THE CONWAY VASSILIEV INVARIANTS ON TWIST KNOTS

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Abstract. We prove that any non-trivial primitive Vassiliev invariant expressible as a polynomial in the coefficients of the Conway polynomial vanishes only on finitely many twist knots.

1 Introduction and result

Vassiliev invariants have in recent years attracted knot theorists' interests because of their important connections to various other fields of matheamtics, see for example [BS]. It turns out, however, that these invariants, although by [BL] directly related to the knot polynomials, offer little understanding about the topology of knots. As an example, a discussion of this problematics w.r.t. knot cobordism can be found in [Ng].

In [St] I initiated a search for some connections between Vassiliev invariants and the genus of a knot, at least in the cases where latter can be handled combinatorially. *Inter alia*, therein I proposed the following problem:

Problem. Is there a (non-trivial) primitive Vassiliev knot invariant vanishing on genus one knots?

Primitive Vassiliev invariants are those which are additive under connected sum, that if v is primitive iff for any two knots $K_{1,2}$, we have $v(K_1\#K_2) = v(K_1) + v(K_2)$. They are of fundamental importance because all Vassiliev invariants are isomorphic to the symmetric algebra of the primitive ones, see [BN].

The aim of this note is to give a negative answer to the above problem for the Vassiliev invariants coming in the usual way from the Conway polynomial ∇ [Co]. In fact, we have the following

Theorem 1.1 Any non-trivial primitive Vassiliev invariant expressible as a polynomial in the coefficients of the Conway polynomial vanishes only on finitely many twist knots (i. e., knots with Conway notation $C(n,2)$, $n \in \mathbf{N}$).

The reason we hope this to be of some interest is that the picture changes strikingly when modifying the conditions.

Fact 1.1 (see [St2]) Let $\tilde{K}_g$ be the set of knots having diagrams on which the Seifert algorithm renders a Seifert surface of genus at most g. Then there are more than polynomially many in the degree linearly independent primitive Vassiliev invariants vanishing on $\tilde{K}_g$.

Fact 1.2 Let K_g be the set of knots of genus at most g. Then there are infinitely many Vassiliev invariants of the Conway polynomial vanishing on K_g for any $g \in \mathbf{N}$, namely its coefficients (see [BL, BN, Ro]).

2 The proof

To prove our theorem, we start by a number theoretic lemma, whose idea of proof is due to Lev Borisov.

Lemma 2.1 For any $k \in \mathbf{N}$ there is a $0 \neq m \in \mathbf{Z}$ such that the power series $1 + mx^k + O(x^{k+1})$ can be written as a product of the polynomials $\{1 + ax : a \in \mathbf{Z}\}$ (regarded formally as power series with almost all coefficients zero) and their inverse series $\{\, 1/(1+bx) = 1 - bx + (bx)^2 - \dots : b \in \mathbf{Z} \,\}$.

Proof. The statement is equivalent to asking for integers $\{a_i\}$ and $\{b_j\}$ such that the coefficients of x^n in $\prod_i (1 + a_i x)$ and $\prod_j (1 + b_j x)$ do not differ for $n < k$ but do so for $k = n$.

As the elementary symmetric polynomial of degree n in the a_i (resp. b_j) is equivalent modulo a function of the elementary symmetric polynomials of lower degree to $\sum_i a_i^n$ (resp. $\sum_j b_j^n$), we look for numbers a_i and b_j with $\sum_i a_i^n = \sum_j b_j^n$ for $n < k$ but inequality for $k = n$.

Now, consider the linear equation system with k equations and k variables c_s:

$$\sum_{s=1}^{k} c_s s^n = \begin{cases} 0 & 1 \leq n < k \\ p & k = n \end{cases} \tag{1}$$

The matrix of the equation system is the Vandermonde matrix, which, as well-known, is regular. Hence, letting p to be a multiple of its determinant, we obtain (by Cramer's rule) integer solution for c_s. Then, bring the negative c_i to the other side of (1), and reinterpreting the term $c_s s^k$ as the contribultion of c_s copies of $a_i := s$ (resp. $b_j := s$), you obtain the desired collection of integers $\{a_i\}$ and $\{b_j\}$. □

Remark 2.1 The number of factors of this construction is clearly very large. Lev Borisov conjectures (using algebraic geometric heuiristics) that the minimum number of factors required will be no more than a quadratic function in k.

Proof of theorem. When a primitive Vassiliev invariant v vanishes on all twist knots then, using their Conway polynomials (which have the form $\nabla(z) = 1 + nz$, $n \in \mathbf{Z}$), we see that v is a polynomial in $v_0, v_2, \dots, v_k$ for some even integer k, where $v_i = [\nabla(z)]_{z^i}$ is the coefficient of z^i in ∇, which vanishes on tuples of the form $(1, n, 0, 0, \dots, 0)$.

Hence by primitivity of v this polynomial must vanish on the coefficient lists of arbitrary connected sums and Gousarov $(k+1)-$inverses[1] of such knots, and connected sums and Gousarov $(k+1)-$inverses correspond to multiplication and inversion of the tuples $(v_0, v_2, \dots, v_k)$, where each one is regarded as coefficient list of a formal power series $\sum_{i=0}^{k} v_i x^i + O(x^{k+1})$ in x modulo x^{k+1}.

But then the lemma says that v is a polynomial vanishing on tuples of the form $(v_0, v_2, \dots, v_k)$, in which any v_i runs over a rest class c_i modulo some number m_i, where both c_i and m_i may depend on $\{c_j, m_j : j < i\}$.

You can see this inductively over i, where the freedom to change v_{2i} by a multiple of some number m_i, fixing all v_j for $j < i$, is given by multiplying the power series with powers of the product whose existence is shown in the lemma for $k = i$ (or on the level of knots, taking the connected sum with multiple copies of the corresponding knots with factors twist knots and their Gousarov inverses).

But a polynomial vanishing on tuples $(v_0, v_2, \dots, v_k)$ that vary in this way, is zero (again you see it inductively over i, using that a polynomial with infinitely many zeros is zero), and hence so would be v.

Therefore, v does not vanish on some twist knot, and then it does not either on almost all twist knots by Trapp's observations [Tr], because in Trapp's terminology twist knots form a 'twist sequence', and v beahves polynomially on this sequence, but a non-trivial polynomial has only finitely many zeros. □

[1] In [G], Gousarov proved that for any $k \in \mathbf{N}$ and any knot K there is a knot $K^{-1,k}$, called K's Gousarov $k-$inverse, such that $K\#K^{-1,k}$ cannot be distinguished from the unknot by any Vassiliev invariant of degree less than k.

Remark 2.2 Although the most natural one, it is not clear whether building a polynomial is the only way to produce further Vassiliev invariants out of the coefficients of the Conway polynomial, and so it would be interesting to extend the theorem to *arbitrary* (not just polynomial) dependence of v on the v_i.

Moreover, Gousarov's proof of existence of $k-$inverses is rather difficult, so it would be favourable to avoid also their use.

However, the argument presented here does not seem to work to prove this improvement of the result. To avoid polynomial dependence, we would need to have always $m = 1$ in the lemma 2.1.

But one can see that a polynomial P (over $\mathbf{R}$), whose top degree coefficients are $1, 0, m, \dots$ with $m \geq 0$ always has a complex zero, unless $m = 0$ and P is a monomial. Namely, if $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ were all roots of P, then $\sum \lambda_i = 0$ and $\sum_{i \neq j} \lambda_i \lambda_j = m \geq 0$, whence

$$0 = \Big(\sum_i \lambda_i\Big)^2 = \sum_i \lambda_i^2 + 2 \sum_{i \neq j} \lambda_i \lambda_j \geq 2 \sum_{i \neq j} \lambda_i \lambda_j = 2m \geq 0\,,$$

which is only possible for $m = \lambda_1 = \dots = \lambda_n = 0$.

So without use of Gousarov inverses we would already fail for $k = 2$. But even taking inverse series, one can calculate by similar but slightly longer arguments that if the series $1 + mx^3 + O(x^4)$ is a product of the kind we need, then m must be even, so we cartainly fail for $k = 3$.

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