DECIDING MUTATION WITH THE COLORED JONES POLYNOMIAL

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Abstract. We show examples of knots with the same polynomial invariants, colored Jones polynomial and same hyperbolic volume, which are not mutants.

1. Introduction

Mutation was introduced by Conway [Co]. It is a procedure of altering a knot into a (possibly) other knot, which "resembles" the original one. Most of the common (efficiently computable) invariants coincide on mutants, and so mutants are difficult to distinguish. A basic exercise in skein theory shows that mutants have the same Alexander polynomial. The same argument applies to the later discovered Jones, HOMFLY, BLMH and Kauffman polynomial [J, F&, LM, BLM, Ka]. The cabling formula for the Alexander polynomial (see for example [Li, theorem 6.15]) shows also that Alexander polynomials of all cable knots of mutants coincide. While the Jones polynomial was known not to satisfy a cabling formula (because it distinguishes some cables of knots with the same polynomial), nontheless Morton and Traczyk [MT] showed that Jones polynomials of all cables of mutants are equal. A further coincidence result was proved by Lickorish and Lipson [LL] and Przytycki [P] for the HOMFLY and Kauffman polynomials of 2-cables of mutants. 3-cable HOMFLY polynomials can generally distinguish mutants (for example the K-T and Conway knot; see [CM]), but require a calculation effort that can hardly be considered reasonable. As a follow-up to the Morton-Traczyk result, Przytycki raised a point (see [Ki, problem 1.91(2)]) as to the possibility that the Jones polynomial of all cables might be a *complete* mutation invariant, i.e. distinguish all knots which are not mutants or their cables. Since the Jones polynomial of all cables is (equivalent to) what is now known as the "colored Jones polynomial", Przytycki's problem must be seen in the light of (its considerable impact on) new developments concerning this invariant.

The colored Jones polynomial plays a central role in several important recent problems in knot theory. It is conjectured, for example, that it determines the A-polynomial ("AJ-conjecture" [DG]), and the Gromov norm (Volume conjecture [MM]), and it was proved to determine the Alexander polynomial (Melvin-Morton-Rozansky Conjecture [BG]). Latter fact would follow alternatively from a full mutation invariance property of the colored Jones polynomial. Since Ruberman [Ru] showed that mutants have equal volume in all hyperbolic pieces of the JSJ decomposition, we would also obtain (a qualitative version of) the Volume conjecture. Such a property is also consistent with the AJ-conjecture and a recent result of Tillmann [Ti, corollary 3] on coincidence of factors of A-polynomials of mutants. Note also that the Volume conjecture[1], as well as the AJ-conjecture, in turn imply

[1]The second author [T] has shown that it is in fact sufficient the Volume conjecture to hold for doubled knots.

that the colored Jones polynomial (and hence Vassiliev invariants [BN]) detect the unknot.

In our talk we discussed several pairs of knots with equal colored Jones polynomial. In particular, with a pair of non-mutants, we have an answer to Przytycki's question.

2. Mutations and Przytycki's question

Consider a knot being formed from two tangles T_1 and T_2. A *mutation* is the following operation. Cut the knot open along the endpoints on each of the four strings coming out of T_2. Then rotate T_2 by π along some of the 3 axes - horizontal in, vertical in, or perpendicular to the projection plane. This maps the tangle ends onto each other. Finally, glue the strings of $T_{1,2}$ back together (possibly altering orientation of all strings in T_2).

We consider the following question due to J. Przytycki.

Question (Problem 1.91(2)[Ki]). Let K be a prime, simple, unoriented knot. Is there any knot, other than mutations of K, which cannot be distinguished from K by the Jones polynomial of K and its satellites?

The knots 14_{41721} and 14_{42125} from [HT] are depicted in Figure 1.

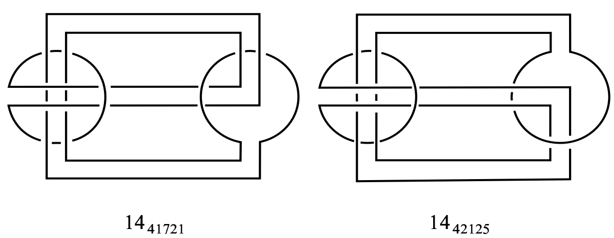

Figure 1

These are ribbon knots, and have trivial Alexander polynomial. 14_{41721} and 14_{42125} have the same HOMFLY and Kauffman polynomial invariants, the same (parallel) 2-cable HOMFLY polynomial, and the same hyperbolic volume. For this pair of knots, we can show the following.

Theorem 2.1. 14_{41721} *and* 14_{42125} *are not mutants.*

Theorem 2.2. 14_{41721} *and* 14_{42125} *have the same colored Jones polynomial.*

Theorems 2.1 and 2.2 combinedly give an answer to the question above.

The non-mutant status of this pair could be shown by calculation of Whitehead double HOMFLY polynomials. The coincidence of the colored Jones polynomial was established using the graphical calculus of Masbaum and Vogel [MV] (or see also [BHMV]). Details will be given in a forthcoming paper.

3. Undecided mutations

The pair we presented came up in the first author's project to determine mutations among low crossing knots in [HT]. Up to 13 crossings this task was completed by tracking down coincidences of Alexander, Jones polynomial and volume on the one hand, and then exhibiting the mutation in minimal crossing diagrams on the other hand. A (non-exhaustive) verification of 14 and 15 crossing knots turned up several more difficult cases (discussed in [St]), one of which provided the example we showed here.

Even more problematic is the pair 14_{41739} and 14_{42126}, and a number of pairs of 15 crossing knots, for which no mutations could be found in diagrams up to 16 crossings, but (along with all the invariants that do so for our example) Whitehead double HOMFLY polynomials were also found to coincide. The complexity of the 2-cable Kauffman polynomial makes its evaluation very difficult, and we succeeded only for three of the pairs, the 14 crossing knots and two 15 crossing pairs. The 2-cable Kauffman polynomials, too, failed to distinguish the knots. By a similar calculation to the proof of Theorem 2.2, we managed to verify that for the 14 crossing knots and one of the two 15 crossing pairs we mentioned, $(15_{148731}, 15_{156433})$, the colored Jones polynomials are also equal. Thus these pairs satisfy all polynomial coincidence properties known for mutants. Since we could not find diagrams exhibiting the mutation, this deepens the decision problem whether these knots are mutants or not.

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