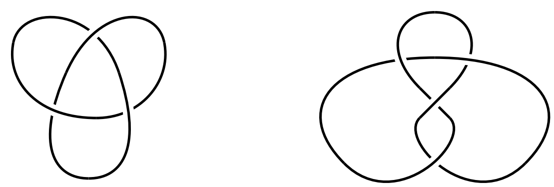
For many years the nature of knots has fascinated many people, especially mathematicians. Trying to rigorously understand and describe this phenomenon led to the development of knot theory as a branch of mathematics, more precisely of topology.

Initiated by Vassiliev's considerations of the homology of knot spaces, in the last years the theory of Vassiliev invariants has become an interesting and intensively studied field of knot theory. It was related by the deep work of Kontsevich and Drinfel'd to an algebraic structure consisting of combinatorial objects called chord diagrams (CD). In this way this many-sided new field opens interesting relations to classical fields of mathematics, such as the homological algebra, the theory of Lie groups, modular forms, graph theory and topological quantum field theories (TQFT).

Gauß diagrams are a recently introduced refinement of chord diagrams and used to write down formulas for Vassiliev invariants. These formulas are of particular interest for positive knots, a class of knots related to singularities of algebraic curves, and give a link between the Bennequin type inequalities coming from the proof of the Thom conjecture and properties of the Jones type polynomials of such knots.

2 The classification problem of knots

Consider a knot, i. e. an oriented embedding of S^1 in $\mathbb{R}^3$. Let's assume, that this embedding is C^1, which particularly means tame, i. e. the knot is isotopic to a piecewise linear embedding. This does not constrain too much the complexity of knots, it only excludes some ugly pathological cases. The following picture shows the two simplest, but non-trivial knots.



trefoil figure 8 knot

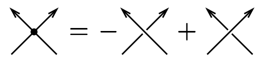
Thinking of a knot as of a real object, say a closed piece of rope, one gets an idea of transforming one knot into another by a sequence of pulling and twisting its strands, but not cutting the knot somewhere. One might consider knots as equal, if they are in this sense transformable into each other. This is mathematically described by the notion of (ambient) isotopy. So, in mathematical terms, knots are called topologically equal, if they are ambient isotopic, and to topologically classify a given knot means to determine uniquely its isotopy class.

Trying to classify knots led to the search for knot invariants, which, at least in some particular cases, can distinguish knots. Some approaches have been made to construct such invariants topologically. The most famous invariants of this kind are the polynomial invariants like the Alexander/Conway [Al, Co, Ka3] and HOMFLY [H, LM] polynomials, and also Casson's invariant [AM]. Unfortunately, these topological approaches often meet major difficulties, resulting from the analytical methods they involve.

On this background the main advantage of the theory of Vassiliev invariants appears to be, that it offers a new view of the topological structure of knots in a combinatorial and therefore more discrete and directly accessible way and makes it possible to apply simpler algebraic methods for exploring them.

3 The filtration of the knot space

Consider the linear space $\mathcal{V}$, (freely) generated by all the (isotopy classes of) knot embeddings. There is a self-suggesting way how to denote them. Every embedding class of a knot is determined by its projection on a 2-plane in $\mathbb{R}^3$, where all the crossings are transversal and equipped with the additional information, whether they are over- ( ) or under- ( ) crossings. Let $\mathcal{V}^p$ be the space of singular knots with exactly p double points (up to isotopy). $\mathcal{V}^p$ can be identified with a linear subspace of $\mathcal{V}$ by resolving the singularities into the difference of an over- and an undercrossing via the rule



(1)

where all the rest of the knot projections are assumed to be equal (one can show that the result does not depend on the order in which all the double points are resolved). This yields a filtration of $\mathcal{V}$

$$\mathcal{V} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \mathcal{V}^2 \supset \mathcal{V}^3 \supset \dots \tag{2}$$

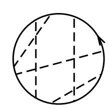
There exists a combinatorial description of the graded vector space,

$$\bigoplus_{i=0}^{\infty} \left(\mathcal{V}^i \Big/ \mathcal{V}^{i+1} \right) \tag{3}$$

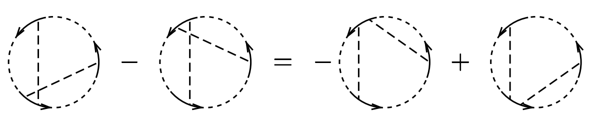
associated to this filtration, namely

$$\mathcal{V}^i \Big/ \mathcal{V}^{i+1} \simeq \mathcal{L}in\{\text{ chord diagrams of degree } i \,\} \Big/ \begin{smallmatrix} 4T \text{ relation} \\ FI \text{ relation} \end{smallmatrix}, \tag{4}$$

where the chord diagrams (CD's) are objects like this (an oriented circle with finitely many dashed chords in it)



(up to isotopy) and are graded by the number of chords. The $4T$ (4 term) relations have the form



and the FI (framing independence) relation requires, that each CD with an *isolated* chord (i. e., a chord not crossed by all the others) is zero.

The map which yields this isomorphism is a simple way how to assign a CD D_K to a singular knot K. Connect in the parameter space of K (which is an oriented S^1) pairs of points with the same image by a chord. Actually, the idea to describe singular knots in this way led to the representation (4), and, more generally, many of the further algebraic statements are based on this idea. Once introduced the CD's, the $4T$ and FI relations become self-suggesting. Look e. g. at the FI relation. If one takes a singular knot corresponding to a diagram with an isolated chord and resolves the singularity corresponding to this chord, one gets exactly an ambient isotopy relation of (singular) knots.

Note, that (4) implies the finite dimensionality of the filtration (2), which considerably simplifies further algebraic considerations.

Definition 3.1 *A Vassiliev invariant of degree m is an element $V \in \mathcal{V}^*$, s. t.*

$$V\Big|_{\mathcal{V}^{p+1}} \equiv 0 \quad \textit{and} \quad V\Big|_{\mathcal{V}^p} \not\equiv 0 .$$

It is possible in some sense to consider (1) as a way to "differentiate" a knot, and in this language Vassiliev invariant corresponds to a polynomial invariant, i. e. a function with a vanishing derivative.

This notion is interesting in connection with many other known knot invariants, such as the Conway and the HOMFLY polynomials, which have originally come about out by topological considerations, but which fulfill certain relations closely connected to (1). E. g., the Conway polynomial $C_K(z)$ of a knot K satisfies the (skein) relation

$$C\big(\times_{\bullet}\big)(z) := C\big(\times_{+}\big)(z) - C\big(\times_{-}\big)(z) = zC\big(\,)\,(\,\big)(z)\,, \tag{5}$$

from which one concludes, that

Lemma 3.2 *The coefficient* $\mathrm{coeff}_{z,k}C_\star(z)$ *is a Vassiliev invariant of degree at most* k.

By similar arguments the same is true for the HOMFLY polynomial in his reparametrization introduced by Jones [J] $F_K(N,e^z)$ (and therefore, setting $N=2$, for the Jones polynomial [J2] as well) and the Kauffman polynomial [Ka] in his version called by Kauffman the Dubrovnik polynomial [Ka2]. See [BL].

In this way Vassiliev invariants generalize many known (topological) invariants, and that is another reason why Vassiliev invariants are so interesting.

Note, that the grading of the CD's is preserved by $4T$, so the linear space $\mathcal{A}^r$ obtained by taking the direct sum over all i of the right hand side of (4) is a finite dimensionally graded space. Let G_m denote the degree-m-piece of $\mathcal{A}^r$.

By definition all Vassiliev invariants of degree $\le m$ are sensitive with respect to knot classification only *maximally* up to $\mathcal{V}^{m+1}$. However, one does not know yet, whether *all* Vassiliev invariants are capable of a *complete* topological classification of knots.

Conjecture 3.3 *Vassiliev invariants separate knots.*

By now conjecture 3.3 has been proved at least for pure braids [BN3, BN4] (see section 6), but for knots we don't know much about it. As it stands it sounds very appealing, but unfortunately, we cannot even yet affirm the following weaker

Question 3.1 *(see [BN2, sect. 7.2]) Do Vassiliev invariants distinguish knot orientation?*

This is one of the hardest problems in knot theory. Yet, there are no easily definable invariants, as quantum and skein invariants, which distinguish knot orientation. As pointed out by Birman [Bi], the fact itself that non-invertible knots exist, although previously conjectured for a long time, has been proved only in the 60's by Trotter [Tro]. In my paper [St4] I tried to enlighten the problems with detecting orientation with Vassiliev invariants in an independent way from Bar-Natan's computational arguments (see section 8).

With regard to question 3.1, one can also pose the weaker question of how much information can be obtained from invariants of bounded degree. For that situation, much more is known. Threre exist constructions that preserve Vassiliev invariants of bounded degree and make a knot into an alternating one [Sa] or slice one (up to the Arf invariant) [Ng]. Also, one can obtain any possible value of a Tristram-Levine signature [St7] and unknotting number [St15].

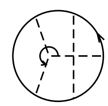
4 The Algebra $\mathcal{A}$

Let us for a moment forget about the FI relation and consider $\mathcal{A}$ as the space, obtained from $\mathcal{A}^r$ by not factoring out the FI relation.

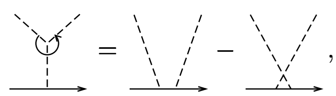
For this space $\mathcal{A}$ at least 3 other descriptions are known (for the proof, that they are all isomorphic, see [BN2]).

1) $$\mathcal{A} = \mathcal{L}in \left\{ \begin{matrix} \text{UTD's (unitrivalent or} \\ \text{Feynman diagrams)} \end{matrix} \right\} \Big/ STU \text{ relation},$$

where a UTD is something like

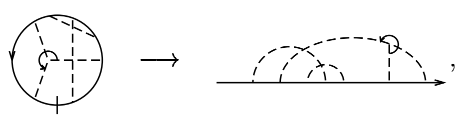


(like a chord diagram, but with oriented internal trivalent vertices allowed, which represent singular triple points) and the STU relation is



2) $$\mathcal{A} = \mathcal{L}in \{ \text{ linearized UTD's } \} \Big/ STU \text{ relation},$$

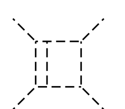
where a linearized UTD is a UTD, with the solid line cut somewhere



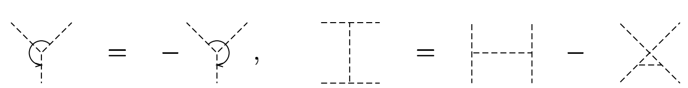
and both spaces are graded by half the number of trivalent vertices (internal or on the solid line) and

3) $$\mathcal{A} = \mathcal{L}in \{ \text{ UTG's (unitrivalent graphs) } \} \Big/ AS \text{ and } IHX \text{ relation},$$

where the UTG's are objects like this



(UTD's with the solid line removed) and graded by half the number of vertices (univalent or trivalent), and the AS (antisymmetry) and the IHX relations are



Using the different representations of $\mathcal{A}$ one can define multi- and comultiplication, which make $\mathcal{A}$ into a graded commutative and cocommutative Hopf algebra [MiM], including also a sort of Adams operations ψ^q, $q \in \mathbb{Z}$.

5 Weight systems

We have defined the Vassiliev invariants in terms of spaces of knot embeddings, but we found an easier description of these spaces by diagrams. Then is it possible to describe Vassiliev invariants by combinatorical objects defined entirely with the help of our diagram spaces?

Definition 5.1 *A weight system of degree m is an element $W \in (G_m\mathcal{A})^*$. Let $\mathcal{W}^m$ be the linear space of all weight systems of degree m.*

It is easy to assign to a Vassiliev invariant V of deg $\leq m$ a weight system W of degree m. Consider the (graded) map W_* given by

$$\mathcal{V}_m := (\mathcal{V}^m)^* \quad \xrightarrow{W_m} \quad (G_m\mathcal{A})^* =: \mathcal{W}^m \tag{6}$$

$$W_m(V)\,(\text{diagram of deg } m) \quad := \quad V\left(\begin{array}{c}\text{one singular knot with } m \text{ singula-}\\ \text{rities, that represents this diagram}\end{array}\right) \tag{7}$$

The kernel of this map is by definition $\mathcal{V}_{m-1} = (\mathcal{V}^{m-1})^*$, i. e. the space of Vassiliev invariants of deg $\leq m-1$, so we have the short exact sequence

$$0 \longrightarrow \mathcal{V}_{m-1} \hookrightarrow \mathcal{V}_m \xrightarrow{W_m} \mathcal{W}_m \longrightarrow 0\,.$$

Theorem 5.2 (Fundamental Theorem of Vassiliev invariants) *This exact sequence splits, i. e. there is a (graded) map V*

$$V_m : \mathcal{W}_m \longrightarrow \mathcal{V}_m\,,$$

such that

$$Im(Id_{\mathcal{V}_m} - V \circ W) \subset \mathcal{V}_{m-1} \quad \textit{and} \quad W_m \circ V_m = Id_{\mathcal{W}_m}\,.$$

In some sense the map V can be considered as "integrating" a weight system. If we want to do this, we need to consider another crucial object, closely related to this theorem. Namely, it turns out, that proving that theorem is equivalent to constructing a *universal Vassiliev invariant*, i. e. a map

$$\mathcal{V}^* \xrightarrow{V'} \mathcal{A}_*^{**}\,,$$

where $\mathcal{A}^{**}$ is the graded completion of $\mathcal{A}$, such that for every knot $K \in \mathcal{V}^m$ and any Vassiliev invariant $V \in \mathcal{V}_m$ there holds

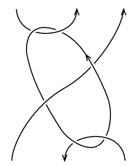
$$\big(W(V)\big)\,(V'(K)) \quad = \quad V(K)\,.$$

Therefore, universal Vassiliev invariants play a basic role in the theory of Vassiliev invariants. Originally the existence of universal Vassiliev invariants followed from the deep work of Kontsevich [Ko] and Drinfel'd [Dr]. Their ideas have been subsequently worked out by Bar-Natan in [BN2] and [BN], where the Kontsevich integral and the category theoretic approach of Drinfel'd were explained. After that, universal Vassiliev invariants have been constructed by many people an in different ways, e. g. [Pi]. Though it is known, that the choice of a universal Vassiliev invariant is not unique, surprisingly many of the ansatzes seem to generate the same special one.

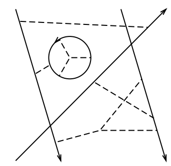
6 Vassiliev invariants for braids and string links

In the same way as for knots, the idea of constructing Vassiliev invariants can be generalized to other "knot-like objects", i. e. certain classes of 1-dimensional embeddings into $\mathbb{R}^3$, factorized by an appropriate notion of isotopy, such as tangles, (pure) braids and string links. The resulting diagram spaces **AT** (for tangles), **AB** (for braids), $\mathbf{AP} = \mathcal{A}^{sl}$ (for string links) and $\mathcal{A}^{pb} = \mathbf{AP}^{hor}$ (for pure braids) are constructed in the same way as for knots and have a similar algebraic structure (here the multiplication is given by stacking up) and properties as the knot diagram algebra $\mathcal{A}$. Here are some typical examples for the several embedding classes and for diagrams, corresponding to these classes.

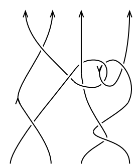
A tangle is an oriented embedding $T : [0,1] \times \mathbb{Z}_n \cup S^1 \times \mathbb{Z}_m \hookrightarrow [0,1] \times \mathbb{R}^2$, such that $T(\{0,1\} \times \mathbb{Z}_n) \subset \{0,1\} \times \mathbb{R}^2$.



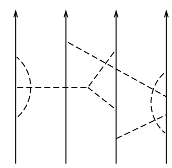
Here is a singular tangle diagram.



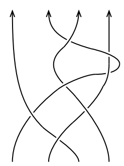
A string link is an oriented embedding $S : [0,1] \times \mathbb{Z}_n \hookrightarrow [0,1] \times \mathbb{R}^2$, such that $S(\{i\} \times \mathbb{Z}_n) \subset \{i\} \times \mathbb{R}^2$ for $i = 0,1$.



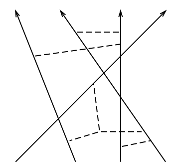
The diagram of a string link contains no cyclic full lines, and all strands are vertical and point to the top.



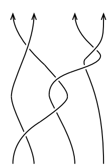
Here is a braid. It is a string link with monotonous 1st component.



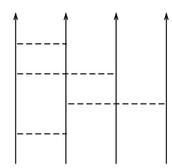
The diagram of a braid contains no chords ending on one strand only.



A pure braid is a braid, which preserves the order of the strands.



The diagram of a pure braid can be simplified to a diagram only with horizontal chords.



$\mathcal{A}^{sl}$ is known to have also a formulation as a space of coloured UTG's, i. e. UTG's with coloured univalent vertices [BN4].

In the case of pure braids, it is more convenient to use the STU relation to resolve all trivalent vertices of dashed lines. Note, that in this case the height function is monotonous, so in distinction from $\mathcal{A}^{sl}$ there are no self-intersections of the strands, and the resulting diagrams can be drawn as made up only of horizontal chords. This makes it possible, by setting t_{ij} to be the chord connecting strands i and j, to describe $\mathcal{A}_n^{pb}$ in purely algebraic terms, namely as the algebra in $\frac{n(n-1)}{2}$ variables with 2 relations (the isotopy and the $4T$ relation)

$$\mathcal{A}_n^{pb} = \left\langle \; t_{ij}, \quad 1 \leq i \neq j \leq n \;\middle|\; \begin{array}{ll} t_{ij} = t_{ji} & \\ [t_{ij}, t_{kl}] = 0 & |\{i,j,k,l\}| = 4 \\ [t_{ij}, t_{ik} + t_{jk}] = 0 & |\{i,j,k\}| = 3 \end{array} \right\rangle$$

This description allowed Hutchings [Hu] to find a purely algebraic-combinatorial proof of the Fundamental Theorem for (pure and ordinary) braids.

The picture for string links is more complicated, but the Kontsevich integral was constructed for them as well. In a similar way to knots, duals of certain quotients of the diagram algebra correspond to Vassiliev invariants of string links up to homotopy [BN4] and concordance [HM]. Latter class was shown to be equivalent to Milnor's $\bar{\mu}$ invariants [Mi], which have been previously shown to be Vassiliev invariants of string links by Bar-Natan [BN4] and Lin [L].

7 Vassiliev invariants and Lie algebras

Reshetikhin and Turaev [RT] introduced a way how to construct a framing-dependent weight system $W_{\mathfrak{g},R}$ (an element in $\mathcal{A}^*$ instead of $(\mathcal{A}^r)^*$) given an irreducible representation R of a semi-simple Lie algebra $\mathfrak{g}$. It is easy to "untwist" such a $W_{\mathfrak{g},R}$, i. e. to make it into a (framing-independent) weight system $\hat{W}_{\mathfrak{g},R}$. Integrating $\hat{W}_{\mathfrak{g},R}$ one obtains special Vassiliev invariants, called Reshetikhin-Turaev or quantum invariants. It seemed for a while, that these invariants, constructed to the most common Lie algebras, already exhaust all Vassiliev invariants. Some explicite computations in lower degrees (see section 8), led Bar-Natan [BN2] to the following

Conjecture 7.1 *All Vassiliev invariants are Reshetikhin-Turaev invariants.*

However, it was already known, that conjecture 7.1 contradicts the not less plausibly looking conjecture 3.3, because quantum invariants are known not to detect orientation (see question 3.1). Later, Vogel [Vo] indeed proved that there are Vassiliev invariants which do not come out of the Reshetikhin-Turaev construction for semi-simple Lie algebras. Recently, he announced [Vo2] a counterexample to conjecture 7.1 for *all* (not necessarily semi-simple) Lie algebras.

Nevertheless, Lie algebra invariants form a substantial part of Vassiliev theory. A recent nice application of this was the proof by Bar-Natan and Garoufalidis [BG] of a conjecture of Melvin and Morton [MeM] about a relation between the weight systems coming from the Alexander and (cables of the) Jones polynomial.

8 Dimension bounds, modular forms and hyperbolic volume

Though we have a nice description of the spaces introduced above, some simple algebraic features seem hopelessly hard to examine.

As a remarkable example in this regard, Bar-Natan observed [BN5], that a certain degeneracy statement of the sl_n weights of a UTD is equivalent to the Four Color Theorem, one of the most complicatedly proven theorems in mathematical history [AH, SK]!

Another illustration: one does not know any way to compute the graded dimension of $\mathcal{A}$. This has only been done for degrees ≤ 9 by Bar-Natan, and later by Kneissler in [Kn] up to degree 12 at the cost of a huge computer effort.

There only exist some special statements about the asymptotical behaviour of there dimensions as degree $\to \infty$, mainly due to Chmutov and Duzhin [CD, CD2] and myself [St2]. They, however, are not sufficient to give a satisfying answer to this question. Only in the case of pure braids it is possible to reduce this question to some numerical combinatorics (see [BN3]). This shows the fascinating complexity of these simple looking objects.

Nevertheless, the enumeration problems of [St2] turned out to be interesting in a different context. Don Zagier [Za] discovered that they give the coefficients of an asymptotic expansion near 1 of a "derivative" of the Dedekind eta-function, a classical modular form of weight $1/2$, thus indicating the possible existence of a theory of Eichler integrals for modular forms of half-integral weight. Using the theory of Dirichlet series, Zagier found the exact asymptotical behaviour of the numbers of [St2], thus establishing the currently best upper bound for the graded dimension of $\mathcal{A}$.

A similar phenomenon occurs in the asymptotic expansion of Witten-Reshetikhin-Turaev-invariants [Wi, RT2] of certain 3-manifolds, as the Poincaré homology sphere [LZ], and is therefore of great number theoretic interest. The existence of such expansions was established by Ohtsuki [Oh], and they form the 3-manifold counterpart of Vassiliev theory for knots [LMO].

Recently the same type of power series appeared in yet another context related to Vassiliev invariants. They give limits of evaluations at roots uf unity of (some projective version of) the colored Jones polynomials, invariants, which can be expressed by the cablings of the Jones polynomial and by the method of [BL] can be transformed into power series, whose coefficitns are Vassiliev invariants with weight systems coming via the construction in §7 from the n-dimensional irreducible representation of sl_2.

These evaluations have recently caused much excitement, because Murakami and Murakami [MM] showed that they coincide with Kashaev's quantum dilogarithm invariants [Ks], for which he conjectured that the asyptotic behaviour of their evaluations at $e^{2\pi i/N}$ as $N \to \infty$ can be used to obtain the hyperbolic volume (or, if the knot is non-hyperbolic, the Gromov norm [Gr]) of the complement of the knot. The Kashaev-Murakami-Murakami conjecture has been established for the figure eight knot, but for more complicated examples or even the general case it will require much number theoretic work. It has been noted by Murakami and Murakami, that the conjecture implies that a knot is trivial if and only if all of its Vassiliev invariants are trivial, that is, a partial case of conjecture 3.3 for the unknot.

9 Braiding sequences

The combinatorial structure of chord diagrams and unitrivalent diagrams considerably simplified our understanding of Vassiliev invariants and was the main tool in the proof of a series of results [BG, Vo]. Despite being therefore much celebrated, this approach has some serious defects. Although many ways exist to prove the Fundamental theorem [BS], they are all rather complicated and at some point unnatural, and their connections are not yet completely understood. So the integration of the (series of) weight system(s) to a Vassiliev invariant is far from being routine work.

But even for itself, although simpler and much friendlier to work with, the combinatorial structure of chord diagrams is far from being easily understandable.

In an attempt to create an alternative to the (defects of the) classical approach and generalizing some ideas of Dean [De], Trapp [Tr] and Stanford [Sa], in [St4] I introduced the notion of a braiding sequence. It offered a simple direct understanding of the behaviour of Vassiliev invariants on special knot classes, something, which was never worked out using the classical approach. Beside some other facts, it gave relatively simple proofs that the dimension of the space of Vassiliev invariants of degree $\leq n$ on certain knot classes is finite (arborescent knots), and in some cases even exponential upper bounds in n for this dimension (e. g., rational knots, closed 3 braids), something, which was not yet achieved by chord diagrams.

Moreover, while the Kontsevich-Drinfel'd approach (used in [CD]) works only over zero (field) characteristic, our arguments with braiding sequences hold for any zero divisor free ring, in particular the fields $\mathbb{Z}_p$, p prime.

Definition 9.1 *For some odd $k \in \mathbb{Z}$, a k-braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k (see figure 1). A braiding sequence (associated to a numbered set P of crossings in a diagram D; all crossings by default) is a family of diagrams, parametrized by $|P|$ odd numbers $x_1,\dots,x_{|P|}$, each one indicating that at crossing number i an x_i-braiding is done.*

Any Vassiliev invariant v of degree at most k behaves on a braiding sequence as a polynomial of degree at most k in $x_1,\dots,x_{|P|}$ (see [St4] and [Tr]), and this polynomial is called the braiding polynomial of v on this braiding sequence.

In [St9], I use the approach of braiding sequences to prove exponential upper bounds for the number

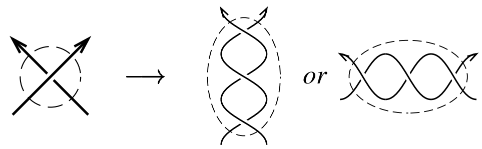


Figure 1: Two ways to do a -3-braiding at a crossing.

of Vassiliev invariants *inter alia* on knots with bounded braid index and arborescent knots.

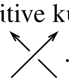
10 The approach of Gauß sums, Thom conjecture and positive knots

Before Vassiliev, the common approach for finding knot invariants, as applied for the polynomials, is to fix certain initial values and a relation of the values of the invariant on links equal except near one crossing. Except for the Kauffman polynomial this relation involves triples as depicted in (5) and such a relation is called *skein relation*.

An alternative approach was initiated by Fiedler [Fi2] and Polyak-Viro [PV]. They look at an n-tuple of crossings in a fixed knot projection satisfying certain conditions and sum over such choices terms associated to each choice. The choices are indicated by the preimages of the crossings on the S^1 parametrizing the knot, thus giving chord diagrams, but often with the orientation of the under-/over-crossing indicated by replacing the chord by an arrow. The objects arising (a solid line with arrows connecting pairs of points on it) are called *Gauß diagrams*, and the formulas Gauß sum formulas. If the summation in such a formula is well chosen, then it is invariant under the Reidemeister moves and gives a knot invariant. It was proved [PV], see also [St4], that the Polyak-Viro invariants are Vassiliev invariants.

These invariants can be generalized also to other 3-manifolds, for example the solid torus [FS, Fi] (or more generally to line bundles over closed surfaces), and give this way a partial solution to the mutation problem for 2-component links there [St14].

The Polyak-Viro-Fiedler invariants lead to a series of new results on the values of the Jones polynomial [St13] and on positive [St12] and almost positive knots [St11].

Positive knots are knots which admit positive diagrams, that is, diagrams with all crossings looking like . These knots are of particular interest because of the variety of tools applicable for their examination.

They form a subclass of the quasi-positive knots, introduced by Rudolph [Ru] to describe the knots coming from neighborhoods of singularities of complex algebraic curves, and thus are relevant to singularity theory, see, e. g., [A]. Therefore, on these knots where the genus inequality of Bennequin [Be, theorem 3], and its subsequent generalizations [Ru2] based on the proof of the Thom conjecture [KM], are particularly strong.

On the other hand, combinatorial methods related to the theory of link polynomials [J, Ka2, Ka4, H] lead to proofs of properties of these invariants for positive knots, see e. g., [Cr, CM, N, Zu].

The Gauß diagram formulas give inequality relations between the Vassiliev invariants, expressible by [BL] by the knot polynomials, and the genus and unknotting number of a positive knot. This way, e. g., the Kronheimer-Mrowka-Rudolph results can be used to prove the following self-contained property of the Jones polynomial $V_K \in \mathbb{Z}[t,t^{-1}]$ of a positive knot K:

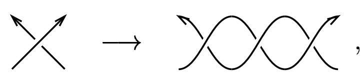
Theorem 10.1 *(see [St12, §6]) If a knot K is positive, then $-5V_K''(1) \geq 6\max\deg V_K$.*

Further properties of the Gauß sum formulas can be used together with considering the genus to classify all positive diagrams of some positive knots, as done in [St5]. As an example, it was shown there that the knot $!10_{120}$ of [Ro, appendix A] (where '!' denotes the mirror image) has just one positive diagram (the mirror image of the diagram in Rolfsen's table).

11 A new relation between braiding sequences, weight systems and hyperbolic volume

Braiding sequences appear in a particular case in the classification of knot diagrams by canonical genus.

Theorem 11.1 *([St16, theorem 3.1]) (Reduced) knot diagrams of given genus g decompose into finitely many equivalence classes under $\bar{t}_2'$ twists*



This approach was begun in [St16], and then set forth in [St5], and later in [SV], where some algebraic tools were discovered to have new applications. Independently, the structure theorem for canonical genus was discovered by Brittenham [Br], who gave as application an upper bound on the hyperbolic volume in terms of canonical genus.

Later, using work of [St4], we can state Brittenham's result as follows. Let $n(L)$ be the number of components of a(n oriented) link L, and $\chi_c(L)$ the maximal Euler characteristic $\chi(D)$ of a diagram D of L.

Set for $\chi < 0$

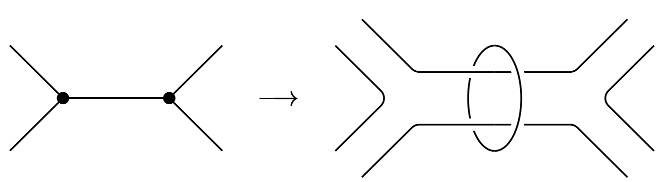
$$v_{n,\chi} = \sup \{ \operatorname{vol}(L) : n(L) = n, \chi_c(L) = \chi \},$$

and

$$v_\chi = \sup \{ \operatorname{vol}(L) : \chi_c(L) = \chi \} = \max_n v_{n,\chi}, \tag{8}$$

with the maximum taken over all $1 \leq n \leq 2-\chi$ with $n+\chi$ even. (Note that the upper bound for n follows from the fact that a canonical Seifert surface of a non-split diagram is connected.)

Define for a 3-connected 3-valent planar graph G the (*unoriented*) link L_G by



Theorem 11.2 *For $\chi < 0$,*

$$v_\chi = \max_{\chi(G)=\chi} \operatorname{vol}(L_G) = \sup \{ \operatorname{vol}(L) : \chi(L) = \chi,\ L \textit{ special alternating} \},$$

with the maximum taken over 3*-connected* 3*-valent planar graphs G, considered up to isotopy in the sphere.*

There is, on the other hand, some relation between the enumeration problems in [SV], and the sl_N weight system polynomial W_N of G. Computations have then led to the following remarkable

Conjecture 11.3 (Weight system-Volume-Conjecture)
For every irreducible polynomial $P_i(x) \in \mathbb{Z}[x]$ exists a number $c_i = c_{P_i} = c_{P_i(x)} \in \mathbb{R}$ with the following property: when G is a trivalent, 3-connected planar graph of odd χ, and

$$\frac{W_N(G)}{2N(N^2-1)} = \prod_{j=1}^{k} P_{i_j}(N^2)\,, \tag{9}$$

then

$$\mathrm{vol}(L_G) = \frac{c_x}{3} + \sum_{j=1}^{k} c_{i_j}\,. \tag{10}$$

In subsequent work, we will try to explain what we know, and investigate what we do not know about this type of relation.

12 Personal projects

There are several challenging projects arising from the present state of research. Here are some on which I particularly concentrate.

- Prove an exponential upper bound for the number of Vassiliev invariants on all knots. In view of the problems of [St2], it is unlikely that a reasonable proof exists via CD's. On the other hand, braiding sequence arguments of [St9] quickly come close to a solution (and succeed in many special cases), but still fail at some point in the general case.

- Understand better the structure of Gauß diagrams and sharpen the inequalities of [St12]. As simple examples show, except for the low crossing number cases and connected sums thereof these inequalities are far from being sharp, so significant improvement seems possible.

- Find solid torus Gauß sum invariants detecting 2-component link orientation. Note that yet no purely diagrammatic invariant is known which does so (without involving knot groups). This is the best we can hope for after we found out in [St4], that solid torus Gauß sum invariants, at least in reasonably low degree, cannot detect knot orientation.

- Explore the relations between hyperbolic volume and various combinatorial structures, in particular the mystery behind the above Weight system-Volume-Conjecture.

13 Publications on Vassiliev invariants

In the last years a lot of literature appeared on Vassiliev invariants. Two papers are probably best appropriate as an introduction to this field, Bar-Natan's [BN2] and especially Birman's [Bi]. The relation to the classical knot invariants is discussed in [BL]. For more on the different approaches to the proof of the Fundamental Theorem, see [BS]. Finally, there recently appeared 2 books by Kassel [K] and Turaev [T] treating the relation of Vassiliev invariants to quantum topology.

An extensive survey on authors and publications may be found in [BN6] and loc. cit. in [Bi].

References

[A] N. A'Campo, *Das Generic immersions of curves, knots, monodromy and gordian number*, Inst. Hautes Études Sci. Publ. Math. No. **88** (1998), 151–169.

[Al] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), 275–306.

[AM] S. Akbulut and J. D. McCarthy, *Casson's invariant for oriented 3-spheres*, Mathematical notes **36**, Princeton, 1990.

[AH] K. I. Appel and W. Haken, *Every planar map is four colorable*, Contemp. Math. **98**, Amer. Math. Soc., Providence, 1989.

[BN] D. Bar-Natan, *Non-associative tangles*, in "Geometric topology" (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[BN2] ——— " ———, *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[BN3] ——— " ———, *Vassiliev and quantum invariants for braids*, in Proc. of Symp. in Appl. Math. **51** (1996), 129–144, "The interface of knots and physics", (L. H. Kauffman, ed.), Amer. Math. Soc., Providence.

[BN4] ——— " ———, *Vassiliev Homotopy String Link Invariants*, Jour. of Knot Theory and its Ramifications **4** (1995), 13–32.

[BN5] ——— " ———, *Lie algebras and the Four Color Theorem*, Combinatorica **17(1)** (1997), 43–52. See also `q-alg/9606016`.

[BN6] ——— " ———, *Bibliography of Vassiliev invariants*, available on the internet site `http://www.math.toronto.edu/drorbn/VasBib/VasBib.html`.

[BG] ——— " ——— und S. Garoufalidis, *On the Melvin-Morton-Rozansky Conjecture*, Invent. Math. **125** (1996), 103–133.

[BS] ——— " ——— and A. Stoimenow, *The Fundamental Theorem of Vassiliev invariants*, "Geometry and Physics", Lecture Notes in Pure & Appl. Math. **184**, M. Dekker, New York, 1996, 101–134.

[Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Soc. Math. de France, Astérisque **107–108** (1983), 87–161.

[Bi] J. S. Birman, *New Points of View in Knot Theory*, Bull. Amer. Math. Soc. **28** (1993), 253–287.

[BL] ——— " ——— and X-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225–270.

[Br] M. Brittenham, *Bounding canonical genus bounds volume*, preprint (1998), available at `http://www.math.unl.edu/~mbritten/personal/pprdescr.html`.

[CD] S. V. Chmutov and S. V. Duzhin, *An upper bound for the number of Vassiliev knot invariants*, Jour. of Knot Theory and its Ramifications **3(2)** (1994), 141–151.

[CD2] ——— " ——— and ——— " ———, *A lower bound for the number of Vassiliev knot invariants*, Topol. Appl. **92(3)** (1999), 201–223.

[Co] J. H. Conway, *On enumeration of knots and links*, in "Computational Problems in abstract algebra" (J. Leech, ed.), 329–358. Pergamon Press, 1969.

[Cr] ——— " ———, *Homogeneous links*, J. London Math. Soc. (series 2) **39** (1989), 535–552.

[CM] ——— " ——— and H. R. Morton, *Positivity of knot polynomials on positive links*, J. Knot Theory Ramif. **1** (1992), 203–206.

[De] J. Dean, *Many classical knot invariants are not Vassiliev invariants*, Jour. of Knot Theory and its Ramifications, **3(1)** (1994), 7–9.

[Dr] V. G. Drinfel'd, *On quasitriangular Quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991), 829–860.

[Fi] Th. Fiedler, *Gauss Diagram Invariants for Knots and Links*, Kluwer Academic Publishers, Mathematics and Its Applications Vol **532** (2001).

[Fi2] ——— " ———, *A small state sum for knots*, Topology **32 (2)** (1993), 281–294.

[FS] ——— ” ——— and A. Stoimenow, *New knot and link invariants*, Proceedings of the International Conference on Knot Theory “Knots in Hellas, 98”, Series on Knots and Everything **24**, World Scientific, 2000.

[Gr] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1983), 5–99.

[HM] N. Habegger and G. Masbaum, *The Kontsevich integral and Milnor’s invariants*, Topology **39(6)** (2000), 1253–1289.

[H] P. Freyd, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu and D. Yetter, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–246.

[Hu] M. Hutchings, *Integration of singular braid invariants and graph cohomology*, Trans. Amer. Math. Soc. **350(5)** (1998), 1791–1809.

[J] V. F. R. Jones, *On knot invariants related to some statistical mechanical models*, Pacific. J. Math. **137(2)** (1989), 311–334.

[J2] ——— ” ———, *A polynomial invariant of knots and links via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[K] C. Kassel, *Quantum Groups*, Springer-Verlag GTM **155**, Heidelberg 1994.

[Ka] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[Ka2] ——— ” ———, *Knots and physics* (second edition), World Scientific, Singapore 1993.

[Ka3] ——— ” ———, *The Conway polynomial*, Topology **20** (1981), 101–108.

[Ka4] ——— ” ———, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.

[Ks] R.M. Kashaev, *A Link Invariant from Quantum Dilogarithm*, Modern Phys. Lett. A **10(19)** (1995), 1409–1418.

[Kn] J. Kneissler, *The number of primitive Vassiliev invariants up to degree twelve*, University of Bonn preprint, June 1997. See also `q-alg/9706022`.

[Ko] M. Kontsevich, *Vassiliev’s knot invariants*, Adv. in Sov. Math., **16(2)** (1993), 137–150.

[KM] P. B. Kronheimer and T. Mrowka, *On the genus of embedded surfaces in in the projective plane*, Math. Res. Lett. **1** (1994), 797–808.

[LZ] R. Lawrence and D. Zagier, *Modular forms and quantum invariants of 3-manifolds*, Sir Michael Atiyah: a great mathematician of the twentieth century, Asian J. Math. **3(1)** (1999), 93–107.

[LMO] T. Q. T. Le, J. Murakami and T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds*, Topology **37(3)** (1998), 271–291.

[LM] W. B. R. Lickorish and K. C. Millett, *A polynomial invariant for oriented links*, Topology **26 (1)** (1987), 107–141.

[L] X-S. Lin, *Power series expansions and invariants of links*, in “Geometric topology” (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 184–202, Amer. Math. Soc. and International Press, Providence, 1997.

[MeM] P. M. Melvin and H. R. Morton, *The coloured Jones function*, Comm. Math. Phys. **169** (1995), 501–520.

[Mi] J. Milnor, *Link groups*, Annals of Math. **59** (1954), 177–195.

[MiM] ——— ” ——— and J. Moore, *On the structure of Hopf algebras*, Annals of Math. **81** (1965), 211–264.

[MM] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186(1)** (2001), 85–104.

[N] T. Nakamura, *Positive alternating links are positively alternating*, J. Knot Theory Ramifications **9(1)** (2000), 107–112.

[Ng] K. Y. Ng, *Groups of ribbon knots*, Topology **37** (1998), 441–458.

[Oh] T. Ohtsuki, *A polynomial invariant of rational homology 3-spheres*, Inv. Math. **123** (1996), 241–257.

[Pi] S. Piunikhin, *Combinatorial expression for universal Vassiliev link invariant*, Comm. Math. Phys. **168(1)**, 1–22.

[PV] M. Polyak and O. Viro, *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Notes **11** (1994), 445–454.

[RT] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990), 1–26.

[RT2] ——— ” ——— and ——— ” ———, *Invariants of 3-manifolds via link polynomials and quantum groups*, Inv. Math. **103** (1991), 547–597.

[Ro] D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[Ru] L. Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. **58** (1983), 1–37.

[Ru2] ——— ” ———, *Positive links are strongly quasipositive*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr. **2**, 555–562.

[Sa] T. Stanford, *The functoriality of Vassiliev-type invariants of links, braids, and knotted graphs*, Jour. of Knot Theory and its Ramifications **3(3)** (1994), 247–262.

[Sa2] ——— ” ———, *Braid commutators and Vassiliev invariants*, Pacific Jour. of Math. **174 (1)** (1996), 269–276.

[St] A. Stoimenow, *Über Harrison-Kohomologie und die Drinfel'd-Vermutung*, diploma thesis, Humboldt University, Berlin, 1995

[St2] ——— ” ———, *Enumeration of chord diagrams and an upper bound for Vassiliev invariants*, J. Of Knot Theory and Its Ram. **7(1)** (1998), 93–114.

[St3] ——— ” ———, *On enumeration of chord diagrams and asymptotics of Vassiliev invariants*, Doctor thesis, Freie University Berlin, 1998.

[St4] ——— ” ———, *Gauß sum invariants, Vassiliev invariants and braiding sequences*, Jour. of Knot Theory and its Ramifications **9(2)** (2000), 221–269.

[St5] ——— ” ———, *Knots of genus two*, preprint `math.GT/0303012`.

[St6] ——— ” ———, *On the number of chord diagrams*, Discr. Math. **218** (2000), 209–233.

[St7] ——— ” ———, *Some applications of Tristram-Levine signatures*, preprint.

[St8] ——— ” ———, *Stirling numbers, Eulerian idempotents and a diagram complex*, J. Of Knot Theory and Its Ram. **7(2)** (1998), 231–256.

[St9] ——— ” ———, *The braid index and the growth of Vassiliev invariants*, J. Of Knot Theory and Its Ram. **8(6)** (1999), 799–813.

[St10] ——— ” ———, *A Survey on Vassiliev Invariants for knots*, “Mathematics and Education in Mathematics”, Proceedings of the XXVII. Spring Conference of the Union of Bulgarian Mathematicians, 1998, 37–47.

[St11] ——— ” ———, *Gauss sums on almost positive knots*, `math.GT/9803073`, Compositio Math. **140(1)** (2004), 228–254.

[St12] ——— ” ———, *Positive knots, closed braids and the Jones polynomial*, `math/9805078`, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **2(2)** (2003), 237–285.

[St13] ——— ” ———, *On some restrictions to the values of the Jones polynomial*, preprint.

[St14] ——— ” ———, *Mutant links distinguished by degree 3 Gauss sums*, Proceedings of the International Conference on Knot Theory “Knots in Hellas, 98”, Series on Knots and Everything **24**, World Scientific, 2000.

[St15] ——— ” ———, *Vassiliev invariants and rational knots of unknotting number one*, `math/9909050`, Topology **42(1)** (2003), 227–241.

[St16] ——— ” ———, *Knots of genus one*, Proc. Amer. Math. Soc. **129(7)** (2001), 2141–2156.

[SV] ——— ” ——— and A. Vdovina, *Counting alternating knots by genus*, to appear in Math. Ann.

[SK] T. L. Saaty and P. C. Kainen, *The four color problem, assaults and conquest*, McGraw-Hill, New-York, 1977.

[T] V. G. Turaev, *Quantum invariants for knots and 3-manifolds*, de Gruyter, Berlin, 94.

[Tr] R. Trapp, *Twist sequences and Vassiliev invariants*, Jour. of Knot Theory and its Ramifications **3(3)** (1994), 391–405.

[Tro] H. F. Trotter, *Non-invertible knots exist*, Topology **2** (1964), 341–358.

[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Vo] P. Vogel, *Algebraic structures on modules of diagrams*, to appear in Inv. Math.

[Vo2] ——— ” ———, to appear.

[Wi] E. Witten, *Quantum field theory and the Jones polyniomial*, Commun. Math. Phys. **121** (1989), 351–394.

[Za] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40(5)** (2001), 945–960.

[Zu] L. Zulli, *The rank of the trip matrix of a positive knot diagram*, J. Knot Theory Ramif. **6 (2)** (1997), 299–301.