ON FINITENESS OF VASSILIEV INVARIANTS AND A PROOF OF THE LIN-WANG CONJECTURE VIA BRAIDING POLYNOMIALS

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Abstract. Using the new approach of braiding sequences we give a proof of the Lin-Wang conjecture, stating that a Vassiliev invariant v of degree k has a value $O_v(c(K)^k)$ on a knot K, where $c(K)$ is the crossing number of K and O_v depends on v only. We extend our method to give a quadratic upper bound in k for the crossing number of alternating/positive knots, the values on which suffice to determine uniquely a Vassiliev invariant of degree k. This also makes orientation and mutation sensitivity of Vassiliev invariants decidable by testing them on alternating/positive knots/mutants only.

We give an exponential upper bound for the number of Vassiliev invariants on a special class of closed braids.

Keywords: mutation, orientation, alternating knot, positive knot, Vassiliev invariant, braid.

1 Introduction

Vassiliev invariants [BL, BN, BN2, BS, Va, Vo] are a class of knot invariants, which can be associated in many ways to polynomials. One way is to think of singularity resolutions as a way to differentiate a knot invariant and in this setting the Vassiliev invariants are (as polynomials) functions with vanishing derivative.

Another such similarity was conjectured by Lin and Wang [LW] and proven by Bar-Natan [BN3], see also Stanford [S]. It asserts that (the values of) Vassiliev invariants are polynomially bounded in the crossing number of knots.

Theorem 1.1 ([BN3]) Let v be a Vassiliev invariant of degree k. Then $\exists C > 0$ with

$$\max_{c(K)\le n} |v(K)| \le C\cdot n^k,$$

where $c(K)$ is the crossing number of the knot K.

A further similarity (which is an extension of the first one) is the approach of braiding sequences and braiding polynomials. It was initiated in a special case in [Tr] and later rediscovered and generalized in [St]. It provides a method to study Vassiliev invariants via their polynomial behaviour on certain sequences of knots. This approach works directly on knots and so it is a counterpart to the classical diagrammatic approach.

The approach of braiding sequences has the advantage that it easily brings special knot features into the game and so it led to quick proofs of several results on Vassiliev invariants on special classes of knots [St2, St4], whereas the diagrammatic approach meets considerable difficulties in such cases. (See Birman and Trapp [BT] for such partial results for closed 3 braids.) Furthermore, it allows us to work on fields of non-zero characteristic.

On the other hand, although telling very much about the limits of Vassiliev invariants, it fails to say much about their actual capabilities in these special cases.

One motivation for the present note is to join the two similarities explained above. It appears appealing to think that the "polynomial" approach of braiding sequences can be used to prove the "polynomial" statement theorem 1.1. Here we shall do so and give an independent proof from [BN3] using our approach.

However, the result for itself will hardly justify the interest of a new proof. The motivation for proposing it, is that it produces a series of new aside consequences, some of which extend results of Stanford [S], Birman and Lin [BL] and [St2], and which may be considered as more important than the proof itself.

In particular, our method will give us the possibility to give a linear (in the degree of the invariant) upper bound for the braid index and a quadratic upper bound for the crossing number of alternating (or positive) knots, which suffices to determine uniquely a Vassiliev invariant, making its orientation and mutation sensitivity decidable by testing it on alternating (or positive) knots/mutants only. That this is possible, does not follow neither from the Kontsevich-Drinfel'd [Ko, Dr], nor from the Birman-Lin [BL] approach.

Acknowledgement. I would wish to thank to D. Bar-Natan, on whose webpage I learned about the Lin-Wang conjecture, and to R. Trapp for sending me a reprint of [Tr]. I would also wish to thank to the referee for some helpful remarks.

2 Braiding polynomials and recursive relations

The technique of recursive relations we apply to prove the theorem was outlined in [St] and set forth in [St2] to prove exponential upper bounds for the number of Vassiliev invariants on knots with bounded braid index and arborescent knots. We hope to be able to extend this result in a subsequent paper to all knots, giving the first (in our opinion) main application of the braiding sequences approach.

Definition 2.1 For some odd $k \in \mathbf{Z}$, a k-braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k (see figure 1). A braiding sequence $\mathcal{S}_{D,P}$, associated to a numbered set P of crossings in a diagram D (all crossings by default, if P is omitted) is a family of diagrams, parametrized by $|P|$ odd numbers $x_1,\dots,x_{|P|}$, each one indicating that at crossing number i an x_i-braiding is done. So the diagram $S_{D,P}(x_1,\dots,x_{|P|}) \in \mathcal{S}_{D,P}$ indexed by $(x_1,\dots,x_{|P|})$ is the one obtained from D by doing a x_i-braiding of the crossing numbered by i for all $1 \le i \le |P|$.

According to [Tr] (for one variable) and [St] (for the general case), any Vassiliev invariant v of degree k behaves on a braiding sequence $\mathcal{S}_{D,P}$ as a polynomial of degree[1] at most k in $x_1,\dots,x_{|P|}$, and this polynomial is called braiding polynomial $L_{D,P}(v)$ of v on this braiding sequence. Formally

$$L_{D,P}(v)(x_1,\dots,x_{|P|}) := v\big(S_{P,D}(x_1,\dots,x_{|P|})\big) \in \mathbf{Q}[x_1,\dots,x_{|P|}]\,.$$

Remark 2.1 The definition 2.1 of a braiding sequence is more restrictive than the one we used in [St]. However, the braiding sequences considered here are simplest and most interesting ones and suffice for doing what we intend, so, at least here, it does not appear useful to reintroduce the concept in full generality.

Remark 2.2 Figure 1 illustrates 2 different ways to perform a braiding – a parallel and an antiparallel one, so that on a diagram D of n crossings, there are *a priori* 2^n possible ways to build up braiding sequences. Except for the result on braids (proposition 3.2), there will be of no importance which one we use, but let us anyway for conformity assume to apply everywhere parallel braidings (which will preserve the property of a diagram to be a closed braid diagram).

[1] Here the *total* degree, and not the maximal of the degrees w.r.t. any of the variables, is meant, e.g. $\deg x_1x_2 = 2$ and not 1.

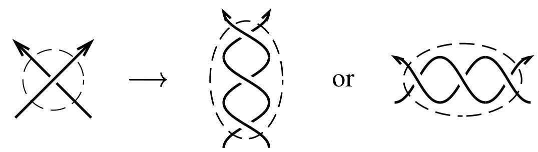
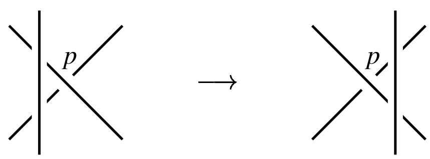


Figure 1: Two ways to do a -3-braiding at a crossing.

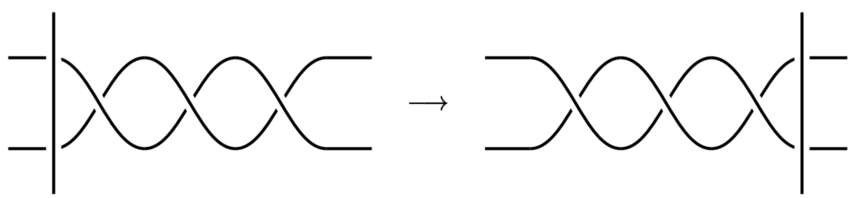
We start by explaining the idea of recursive relations. For this we need two more definitions.

Definition 2.2 A simplification of a diagram D is a sequence of Reidemeister moves applied on this diagram, such that the crossing number of the diagram D before the first move is greater than the crossing number of the diagram after the last move.

Definition 2.3 A simplification of a diagram D commuting with the braiding in a crossing p of D is a simplification of D, such that p is involved only in Reidemeister III moves and each such move can be replaced by three new Reidemeister III moves after a t_2 move is applied to p. Pictorially,



before the t_2 move at p turns to



after the t_2 move (the strand passing from left to right may also be below the rest of the diagram fragment).

Consider now a diagram D and let B be a set of crossings of D such that fixing the sign of these crossings (i.e., switching them in an appropriate way) we can simplify D by Reidemeister moves to a diagram D', such that this procedure commutes with the braidings in all other crossings in D outside of B (in particular, these crossings are not removed in D'). Then the set B in D has reduced to some smaller (possibly empty) set B' in D', and for any Vassiliev invariant v we have an identity between specializations of $L_D(v)$ and $L_{D'}(v)$:

$$L_D(v)(x_1,\dots,x_{|D|})\Big|_{\{x_i=\pm1\}_{i\in B}} = L_{D'}(v)(x'_1,\dots,x'_{|D'|})\Big|_{\{x'_i=\pm1\}_{i\in B'}}. \tag{1}$$

Here the x_i and x'_i are to be chosen between 1 and -1 according to the writhe of the crossing they represent in D resp. D'.

A relation of the kind (1) we will call a *recursive relation*. It allows in particular to relate coefficients of L_D and $L_{D'}$ to each other.

The basic idea to prove theorem 1.1 is to prove that if D has n crossings and v is a Vassiliev invariant of degree k, then the coefficient $[L_D(v)]_X$ for a monomial X of degree $k-l$ is $O_v(n^l)$ with 'O_v' depending on v only. This will be established in lemma 2.2 (the appalling denominator appearing there is a technicality needed to facilitate the estimates).

For this we proceed by induction on the complexity (=crossing number) of D and we use estimates on the coefficients of $L_D(v)$ coming from recursive relations.

The first rather technical lemma we prove is needed to ensure that, when starting the induction at sufficiently complicated diagrams D, there are always enough recursive relations applicable to estimate *all* coefficients of $L_D(v)$ from such of $L_{D'}(v)$ for simpler D', where the property we require we can assume to hold by induction.

Lemma 2.1 $\forall k \geq 2 \exists n_0 \forall n \geq n_0$ and for all diagrams D of $k+n$ crossings and each choice A of (a set of) not more than k crossings in D there exists a set B of crossings in D with $B \cap A = \varnothing$ and the following property: fixing appropriately the crossings in B (and the value of the corresponding variables in the braiding polynomial $L_D(v)$ of some Vassiliev invariant v associated to D) we can achieve a simplification (i. e. reduction of the crossing number of D by Reidemeister moves) commuting with the braidings in the crossings outside of B, such that *either*

1) You have a loop with p crossings on it (including the one self-crossing of the loop), no one of them being in A, and you reduce the number of crossings of D from $n+k$ to $n+k-p$, where $p \leq 2k$, *or*
2) You have a region to apply a reducing Reidemeister II move, where on the segments there are p resp. q crossings, with $p \leq 2k-1$, and by fixing these $(p+q)$ crossings and the 2 mutual crossings of the segments you reduce the number of crossings of D lying on these segments from $p+q+2$ to $2\min(p,q)$.

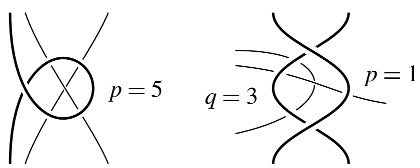


Figure 2: The 2 cases of lemma 2.1. The numbers attached at the segments indicate the number of crossings of each segment with other parts of the solid line (which are drawn with a thinner line). In the first case the self-crossing is counted. On all 3 segments there is no crossing in A.

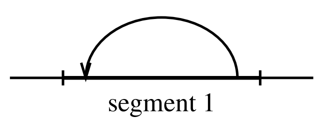
Proof. Consider the Gauß diagram [PV, FS] of the knot diagram D of k arrows of crossings in A (called distinguished arrows) and n arrows of crossings in outside of A (called non-distinguished arrows).

Then the $2k$ endpoints of the distinguished arrows separate the solid line into $2k$ segments of non-distinguished basepoints. Their total length (where the length of a segment is the number of basepoints on it) is the number $2n$ of non-distinguished basepoints.

So there is a segment (called segment 1) with len $\geq {}^n\!/_k$ basepoints on it.

Let $n \geq n_0 := 2k^2 - k + 1$. Then ${}^n\!/_k > 2k-1$ and len(segment 1) $\geq 2k$. First assume it is more than $2k$. Consider now only the leftmost $2k+1$ basepoints on segment 1. In segment 1 then you have *either*

1) an arrow a with both basepoints among the leftmost $2k+1$ basepoints on segment 1



i. e., you have a loop as in case 1 of lemma 2.1 with $p = \mathrm{len}(a) \leq 2k$ crossings on it. Choose the arrow to be minimal, i. e. not to enclose another arrow with the same property. Fix these p crossings so that the loop is above the rest of the diagram. Then the loop can be eliminated (you reduce D by p crossings) and you land in case 1. *or*

2) all leftmost $2k+1$ basepoints on segment 1 are of different arrows. Then there is one of the other $2k-1$ segments or the remaining righter part of segment 1 (called segment 2) with ≥ 2 basepoints of the left-most $2k+1$ arrows ending on segment 1. If > 2 such arrows end on segment 2, then choose 2 such that there is not a third one between them (i. e., with *both* basepoints between those of the other two on both segment 1 and 2, see figure 3). Clearly, on segment 1 the endpoints of both chords have $\leq 2k-1$ other endpoints between them (see figure 4).

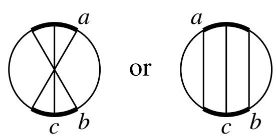


Figure 3: The chord c is between a and b w.r.t. to marked segments.

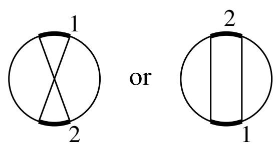


Figure 4: The position of the chords (=arrows with both orientations allowed) in the Gauß diagram in case 2 of lemma 2.1. The thickened parts do not contain a basepoint of a marked arrow, and one of them has $\leq 2k-1$ basepoints on it.

Change the $p+q+2$ crossings to pull above the rest of the diagram, move the arc with higher number (that is $\max(p,q)$; if $p=q$ it doesn't matter which one) of crossings towards the one with lower number of crossings and perform a Reidemeister II move (see figure 5). Now on your segments you have reduced the $p+q+2$ crossings before to $2\min(p,q)$ (note that there can be more crossings within the region bounded by the arcs in the original diagram, which, however, are not affected by this procedure).

If len(segment 1) $= 2k$, then consider the $2k$ basepoints on segment 1 and apply the same arguments, except that in case 2 there remain only $2k-1$ segments for $2k$ arrows. □

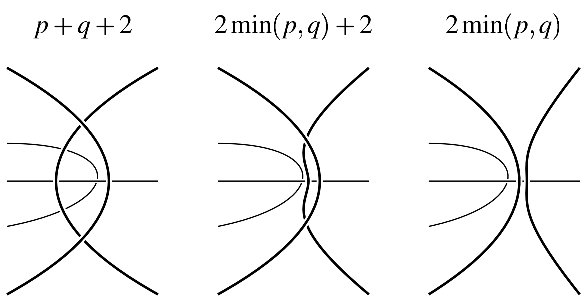


Figure 5

Remark 2.3 If the segment with q crossings on it (called for simplicity q as well) is knotted itself (i. e., it has self-crossings; note that by construction neither p has self-crossings nor there is another crossing of p and q). Then the number q counts also self-crossings (even twice). In that case, after the $p+q+2$ crossing fixes all the self-knotting of q can be trivialized and so you can eliminate even more crossings than you would do (with the same numbers p and q), if there was no self-knotting of q. Therefore we may think of q as being unknotted.

Once we have ensured the existence of recursive relations, the rest is just a proper choice of bounds and an (somewhat tedious but) elementary algebraic calculation.

Lemma 2.2 Let v be a Vassiliev invariant of $\deg = k$. Then $\exists C > 0 : \forall D$ diagram of $|D| = n$ crossings ($\forall n$) we have

$$\big|[P_D]_X\big| \leq \frac{C \cdot n^{k-\deg X}}{(12k^2 2^k)^{\deg X}},$$

where $P_D = L_D(v)$ is the braiding polynomial of v on the braiding sequence associated to D (recall, that this is a polynomial of deg $\leq k$ in n variables $x_1, \dots, x_n$), X is a monomial in $x_1, \dots, x_n$ of degree $\leq k$, and $[P]_X$ is the coefficient of X in P.

Proof. As for $k = 0, 1$, v is constant, assume always in the sequel $k \geq 2$. Fix n_0 from lemma 2.1 for the given k. For $n < n_0$ such C exists (as you have only finitely many braiding polynomials and coefficients). We will show that this C does the job for all higher n, inductively over n and for fixed n inductively top-down over $m = \deg X$ (starting with $m = k$).

In the course of the proof we may need to augment our n_0 (depending on k, which is fixed), to make our n sufficiently large, but this does not spoil the argument.

Let D be a diagram of n crossings, $n \geq n_0$. Introduce a map

$$\Phi : \{ \text{ monomials in } x_1, \dots, x_n \} \longrightarrow \mathcal{P}(\{1, \dots, n\})$$

by $\Phi(x_i) := \{i\}$ and $\Phi(a \cdot b) := \Phi(a) \cup \Phi(b)$. Φ just indicates which indices are contained in the monomial, e. g. $\Phi(x_1^2 x_2 x_5 x_6) = \{1, 2, 5, 6\}$.

Consider a monomial X in P_D. We have $|\Phi(X)| \leq k$. Fix crossings corresponding to $A := \Phi(X)$ and choose B from lemma 2.1.

1. Case. You have in the Gauß diagram a non-linked arrow[2] (i. e., case 1 of lemma 2.1, i. e. a crossing in the diagram, bounding a loop with no crossing of A on it). Then from lemma 2.1 we have $B = \{$ crossings on loop $\}$ with $|B| = p \leq 2k$ and

$$P_{D'} = P_D\Big|_{\{x_i = \pm 1\}_{i \in B}},$$

and so

$$\big|[P_D]_X\big| \leq \big|[P_{D'}]_X\big| + \sum_{X' \succ_B X} \big|[P_D]_{X'}\big|,$$

where $X' \succ_B X$ means that X'/X is a non-trivial monomial and contains only variables with indices in $\Phi(B)$. This is, because

$$[P_{D'}]_X = \sum_{X' \succeq X} \pm [P_D]_{X'},$$

(as all x's with indices in $\Phi(B)$ are set to ± 1 in P_D).

Note, that

$$\big|\{ X' \succ X : \deg(X'/X) = l \}\big| = \binom{l + |B| - 1}{l},$$

with $|B| = p \leq 2k$. Let $m = \deg X$. Then

$$\begin{aligned}\big|[P_D]_X\big| \quad \leq \quad & C \cdot \frac{(n-p)^{k-m}}{(12k^2 2^k)^m} + p \cdot C \cdot \frac{n^{k-m-1}}{(12k^2 2^k)^{m+1}} + \binom{p+1}{2} \cdot C \cdot \frac{n^{k-m-2}}{(12k^2 2^k)^{m+2}} + \\ & \dots + \binom{p+k-m-1}{k-m} \cdot C \cdot \frac{n^0}{(12k^2 2^k)^k}.\end{aligned}$$

[2] This terminology is used by Fiedler [Fi2, FS].

Now by $\frac{n}{2} < n - 2k \leq n - p$

$$\frac{n^{k-m-l}}{2^{k \cdot l}} \leq (n-p)^{k-m-l}$$

and as $\binom{p+l-1}{l} \leq p^l$ you obtain

$$\begin{aligned} \left|[P_D]_X\right| &\leq \frac{C}{(12k^2 2^k)^m} \cdot \left[(n-p)^{k-m} + p \cdot (n-p)^{k-m-1} + p^2 \cdot (n-p)^{k-m-2} + \dots\right] \\ &\leq \frac{C}{(12k^2 2^k)^m} n^{k-m}. \end{aligned}$$

For the next 3 cases assume case 2 in lemma 2.1, $p \leq 2k-1$. In this case for given X we have

$$P_{D'}\Big|_{\{x_i = \pm 1\}_{i \in B'}} = P_D\Big|_{\{x_i = \pm 1\}_{i \in B}},$$

for some B' with $|B'| < |B|$ and D' with $|D'| < |D|$, where D' is the simplified diagram resulting from D after fixing the crossings in B (again the one yielded by lemma 2.1 for $A := \Phi(X)$) and B' are the crossings in D', which have not been eliminated after the simplification. Then

$$\left|[P_D]_X\right| \leq \left|[P_{D'}]_X\right| + \sum_{X' \succ_{B'} X} \left|[P_{D'}]_{X'}\right| + \sum_{X' \succ_B X} \left|[P_D]_{X'}\right|.$$

As $|B'| < |B|$ and $|D'| < |D|$ we will use the upper estimate for the third term as well as for the second one.

2. Case. $q \leq 4k+1$. Then by $p+q+2 \leq 6k+2$ crossing changes lead to a simplification of D by ≥ 2 crossings. So in the same manner as above we have

$$\begin{aligned} \left|[P_D]_X\right| &\leq C \cdot \frac{(n-2)^{k-m}}{(12k^2 2^k)^m} + 2\left\{(6k+2) \cdot C \cdot \frac{n^{k-m-1}}{(12k^2 2^k)^{m+1}} + \binom{6k+3}{2} \cdot C \cdot \frac{n^{k-m-2}}{(12k^2 2^k)^{m+2}} + \right. \\ &\quad \left. \dots + \binom{7k-m+1}{k-m} \cdot C \cdot \frac{n^0}{(12k^2 2^k)^k}\right\} \end{aligned}$$

Now, as $\frac{n}{2} < n-2$, we have $2n^{k-m-l} < 2^{k \cdot l}(n-2)^{k-m-l}$, so

$$\begin{aligned} \left|[P_D]_X\right| &\leq \frac{C}{(12k^2 2^k)^m} \cdot \left[(n-2)^{k-m} + (6k+2) \cdot \frac{(n-2)^{k-m-1}}{12k^2} + \binom{6k+3}{2} \frac{(n-2)^{k-m-2}}{(12k^2)^2} + \dots\right] \\ &\leq \frac{C}{(12k^2 2^k)^m} n^{k-m}. \end{aligned}$$

3. Case. $q > 4k+1$ and $q < \frac{n}{2}$. By $p+q+2 \leq q+2k+1$ crossing changes we transform our diagram D of n crossings into one of $n-q+p+2 \leq n-q+2k+1$ crossings (note that $q > 2k \geq p$).

We have as before

$$\begin{aligned} \left|[P_D]_X\right| &\leq C \cdot \frac{(n-q+2k+1)^{k-m}}{(12k^2 2^k)^m} + 2\left\{C \cdot (q+2k+1) \frac{n^{k-m-1}}{(12k^2 2^k)^{m+1}} + C \cdot \binom{q+2k+2}{2} \cdot \right. \\ &\quad \left. \cdot \frac{n^{k-m-2}}{(12k^2 2^k)^{m+2}} + \dots + C \cdot \binom{q+3k-m}{k-m} \frac{n^0}{(12k^2 2^k)^k}\right\}. \end{aligned} \tag{2}$$

Now as $q \geq 4k+2$ we have $\frac{q+2k+1}{3} \leq q-2k-1$, so

$$\frac{1}{3^l}\binom{q+2k+l}{l} \leq (q-2k-1)^l,$$

and as $q < \frac{n}{2}$, also

$$n < 2(n-q) < 2(n-q+2k+1) \Rightarrow n^{k-m-l} < 2^{k-m-l}(n-q+2k+1)^{k-m-l} \leq 2^{k\cdot l}(n-q+2k+1)^{k-m-l}.$$

The rest of the denominator, namely $\left(\frac{12k^2 2^k}{3\cdot 2^k}\right)^l = 4^l k^{2l}$, use to get rid of the '2' before the braces in (2), and so

$$\begin{aligned}\left|[P_D]_X\right| \quad &\leq \quad \frac{C}{(12k^2 2^k)^m}\cdot\Big[(n-q+2k+1)^{k-m} + (q-2k-1)(n-q+2k+1)^{k-m-1} + \\ &\qquad (q-2k-1)^2(n-q+2k+1)^{k-m-2} + \ldots + (q-2k-1)^{k-m}\Big] \leq \frac{C}{(12k^2 2^k)^m} n^{k-m}.\end{aligned}$$

4. Case. $q \geq \frac{n}{2}$. Clearly, $q \leq 2(k+n)$, so at fixed $k \geq 2$ for n sufficiently large $q < 2k(n-1)-1$. Then use $\frac{q+2k+1}{2k} < n$ in (2) (and $2^k \geq 2$ to get rid of the '2' before the braces) to obtain

$$\left|[P_D]_X\right| \leq \ldots \leq \frac{C}{(12k^2 2^k)^m}\cdot\left[(n-q+2k+1)^{k-m} + n\cdot\frac{n^{k-m-1}}{6k} + n^2\cdot\frac{n^{k-m-2}}{(6k)^2} + \ldots\right].$$

Now, as $q \geq \frac{n}{2}$, for $n > 12k+6$ we have $n-q+2k+1 < \frac{2n}{3}$, and so

$$\left|[P_D]_X\right| \leq \frac{C}{(12k^2 2^k)^m}\cdot\underbrace{\left[\left(\frac{2}{3}\right)^{k-m} + \sum_{i=1}^{k-m}\frac{1}{(6k)^i}\right]}_{=:\,\mu_m} n^{k-m}$$

Then $\mu_k = 1$ and for $m < k$

$$\mu_m \leq \frac{2}{3} + \frac{1}{6k-1} \leq \frac{2}{3} + \frac{1}{5} < 1,$$

and so you are done. □

3 The proof and more theorems

With lemma 2.2 we have done the main work.

Proof of theorem 1.1. Let v be a Vassiliev invariant of $\deg = k$. Then by lemma 2.2

$$\exists C : \left|[P_D]_X\right| \leq C\cdot n^{k-\deg X}$$

for all D diagram of n crossings and X monomial in $x_1,\ldots,x_n$. Then D corresponds to some choice of parameters ± 1 (according to the writhe of its crossings) in the braiding sequence, associated to itself, and, using that the number of monomials in $x_1,\ldots,x_n$ of degree $\leq d$ is $\binom{n+d-1}{d}$, we have

$$\begin{aligned}|v(D)| = |P_D(\pm 1,\ldots,\pm 1)| &\leq \sum_{d=\deg X\leq k}\left|[P_D]_X\right| \leq \\ &\sum_{d=0}^{k}\underbrace{\binom{n+d-1}{d}}_{\leq n^d}\cdot C\cdot n^{k-d} \leq \underbrace{C\cdot(k+1)}_{C'}\cdot n^k = C'\cdot n^k.\end{aligned}$$

So, setting $C' := C\cdot(k+1)$, we are done. □

Remark 3.1 Theorem 3 of [BN3] follows in the same way by building difference sequences of P_D according to the variables whose corresponding crossings we make singular.

But now we can prove a little more using our method.

Theorem 3.1 Any Vassiliev invariant of degree $\leq k$ is uniquely determined by its values on the braiding sequences associated to diagrams with $\leq 2k^2$ crossings.

Proof. It suffices to prove, that a Vassiliev invariant vanishing on these braiding sequences is zero at all. For this just note, that we can choose $C = 0$ in the proof of lemma 2.2 (without having to make any case distinction or augmenting assumptions for n_0). □

Corollary 3.1 Any Vassiliev invariant of degree $\leq k$ is uniquely determined by its values on alternating knots with $\leq 2k^2 + 2k$ crossings.

In the terminology of [St2], this means that Vassiliev invariants are finitely-determined on all knots.

Imposing only alternation, the result is due to Stanford [S2] (see also [St, St5]). Imposing only a quadratic upper bound for the crossing number, the result should follow from Birman and Lin [BL]. So it is important the we here link both conditions.

Corollary 3.2 Any Vassiliev invariant of degree $\leq k$ is uniquely determined by its values on positive knots with $\leq 2k^2 + 2k$ crossings.

Caution: as the existence of minimal positive diagrams of positive knots is not yet shown (see [St3, question 5.1]), here a "positive knot with $\leq n$ crossings" means a positive knot with a positive diagram of $\leq n$ crossings.

Proof of corollaries 3.1 and 3.2. Both corollaries follow from theorem 3.1 and the arguments used in the proof of theorem 4.1 of [St2]. □

As a consequence, the space of Vassiliev invariants of degree $\leq k$ is finite-dimensional. This fact is classic, but our proof is independent from [Ko, Dr, BN] or [BL]. This result means in particular, that we can (well, at least theoretically) decide, completely without use of chord diagrams, by applying the invariant on finitely many knots, whether it detects orientation (that is, the Vassiliev invariant $v(K) - v(-K)$ is zero or not).

Unfortunately, in the way it is, corollary 3.1 gives an unattractive upper bound for this dimension, namely C^{k^2} for some $C > 1$, as it is known [Ad, §2.1], that the number of alternating knots grows exponentially in the crossing number. So the testing procedure is not yet economical.

The challenge of future work is to reduce the maximal crossing number in corollary 3.1 to something linear in k.

However, unlike in [BL], we proved that it suffices to consider alternating or positive knots only. Furthermore, applying it for tangles, the argument can be used to prove that we can decide, entirely on the level of knots, and completely without use of chord diagrams and the related heavy machinery of Kontsevich's integral [Ko] or a Drinfel'd associator [Dr] whether a Vassiliev invariant detects mutation [Co, CDL, MC, MR].

Exercise 3.1 Use the above arguments to show that there exists a $C > 0$, s.t. if a Vassiliev invariant v of deg $\leq k$ does not distinguish any pair of knots with alternating (or positive) mutated diagrams with $\leq Ck^2$ crossings, then it does not detect mutation at all.

A similar argument can be used to prove the following additivity test:

Exercise 3.2 Show, that if a Vassiliev invariant v of deg $\le k$ satisfies $v(K_1\#K_2) = v(K_1) + v(K_2)$ for any pair K_1, K_2 of alternating (or positive) knots with $\le 2k^2 + 2k$ crossings, then it is additive at all.

A further consequence (which is provable also by [Ko] or [BL]) is, that rational Vassiliev invariants are modulo scaling the same as integral Vassiliev invariants.

Proposition 3.1 For any $\mathbf{Q}$-valued Vassiliev invariant v of degree k there exists a $n \in \mathbf{Z}$, s.t. nv is a $\mathbf{Z}$-valued Vassiliev invariant.

Proof. Choose n so that the braiding polynomials of nv associated to diagrams of $\le 2k^2$ crossings have integral coefficients. Then by the proof of lemma 2.2, integrality is inherited by all coefficients of all other braiding polynomials of nv. $\square$

Finally, we have a divisibility property for Vassiliev invariants.

Theorem 3.2 If a $\mathbf{Z}$-valued Vassiliev invariant v of degree k is divisible by some $p \in \mathbf{N}$ on alternating (or positive) knots with $\le 2k^2 + 2k$ crossings, then it is divisible by p on all knots.

Proof. For p prime consider v mod p as a Vassiliev invariant over[3] $\mathbf{Z}_p$ and use corollaries 3.1 and 3.2. For any other p argue inductively over its prime factorization. $\square$

The last 2 results imply the following

Corollary 3.3 If a $\mathbf{Q}$-valued Vassiliev invariant of degree k is integral on alternating (or positive) knots with $\le 2k^2 + 2k$ crossings, then it is integral-valued on all knots.

Proof. Choose for v the number n from proposition 3.1 and apply theorem 3.2 with $p := n$ to nv. $\square$

Remark 3.2 By a similar argument one can show that any Vassiliev invariant of degree $\le k$ is uniquely determined by its values on knots of braid index $\le 2k$, and deduce from that corollary 3.1 in an independent way using the results of [St2]. The argument is: if you have $> 2k$ strands, in which way you ever choose k crossings in the braid, there is always one strand with no chosen crossing on it. Then, fixing all its crossings, pull it to the top and remove the strand, reducing the strand number. (You can assume w. l. o. g., that the induced permutation of the braid is $(1\,2\dots n)$ and then you do not even create a new crossing, while pulling the strand to the top, and preserving this kind of induced permutation after reducing the strand number.) Note, however, that this argument does not extend to (all) tangles.

Using this argument, we finish by an extension of the exponential upper bound results of [St2].

Proposition 3.2 Vassiliev invariants are exponentially bounded on all knots, which are closed braids β with

$$\beta = \prod_{i=1}^{k} \sigma_{p_i}^{r_i}, \qquad k, p_i \in \mathbf{N}, r_i \in \mathbf{Z},$$

(σ_i denote the Artin generators) such that $|\{\, 1 \le i < k : p_{i+1} < p_i + 2 \,\}| < C$ for any given constant $C \in \mathbf{N}$.

Here "exponential upper bound on a class L of knots" means: there exists some $C > 1$ with

$$\dim\{\text{ Vassiliev invariants of degree} \le k \text{ on } L \,\} \le C^k.$$

Proof sketch. First note, that such β admit (a presentation as) "generalized" braid schemes of height (=number of rows) $\le C$. Here a braid scheme is the one introduced in [St2] and "generalized" means, that we no longer require in each row entries corresponding to generators (with indices) of given parity, but only that the mutual distance of any pair of indices in a row is ≥ 2. That is, β is a stack up of $\le C$ "blocks"

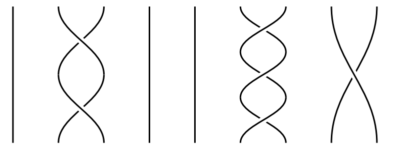


Figure 6: The block corresponding to $\sigma_2^{-2}\sigma_6^3\sigma_8^{-1}$

(of arbitrary but equal width) of the form shown on figure 6. Now use the argument in remark 3.2, and the observation, that removing a strand from such a braid does not augment its generalized scheme height, to deduce that it suffices to consider $\le 2n$ strands only, where n is the degree of the Vassiliev invariant. The number k of entries in such schemes (=number of generators) is then linearly bounded in n. The rest of the argument is as in the proof of theorem 4.1 of [St2]. □

Remark 3.3 An even much more immediate proof of the Lin-Wang conjecture would follow from the representability of any Vassiliev invariant of deg $\le k$ by a Gauß sum [PV, Fi, Fi2, FS] of degree k (where the degree of a Gauß sum is the maximal number of arrows appearing in any of the Gauß diagrams; it is straightforward to check that such invariants fulfill the assertion of theorem 1.1). It was recently announced by Polyak and Viro [GPV] a proof by Gousarov that any Vassiliev invariant has a Gauß sum formula. However, at present I do not know whether the above degree condition has been proved as well to be able to quote Gousarov's result as a third proof of the conjecture.

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