WEAKLY SUCCESSIVELY ALMOST POSITIVE LINKS I: POLYNOMIAL INVARIANTS

TETSUYA ITO AND ALEXANDER STOIMENOW

Abstract. We introduce a weakly successively almost positive links as an extension of positive and almost positive diagrams and links, We prove various properties of polynomial invariants and signatures of weakly successively almost positive link extending previous results about positive or almost positive links, which includes answers of several open questions. We discuss their minimal genus and fibering property and for the latter prove a fibering extension of Scharlemann-Thompson's theorem (valid for general links).

1. Introduction

A link diagram is *positive* if all the crossings are positive, and a *positive link* is a link that can be represented by a positive diagram. Positive links have various nice properties and form an important class of links. In particular, reflecting the positivity of diagrams, it has been shown that various invariants of positive links are 'positive' in some sense.

An *k-almost positive diagram* is a diagram such that all but k crossings are positive, and a *k-almost positive link* is a link that is represented by a k-almost positive diagram. A 1-almost positive diagram (resp. link) are usually called *almost positive diagram* (resp. link). It is known that almost positive links share various properties with positive links. Thus it is fair to say that the notion of almost positive is an appropriate generalization of positive.

However, for $k \geq 2$, k-almost positive links fail to have nice properties similar to positive links. This is simply because every link is k-almost positive for sufficiently large k. For example, the figure-eight knot is 2-almost positive but it violates various properties of positive links.

Therefore it is natural to investigate more appropriate generalizations of positive diagrams. In [It3] the first-named author introduced a notion of *successively almost positive link* as a better generalization of positive links. This notion appeared (without name) in [IMT, Theorem 5.3], a joint work of the first author, designed so that the technique of constructing generalized torsion elements developed therein can be applied. In [It3] it is shown that successively almost positive links share various nice properties of positive links.

The aim of this paper is to introduce *weakly successively almost positive link* as a better and more appropriate generalization of positive links.

Definition 1.1. We say that a link diagram D is *weakly successively k-almost positive* if all but k crossings of D are positive, and the k negative crossings appear along a single overarc (see Figure 1). We call this the *negative overarc*.

2020 *Mathematics Subject Classification.* Primary 57K10; Secondary 57K14, 05C10, 57K33, 57K35.

A *weakly successively almost positive link* is a link which is represented by a weakly successively k-almost positive diagram for some k.

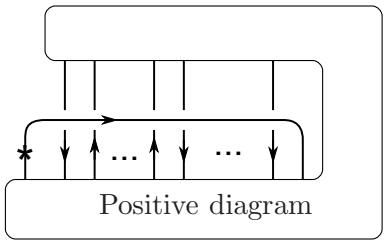


FIGURE 1. Weakly successively almost positive diagram. Here $*$ denotes its standard base point.

As we will show in the paper, weakly successively almost positive links give a unified and satisfactory framework to treat the positivity of diagrams.

Clearly, positive links and almost positive links are weakly successively almost positive. The aforementioned *successively k-almost positive diagram* in [It3] is a weakly successively k-almost positive diagram such that the negative overarc contains no positive crossings. That is, its k negative crossings appear successively along a single arc, which explains our terminology. Thus weakly successively almost positive links subsume the previously proposed generalizations of positive links.

In this paper we show that invariants of weakly successively almost positive links have several specific properties which extend known results of (almost) positive links. Our results are new or stronger even for positive links, or, answer questions about (almost) positive links that appeared in literature. Furthermore, our proofs are often much simpler than known proofs for the (almost) positive link case, despite its strength.

1.1. **Summary of results and organization.** Although we will prove number of results, we give a quick summary of our main results and contents of the paper.

In Section 2 we begin with the discussion by introducing a notion of *weakly positive diagram* (Definition 2.1). A weakly positive diagram is the weakest form of positivity that enjoys several remarkable properties. It contains weakly successively almost positive as its special case.

Among them, we prove the following, which means that a weakly positive diagram represents a split link if and only if it is obviously so. Here a diagram is *height-split* means that the diagram is obtained by piling two diagrams at different heights (see Definition 2.7 for details).

Theorem 2.8. *Let D be a weakly positive diagram. Then D represents a split link if and only if D is height-split.*

This property allows us to show the following.

Corollary 2.12. *If a weakly positive link L is non-split, then every smoothly embedded surface in B^4 without closed components that bounds L is connected.*

For a link L, we denote by $\#L$ the number of its components. We define the *genus* and *the slice genus (smooth 4-ball genus)* of a link L by

$$g(L) = \frac{-\chi(L) - \#L + 2}{2}, \quad g_4(L) = \frac{-\chi_4(L) - \#L + 2}{2} \tag{1.1}$$

Here $\chi(L)$ (resp. $\chi_4(L)$) is the maximum Euler characteristic of Seifert surface of L (resp. smoothly embedded surface in B^4 whose boundary is L), where surfaces can be disconnected. The definition (1.1) assumes that surfaces are connected. Corollary 2.12 ensures that this is always satisfied for a non-split weakly positive link.

In Section 3 we study the fundamental properties of weakly successively almost positive diagrams. We introduce a notion of *complexity* $\mathcal{C}(D) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ of a weakly successively almost positive diagram (Definition 3.1) so that $\mathcal{C}(D) = (0,0)$ if and only if D consists of disjoint circles.

A *skein triple* is a triple of diagrams (D_+, D_0, D_-) (or, their corresponding links (L_+, L_0, L_-)) which are equal except around one crossing point as



D_+ D_0 D_-

We define a *standard skein triple*, a skein triple canonically defined for a weakly successively almost positive diagram D (Definition 3.2). The following is a crucial feature of weakly successively almost positive link.

Theorem 3.3. *Let D be a weakly successively almost positive diagram and let $(D = D_+, D_-, D_0)$ be the standard skein triple. Then both D_- and D_0 are weakly successively almost positive and $\mathcal{C}(D_0), \mathcal{C}(D_-) < \mathcal{C}(D)$.*

This means that one can simplify weakly successively almost positive diagrams inside the class of weakly successively almost positive diagrams by skein relations. This makes a sharp contrast to positive diagrams, where the skein relation often leads to non-positive diagrams.

We will also introduce a notion of standard unknotting/unlinking sequence of weakly successively almost positive diagrams (Proposition-Definition 3.5). This allows us to relate various invariants and unknotting numbers.

As a demonstration of the usefulness of standard skein triple argument, in Section 4 we show that the Conway polynomial

$$\nabla_L(z) = \sum_{i=\#L-1}^{d} a_{2i+\#L-1} z^{2i}$$

of a non-split, weakly successively almost positive link L has the following properties.

Theorem 4.2. *Assume that L is a non-split weakly successively almost positive link.*

(i) $a_{\#L-1+2i}(L) \geq \binom{g_4(L)}{i}$.
(ii) $\max\deg_z \nabla_L(z) = 1 - \chi(L)$.
(iii) *The Conway polynomial is strictly positive, i.e., $a_{\#L-1+2i}(L) > 0$ for all $i = 0, \ldots, d$.*

(iv) *L is fibered if and only if the Conway polynomial is monic, i.e., its leading coefficient* $a_{1-\chi(L)}(L) = 1$.

As far as the authors know, Theorem 4.2 subsumes all known results for the Conway polynomial of positive or almost positive links. For positive links, the assertion (ii) and (iv) are proven in [Cr]. In [Cr], the non-negativity $a_{\#L-1+2i}(L) \geq 0$, an assertion a bit weaker than (iii), was also shown, but no counterpart of (i), a quantitative bound, was given. For almost positive links, the assertion (ii) was proven in [St7] with great effort. The property (iv) has remained a question there.

Since $g(L) = g_4(L)$ holds for non-split positive or almost positive links (see, e.g., [AT, Theorem 1.6]), Theorem 4.2 (i) leads to the following (where we state for knots for simplicity).

Corollay 1.1. *If K is a positive or almost positive knot, then*

$$a_{2i}(K) \geq \binom{g(K)}{i}.$$

The inequality is known for braid-positive links, knots represented by the closure of positive braids [Bu], but has been widely open of positive links. The inequality $a_2(K) \geq g(K)$ for a positive knot is proven in [St2, Theorem 6.1]. For the (non-trivial) almost positive case, $a_2(K) > 0$ was proven in [St3, Theorem 4.1]. The corollary is an ostensible improvement of [St9, Proposition 4.1].

In Section 5, we proceed to study the *HOMFLY(-PT) polynomial* $P_L(v, z)$ of a link L defined by the skein relation

$$v^{-1}P_{L_+}(v, z) - vP_{L_-}(v, z) = zP_{L_0}(v, z) \qquad P_{\mathsf{Unknot}}(v, z) = 1.$$

Theorem 5.1. *Assume L is a weakly successively almost positive link. Then the HOMFLY polynomial $P_L(v, z) = \sum_{i,j} c_{i,j} v^i z^j$ has the following properties.*

(i) *The coefficients $c_{i,j}$ satisfy the following.*
  (i-a) $c_{i,j} = 0$ *if* $i < j$.
  (i-b) $c_{j,j} \geq 0$ *for all* j.
  (i-c) $\sum_i c_{i,i} = 1$. *Thus* $c_{j,j} = 0$ *for all* j *except exactly one* j, *where* $c_{j,j} = 1$.
  (i-d) *If L is non-split,* $c_{i,1-\chi(L)} \geq 0$ *for all* i.
  (i-e) *L is fibered if and only if* $c_{i,1-\chi(L)} = 0$ *for all i except exactly one i, where* $c_{i,1-\chi(L)} = 1$.

(ii) $\max\deg_z P_L(v, z) = 1 - \chi(L)$.

(iii) *If L is non-split, then* $\min\deg_v P(L) \geq \#L - 1$. *Moreover, if L is a non-trivial knot, then* $\min\deg_v P(L) \geq 2$.

Property (i-d), (ii) for positive links are proven in [Cr, Corollary 4.3] but the other properties seem to be new, even for positive links.

By exploiting the fact that the Jones polynomial is recovered from the HOMFLY polynomial, we also prove the following.

Theorem 5.4. *Let L be a weakly successively almost positive link, and let $V_L(t)$ be its Jones polynomial.*

(i) *The sign of the coefficient* $t^{\min\deg_t V_L(t)}$ *is* $(-1)^{\#L-1}$.

(ii) $\min\deg_v P_K(v, z) \leq 2\min\deg_t V_K(t) \leq \max\deg_z P_K(v, z) = 1 - \chi(K)$.

(iii) *If L is non-trivial and non-split,* $\min\deg_t V_K(t) \geq \begin{cases} 1 & (\#L = 1) \\ \frac{\#L-1}{2} & (\#L > 1). \end{cases}$

In Section 6 we prove Theorem 6.3, an improvement of results for polynomial invariants, under the additional assumption that the link is *Bennequin-sharp*, that is, under the assumption that $\overline{sl}(L) = -\chi(L)$ holds. Here $\overline{sl}(L)$ is the maximum self-linking number.

In Section 7 we discuss the positivity of signature and related invariants. Let $\sigma(L)$ be the signature of a link L, where we use the convention that $\sigma(\mathsf{positive\ trefoil}) = 2$. We show the following, which was proven in [CG, Tr] for positive links and in [PT] for almost positive links.

Theorem 7.1. *If L is a non-trivial weakly successively almost positive link, then $\sigma(L) > 0$. In particular L is not (algebraically) slice, and is not amphicheiral.*

We will also prove the following stronger result that leads to the positivity of many other invariants.

Theorem 7.6. *Let L be a non-split weakly successively almost positive link.*

(i) *If L is a knot K, then by at most $a_2(K) - 1$ positive-to-negative crossing changes, K can be made to the positive trefoil knot.*
(ii) *If $\#L > 1$, then by at most $a_{\#L+1}(L) + a_{\#L-1} - 1$ positive-to-negative crossing changes, L can be made to the connected sum of $(\#L - 1)$ positive Hopf links.*

This was proven in [PT] for almost positive links without evaluation of the number of crossing changes.

In Section 8 we give several examples how our established properties rule out weakly successively almost positivity. We determine the weakly successively almost positive knots up to 12 crossings, with only four exceptions.

Theorem 8.2. *Up to 12 crossings, a prime knot K is weakly successively almost positive if and only if it is either positive or almost positive, possibly except $12_{1408} = 12n_{280}$, $12_{1487} = 12n_{259}$, $12_{1837} = 12n_{709}$, $12_{2037} = 12n_{909}$.*

This raises the question whether weakly successively almost positive knots form a strictly larger class than almost positive knots. This problem, and comparison of various classes concerning positivity, will be discussed in a separate paper [IS].

In Appendix A we prove the following, which is used to prove the assertion concerning fiberedness in Theorem 4.2 (iv) and Theorem 5.1 (iii).

Theorem A.13. *Let (L_+, L_-, L_0) be a skein triple.*

(i) *Assume that $\chi(L_+) = \chi(L_-) < \chi(L_0) - 1$, and that L_- is fibered. Then L_+ is fibered.*
(ii) *Assume that $\chi(L_\pm) = \chi(L_0) - 1 < \chi(L_\mp)$, and that L_0 is fibered. Then $L_\pm$ is fibered.*

This is a slight generalization of Scharlemann-Thompson's Theorem (Theorem 4.1) of maximum Euler characteristic of skein triples, and is interesting in its own right.

Acknowledgement

The first author would like to thank Kimihiko Motegi and Masakazu Teragaito for stimulating conversations and discussions. He also thanks Keiji Tagami for various comments, and informing of several related results and errors in an earlier version

of the paper. He was partially supported by KAKENHI Grant Number 19K03490 and 21H04428. The second author was partially funded by the National Research Foundation of Korea (grants NRF-2017R1E1A1A03071032) and the International Research & Development Program of the Korean Ministry of Science and ICT (grant NRF-2016K1A3A7A03950702).

2. Weakly positive diagram

In this section, we introduce the notion of weakly positive diagram. By looking at the linking number, it is clear that a positive diagram D represents a split link if and only if the diagram D is split, namely, D is disconnected as a subspace of $\mathbb{R}^2$.

One motivation to introduce weakly positive diagrams is to justify the assertion

"*a diagram represents a split link if and only if it is obviously so*"

for a wider class.

The weakly positive links form a class broader than weakly successively almost positive links. The results in this section are interesting in their own right and will be used in later sections.

2.1. **Link diagrams.** A *link diagram* D of a link $L \subset \mathbb{R}^3$ is an image $p(L)$ of the projection to the plane $p : \mathbb{R}^3 \to \mathbb{R}^2$ having only (transverse) double point singularities, equipped with over-and-under information at each double point. We denote by $c(D)$ the number of crossings of D, and we denote by $c_+(D)$ (resp. $c_-(D)$ the number of positive (resp. negative) crossings of D.

We call the image $D_i := p(L_i)$ of a component L_i of L a *component* of the diagram D (do not confuse with the *connected components* of D as a subset of $\mathbb{R}^2$). By a *sub diagram* of D we mean an image $p(L')$ of a sublink L' of L.

An *ordered based link* is an oriented ℓ-component link $L = L_1 \cup \cdots \cup L_\ell$ such that the set of the components $\{1, \ldots, \ell\}$ is ordered, and that for each component L_i, a base point $*_i \in L_i$ is assigned.

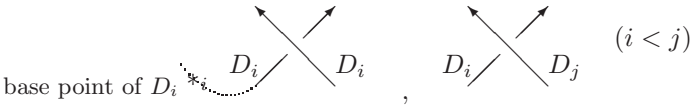
An *ordered based link diagram* D is a link diagram of an ordered based link L, taken so that none of the base points $*_i$ is a crossing point. That is, an ordered based link diagram is a link diagram such that

- the ordering of components of D is given, and that,
- for each component D_i of D, a base point $*_i$ is given.

In the following, we usually assume that a link L and its diagram D is always oriented.

2.2. **Definition and simple properties.**

Definition 2.1 (Weakly positive diagram and links)**.** An ordered based link diagram D is *weakly positive* if there are no negative crossings of the following types.



That is, D is weakly positive if

- For a negative crossing between the same component D_i, the crossing first appears as over crossing if we start from the base point $*_i$ of D_i.

- For a negative crossing between different components D_i and D_j with $i < j$, D_i always appears as an overarc.

We say that a link L is *weakly positive* if L is represented by a weakly positive diagram.

In the following, by abuse of notation we say that a link diagram D is *weakly positive* if D is a weakly positive diagram with a suitable choice of ordering and base points.

Example 2.2. A 2-almost positive diagram D of a *knot* with two negative crossings c_1, c_2 is a weakly positive diagram by suitably taking its base point.

Contrarily, a 2-almost positive diagram of a *link* is not always weakly positive; the standard diagram of the negative Hopf link gives such an example. Similarly, a 3-almost positive diagram of a knot is not always weakly positive; the standard diagram of the negative trefoil gives such an example.

Example 2.3. A weakly successively almost positive diagram is weakly positive. We take the ordering of the component so that the component that contains the negative overarc is the smallest (say, L_1), and take a base point $*_1$ near the initial point of the negative overarc. The other choices, the orderings and base points on other components are arbitrary.

The definition immediately leads to the following positivity property.

Proposition 2.4. *If L is weakly positive, then L can be made into an unlink by positive-to-negative crossing changes.*

Proof. We prove the theorem by induction on $(c(D), c_+(D))$. We take the first positive crossing c which we pass along an underarc. Let D_- be the diagram obtained by changing c to a negative crossing. Then D_- is weakly positive, hence by induction the link L_- represented by D_- can be made into the unlink by positive-to-negative crossing changes. □

This simple property allows us to derive the non-negativity of several integer-valued link invariant.

Corollary 2.5. *Let v be an invariant of links. Assume v satisfies the property that*

$$v(\mathsf{Unlink}) = 0, \quad v(L_+) \geq v(L_-)\,,$$

where L_- is a link obtained by a positive-to-negative crossing change from L_+. Then $v(L) \geq 0$ for all weakly positive link L.

Example 2.6. There are many invariants that satisfy the assumption of the Corollary.

- The Levine-Tristram signature σ_ω for every $\omega \in \{z \in \mathbb{C} \mid |z| = 1\}$.
- Slice-torus invariants as defined in [Le] for knots and in [CC] for links.

2.3. **Splitness criterion for weakly positive links.**

Definition 2.7 (Height-split diagram)**.** A link diagram D is *height-split* if D is decomposed as a union of sub diagrams $D = D_1 \cup D_2$ such that the sub diagram D_1 lies above D_2; at each crossing of D formed by D_1 and D_2, the component in D_1 always appears as an overarc.

This is a generalization of split diagram in the sense that a height-split diagram obviously represents split link; when $D_1 = p(L_1)$ and $D_2 = p(L_2)$, then L is the split union of L_1 and L_2.

Theorem 2.8. *Let D be a weakly positive diagram. Then D represents a split link if and only if D is height-split.*

The theorem essentially comes from the following non-triviality of the linking numbers.

Lemma 2.9. *Let $L = L_1 \cup \cdots \cup L_\ell$ be an ℓ-component link, and let D be a weakly positive diagram of L. Then for $i < j$, we have $lk(L_i, L_j) \geq 0$, and $lk(L_i, L_j) = 0$ occurs if and only if the component D_i lies above of D_j.*

Proof. By the definition of a weakly positive diagram, D has no negative crossing of the form  for $i < j$. Thus

$$lk(L_i, L_j) = \#$$  $$- \#$$  $$= \#$$  $$\geq 0$$

Furthermore, when $lk(L_i, L_j) = 0$, this means that D has no crossings of the form  either. Therefore the crossings between D_i and D_j are either of the form  or , and hence D_i always lies above of D_j.

□

To utilize the information of linking numbers, we use the following graph.

Definition 2.10. The *linking graph* $\Lambda(L)$ of a link L is a (labelled) graph, such that the set of vertices $V(\Lambda(L))$ is the set of the components of L, and two vertices L_i and L_j are connected by an edge if and only if their linking number $lk(L_i, L_j)$ is non-zero, and we assign a labeling $lk(L_i, L_j)$ to that edge.

Proof of Theorem 2.8. Assume that L is split, so the linking graph $\Lambda(L)$ is disconnected. Let Λ_o be the connected component of the linking graph that contains L_1, and let Λ_u be the union of the rest of the components. Let $L_o = \bigsqcup_{i \in V(\Lambda_o)} L_i$ and $L_u = \bigsqcup_{i \in V(\Lambda_u)} L_i$, and, let D_o and D_u be the subdiagrams of D that correspond to L_o and L_u.

Since for any component L_i of L_o and L_j of L_u, we have $lk(L_i, L_j) = 0$, by Lemma 2.9 the diagram D_i lies above of D_j if $i < j$. Thus, to see that D_o lies above of D_u, it is sufficient to show that $i < j$ holds for all $i \in V(\Lambda_o)$ and $j \in V(\Lambda_u)$.

We prove $i < j$, for a fixed j, by induction on the edge distance $\delta(i, 1)$ between D_i and D_1 in the the same connected component Λ_o of the linking graph.

For $\delta(i, 1) = 0$, we have $i = 1$, and obviously $1 < j$. Thus the induction start holds.

To do the induction step, fix a vertex i' adjacent to i in Λ_0 (i.e., with $lk(D_i, D_{i'}) \neq 0$), so that the distance $\delta(i', 1) = \delta(i, 1) - 1$ in Λ_0. Assume, to the contrary that

$i > j$ happens. Since $i' < j$ by induction, this implies that $D_{i'}$ lies above of D_j, and (because of $i > j$ and $lk(D_i, D_j) = 0$) that D_j lies above of D_i. This means that $D_{i'}$ lies above of D_i, so $lk(D_i, D_{i'}) = 0$. However, we assumed $lk(D_i, D_{i'}) \neq 0$. This is a contradiction.

□

The linking number determines the lowest coefficient $a_{\#L-1}(L)$ of the Conway polynomial by the following Hoste-Hosokawa formula

$$a_{\#L-1}(L) = \sum_{g \in T} w(g)\,, \tag{2.1}$$

where T is the set of the spanning trees of the linking graph $\Lambda(L)$, a subtree having exactly $\#L - 1$ distinct edges, and $w(g)$ is the product of linking numbers that appear as an edge of g.

As a corollary we get the following splitness criterion for weakly positive links.

Corollary 2.11. *For a weakly positive link L, the following are equivalent.*

(i) *L is split.*
(ii) *The linking graph $\Lambda(L)$ is disconnected.*
(iii) *The coefficient of $z^{\#L-1}$ of $\nabla_L(z)$ is zero.*

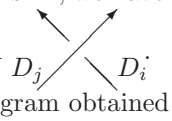
Proof. The equivalence of (i) and (ii) follows from Theorem 2.8, and the implication (i) $\Rightarrow$ (iii) is obvious. To see (iii) $\Rightarrow$ (ii), in Hoste-Hosokawa's formula (2.1), $w(g) > 0$ for every spanning tree g since $lk(L_i, L_j) \geq 0$ for every $i \neq j$. Thus $a_{\#L-1}(L) = 0$ means that the linking graph $\Lambda(L)$ has no spanning tree, which means that $\Lambda(L)$ is disconnected. □

A non-split weakly positive link having a connected linking graph implies the following.

Corollary 2.12. *If a weakly positive link L is non-split, then every smoothly embedded surface in B^4 without closed components that bounds L is connected.*

Corollary 2.11 gives a close connection between the linking graph $\Lambda(L)$ and splitness. As for the primeness, we have the following.

Lemma 2.13. *Let L be a weakly positive, non-split link. If the linking graph $\Lambda(L)$ has an edge e with labeling '1' such that removing the edge e makes the graph disconnected, then L is a connected sum of the positive Hopf link and some other weakly positive links.*

Proof. Assume that the edge e is an edge connecting L_i and L_j.$(i < j)$. Since the labeling of the edge e is '1', we have $lk(L_i, L_j) = 1$. Thus there is a unique positive crossing c of the form 

Let D_- be the diagram obtained by changing c into a negative crossing. Then D_- still remains weakly positive. The linking graph $\Lambda(D_0)$ is the obtained from $\Lambda(D)$ by removing the edge e so $\Lambda(D_0)$ is disconnected. Therefore by Corollary 2.11, D_- is height-split.

Thus we may write $D_- = D_o \cup D_u$, where D_o lies above of D_u. Since D and D_- are the same except the crossing c, by separating D_o and D_u preserving the crossing c, we get a diagram D' which is 'almost' split, in the sense that the D_o

and D_u are disjoint except at the crossing c and the other positive crossing c', as shown in Figure 2. In particular, the diagram D' is the connected sum of the Hopf link diagram H and some other link diagrams.

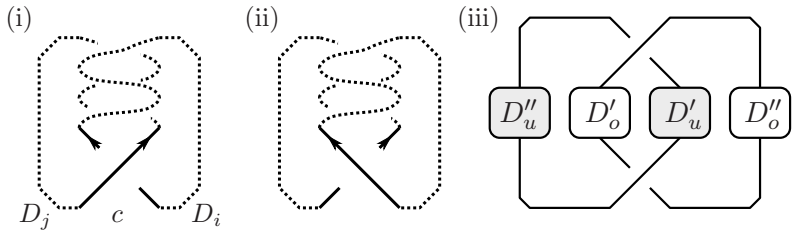


FIGURE 2. (i) The diagram D; the component L_i lies above of L_j, except at c. (ii) The height-split D_- obtained by crossing change at c. (iii) The diagram D' which is 'almost' split.

□

This observation leads to the following estimate of the lowest coefficient of the Conway polynomial.

Proposition 2.14. *When L is a non-split prime weakly positive link, then*

$$a_{\#L-1}(L) \geq lk(L) \geq \#L$$

unless L is the Hopf link. (Here $lk(L) = \sum_{i<j} lk(L_i, L_j)$ is the total linking number.)

Proof. To prove the theorem, we use the following rephrasing of Hoste-Hosokawa's formula (2.1). Let $\Lambda'(L)$ be a graph such that the set of vertices is the set of the components of L, and two vertices L_i and L_j are connected by $lk(L', L'')$ parallel edges. Then one can rewrite (2.1) as

$$a_{\#L-1}(L) = \#\text{spanning trees of } \Lambda'(L). \tag{2.2}$$

Lemma 2.13 implies that removing edges of $\Lambda'(L)$ does not make the graph disconnected. Then the assertion follows from the standard fact of graph theory that such a graph has at least as many spanning trees as edges (and at least as many edges as vertices). □

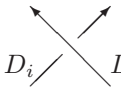
As a complementary result, we mention the following.

Proposition 2.15. *If a non-split weakly positive link L satisfies $a_{\#L-1}(L) = 1$, then L is the connected sum of $\#L - 1$ Hopf links and some other (weakly positive) knots.*

Proof. We prove the assertion by induction on $\#L$. The case $\#L = 2$ is proven in Proposition 2.14.

By Hoste-Hosokawa's formula (2.1), the linking graph $\Lambda(L)$ is the path graph with $\#L$ vertices and all the $(\#L - 1)$ edges have weight one. In particular, by Lemma 2.13 L is a connected sum of the Hopf link H and some other weakly positive links L', L''. Since $1 = a_{\#L-1}(L) = a_{\#L'-1}(L')a_1(H)a_{\#L''-1}(L'')$, we have $a_{\#L'-1}(L') = a_{\#L''-1}(L'') = 1$. Thus by induction both L' and L'' are connected sums of Hopf links and some other (weakly positive) knots. □

Remark 2.16. In the proof of Theorem 2.8, Lemma 2.9, Lemma 2.13 and Proposition 2.14, we only used the property that D contains no crossing of the form



$(i < j)$.

3. Weakly successively almost positive diagram

In this section we initiate the study of our main object, a weakly successively almost positive (which we often abbreviate as "w.s.a.p.") diagram and link. In the following, we will always regard a weakly successively almost positive diagram as an ordered based link diagram so that it is a weakly positive diagram.

3.1. **The standard skein triple of weakly successively almost positive diagram.** The *first underarc positive crossing* of a weakly successively almost positive diagram D is the positive crossing c which we first pass along underarc; that is, the positive crossing which is the endpoint of the negative overarc (see Figure 3). If there is no such crossing, then the overarc is an entire (height-)split loop, and such links can be simplified.

Definition 3.1 (Complexity)**.** For a weakly successively almost positive diagram D, we define the *complexity* of D by

$$\mathcal{C}(D) = (c(D), c(D) - \ell(D)),$$

where $\ell(D)$ is the number of crossings that lie on the negative overarc, which we call the *length* of the negative overarc. As usual, we compare the complexity according to the lexicographical ordering.

For a weakly successively almost positive link L, we define the *complexity*

$$\mathcal{C}(L) = \min\{\mathcal{C}(D) \mid D \text{ is a weakly successively almost positive diagram of } L\}$$

We say that a weakly successively almost diagram diagram D of a link L is *minimum* if $\mathcal{C}(D) = \mathcal{C}(L)$.

A key feature of a weakly successively almost positive diagram is that it admits a complexity-reducing skein triple in the realm of weakly successively almost positive diagrams.

Definition 3.2 (Standard skein triple)**.** Let $(D = D_+, D_0, D_-)$ be the skein triple obtained at the first positive crossing c terminating the negative overarc (which passes it as undercrossing). That is, D_0 is a diagram obtained by smoothing the crossing c, and D_- is a diagram obtained by changing the positive crossing c into a negative crossing.

We call $(D = D_+, D_0, D_-)$ (or, the links $(L = L_+, L_0, L_-)$ represented by D, D_0 and D_-) the *standard skein triple* of D.

Theorem 3.3. *Let D be a weakly successively almost positive diagram and let $(D = D_+, D_-, D_0)$ be the standard skein triple. Then both D_- and D_0 are weakly successively almost positive and $\mathcal{C}(D_0), \mathcal{C}(D_-) < \mathcal{C}(D)$.*

Proof. The digram D_- is naturally regarded as an ordered based link diagram. We view D_0 as an ordered based link diagram as follows (see Figure 3).

When smoothing the crossing c connects two distinct components L_1 and L_i to form a component L_{1^*} of L_0, then we just forget the relevant information of the component L_i; the ordering is $1^* < 2 < \cdots$ and the base point of D_{1^*} is just $*_1$.

When the smoothing the crossing separates the component L_1 into two distinct components $L_{1'}$ and $L_{1''}$, then we put the ordering $1' < 1'' < 2 < \cdots$, where $L_{1'}$ is the component that contains the base point $*_1$ of D_1. We take the base point $*_{1'}$ of $D_{1'}$ as $*_1$, and take a base point of $D_{1''}$ arbitrary (however, it is often convenient to take it near the smoothed crossing c).

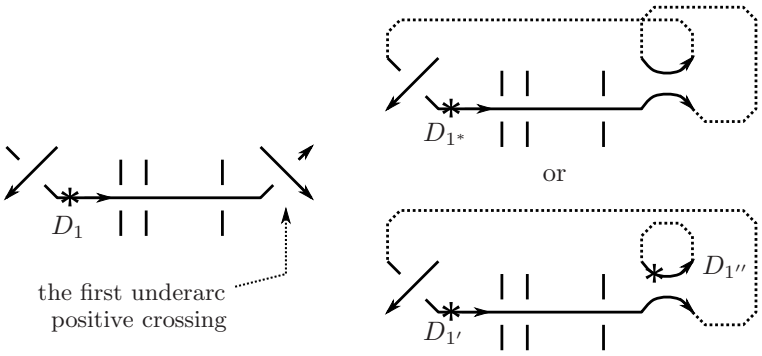


FIGURE 3. The first underarc crossing (left) of D and how to view the diagram D_0 as a based ordered diagram (right)

Then both D_0 and D_- are weakly successively almost positive. By the definition of the complexity, $\mathcal{C}(D_0), \mathcal{C}(D_-) < \mathcal{C}(D)$. □

3.2. **Split links in the standard skein triple.** To use the standard skein triple effectively, we often need to take care of split links. Thanks to Theorem 2.8, we can completely understand when L_- or L_0 becomes a split link, when we use the standard skein triple for a *minimum* weakly successively almost positive diagram.

Lemma 3.4. *Let D be a minimum weakly successively almost positive diagram representing a non-split link L, and let $(L = L_+, L_0, L_-)$ be the standard skein triple of L. Then L_0 is always non-split. Moreover, if L_- is split, then the minimum diagram D can be taken so that is it of the form $D = D' \# H \# D''$ (see Figure 4), where D' is a weakly successively almost positive diagram, D'' is a positive diagram, and H is the standard Hopf link diagram.*

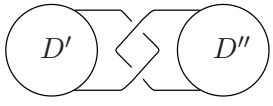


FIGURE 4. Diagram $D = D' \# H \# D''$

Proof. Since D is minimum, D is reduced and the diagram D_0 is non-split.

First, assume to the contrary that L_0 is split. By Theorem 2.8, D_0 is height-split so $D_0 = D_u \cup D_o$ where D_o contains the negative overarc, lying above of D_u. Since the diagram D_0 itself is non-split, there are crossings between D_o and D_u. However,

as D_u lies above of D_o, such a crossing can be removed by suitable isotopy. This leads to a weakly successively almost positive diagram of L with smaller complexity (see Figure 5). This is a contradiction.

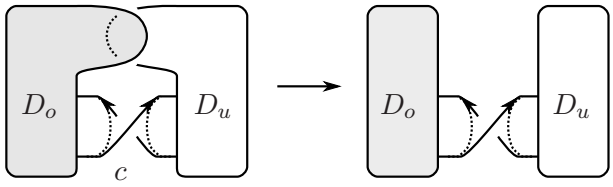


FIGURE 5. When D_0 is height-split, D_o and D_u overlap but their overlaps can be removed without touching the skein crossing c.

Next assume that L_- is split. By Theorem 2.8, D_- is height-split, so we may write $D_0 = D_u \cup D_s$ where D_o contains the negative overarc, lying above of D_u. Since D_- is not split, there are crossings between D_o and D_u. The assumption that D is minimum implies that the number of crossings between D_o and D_u is two. Because otherwise, as in the D_0 case, by removing overlaps between D_o and D_u, we can find a weakly successively positive diagram of L with smaller complexity. Then finally we modify the diagram D in the form $D'\#H\#D''$, without changing the complexity (see Figure 6).

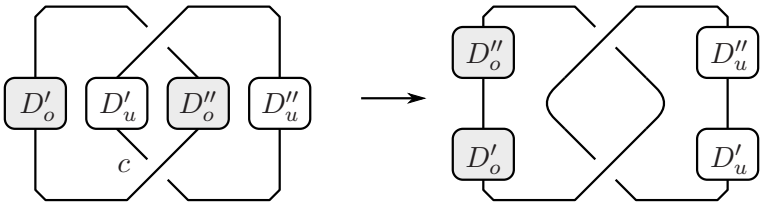


FIGURE 6. The case D_- is height-split

□

3.3. **Standard unknotting/unlinking sequence.** The standard skein triple gives rise to the following quite specific types of unknotting/unlinking sequence of a non-split weakly successively almost positive link L.

Proposition-Definition 3.5 (Standard unknotting/unlinking sequence)**.** *For each non-split weakly successively almost positive link L, there exists a positive-to-negative crossing switch sequence*

$$L = L_0 \to L_1 \to \cdots \to L_m \tag{3.1}$$

having the following properties:

- *L_m is the connected sum of $(\#L - 1)$ positive Hopf links (when L is a knot, L_m is the unknot).*
- *All L_i are non-split weakly successively almost positive.*
- *For each i, the crossing change $L_i \to L_{i+1}$ is realized as the standard skein triple $(D = D_+, D_-, D_0)$ of a minimum weakly successively almost positive diagram D of L_i.*

We call such a crossing change sequence (3.1) *a* standard unknotting/unlinking sequence *of* L.

Proof. To show the assertion, it is sufficient to show that for a non-split weakly successively almost positive link L which is not a connected sum of positive Hopf links, there is a minimum weakly successively almost positive diagram D of L such that its standard resolution D_- (the diagram D_- in the standard skein triple $(D = D_+, D_0, D_-)$) represents a non-split link.

By Lemma 3.4, D_- represents a split link only if D is a diagram connected sum $D = D' \# H \# D''$, where D' is a weakly successively almost positive diagram, H is the standard Hopf link diagram, and D'' is a positive diagram. Let L' and L'' be the link represented by D' and D'', respectively.

If L' is not a connected sum of Hopf links, by induction there exists a minimum w.s.a.p. diagram $\widehat{D}'$ of L' such that its standard resolution, $\widehat{D}'_-$ does not represent a split link. Let $\widehat{D} = \widehat{D'} \# H \# D''$, where the connected sum is taken away from the negative over arc of $\widehat{D}'$. Then $\widehat{D}$ is a minimum w.s.a.p. diagram of L and its standard resolution $\widehat{D_-}$ represents a non-split link.

If L' is a connected sum of Hopf links, D' is a positive diagram, and L'' is not a connected sum of Hopf links, then we replace the base point of D on D'' to interchange the role of D' and D''. Then the same argument gives the desired minimum w.s.a.p. diagram $\widehat{D}$ of L. □

Remark 3.6. Note that there is a way to modify the standard unknotting/unlinking sequence by choosing the standard skein triple not at the end of the negative overarc, but at the beginning. That is, we choose the crossing in D_+ to be the underpass terminating the overarc backwards, and in D_- and D_0 can move the starting point along the negative overarc backward along its respective component. We will not need this freedom to extend the negative overarc backward too often, but this will be used for Theorem 7.6.

Remark 3.7. It is informative to note that the argument in this section can be applied to a wider class of link diagrams. For the proofs of the theorems in this section to work, we actually need a class of diagrams $\mathcal{C}$ such that:

(0) The trivial link diagrams belong to the class $\mathcal{C}$.
(1) Theorem 2.8 holds; namely, a diagram D in the class $\mathcal{C}$ represents a split link of and only if it is height-split.
(2) A minimum complexity diagram D admits a positive skein resolution in the class $\mathcal{C}$; the diagram D has a positive crossing c such that in the skein triple $(D = D_+, D_0, D_-)$, both D_- and D_0 belong to the class $\mathcal{C}$.

In this prospect, a good feature of a weakly successively almost positive diagram is that the standard triple $(D = D_+, D_0, D_-)$ satisfies the property (2).

Thus it an interesting problem to find a useful (and wider) class of diagrams $\mathcal{C}$ that satisfies all the above properties. Since the class of weakly positive diagrams satisfies properties (0) and (1), it is important to study to what extent we can expect the property (2) for weakly positive diagrams.

It is known that a positive diagram does not need to be realizable among minimal crossing number diagrams. As is shown in [St10, Theorem 1], the knot 11_{550} gives such an example.

One can say the same for weakly successively almost positive diagrams.

Example 3.8. The knot 15_{132907} (see Section 8 for our convention of numbering) below has no weakly successively almost positive diagrams of 15 crossings, but it does have a 2-successively almost positive diagram of 17 crossings.

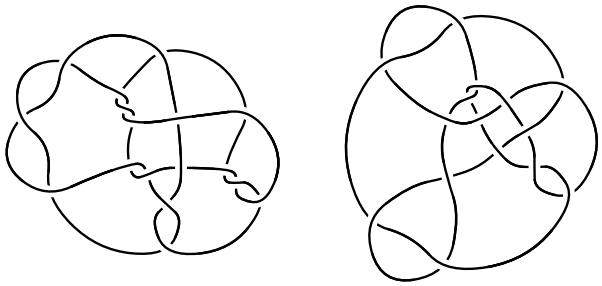


FIGURE 7. The knot 15_{132907} (as appeared in [HT] in the right diagram) and its weakly successively 2-almost positive diagram. Here the negative crossings appear on an underarc.

4. Application of standard skein triple I: Conway polynomial

We proceed to apply the standard skein triple to establish various properties of weakly successively almost positive links.

For a general skein triple $(L = L_+, L_0, L_-)$, we always have the inequality

$$-\chi_4(L_-) \geq \chi_4(L) - 2 \text{ and } -\chi_4(L_0) \geq -\chi_4(L) - 1 \tag{4.1}$$

For $\chi(L)$, we can say much stronger.

Theorem 4.1 (Scharlemann-Thompson [ST])**.** *Let (L_+, L_-, L_0) be a skein triple. Then one of the following occurs.*

- $\chi(L_+) = \chi(L_-) \leq \chi(L_0) - 1$.
- $\chi(L_+) = \chi(L_0) - 1 \leq \chi(L_-)$.
- $\chi(L_-) = \chi(L_0) - 1 \leq \chi(L_+)$.

Furthermore, we can also say about the fiberedness properties from the skein triple, as we will show in Theorem A.13.

Armed with this knowledge, we use the standard skein triple to relate knot polynomials and χ or χ_4.

4.1. Conway polynomial and (4-ball) genus.

The following property of the Conway polynomial nicely reflects the positivity of diagrams.

Theorem 4.2. *Assume that L is a non-split weakly successively almost positive link.*

(i) $a_{\#L-1+2i}(L) \geq \binom{g_4(L)}{i}$.
(ii) $\max\deg_z \nabla_L(z) = 1 - \chi(L)$.
(iii) *The Conway polynomial is strictly positive, i.e., $a_{\#L-1+2i}(L) > 0$ for all $i = 0, \ldots, d$.*

(iv) *L is fibered if and only if the Conway polynomial is monic, i.e., its leading coefficient $a_{1-\chi(L)}(L) = 1$.*

Proof. All the assertions (i)–(iv) are proven simultaneously by induction on the complexity of L. Let $D = D_+$ be a minimum w.s.a.p. diagram of L, and (D_+, D_0, D_-) be its standard skein triple.

Case 1. L_- is split.

By Lemma 3.4, we may assume that the diagram D is of the form $D = D' \# H \# D''$ (where D' is w.s.a.p., H is the standard Hopf link diagram, and D'' is positive).

Let L' and L'' be the links represented by D' and D'', respectively. Then

$$\#L = \#L' + \#L'', \quad g_4(L) \le g_4(L') + g_4(L''), \text{ and } \chi(L) = \chi(L') + \chi(L'') - 2.$$

Moreover both L' and L'' have strictly smaller complexity, so they satisfy the properties (i)–(iv).

Since $\nabla_L(z) = \nabla_{L'}(z)\nabla_H(z)\nabla_{L''}(z) = z\nabla_{L'}(z)\nabla_{L''}(z)$, the property (ii) of L' and L'' shows that

$$\begin{aligned} a_{\#L-1+2i}(L) &= \sum_{k+j=i} a_{\#L'-1+2j}(L') a_{\#L''-1+2k}(L'') \\ &\ge \sum_{k+j=i} \binom{g_4(L')}{j} \binom{g_4(L'')}{k} = \binom{g_4(L') + g_4(L'')}{i} \\ &\ge \binom{g_4(L)}{i}. \end{aligned}$$

Similarly,

$$\begin{aligned} \max\deg_z \nabla_L(z) &= \max\deg_z \nabla_{L'}(z) + \max\deg_z \nabla_{L''}(z) + 1 \\ &= (1 - \chi(L')) + (1 - \chi(L'')) + 1 \\ &= 1 - (\chi(L') + \chi(L'') - 2) \\ &= 1 - \chi(L) \end{aligned}$$

In particular, since

$$a_{\#L-1+2i}(L) = \sum_{k+j=i} a_{\#L'-1+2j}(L') a_{\#L''-1+2k}(L''),$$

this means that $a_{\#L-1+2i}(L) > 0$ for all $i = 0, \dots, 2g(L) = 1 - \chi(L) + 1 - \#L$.

Finally, if $\nabla_L(z)$ is monic, then $\nabla_{L'}(z)$ and $\nabla_{L''}(z)$ are monic. By induction, L' and L'' are fibered, so $L = L' \# H \# L''$ is fibered.

Case 2. L_- is non-split.

To prove the assertion (i), we look at $\#L_0$. If $\#L_0 = \#L - 1$, then

$$g_4(L_0) = \frac{1}{2}(-\chi_4(L_0) - \#L_0 + 2) \ge \frac{1}{2}(-\chi_4(L) - 1 - (\#L - 1) + 2) = g_4(L).$$

Thus by induction and the skein relation,

$$a_{\#L-1+2i}(L) \ge a_{\#L_0-1+2i}(L_0) \ge \binom{g_4(L_0)}{i} = \binom{g_4(L)}{i}$$

Similarly, if $\#L_0 = \#L + 1$, then

$$g_4(L_0) = \frac{1}{2}(-\chi_4(L_0) - \#L_0 + 2) \geq \frac{1}{2}(-\chi_4(L) - 1 - (\#L + 1) + 2) = g_4(L) - 1$$

and

$$g_4(L_-) = \frac{1}{2}(-\chi_4(L_-) - \#L_0 + 2) \geq \frac{1}{2}(-\chi_4(L) - 2 - \#L + 2) = g_4(L) - 1.$$

By induction and the skein relation, we get

$$\begin{aligned} a_{\#L-1+2i}(L) &= a_{\#L_0-1+2(i-1)}(L_0) + a_{\#L_--1+2i}(L_-) \\ &\geq \binom{g_4(L)-1}{i-1} + \binom{g_4(L)-1}{i} = \binom{g_4(L)}{i}. \end{aligned}$$

Similarly, to prove the assertion (ii)–(iv), we look at the maximal degree of the Conway polynomial. Since both $\nabla_{L_0}(z), \nabla_{L_-}(z)$ are strictly positive by induction, we consider the following three cases.

Case A: $\max\deg_z \nabla_L(z) = \max\deg_z \nabla_{L_-}(z) = \max\deg_z \nabla_{L_0}(z) + 1$

In this case $\chi(L_-) = \chi(L_0) - 1$, hence by Theorem 4.1, $\chi(L_-) = \chi(L_0) - 1 \leq \chi(L)$. Therefore

$$1 - \chi(L) \leq 1 - \chi(L_-) = \max\deg_z \nabla_{L_-}(z) = \max\deg_z \nabla_L(z) \leq 1 - \chi(L)\,,$$

so we get the desired equality $1 - \chi(L) = \max\deg_z \nabla_L(z)$.

The strict positivity of $\nabla_L(z)$ follows from the strict positivity of $\nabla_{L_-}(z)$. In this case $\nabla_L(z)$ cannot be monic.

Case B: $\max\deg_z \nabla_L(z) = \max\deg_z \nabla_{L_-}(z) > \max\deg_z \nabla_{L_0}(z) + 1$

Since $\chi(L_-) = 1 - \max\deg_z \nabla_{L_-}(z) < 1 - \max\deg_z \nabla_{L_0}(z) = \chi(L_0) - 1$, we have by Theorem 4.1 $\chi(L_+) = \chi(L_-) < \chi(L_0) - 1$. Therefore, $1 - \chi(L) = 1 - \chi(L_-) = \max\deg_z \nabla_{L_-}(z) = \max\deg_z \nabla_L(z)$.

The strict positivity of $\nabla_L(z)$ follows from the strict positivity of $\nabla_{L_-}(z)$.

If $\nabla_L(z)$ is monic, then $\nabla_{L_-}(z)$ is monic, so by induction L_- is fibered. By Theorem A.13, we conclude L is fibered.

Case C: $\max\deg_z \nabla_L(z) = \max\deg_z \nabla_{L_0}(z) + 1 > \max\deg_z \nabla_{L_-}(z)$

Since $\chi(L_0) - 1 = -\max\deg_z \nabla_{L_0}(z) = 1 - \max\deg_z \nabla_L(z) < 1 - \max\deg_z \nabla_{L_-}(z)$, by Theorem 4.1 $\chi(L_0) - 1 = \chi(L) < \chi(L_-)$. Therefore, $1 - \chi(L) = 2 - \chi(L_0) = 1 + \max\deg_z \nabla_{L_0}(z) = \max\deg_z \nabla_L(z)$.

To see $\nabla_L(z)$ is strictly positive, we note that by Corollary 2.11,

$$\min\deg_z \nabla_{L_0}(z) = \#L_0 - 1 = \#L - 2 \text{ or, } \#L.$$

Thus for $i > 0$, we have $\#L - 1 + 2i \geq \#L_0 - 1$. Therefore for $i = 0, \ldots, \frac{1}{2}(\max\deg_z \nabla_L(z) - \#L + 1)$,

$$a_{\#L-1+2i}(L) = a_{\#L_0+2i}(L_0) + a_{\#L_--1+2i}(L_-) \geq \begin{cases} a_{\#L-1}(L_-) > 0 & (i = 0) \\ a_{\#L-1+2i}(L_0) > 0 & (i > 0) \end{cases}$$

If $\nabla_L(z)$ is monic, then $\nabla_{L_0}(z)$ is monic, so by induction, L_0 is fibered. From Theorem A.13, we conclude L is fibered.

□

As we have mentioned, since $g_4(L) = g(L)$ for positive or almost positive links, Theorem 4.2 (i) gives the inequality

$$a_{\#L-1+2i}(L) \geq \binom{g(L)}{i} \tag{4.2}$$

for non-split (almost) positive links L.

It is interesting and natural to discuss to what extent Theorem 4.2 (i) or Corollary 1.1 is optimal.

The following examples and observations on other related results show the (limited) scope of (possible) further improvements. This pertains to some kind of optimality of Theorem 4.2 (i), at least as far as positive links are concerned.

Example 4.3. (i) The connected sums of positive Hopf links and trefoils make (4.2) exact for arbitrary $\#L, i$. Thus without adding further assumptions such as primeness, the bound (4.2) is optimal.

(ii) For the leading coefficient case, $i = g$, the presence of fibered links makes (4.2) exact in a trivial way, even though we add an assumption that the link is *prime*. Positive prime fibered links occur for all $g > 0$ and $\#L$. The (positively oriented) Montesinos link

$$N_{\ell,g} = M\Big(-1, -\frac{1}{2g-1}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{\ell}\Big) \quad (\ell, g > 0).$$

is prime, fibered with $\#N_{\ell,g} = \ell$ and $g(N_{\ell,g}) = g$ (see [St6, Section 2.7] for explanation and convention).

(iii) For $i = g-1$ and $i = g-2$, in [It2, Corollary 1.2], it is proven that

$$2g(K) - 1 \leq a_{2g-2}(K), \qquad (2g(K)-1)(g(K)-3) \leq a_{2g-4}(K)$$

for a prime, positive braid knot.

(iv) If one considers $i \ll g$, a simple skein (module) calculation for the positive Montesinos links

$$M_{\ell,g} = M\Big(\underbrace{\frac{2}{3}, \ldots, \frac{2}{3}}_{g}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{\ell}\Big) \quad (\ell, g > 0)\,,$$

satisfying $\#M_{\ell,g} = \ell$, $g(M_{\ell,g}) = g$, shows that

$$a_{m-1+2i}(M_{m,n}) = \binom{n}{i} \cdot (m+i)\,. \tag{4.3}$$

For example, $g = 2$ and $\ell = 1$ gives $M_{1,2} = 8_{15}$ with $\nabla(8_{15}) = 1 + 4z^2 + 3z^4$. Thus, even if we consider prime links L, at least when $\#L$ and i are fixed, (4.2) cannot be improved by more than the factor $\#L + i$ independent of g.

(v) For $i = 1$ and K a positive knot, a substantial study was conducted in [St2]. Along with $a_2(K) \geq g(K)$ ([St2, Theorem 6.2]), we knew some modifications and improvements.

(vi) For $i = 0$, the impact of $\#L$ under primeness, suggested by (iv) above, is indeed widely present in a much weaker setting. As we have seen in Proposition 2.14, for a weakly positive prime link which is not the Hopf link, $a_{\#L-1} \geq \#L$. However, Theorem 4.2 only shows that $a_{\#L-1} \geq 1$.

Since the notion of (weakly) successively almost positive link comes from a study of generalized torsion elements motivated from the bi-orderability of the link group, it deserves to mention the following corollary.

Corollary 4.4. *If L is weakly successively almost positive, then the link group $\pi_1(S^3 \setminus L)$ is not bi-orderable.*

Proof. Since $\nabla_L(z)$ is strictly positive, the Alexander polynomial $\Delta_L(t) = \nabla_L(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ cannot have a positive real root. On the other hand, for a link L with $\max\deg_z \nabla_L(z) = 1 - \chi(L)$ (such a link is called *rationally homologically fibered*) if its link group $\pi_1(S^3 \setminus L)$ is bi-orderable, then its Alexander polynomial $\Delta_L(t)$ has at least one positive real root [It1]. Thus $\pi_1(S^3 \setminus L)$ is not bi-orderable. □

4.2. **Unknotting number and the Conway polynomial.** From the standard unknotting/unlinking sequence, we can also relate the unknotting/unlinking numbers and the Conway polynomial.

Proposition 4.5.

(i) *For each weakly successively almost positive knot K,*

$$u(K) \leq u_+(K) \leq a_2(K)\,. \tag{4.4}$$

Moreover, if $u(K) = a_2(K)$ or $u_+(K) = a_2(K)$, then the length of the standard unknotting sequence (3.1) *is equal to $u(K)$, and $a_2(K_i) = u(K) - i$.*

(ii) *For each weakly successively almost positive non-split link L with $\#L > 1$, L can be converted to the connected sum of $\#L - 1$ positive Hopf links by at most $a_{\#L+1}(L) + a_{\#L-1}(L) - 1$ positive-to-negative crossing changes. Furthermore,*

$$\begin{aligned} sp(L) &\leq \#L - 2 + a_{\#L-1}(L) \\ u^{comp}(L) &\leq a_{\#L+1}(L) \\ u(L) &\leq \#L - 2 + a_{\#L-1}(L) + a_{\#L+1}(L) \end{aligned}$$

Here $u_+(K)$ be the *positive unknotting number* of K, the minimum number of positive-to-negative crossing changes needed to make K into the unknot (if such a sequence of crossing change does not exist, we put $u_+(K) = \infty$). For link L, we say $u(L)$ is the unlinking number, the minimum number of crossing changes needed to convert L into the $\#L$-component unlink.

Furthermore, $sp(L)$ is the *splitting number*, the minimum number of crossing changes *between different components of* L to make L into the split union of $\#L$ knots. (Note that in general there can be a shorter sequence of crossing changes when we allow crossings of the same component of L to be switched. $u^{comp}(L) = \sum_{i=1}^{\#L} u(L_i)$ is the total componentwise unknotting number.

Proof. (i) Let $K = K_0 \to K_1 \to \cdots \to K_m = \bigcirc$ be a standard unknotting sequence. Since each crossing change $K_i \to K_{i+1}$ is a part of the standard skein triple (D_+, D_0, D_-) of a minimum w.s.a.p. diagram $D = D_+$ of K_i, by Lemma

3.4 the 2-component link K_i' represented by D_0 is not split. Thus $a_1(K_i') \geq 1$. Therefore

$$\begin{aligned} a_2(K) &= a_2(K_1) + a_1(K_0') \\ &= a_2(K_2) + a_1(K_1') + a_1(K_0') \\ &= \cdots \\ &= a_2(K_m) + \sum_{i=0}^{m-1} a_1(K_m') = \sum_{i=0}^{m-1} a_1(K_m') \geq m \end{aligned}$$

In particular, when $a_2(K) = u(K)$ or $a_2(K) = u_+(K)$, this implies that the length $m = a_2(K)$ and $a_2(K_i) = u(K) - i$.

(ii) For a standard unlinking sequence $L = L_0 \to L_1 \to \cdots \to L_m$ each crossing change $L_i \to L_{i+1}$ is a part of the standard skein triple (D_+, D_-, D_0) of a minimum w.s.a.p. diagram $D = D_+$ of L_i. By Lemma 3.4 the link L_i' represented by D_0 is not split.

When the crossing change $L_i \to L_{i+1}$ is a self-crossing change (the two strands at the crossing belong to the same component of L_i), then $\#L_i' = \#L + 1$. Therefore $a_{\#L}(L_i') = a_{\#L_i'-1}(L_i') \geq 1$, and thus

$$\begin{aligned} a_{\#L-1}(L_i) &= a_{\#L-1}(L_{i+1}), \\ a_{\#L+1}(L_i) &= a_{\#L+1}(L_{i+1}) + a_{\#L}(L_i') \geq a_{\#L+1}(L_{i+1}) + 1 \end{aligned}$$

Similarly, when the crossing change $L_i \to L_{i+1}$ is a non-self-crossing change (the two strands at the crossing belong to different components of L_i), then $\#L_i' = \#L - 1$. Thus $a_{\#L-2}(L_i') = a_{\#L_i'-1}(L_i') \geq 1$ and therefore

$$\begin{aligned} a_{\#L-1}(L_i) &= a_{\#L-1}(L_{i+1}) + a_{\#L-2}(L_i') \geq a_{\#L-1}(L_{i+1}) + 1. \\ a_{\#L+1}(L_i) &= a_{\#L+1}(L_{i+1}) + a_{\#L}(L_i') \geq a_{\#L+1}(L_{i+1}) \end{aligned}$$

Thus if in the standard unknotting sequence, there are m' self-crossing changes and m'' non-self-crossing changes, we have

$$a_{\#L-1}(L) \geq m'' + 1, \ a_{\#L+1}(L) \geq m'$$

Thus L can be converted to L_m, the connected sum of $(\#L - 1)$ Hopf links, by $m' + m'' \leq a_{\#L+1}(L) + a_{\#L-1}(L) - 1$ positive-to-negative crossing changes. Since the link L_m is made into the $\#L$-component unlink by $\#L - 1$ non-self crossing changes, we have $m' + m'' + \#L - 1 \geq u(L)$. Moreover, since each component of L_m is already the unknot,

$$m'' + \#L - 1 \geq sp(L), \quad m' \geq u^{comp}(L)$$

The bound in the first equation is obtained by adding $\#L - 1$ to the value m'' above, where these extra crossing changes are needed to undo the Hopf link factors. Therefore, we conclude

$$\begin{aligned} sp(L) &\leq \#L - 2 + a_{\#L-1}(L), \\ u^{comp}(L) &\leq a_{\#L+1}(L), \\ u(L) &\leq \#L - 2 + a_{\#L-1}(L) + a_{\#L+1}(L)\,. \end{aligned}$$

□

The inequality (4.4) was known for positive knots from [St2, Theorem 6.2]. More precisely, it can be seen from the proof there (compare also with [St2, Theorem 6.4]) that $a_2(K) \geq u(D)$ for the unknotting number of a positive diagram D of K. The unknotting number $u(D)$ of a diagram D is the minimum number of crossing changes needed to convert D into a diagram that represents the unknot. (Working with $u(D)$ here is less convenient, when we choose minimal w.s.a.p. diagrams.)

We next have a property of the signature σ (as explained in the introduction); it is treated in more detail in Section 7.

Proposition 4.6. *If L is a non-split weakly successively almost positive link, then*

$$\min\deg_z \nabla_L(z) = \#L - 1 \leq \sigma(L) \leq \max\deg_z \nabla_L(z).$$

Proof. From a standard unlinking sequence (3.1), by positive-to-negative crossing changes, we may convert L into a connected sum of $\#L - 1$ (positive) Hopf links L_m, so $\sigma(L) \geq \sigma(L_m) = \#L - 1$.

For every link L, by [GiLv, Corollary 2.2]

$$|\sigma(L)| + 1 - \#L + m_1/2 \leq m_2$$

where m_1 is the multiplicity of -1 as a root of the Alexander polynomial $\Delta_L(t) = \nabla(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, and m_2 is the number of roots of $\Delta_L(t)$ (counted with multiplicity) away from 1. We have

$$m_1 \geq 0 \quad \text{and} \quad m_2 \leq \max\deg_z \nabla_L(z) - \min\deg_z \nabla_L(z)\,.$$

Since $\sigma(L) \geq 0$ and $\#L - 1 = \min\deg_z \nabla_L(z)$ for a w.s.a.p. link L, this implies

$$\sigma(L) \leq \#L - 1 + (\max\deg_z \nabla_L(z) - \min\deg_z \nabla_L(z)) = \max\deg_z \nabla_L(z).$$

□

5. Application of standard skein triple II: HOMFLY polynomial

A similar argument proves various special properties of the HOMFLY polynomial.

Theorem 5.1. *Assume L is a weakly successively almost positive link. Then the HOMFLY polynomial $P_L(v,z) = \sum_{i,j} c_{i,j} v^i z^j$ has the following properties.*

- (i) *The coefficients $c_{i,j}$ satisfy the following.*
  - (i-a) *$c_{i,j} = 0$ if $i < j$.*
  - (i-b) *$c_{j,j} \geq 0$ for all j.*
  - (i-c) *$\sum_i c_{i,i} = 1$. Thus $c_{j,j} = 0$ for all j except exactly one j, where $c_{j,j} = 1$.*
  - (i-d) *If L is non-split, $c_{i,1-\chi(L)} \geq 0$ for all i.*
  - (i-e) *L is fibered if and only if $c_{i,1-\chi(L)} = 0$ for all i except exactly one i, where $c_{i,1-\chi(L)} = 1$.*
- (ii) *$\max\deg_z P_L(v,z) = 1 - \chi(L)$.*
- (iii) *If L is non-split, then $\min\deg_v P(L) \geq \#L - 1$. Moreover, if L is a non-trivial knot, then $\min\deg_v P(L) \geq 2$.*

Proof. The proof is similar to the Conway polynomial case and is done by induction on the complexity of L. Let D be a minimum weakly successively positive diagram of L, and let $(D = D_+, D_0, D_-)$ be the standard skein triple.

(i-a), (i-b): They follow from the skein relation

$$P_L(v,z) = v^2 P_{L_-}(v,z) + vz P_{L_0}(v,z).$$

(i-c): It follows from the identity[1]

$$P(v, v^{-1} - v) = 1\,. \tag{5.1}$$

Indeed, this implies

$$1 = P(v, v^{-1} - v) = \sum_{i \geq j} c_{i,j} v^i (v^{-1} - v)^j = \sum_j c_{j,j} + (\text{terms of degree} > 0 \text{ in } v).$$

(i-d): The assertion is easy to see if L_- is non-split. If L_- is split, then by Lemma 3.4 L is a connected sum $L' \# H \# L''$ where L', L'' are non-split weakly successively almost positive links of smaller complexity and H is the positive Hopf link. The HOMFLY polynomial of the positive Hopf link H is $P_H(v, z) = z^{-1}(v - v^3) + vz$, hence H satisfies (i-d). By induction, both L' and L'' satisfy (i-d) and $P_L(v, z) = P_{L'}(v, z) P_H(v, z) P_{L''}(v, z)$, thus we conclude that L also satisfies (i-d).

(i-e): It follows from (i-d) and the observation that $a_{1-\chi(L)}(L) = \sum_i c_{i,1-\chi(L)}$.

(ii): The proof for the non-split link case is the same as for the Conway polynomial case. Assume thus that L is a split link, hence L is a split union of non-split links $L = L_1 \sqcup \cdots \sqcup L_m$. (That is, L_i are the split components of L.) Thus $1 - \chi(L_i) = \max\deg_z P_{L_i}(v, z)$. Since $P_L(v, z) = \left(\frac{v^{-1} - v}{z}\right)^{m-1} P_{L_1}(v, z) \cdots P_{L_m}(v, z)$, we conclude

$$\begin{aligned} 1 - \chi(L) &= 1 - (\chi(L_1) + \cdots + \chi(L_m)) \\ &= (1 - m) + (1 - \chi(L_1)) + \cdots + (1 - \chi(L_m)) \\ &= (1 - m) + \max\deg_z P_{L_1}(v, z) + \cdots + \max\deg_z P_{L_1}(v, z) \\ &= \max\deg_z P_L(v, z). \end{aligned}$$

(iii): The first assertion is obviously true for connected sums of (positive) Hopf links (meant to include the unknot if $\#L = 1$), so the standard unknotting/unlinking sequence and the induction over the complexity prove the first assertion.

For a non-trivial knot K let $(K = K_+, K_0, K_-)$ be the standard skein triple. Since K_0 is a non-split 2-component link, $\min\deg_v P_{K_0}(v, z) \geq 1$. Thus from the skein relation

$$P_K(v, z) = vz P_{K_0}(v, z) + v^2 P_{K_-}(v, z)$$

we conclude $\min\deg_v P_K(v, z) \geq 2$.

□

As an application, we show that homogeneous weakly successively almost positive link is always a positive link. This extends [AT, Corollary 1.7] that says that almost positive homogeneous links are positive (see [Cr] for the definition of homogeneous link).

Corollary 5.2. *A homogeneous link L is weakly successively almost positive if and only if it is positive.*

[1]Note that the coefficient polynomial in v of z^{-m} in P is always divisible by $(v^{-1} - v)^m$, for the substitution to yield a genuine polynomial. A similar remark applies to the the HOMFLY-Jones substitution $V(t) = P(t, t^{1/2} - t^{-1/2})$.

Proof. It is sufficient to show the case L is non-split. For a (non-split) homogeneous link which is not a positive link, strict inequality

$$\min\deg_v P_L(v,z) < 1-\chi(L)$$

holds [Cr, Theorem 4 (b)]. Indeed, Cromwell actually showed that for a non-split, non-positive homogeneous link, there is $i < 1-\chi(L)$ such that $c_{i,1-\chi(L)} \neq 0$ (see [It4, Theorem 2.4]). Therefore, by Theorem 5.1 (i-a), L is not weakly successively almost positive.

□

As for the assertion (i-e) in Theorem 5.1, for the positive link case this single non-zero $c_{i,1-\chi(L)}$ is $c_{1-\chi(L),1-\chi(L)}$, so we expect the following.

Conjecture 5.3. *For a fibered weakly successively almost positive link L, the unique non-zero coefficient $c_{i,1-\chi(L)}$ in (i-e) is $c_{1-\chi(L),1-\chi(L)} = 1$.*

Since the Jones polynomial is recovered from the HOMFLY polynomial, as a consequence we get the following properties of the Jones polynomial.

Theorem 5.4. *Let L be a weakly successively almost positive link, and let $V_L(t)$ be its Jones polynomial.*

(i) *The sign of the coefficient $t^{\min\deg_t V_L(t)}$ is $(-1)^{\#L-1}$.*
(ii) $\min\deg_v P_K(v,z) \leq 2\min\deg_t V_K(t) \leq \max\deg_z P_K(v,z) = 1-\chi(K)$.
(iii) *If L is non-trivial and non-split,* $\min\deg_t V_K(t) \geq \begin{cases} 1 & (\#L=1) \\ \frac{\#L-1}{2} & (\#L>1). \end{cases}$

Proof. Let $P_L(v,z) = \sum_{i,j} c_{i,j} v^i z^j$ be the HOMFLY polynomial of L, and let

$$m = \min\Big\{\frac{1}{2}(2i-j) \mid c_{i,j} \neq 0\Big\}. \tag{5.2}$$

Since

$$V_L(t) = P_L(t, t^{1/2}-t^{-1/2}) = \sum_{i\geq j} c_{i,j} t^i (t^{1/2}-t^{-1/2})^j\,,$$

$$\min\deg_t V_L(t) = \min\deg_t P_L(t, t^{1/2}-t^{-1/2}) \geq m \tag{5.3}$$

Since $P_L(v,z) \in (vz)^{1-\#L}\mathbb{Z}[v^{\pm 2}, z^2]$, the equality happens when the coefficients $c_{i,j}$ satisfy the property

$$c_{i,j} > 0 \text{ whenever } \frac{1}{2}(2i-j) = m \tag{5.4}$$

Now thanks to the skein relation $P_L(v,z) = v^2 P_{L_-}(v,z) + vz P_{L_0}(v,z)$, the property (5.4) holds for L when it holds for both L_- and L_0. Since the HOMFLY polynomial of the unlink satisfies the property (5.4), we conclude that

$$\min\deg_t V_L(t) = m$$

holds for all weakly successively positive links.

Since $c_{i,j} > 0$ in (5.4), their sum is also positive, meaning that the coefficient of t^m in $V_L(t)$ is positive (resp. negative) if even (resp. odd) powers of v occur in $P_L(v,z)$, which is equivalent to $\#L$ being odd (resp. even). Note that the sign comes from the $-$ for $t^{-1/2}$ in (5.3). This shows (i).

Consider (ii). By Theorem 5.1 (i-c), there is an $i_0 = j_0$ with $c_{i_0,j_0} \neq 0$. In (5.2), this gives $2i - j = j_0$, and therefore we conclude

$$m \leq \frac{1}{2} j_0 \leq \frac{1}{2} \max\deg_z P_L(v,z)\,.$$

Then, by Theorem 5.1 (i-a), we have in (5.2) that $i \geq j$, so that $2i - j \geq i$, and we conclude

$$m \geq \frac{1}{2} \min\deg_v P_L(v,z)\,.$$

This proves (ii). Finally, (iii) similarly easily follows from (ii) and Theorem 5.1 (iii). □

6. Bennequin-sharpness and weakly successively almost positive links

The *maximum self-linking number* $\overline{sl}(L)$ is the maximum of self-linking numbers of a transverse link topologically isotopic to L. In terms of a link diagrams, the maximum self-linking number is given by

$$\overline{sl}(L) = \min\{-s(D) + c_+(D) - c_-(D) \mid D \text{ is a diagram of } L\}\,,$$

where $s(D)$ is the number of Seifert circles of D. In [Be], Benneuqin proved the famous *Bennequin inequality*

$$-\chi(L) \geq \overline{sl}(L)$$

The Bennequin-sharp link is a link with sharp Bennequin inequality.

Definition 6.1. We call a link L *Bennequin-sharp* if $-\chi(L) = \overline{sl}(L)$.

It is known that positive and almost positive links are Bennequin sharp.

The Bennequin inequality later had $\chi(L)$ replaced by $\chi_4(L)$, giving the *slice Bennequin inequality*

$$-\chi_4(L) \geq \overline{sl}(L)$$

(see for example [Rud]). Thus if L is Bennequin-sharp then

$$\overline{sl}(K) \leq -\chi_4(L) \leq -\chi(L) = \overline{sl}(K)$$

hence we conclude the following

Proposition 6.2. *For a Bennequin-sharp link L, $-\chi(L) = -\chi_4(L) = \overline{sl}(L)$ holds.*

The results on link polynomials of weakly successively almost positive links can be improved if we further assume that L is Bennequin-sharp.

Theorem 6.3. *Let L be a weakly successively almost positive link. If L is Bennequin-sharp, then the following holds.*

(i) *If L is non-split, $a_{\#L-1+2i}(L) \geq \binom{g(L)}{i}$.*
(ii) $\min\deg_v P_L(v,z) = 2\min\deg_t V_L(t) = \max\deg_z P_L(v,z) = 1 - \chi(L)$.
(iii) *For $j \geq 0$, we have $c_{j,j} = 0$, unless $j = 1-\chi(L)$. Furthermore, $c_{1-\chi(L),1-\chi(L)} = 1$.*
(iv) *If L is non-split, then $c_{1-\chi(L),-1-\chi(L)} \leq 1 - \chi(L)$. Equality holds when L is fibered.*

(v) *If L is non-split, then*

$$V_L(t) = (-1)^{\#L-1}\Big(t^{\frac{1}{2}(1-\chi(L))} + \big(\chi(L) - 1 + c_{1-\chi(L),-1-\chi(L)}\big)t^{\frac{1}{2}(3-\chi(L))} + (\textit{higher degree terms})\Big).$$

In particular, if L is fibered then

$$V_L(t) = (-1)^{\#L-1}t^{\frac{1}{2}(1-\chi(L))} + \big(\textit{terms of } t\textit{-degree at least } \tfrac{1}{2}(5-\chi(L))\big)$$

Proof. (i): It follows from Proposition 6.2 and Theorem 4.2 (i).

(ii): Since L is Bennequin-sharp, by Proposition 6.2

$$1 - \chi(L) = \overline{sl}(L) + 1 \leq \min\deg_v P_L(v,z)\,,$$

where the second inequality is Morton's inequality [Mo]. Thus by Theorem 5.4 (ii),

$$1 - \chi(L) \leq \min\deg_v P_L(v,z) \leq 2\min\deg_t V_L(t) \leq \max\deg_z P_L(v,z) = 1 - \chi(L).$$

(iii): By (ii) and Theorem 5.1 (i-a), we have $c_{i,i} = 0$ unless $i \neq 1-\chi(L)$. Hence by Theorem 5.1 (i-c), $c_{1-\chi(L),1-\chi(L)} = 1$.

(iv): By looking at the coefficient of v^2 in $P_K(v, v^{-1} - v) = 1$ we get

$$-(1-\chi(L))c_{1-\chi(L),1-\chi(L)} + c_{3-\chi(L),1-\chi(L)} + c_{1-\chi(L),-1-\chi(L)} = 0$$

Thus if L is non-split, by Theorem 5.1 (i-d)

$$(1-\chi(L)) - c_{1-\chi(L),-1-\chi(L)} = c_{3-\chi(L),1-\chi(L)} \geq 0 \tag{6.1}$$

as desired. Moreover, when L is fibered (in this case L is automatically non-split), the Conway polynomial $\nabla_L(z) = P_K(1,z)$ is monic. By (iii)

$$1 = a_{1-\chi(L)}(L) = \sum_{j \geq 1-\chi(L)} c_{j,1-\chi(L)} = 1 + \sum_{j > 1-\chi(L)} c_{j,1-\chi(L)}.$$

Since $c_{j,1-\chi(L)} \geq 0$ by Theorem 5.1 (i-d), this implies that $c_{j,1-\chi(L)} = 0$ for all $j > 1-\chi(L)$. Therefore

$$(1-\chi(L)) - c_{1-\chi(L),-1-\chi(L)} = c_{3-\chi(L),1-\chi(L)} = 0.$$

(v): Since $V_L(t) = P_L(t, t^{1/2} - t^{-1/2}) = \sum_{i \geq j} c_{i,j}t^i(t^{1/2} - t^{-1/2})^j$, we immediately get

$$V_L(t) = (-1)^{1-\chi(L)}\left(t^{\frac{1}{2}(1-\chi(L))} + (-1 + \chi(L) + c_{1-\chi(L),-1-\chi(L)})t^{\frac{3}{2}(3-\chi(L))} + \cdots\right)$$

□

Remark 6.4. It is known that when L is a positive link, then the Theorem 6.3 (iv),(v) is an equivalence [St8]; $c_{1-\chi(L),-1-\chi(L)} = 1-\chi(L)$ (equivalently, the coefficient of $t^{\frac{1}{2}(3-\chi(L))}$ is zero) if and only if L is fibered.

This equivalence can be extended to Bennequin-sharp weakly successively almost positive links, if one can show the following *gap-free property* of the maximal z-degree term of $P_L(v,z)$;

$$c_{i,1-\chi(L)} \neq 0, c_{j,1-\chi(L)} \neq 0 \Rightarrow c_{k,1-\chi(L)} \neq 0 \text{ for all } k = i, i+2, \ldots, j \tag{6.2}$$

i.e., the coefficient of $z^{1-\chi(L)}$ is of the form

$$c_{m,1-\chi(L)}v^m + c_{m+2,1-\chi(L)}v^{m+2} + \cdots + c_{M,1-\chi(L)}v^M$$

for some m, M, where all the coefficients $c_{k,1-\chi(L)}$ are non-zero. Compare with Theorem 4.2 (iii), which can be understood as the assertion that the Conway polynomial of weakly successively almost positive link is gap-free.

Indeed, the inequality (6.1) in the proof of (iv) shows that $c_{1-\chi(L),-1-\chi(L)} = 1-\chi(L)$ if and only if $c_{3-\chi(L),1-\chi(L)} = 0$.

When we assume that $P_L(v,z)$ satisfies the gap-free property (6.2), $c_{3-\chi(L),1-\chi(L)} = 0$ means that $c_{m,1-\chi(L)} = 0$ for all $m \geq 3-\chi(L)$. Therefore

$$a_{1-\chi(L)}(L) = \sum_i c_{i,1-\chi(L)} = c_{1-\chi(L),1-\chi(L)} = 1\,.$$

By Theorem 4.2 (iv), L is fibered.

Since almost positive links are Bennequin-sharp, Theorem 6.3 (ii) positively answers [St4, Question 2].

Corollary 6.5. *If L is almost positive link,*

$$\min\deg_v P_L(v,z) = \max\deg_z P_L(v,z) = 1-\chi(L).$$

7. Positivity of the signature and related invariants

In this section we extend the positivity of the signature to weakly successively almost positive links and derive some consequences.

Theorem 7.1. *If L is a non-trivial weakly successively almost positive link, then $\sigma(L) > 0$. In particular L is not (algebraically) slice, and is not amphicheiral.*

In the proof we use the following properties of the signature.

$$|\sigma(L) - \sigma(L_0)| \leq 1 \text{ if } L_0 \text{ is obtained from } L \text{ by smoothing a crossing.} \tag{7.1}$$

$$\sigma(L) - \#L \equiv \begin{cases} -1 \pmod 4 & \text{if } (2\sqrt{-1})^{1-\#L}\nabla_L(2\sqrt{-1}) > 0 \\ 1 \pmod 4 & \text{if } (2\sqrt{-1})^{1-\#L}\nabla_L(2\sqrt{-1}) < 0 \end{cases} \tag{7.2}$$

The first property is well-known, and the second property is [Shi, Theorem 4]. Let

$$\det(L) = |\nabla_L(2\sqrt{-1})|$$

be the *determinant* of a link L. When K is a knot, then because of $\nabla_0(K) = 1$, we have that $\det(K)$ is always odd (and non-zero). In particular, (7.2) gives a well-known relationship (see [MS], for example), that for a knot K,

$$\sigma(K) \equiv 2 \bmod 4 \text{ if and only if } \det(K) \equiv 3 \bmod 4. \tag{7.3}$$

Proof of Theorem 7.1. Since we have already seen $\sigma(L) \geq \#L - 1$ in Proposition 4.6 for non-split links, it remains to show that $\sigma(K) \neq 0$ for a non-trivial weakly successively almost positive knot K. Assume to the contrary that there is a non-trivial w.s.a.p. knot K with $\sigma(K) = 0$. Let

$$d = \min\{\det(K) \mid K \text{ is a non-trivial w.s.a.p. knot}, \sigma(K) = 0\},$$

and take a non-trivial w.s.a.p. knot K with $\sigma(K) = 0$ and $\det(K) = d$ so that among such knots, its complexity $\mathcal{C}(K)$ is minimum.

Let D be a minimum w.s.a.p. diagram of K and let $(D = D_+, D_0, D_-)$ be the standard skein triple. Since $0 = \sigma(K_+) \geq \sigma(K_-) \geq 0$, we have $\sigma(K_-) = 0$. By the minimum assumption of the complexity of K, this implies that $\det(K_-) > d$.

By (7.2) $\sigma(K) = \sigma(K_-) = 0$ implies that $\nabla_K(2\sqrt{-1}), \nabla_{K_-}(2\sqrt{-1}) > 0$. Also, K_0 is w.s.a.p., so by Proposition 2.4, we have $\sigma(K_0) \geq 0$. Thus by (7.1) we see that $\sigma(K) = \sigma(K_-) = 0$ implies $\sigma(K_0) \in \{0, 1\}$. Hence by (7.2) $(2\sqrt{-1})\nabla_{K_0}(2\sqrt{-1}) \geq 0$. Therefore

$$\begin{aligned}\det(K_+) &= |\nabla_K(2\sqrt{-1})| = \nabla_K(2\sqrt{-1}) = \nabla_{K_-}(2\sqrt{-1}) + (2\sqrt{-1})\nabla_{K_0}(2\sqrt{-1}) \\ &= |\nabla_{K_-}(2\sqrt{-1})| + (2\sqrt{-1})\nabla_{K_0}(2\sqrt{-1}) \\ &\geq \det(K_-) > d.\end{aligned}$$

This is a contradiction. □

We give a few applications of this signature positivity. First we give a characterization of the simplest weakly successively almost positive links and knots.

Proposition 7.2. *For a weakly successively almost positive link L, the following conditions are equivalent:*

(i) *L is a connected sum of $\#L - 1$ positive Hopf links.*
(ii) *$\nabla_L(z) = z^{\#L-1}$.*
(iii) *$a_{\#L-1}(L) = 1$ and $\sigma(L) = \#L - 1$.*

Proof. (i) implies (ii) and (iii). To see (ii) implies (iii), we note that $\nabla_L(z) = z^{\#L-1}$ implies that L is non-split. Therefore $\sigma(L) \geq \#L - 1$ by Proposition 2.14. If $\sigma(L) > \#L - 1$, then by Corollary 4.6, $\nabla_L(z) \neq z^{\#L-1}$, so $\sigma(L) = \#L - 1$.

We show (iii) implies (i). Again, $a_{\#L-1}(L) = 1$ implies that L is non-split, hence by Proposition 2.14, L is a connected sum of $(\#L - 1)$ positive Hopf links and some other w.s.a.p. knots. Since a non-trivial w.s.a.p. knot has non-trivial signature, $\sigma(L) = \#L - 1$ implies that L is a connected sum of $(\#L - 1)$ positive Hopf links. □

Proposition 7.3. *A weakly successively almost positive knot K is the positive (right-handed) trefoil if and only if $a_2(K) = 1$.*

This conclusion was also well-known for positive knots from [St2], but is not known even for almost positive knots.

Proof. The one direction is obvious, so assume $a_2(K) = 1$. Let D be a minimum w.s.a.p. diagram of K and let (D_+, D_-, D_0) be the standard skein triple. Since K_0 is not split, $a_1(K_0) \geq 1$.

Since $a_2(K) = a_2(K_-) + a_1(K_0) = 1$, we need that $a_2(K_-) = 0$ and $a_1(K_0) = 1$. This implies that K_- is the unknot. Since $1 \leq \sigma(K_0) \leq \sigma(K_-) + 1 = 1$, this implies $\sigma(K_0) = 1$, so by Proposition 7.2, K_0 is the Hopf link. Thus $\nabla_K(z) = \nabla_{K_-}(z) + z\nabla_{K_0}(z) = 1 + z^2$. This shows that K is a fibered knot of genus one, and only the trefoil is possible. □

Next we give a slight improvement of the strictly positive property of the Conway polynomial in Theorem 4.2.

Proposition 7.4. *Let L be a weakly successively almost positive non-split link. Assume $a_j(L) = 1$ for some $j > \#L - 1$. Then $\max\deg_z \nabla(L) = j$ (hence $j = 1 - \chi(L)$ and L is fibered).*

Another way of saying this is that all "intermediate" coefficients $a_{\#L-1+2i}(L)$ of $\nabla_L(z)$ for $0 < i < d = g(L)$ are at least 2 for a non-split w.s.a.p. link L.

This will, of course, readily follow from Theorem 4.2 (i) if we are able to see $g(L) = g_4(L)$, which in turn follows if L is Bennequin-sharp by Proposition 6.2. But as long as Bennequin-sharpness is not fully established (we will discuss related questions in [IS], the proposition retains an own (however modest) merit.

Proof. We prove the assertion by induction on the complexity of L. Let D be a minimum w.s.a.p. diagram of L and (L_+, L_0, L_-) be the standard skein triple. Since

$$1 = a_j(L) = a_j(L_-) + a_{j-1}(L_0),$$

we have two cases.

Case 1. $a_{j-1}(L_0) = 0$ and $a_j(L_-) = 1$

By induction, $\max\deg_z \nabla_{L_-}(z) = j$, and $\max\deg_z \nabla_{L_0}(z) < j-1$. Hence $\max\deg_z \nabla_L(z) = j$, as desired.

Case 2. $a_j(L_-) = 0$ and $a_{j-1}(L_0) = 1$

By induction, $\max\deg_z \nabla_{L_-}(z) < j$. If $j-1 > \#L_0 - 1$, then $\max\deg_z \nabla_{L_0}(z) = j-1$ by induction, so $\max\deg_z \nabla_L(z) = j$ as desired.

Thus we assume that $j - 1 = \#L_0 - 1$. Since $j > \#L - 1$, this happens only if $j = \#L + 1 = \#L_0$.

If $\max\deg_z \nabla_{L_0}(z) = j - 1 = \#L_0 - 1$, then $\max\deg_z \nabla_L(z) = j$ as desired. So we may further assume that $\max\deg_z \nabla_{L_0}(z) > j - 1 = \#L_0 - 1$, namely, $\nabla_{L_0}(z) \neq z^{\#L_0-1}$.

By Proposition 2.15 and Proposition 7.2, $a_{\#L_0-1}(L_0) = 1$ and $\nabla_{L_0}(z) \neq z^{\#L_0-1}$ imply that L_0 is a connected sum of $(\#L_0 - 1)$ Hopf links and *non-trivial* w.s.a.p. knots. By Theorem 7.1, $\sigma(L_0) \geq (\#L_0 - 1) + 2 = \#L + 2$.

On the other hand, by Proposition 4.6

$$\#L_- - 1 \leq \sigma(L_-) \leq \max\deg_z \nabla_{L_-}(z) < j = \#L_- + 1\,,$$

which means that $\sigma(L_-) \in \{\#L-1, \#L\}$. This is a contradiction, because $\sigma(L_-) - \sigma(L_0) \in \{-1, 0, 1\}$.

□

Note that Proposition 7.4 implies that for a weakly successively almost positive knot K,

$$\nabla_K(1) = \sum_{i=1}^{g(K)} a_{2i}(K) \geq 2g(K) \tag{7.4}$$

Equality is possible only if K is fibered, and we know yet no examples except $K = 3_1$ and $3_1\#3_1$. Indeed, (7.4) looks to be far from optimal since it follows from Theorem 4.2 (i) that

$$\nabla_K(1) = \sum_{i=1}^{g(K)} a_{2i}(K) \;\geq\; \sum_{i=1}^{g(K)} \binom{2g_4(K)}{i} = 2^{g_4(K)} \geq 2^{\sigma(K)/2}$$

Finally, we give a slightly different characterization of the trefoil and the unknot. A knot K is *2-trivadjacent* (2-adjacent to the trivial knot) if K admits a diagram D having two distinct crossings c_1, c_2 such that when we apply the crossing change at c_1, c_2, or both c_1 and c_2, we get the unknot (see [AK] for the notion of adjacency).

Corollary 7.5. *A weakly successively almost positive knot is 2-trivadjacent if and only if it is either the unknot or the trefoil.*

Proof. Assume that K is non-trivial and 2-trivadjacent, so there is a diagram D with crossings c_1, c_2 such that crossing change at c_1, c_2, or both c_1 and c_2 yields the unknot (note that D may not be w.s.a.p.).

By Theorem 7.1, we have $\sigma(K) > 0$. Since a negative-to-positive crossing change never decreases the signature, this means that both c_1 and c_2 are positive crossings.

For $*_1, *_2 \in \{+, -, 0\}$, let $D_{*_1,*_2}$ be the diagram obtained by replacing each $*_i$ as positive/negative/smoothed crossing (so $D = D_{++}$). By the skein formula of the Conway polynomial, for every diagram D we have

$$z^2 \nabla_{D_{00}}(z) = \left(\nabla_{D_{++}}(z) - \nabla_{D_{-+}}(z) - \nabla_{D_{+-}}(z) + \nabla_{D_{--}}(z)\right).$$

Since each of D_{+-}, D_{-+}, D_{--} represents the unknot, we get $\nabla_K(z) = 1 + z^2 \nabla_{D_{00}}(z)$. This implies $a_2(K) = 0, 1$.

If $a_2(K) = 0$, then K is the unknot by Theorem 4.2 (iii) and (ii). If $a_2(K) = 1$, then K is the trefoil by Proposition 7.3. □

This observation allows us to prove the following far-reaching generalization of [PT, Theorem 1.4].

Theorem 7.6. *Let L be a non-split weakly successively almost positive link.*

(i) *If L is a knot K, then by at most $a_2(K) - 1$ positive-to-negative crossing changes, K can be made to the positive trefoil knot.*
(ii) *If $\#L > 1$, then by at most $a_{\#L+1}(L) + a_{\#L-1} - 1$ positive-to-negative crossing changes, L can be made to the connected sum of $(\#L - 1)$ positive Hopf links.*

Proof. Since (ii) was already proven in Proposition 4.5 (ii), we prove (i). Let D be a minimum complexity w.s.a.p. diagram of K. We prove the theorem by induction on the complexity $\mathcal{C}$. Let c_1 and c_2 be the positive crossings which are the end points of the negative overarc. Note that $c_1 \neq c_2$, since K does have a Hopf link factor. Since crossing changes at c_1 or c_2 give rise to w.s.a.p. diagrams of knots with smaller complexity (Remark 3.6), if the crossing change at c_1 or c_2 gives a non-trivial knot $\tilde{K}$, then by induction one can make K into the trefoil by at most $a_2(\tilde{K}) - 1$ positive-to-negative crossing changes. Since $a_2(\tilde{K}) \leq a_2(\tilde{K})$, the assertion follows.

Thus we assume that both crossing changes, at c_1 or c_2, give rise to the unknot, i.e., $K_{-+} = K_{+-} = \bigcirc$. Let K_{--} be the w.s.a.p. knot obtained by crossing changes at both c_1 and c_2. Since the signature cannot increase by the positive to negative crossing change, $\sigma(K_{--}) = 0$ which means that K_{--} is also the unknot. Therefore K is 2-trivadjacent. By Corollary 7.5, this means that K is the trefoil. □

Corollary 7.7. *Let v be an invariant of knots that satisfies the property*

$$v(L_-) \leq v(L_+).$$

Then $v(\mathsf{Trefoil\ knot}) \leq v(K) \leq a_2(K)$ *for a non-trivial weakly successively almost positive knot* K*. For example,* $0 < \phi(K)$ *for every slice-torus invariant* ϕ*, such as Rasmussen's invariant* s*, or the Heegaard Floer* τ*.*

For a w.s.a.p. knot with $a_2(K) = u(K)(= u_+(K))$, we get the following more specific property.

Proposition 7.8. *Assume that* K *is a weakly successively almost positive knot of* $\sigma(K) = 2$*. If* $a_2(K) = u(K)$*, then* $\det(K) \leq 4a_2(K) - 1$*. Moreover, if equality occurs, then* $g(K) = 1$*.*

Proof. We prove the proposition by induction on $u(K) = a_2(K)$. If $u(K) = a_2(K) = 1$, then K is the trefoil, and the assertion follows. Thus in the following we assume that $u(K) = a_2(K) > 1$.

Let $K = K_0 \to K_1 \to \cdots \to K_{u(K)-1} \to K_{u(K)}$ be a standard unknotting sequence. By Proposition 4.5 (i), by $u(K_1) = a_2(K_1) = u(K) - 1$ so by induction $\det(K_1) \leq 4a_2(K_1) - 1$.

Since $2 = \sigma(K) \geq \sigma(K_1) > 0$, we have $\sigma(K_0) = \sigma(K_1) = 2$. Thus by (7.2), $\det(K_0) = -\nabla_{K_0}(2\sqrt{-1})$ and $\det(K_1) = -\nabla_{K_1}(2\sqrt{-1})$.

Let K' be the (w.s.a.p.) 2-component link obtained by smoothing the crossing of $K = K_0$ where we apply the crossing change. Then K' is non-split, $a_1(K') = 1$, and $\sigma(K') \in \{1, 2, 3\}$.

By Proposition 2.14, $a_1(K') = 1$ implies that either K' is the Hopf link, or, it is non-prime. If K' is non-prime we put $K' = L_1 \# L_2$, where L_1 is the 2-component w.s.a.p. link and L_2 is a w.s.a.p. knot. Since $\sigma(L_1) \geq 1$ and $\sigma(L_2) \geq 2$, we have $\sigma(K') \geq 3$. Thus in this case $\sigma(K') = 3$. By (7.2) this implies that $(2\sqrt{-1})\nabla_{K'}(2\sqrt{-1}) > 0$. Therefore

$$\begin{aligned}\det(K) &= -\nabla_K(2\sqrt{-1}) = -\nabla_{K_1}(2\sqrt{-1}) - (2\sqrt{-1})\nabla_{K'}(2\sqrt{-1}) \\ &< \det(K_1) \leq 4a_2(K_1) - 1 \leq 4a_2(K) - 1.\end{aligned}$$

In this case the equality is always strict.

If K' is the Hopf link,

$$\begin{aligned}\det(K) &= -\nabla_K(2\sqrt{-1}) = -\nabla_{K_1}(2\sqrt{-1}) - (2\sqrt{-1})\nabla_{K'}(2\sqrt{-1}) \\ &= \det(K_1) + 4 \leq 4a_2(K_1) - 1 + 4 = 4a_2(K) - 1.\end{aligned}$$

The inequality is exact, i.e.,

$$\det(K) = 4a_2(K) - 1 \tag{7.5}$$

if and only if $\det(K_1) = 4a_2(K_1) - 1$. By induction $g(K_1) = 1$, hence $\nabla_{K_1}(z) = 1 + a_2(K_1)z^2$. We conclude that if (7.5) holds then $\nabla_K(z) = \nabla_{K_1}(z) + z\nabla_{K'}(z) = 1 + a_2(K)z^2$, so $g(K) = 1$. □

8. Examples

In this section we explore various examples of (non-)weakly almost successively almost positive knots.

8.1. **2-almost positive versus weakly successively almost positive.** The following example, which can be inspected from [St11], gives further evidence that even for the 2-almost positive case, most properties of weakly successively almost positive links are lost. This illustrates how the "coordination" of the negative crossings is crucial.

Example 8.1.

- The figure-eight knot $K = 4_1$ is 2-almost positive. Its Conway polynomial $\nabla_K(z) = 1 - z^2$, so that properties (i), (iii), (iv) of Theorem 4.2 fail. Furthermore,
$$P_K(z) = (v^{-2} - 1 + v^2) - z^2,$$
so *all* the properties of Theorem 5.1 (1-a,1-b,1-c,1-d,1-e) fail. Also $\sigma(K) = 0$, and Theorem 7.1 fails.
- The 2-almost positive knot $K = 12_{1581}(= 12n_{293})$ has $\nabla_K(z) = 1 + 2z^2$, $g(K) = g_4(K) = 2$, hence the properties (i) and (ii) of Theorem 4.2 fail.
- The 2-almost positive knot $K = 13_{6407}(= 13n_{1529})$ has $\nabla_K(z) = 1+6z^2+z^4$, $g(K) = g_4(K) = 2$. $\nabla_K(z)$ is monic, but K is not fibered. Therefore, property (iv) of Theorem 4.2 fails whereas the conditions Theorem 4.2 (i)–(iii) are satisfied. This knot also fails to have the property of Theorem 5.1 (1-d,1-e).

8.2. **Weakly successively almost positive knots up to 12 crossings.**

Theorem 8.2. *Up to 12 crossings, a prime knot K is weakly successively almost positive if and only if it is either positive or almost positive, possibly except* $12_{1408} = 12n_{280}$, $12_{1487} = 12n_{259}$, $12_{1837} = 12n_{709}$, $12_{2037} = 12n_{909}$.

Proof. By Corollary 5.2 it is sufficient to treat non-alternating knots. Theorem 4.2 (ii) and Theorem 5.1 (i-a) shows that they are not weakly successively almost positive, except the following knots. For these knots, we use other properties of weakly successively almost positive knots. Here we append indices after the alternating ones; for example, $11_{379} = 11n_{12}$ and $12_{1408} = 12n_{280}$.

- $10_{140}, 11_{379}$ fall to Theorem 5.1 (i-b).
- 12_{1581}, 12_{1609}, 12_{2038} and 12_{2118}, fall to Theorem 5.1 (i-d).
- 10_{132}, 12_{1409}, 12_{1488}, 12_{1811}, 12_{1870}, fall to Theorem 7.1.
- The remaining 4 knots 12_{1408}, 12_{1487}, 12_{1837}, 12_{2037} are the simplest prime knots for which we cannot decide yet on the weakly successively almost positive (from the properties we established so far).

□

8.3. **Detecting non weakly successively almost positivity .** We have shown many properties of invariants of weakly successively almost positive knots. The following is a (not complete) list.

Name	Property	Reference
∇-test	various properties of Conway polynomial	Theorem 4.2
P-test	various properties of HOMFLY polynomial	Theorem 5.1
σ-test	$\sigma(K) > 0$	Theorem 7.1
$(a_{2j}=1)$-test	If $a_{2j}(K) = 1$ $(j > 0)$, $g(K) = j$ and K is fibered	Proposition 7.4
(a_2, u)-test	K can be converted to the trefoil by at most $a_2(K) - 1$ positive-to-negative crossing changes	Theorem 7.6
$(a_2, \det)$-test	$\det(K) \leq 4a_2(K) - 1$ if $\sigma(K) = 2$ and $a_2(K) = u(K)$	Proposition 7.8

TABLE 1. Summary of tests for weakly successively positive properties

We give examples of non weakly successively almost positive knots that pass the ∇-, P-, and σ-tests which are detected by other tests. To find such examples we need to use knots with more than 12 crossings. We use the tables of [HT] for knots up to 16 crossings. However, we take the numbering of knots so that non-alternating knots are appended after alternating ones of the same crossing number. Thus $13_{4878} = 13a_{4878}$ is the last alternating 13 crossing knot (the $(2, 13)$-torus knot), and $13_{4879} = 13n_1$ is the first non-alternating one.

For higher crossing knots, $g_4(K)$ and $u(K)$ are often unknown, or, not included in the knot tables [LMo]. We often use possibly non-exact values of $g_4(K)$ or $u(K)$ that can be inspected from its diagram (though sometimes one can determine the exact values by computing other invariants).

All the knots in the following examples except the ones in Example 8.6 and 8.8 pass the ∇-, P-, and σ-tests.

It is straightforward that if two *knots* K_1, K_2 are weakly successively almost positive, then so is their connected sum $K_1\#K_2$. A similar statement holds for links, for *one possible* way of choosing components to perform the connected sum. However, at this moment, we do not know whether the connected sum of knots which are not weakly successively almost positive can be weakly successively almost positive. Thus although we are primary interested in prime knots, it deserves to investigate non-prime knots.

Indeed, even if a knot K fails to pass various tests above, it often happens that by taking connected sum with an appropriate knot K', the composite knot $K\#K'$ can pass the tests (see Example 8.7 (B),(C)). Thus taking connected sum often makes it hard to show its non-weakly successively almost positivity. In practical terms, the properties required for most examples are highly detailed, and it is difficult enough to construct even composite knots illustrating them. (Many of these examples make use of the second author's independent compilation of unknotting number one knots.)

Example 8.3. The knots $K = 14_{26091}$ and 14_{43602} satisfy $a_2(K) > a_4(K) = 1$ and $\max\deg_z \nabla = 6$, so by the $(a_{2j} = 1)$-test they are not weakly successively almost positive.

Example 8.4. The knots $K = 16_{838974}$, 16_{954872} and $16_{1263307}$ have $a_2(K) = 1$. Hence they are not weakly successively almost positive by the (a_2, u)-test (or, by Proposition 7.3).

To discuss further the (a_2, u)-test, we need to evaluate the number of crossing changes needed to convert K to the trefoil or the unknot. By the famous Montesinos trick, when a knot K' is obtained from a knot K by n crossing changes, then the double branched covering $\Sigma_2(K')$ is obtained from Dehn surgery on an n-component link in $\Sigma_2(K)$. Since $\det(K) = |H_1(\Sigma_2(K); \mathbb{Z})|$ and

$$\Sigma_2(K_1\# \cdots \#K_m) = \Sigma_2(K_1)\# \cdots \#\Sigma_2(K_m)\,,$$

the homology group $H_1(\Sigma_2(K); \mathbb{Z})$ gives the following criterion.

Proposition 8.5. *Let $K_1, \ldots, K_n$ be knots and $K = K_1\#K_2\# \cdots \#K_n$.*

(i) *If $\gcd(\det(K_1), \ldots, \det(K_n)) > 1$, then $u(K) \geq n$.*
(ii) *If $\gcd(\det(K_1), \ldots, \det(K_n)) > 3 (= \det(3_1))$, it requires at least n crossing changes to convert K to the trefoil.*

Example 8.6. Finding an example of a knot K with $a_2(K) > 1$ where the (a_2, u)-test is crucial is much more subtle. We present non-prime examples.

In this example, we first do not take into account the P-test.

Take a knot K having the following properties.

(a) $u(K) = 1$ and $\sigma(K) = 2$ (this shows $g_4(K) = 1$).

(b) The Conway polynomial $\nabla_K(z) = \sum_{i=0}^{d} a_{2i} z^{2i}$ satisfies the following[2].

(b-1) $a_{2i}(K) > 0$ for all $i = 1, \dots, d$.

(b-2) $a_2(K) = 1$.

(b-3) $\det(K) > 3$.

(b-4) $d = 2g(K) = \max \deg_z \nabla_K(z)$, and K is fibered if $\nabla_K(z)$ is monic.

There are systematic constructions of knots for any ∇ satisfying (a),(b) [St5, Theorem 3.1]. The reason those knots are suboptimal here is that the HOMFLY polynomial and the crossing change to the trefoil were not controlled. Then for $n \geq 2$, the connected sum $K_n = \#^n K$ passes the ∇-test and σ-test. The $(a_{2j} = 1)$-test is not applicable.

On the other hand, by Proposition 8.5 and the property (b-3), to convert K_n into a trefoil knot requires at least $n(= a_2(K_n))$ crossing changes. Thus the (a_2, u)-test shows that K_n is not weakly successively almost positive.

- The knot $K = 10_{156}$ is the simplest example of such a knot. For 10_{156}, only a weaker form of the (a_2, u)-test suffices, because 10_{156} and its connected sums K_n cannot be changed to the trefoil by positive-to-negative crossing changes. This can be checked by looking at the Levine-Tristram signature. Let $e^{2\pi\sqrt{-1}\alpha}$ be a root of the Alexander polynomial of 10_{156}. Then $\sigma_\omega(3_1) = 2 > 0 = \sigma_\omega(10_{156})$ for $\omega = e^{2\pi\sqrt{-1}t}$ with $\frac{1}{3} < t < \alpha$.
- The knot $K = 14_{28430}$ also satisfies the properties (a) and (b). The knot 14_{28430} is changed to the trefoil by a single positive-to-negative crossing change, so K_n is changed to the trefoil by exactly $n(= a_2(K_n))$ positive-to-negative crossing changes. (Fewer crossing changes are ruled by Proposition 8.5 (ii), because $\det(14_{28430}) = 163 > 3$.) This example suggests that the bound $a_2(K) - 1$ in the (a_2, u)-test is the practically broadest possible that effectively keeps the criterion applicable.

Example 8.7. One can see the knots in Example 8.6 to be not weakly successively almost positive by the P-test. To find examples which also pass the P-test, we need to consider knots with more than 16 crossings. We use the order in Burton's census [Bur] for knots with 17-to-19-crossings[3].

(A) The knot $K = 17_{2,261,825}$ satisfies all the properties (a)-(b) and passes the P-test. Hence the connected sum of n copies of K gives an example where the (a_2, u)-test is substantial – it passes the ∇-, σ-, and P-test but is not weakly successively almost positive by the (a_2, u)-test.

As for the knot $K = 10_{156}$ in the previous example, the example (A) has a drawback that we do not know whether K_n can be converted to the trefoil knot by positive-to-negative crossing changes.

[2]By (7.3), under the assumption that $\sigma(K) = 2$ in (a), (b-3) is the same as saying $\det(K) \equiv 3 \bmod 4$ but is not 3.

[3]The numbering obeys the following rule. Index is always a single integer. For non-alternating knots, we again append indices after the alternating ones. The sublists (whichever are non-empty) of torus, satellite and hyperbolic knots are added in this succession.

To give an example where K_n can be converted to the trefoil knot by positive-to-negative crossing changes, as we have seen in $K = 14_{28430}$ in the previous example, we use connected sums with different factors, and we weaken our requirements. We require the properties (a), (b-2),(b-3),(b-4) but we allow for (b-1) to fail.

(B) Let $K_n = 18_{8,672,698}\#\left(\#^{n-1}19_{40,631,307}\right)$.

For the first factor $K = 18_{8,672,698}$, its Conway polynomial is $\nabla_K(z) = 1 + z^2 - z^4 + 2z^6$. K has the property that by a single positive-to-negative crossing change, it can be converted to both the unknot and the trefoil.

The second factor $K' = 19_{40,631,307}$ has the Conway polynomial $\nabla_{K'}(z) = 1 + z^2 + 2z^4 + z^6$. The knot K' has the property that by a single positive-to-negative crossing change it can be converted to the unknot.

Although K and K' do not pass the ∇-test, for $n \geq 3$, their connected sum K_n passes the ∇-test, P-test, and σ-test (the $(a_{2j}=1)$-test is not applicable for K_n). Furthermore, K_n can be converted to the trefoil knot by $n(= a_2(K))$ positive-to-negative crossing changes, but is not converted by $n-1$ positive-to-negative crossing changes by Proposition 8.5. Thus the (a_2, u)-test shows that K_n is not weakly successively almost positive.

(C) Let $K_n = 19_{51,035,906}\#\left(\#^{n-1}19_{40,631,307}\right)$.

The first factor $19_{51,035,906}$ has $\nabla = 1 + z^2 - 4z^4 + 3z^6$. As for crossing changes, it shares the same property as $18_{8,672,698}$. For $n \geq 4$, the knots K_n give further examples of knots passing the ∇-test, P-test, and σ-test, but for which the (a_2, u)-test does the job.

Example 8.8. Finally we examine the effectivity of the $(a_2, \det)$-test. As in Example 8.6, in this example we do not take into account the P-test.

Using the information on unknotting numbers in [LMo], we found that the knots 10_{117}, 11_{378}, 11_{391}, 11_{494}, $3_1\#6_3$ (for unknotting number 2) and 12_{1435} (for unknotting number 3) give examples that pass the ∇-, σ-test, but fail to the $(a_2, \det)$-test.

For higher unknotting number, one can take $3_1\#\left(\#^{m-1}10_{164}\right)$ (whose unknotting number being equal to $a_2 = m$ follows from Proposition 8.5).

Example 8.9. The knots $K = 15_{128571}$ and 15_{148331} pass the ∇-, P-, and σ-tests. Furthermore, $u(K) = a_2(K) = 2$, hence K pass the (a_2, u)-test. We found their unknotting number using the tau-invariant $\tau(K) = 2$. Since $\sigma(K) = 2$ and $\det(K) > 7$, by the $(a_2, \det)$-test, they are not weakly successively almost positive.

There is not enough scope of prime knot tables and their entries for us to offer examples of knots where the $(a_2, \det)$-test is essential and $u(K) \geq 3$. To present examples where the $(a_2, \det)$-test is essential for general unknotting number, we use connected sums.

The knot $K = 16_{658907}$ satisfies $a_2(K) = 1$, $\sigma(K) = 0$, $u(K) = 1$, and $\det(K) = 29$. Furthermore, $\tau(16_{658907}) = \tau(3_1) = 1$. Thus the knots

$$K_m := 3_1\#(\#^{m-1}16_{658907})\,(m > 1)$$

satisfy $\sigma(K_m) = 2$, $a_2(K_m) = u(K_m) = m$, and pass the (a_2, u)-test. Also, using $g(16_{658907}) = 2$, one can see that they pass the P- and ∇-tests. However, K_m are not weakly successively almost positive by the $(a_2, \det)$-test.

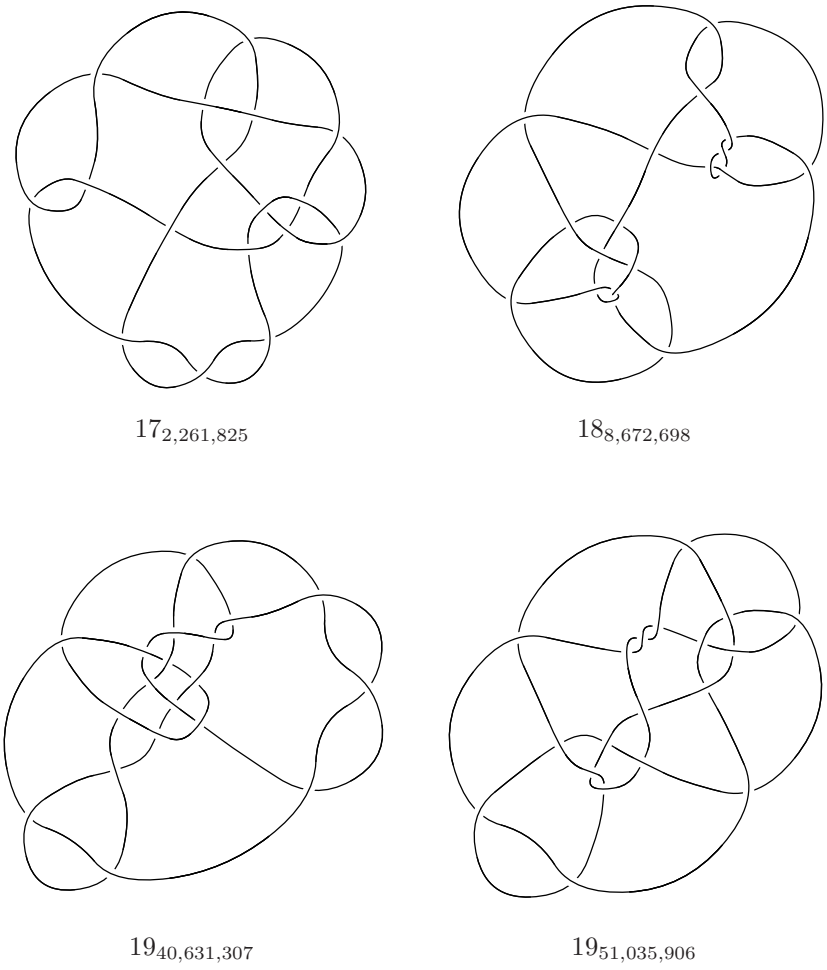


FIGURE 8. Knots used in the construction of Example 8.7

Appendix A. Fibered link enhancement for Scharlemann-Thompson's theorem

In this appendix we prove Theorem A.13. Our proof is a mild extension of Scharlemann-Thompson's proof using sutured manifold theory.

A.1. **Sutured manifold.**

Definition A.1. A *sutured manifold* (M, γ) is a pair of an irreducible oriented compact 3-manifold M and a subset $\gamma \subset \partial M$ consisting of pairwise disjoint annuli such that each annulus contains an oriented core circle.

The *suture* $s(\gamma)$ is the union of the oriented core circles of the annuli γ. We orient each component of $R(\gamma) := \partial M \setminus \gamma$ so that the orientation is coherent with the orientation of the sutures (i.e., the orientation of ∂R coincides with that of the sutures). We denote by $R_+(\gamma)$ (resp. $R_-(\gamma)$) the union of the components of

$R(\gamma)$ such that the orientation agrees (resp. disagrees) with the orientation as a subsurface of ∂M.

Definition A.2. A sutured manifold (M, γ) is *taut* if $R(\gamma)$ is a Thurston norm-minimizing in $H_2(M, \gamma)$.

Example A.3 (Trivial sutured manifold, product sutured manifold)**.** A sutured manifold (M, γ) is a *product sutured manifold* if it is homeomorphic to $(S \times [0, 1], \partial S \times [0, 1])$ for some compact oriented surface S. When $S = D^2$, we say that (M, γ) is the *trivial* sutured manifold.

Example A.4 (Complementary sutured manifold)**.** Let R be a Seifert surface of (non-split) link L in M. The *complementary sutured manifold* (M_R, γ_R) of R is a sutured manifold $(M \setminus (-1, 1) \times R, \partial R \times [-1, 1])$.

For a sutured manifold (M, γ) and a properly embedded surface S that transversely intersects with the sutures $s(\gamma)$, let M' be the 3-manifold obtained from M by cutting along S. Then by connecting sutures naturally, we get a sutured manifold (M', γ'). We refer to such a situation as S *defines a sutured manifold decomposition* $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$.

For example, the complementary sutured manifold (M_R, γ_R) of a Seifert surface R can be seen as a sutured manifold decomposition $(M, \emptyset) \overset{R}{\rightsquigarrow} (M', \gamma')$ defined by R.

A key feature of a sutured manifold decomposition is the following.

Lemma A.5. [Ga1, Lemma 3.5] *Let $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ be a sutured manifold decomposition. If (M', γ') is taut, then (M, γ) is taut.*

Definition A.6. Let (M, γ) be a sutured manifold. A *product annulus* is a properly embedded annulus A in M such that $\partial A \subset R(\gamma)$ and $\partial A \cap R_{\pm}(\gamma) \neq \emptyset$. A *product disk* is a properly embedded disk D in M such that $\gamma \cap D$ is two essential arcs.

A *product decomposition* is a sutured manifold decomposition $(M, \gamma) \overset{\Delta}{\rightsquigarrow} (M', \gamma')$ defined by a product disk Δ.

The following lemma explains a relation between a product disk/annulus and product sutured manifolds.

Lemma A.7. [Ga2, Lemma 2.2, Lemma 2.5] *Let $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ be a sutured manifold decomposition defined by a product disk or a product annulus S. Then (M, γ) is a product sutured manifold if and only if (M', γ') is a product sutured manifold.*

A.2. **Norm-reducing slopes.** Let M be an irreducible 3-manifold whose boundary is a non-empty union of tori, and let P be a component of ∂M. For a slope (unoriented simple closed curve) s of P, we denote by $M(s)$ the Dehn filling on s.

Let S be a properly embedded surface such that $P \cap S = \emptyset$. In this section we review a slope α where S (can) fail to be norm-minimizing in $M(\alpha)$. See [BT] for more general results on how the Thurston norm behaves under Dehn fillings.

Definition A.8. An *I-cobordism* between surfaces S_0 and S_1 is a 3-manifold V whose boundary is $S_0 \cup S_1$ such that the induced maps $(i_j)_* : H_1(S_j) \to H_1(V)$, $(j = 0, 1)$ are both injective.

Let P be a torus boundary component of ∂M and S be a properly embedded surface in M which is disjoint from S. We say that M is *S_P-atoroidal* if, whenever cutting M along a torus T in $M \setminus S$ (and taking a suitable component) gives rise to an I-cobordism between P and T, then T is parallel to P.

To understand norm-reducing slopes, the following existence theorem of sutured manifold hierarchy plays a fundamental role.

Theorem A.9. [Ga3, Step 1 of the proof of Theorem 1.7] *Let R be a norm-minimizing surface which is disjoint from P, and let (M_R, γ_R) be the complementary sutured manifold of R. Then there exists a taut sutured manifold hierarchy*

$$(M_R, \gamma_R) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n), \tag{A.1}$$

such that

(i) *Each separating component of S_i is a product disk.*
(ii) *Each non-separating component of S_i is either a product disk, or a norm-minimizing surface realizing some non-trivial $y \in H_2(H, \partial H)$ which is disjoint from P. Here H denotes the connected component of M_{i-1} that contains P.*
(iii) *(M_n, γ_n) is a union of trivial sutured manifolds and an I-cobordism between a torus T and P with non-empty sutures on T.*

To simplify the notation, in the following arguments we always assume that the surfaces S_i in (A.1) are connected. We will also neglect trivial sutured manifolds components to simply regard (M_n, γ_n) as an I-cobordism between a torus T and P.

Theorem A.10. [Ga3, Theorem 1.8] *Assume that M is irreducible and S_P-atoroidal. Then the surface S remains to be norm-minimizing in $M(s)$, for all but one slope s.*

To give a precise description of the (possible) exceptional slope s, we review the proof.

Proof. Take a sutured manifold hierarchy (A.1) given in Theorem A.9. The assumption that M is S_P-atoroidal says that $M_n \cong P \times [0,1]$. Let s be a slope on P defined by the suture on $T = P \times \{1\}$. We call the slope s the *suture slope* (with respect to the sutured manifold hierarchy (A.1)).

Let us consider the Dehn filling of $M(s')$ of a slope s' on P, for $s \neq s'$. Then by attaching solid tori along P at each stage, the sutured manifold hierarchy (A.1) yields the sutured manifold hierarchy

$$(M_S(s'), \gamma_S) \overset{S_1}{\rightsquigarrow} (M_1(s'), \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n(s'), \gamma_n')$$

Since we have assumed that s' is not equal to the suture slope s, the slope s' does not bound a disk in the solid torus in $M_n(s') \cong D^2 \times S^1$. This means that we can further apply the sutured manifold decomposition

$$(M_n(s'), \gamma_n) \overset{\Delta}{\rightsquigarrow} (M_{n+1}(s'), \gamma_{n+1})$$

along the meridional disk Δ to eventually get the trivial sutured manifold $(M_{n+1}(s'), \gamma_{n+1})$. By Lemma A.5 this implies that the sutured manifold $(M_S(s'), \emptyset)$ is taut, hence $S \subset M(s')$ is norm-minimizing. □

Remark A.11. By some additional arguments, as is stated and proven in [Ga3, Corollary 2.4], the conclusion of Theorem A.10 remains to be true without assuming the S_P-atoroidality assumption[4].

A.3. **Scharlemann-Thompson theorem and its enhancement.** We proceed to prove the fibered link enhancement of Scharlemann-Thompson's theorem.

For the skein triple (L_+, L_0, L_-) formed at the crossing c, let D be the crossing disk at c, and let $K = \partial D$. Let $R = R_+$ be a Seifert surface of $L = L_+$ contained in $S^3 \setminus K$ that attains the maximum Euler characteristic among such Seifert surfaces (so it can happen that $\chi(L_+) \neq \chi(R)$). We can put R so that its intersection $R \cap D$ is a single arc α. By splitting R along α, we get a Seifert surface R_0 of L_0. By (-1) surgery on K we get a link L_-, and R gives rise to a Seifert surface R_- of L_- (see Figure 9).

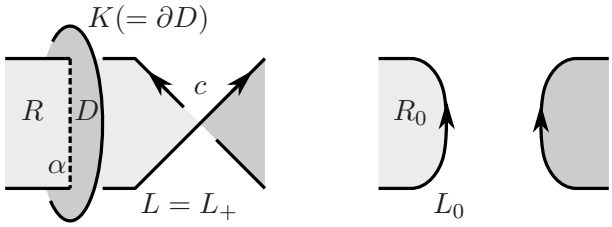


FIGURE 9. Crossing disk D (left) and Seifert surface R_0 of L_0 (right).

Using these terminologies, the Scharlemann-Thompson theorem is stated in the following manner which contains additional information.

Theorem A.12. *(Scharlemann-Thompson theorem, slightly more precise version) At least two of (R_+, R_-, R_0) are norm-minimizing (i.e., attain the maximum Euler characteristic as a Seifert surface in S^3).*

To explain which can be the largest among $\{\chi(R_+), \chi(R_-), \chi(R_0) - 1\}$ (i.e., which can fail to be norm-minimizing) we review the proof.

Proof. Let $M = S^3 \setminus N(L \cup K)$ be the complement of the link $L \cup K$, and let P be the torus boundary component of M which comes from K. By looking at the split factor of L that contains the skein crossing, in the following we always assume that L is non-split and that M is irreducible. Let $M_0, M_-, M_+(= M)$ be the $0, -1, \infty$ surgery on K.

When M is not R_P-atoroidal, Scharlemann-Thompson showed that [ST, Subclaim (a), (b)]

$$M_\pm \text{ are irreducible and } R_\pm \text{ are norm-minimizing.} \tag{A.2}$$

Thus in the following we assume that M is R_P-atoroidal.

In this case, the Seifert surface R_0 of L_0 is related to the surface $R \subset M_0$ in the following manner.

[4]The original assertion [Ga3, Theorem 1.8] states a stronger assertion on the existence of a taut foliation having S as its leaf, which requires the S_P-atoroidality assumption.

Claim 1. [ST, Claim 2] *Let (M,γ) be the complementary sutured manifold for $R \subset M_0$, and let (M_{R_0},γ_{R_0}) be the complementary sutured manifold for $S_0 \subset S^3$. Then there is a product disk decomposition*

$$(M,\gamma) \overset{\Delta}{\rightsquigarrow} (M_{R_0},\gamma_{R_0})$$

Then the assertion follows from Theorem A.10; as we have seen, $R \subset S^3\backslash(L\cup\partial D)$ remains norm-minimizing for every Dehn filling on a slope α of ∂D^2 with at most one exception. The possible exception α is equal to the suture slope s (with respect to some sutured manifold hierarchy (A.1)). □

The fibered link enhancement can be proven in a similar manner, by adding some additional arguments on the sutured manifold hierarchy.

Theorem A.13. *Let (L_+,L_-,L_0) be a skein triple.*

(i) *Assume that $\chi(L_+) = \chi(L_-) < \chi(L_0) - 1$, and that L_- is fibered. Then L_+ is fibered.*
(ii) *Assume that $\chi(L_\pm) = \chi(L_0) - 1 < \chi(L_\mp)$, and that L_0 is fibered. Then $L_\pm$ is fibered.*

Proof of Theorem A.13. (i) Assume that $\chi(L_+) = \chi(L_0) + 1 < \chi(L_-)$ and that L_0 is fibered, hence R_0 is a fiber surface for L_0.

Then L_0 is not split so M_0 is irreducible. If M is not R_P-atoroidal, then by (A.2) $R_\pm \subset M_\pm$ are norm-minimizing. This contradicts the assumption $\chi(L_+) < \chi(L_-)$, so M is R_P-atoroidal.

Let (M_R,γ_R) be the complementary sutured manifold of $R \subset S^3$. Since $\chi(L_+) < \chi(L_-)$, R_- is not norm-minimizing in M_-. Thus by Theorem A.9 there is a sutured manifold hierarchy

$$(M_R,\gamma_R) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n). \tag{A.3}$$

such that its suture slope is -1.

On the other hand, since R_0 is a fiber surface of L_0, by Lemma A.7, the complementary sutured manifold (M_{R_0},γ_{R_0}) of $R_0 \subset S^3$ is a product sutured manifold. By Claim 1 and Lemma A.7, this implies that the complementary sutured manifold (M,γ) of $R \subset M_0$ is a product sutured manifold.

Let us take the taut sutured manifold hierarchy obtained from (A.3) by 0-surgery on each torus P:

$$(M,\gamma) = (M_R(0),\gamma_R) \overset{S_1}{\rightsquigarrow} (M_1(0),\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n(0),\gamma_n).$$

By Theorem A.9 (ii), if S_1 is not a product disk, S_1 is non-separating and $-S_1$ also defines a taut sutured manifold decomposition. Therefore by [Ga2, Corollary 2.7], we have that S_1 is a product annulus. Then S_1 is either a product disk or a product annulus, so by Lemma A.7 $(M_1(0),\gamma_1)$ is a product sutured manifold. Repeating the same argument, we conclude that S_i is either a product disk or a product annulus, and $(M_i(0),\gamma_i)$ is a product sutured manifold for all i.

Therefore the suture of γ_n of $M_n = P\times[0,1]$ consists of two curves of slope -1. This shows that $(M_n(\infty),\gamma_n)$ is also a product sutured manifold. Let us take the taut sutured manifold hierarchy obtained from (A.3) by ∞-surgery on each torus P

$$(M_R(\infty),\gamma_R) \overset{S_1}{\rightsquigarrow} (M_1(\infty),\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n(\infty),\gamma_n).$$

Since we have seen that each S_i is a product disk or a product annulus, $(M_R(\infty), \gamma_R)$, the complementary sutured manifold of $R \subset S^3$, is a product sutured manifold by Lemma A.7. Therefore R is a fiber surface of $L = L_+$.

(ii) We assume that $\chi(L_+) = \chi(L_0) + 1 < \chi(L_-)$ and that L_- is fibered with fiber surface R_-. If M is not R_P-atoroidal, then L_- is a satellite link with winding number zero. However, this contradicts the assumption that L_- is fibered because winding number zero satellite implies that the commutator subgroup of $\pi_1(S^3 \setminus L_-)$ contains the fundamental group of the companion torus $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, so it cannot be the free group.

Thus in this case M is R_P-atoroidal. The rest of the argument is the same; R_0 being not norm-minimizing implies that we can find a sutured manifold hierarchy of the complementary sutured manifold (M_R, γ_R) so that the suture slope is 0. That R_- is a fiber surface implies that each S_i in the sutured manifold hierarchy is a product disk or a product annulus, and the suture of (M_n, γ_n) consists of two curves of slope 0. Hence $(M_n(\infty), \gamma_n)$ is also a product sutured manifold. By Lemma A.7 this implies that $(M_R(\infty), \gamma_R)$ is a product sutured manifold, hence R is a fiber surface of $L = L_+$. □

References

[AT] T. Abe and K. Tagami, *Characterization of positive links and the s-invariant for links*, Canad. J. Math. **69(6)** (2017), 1201–1218.

[AK] N. Askitas and E. Kalfagianni, *On knot adjacency*, Topol. Appl. **126(1-2)** (2002), 63–81.

[BT] K. Baker and S. Taylor, *Dehn filling and the Thurston norm.* J. Differential Geom. **112(3)** (2019), 391–409.

[Be] D. Bennequin, *Entrelacements et équations de Pfaff.* Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982), 87–161, Astérisque, 107-108, Soc. Math. France, Paris, 1983.

[Bur] Benjamin A. Burton, *The next 350 million knots*, 36th International Symposium on Computational Geometry (SoCG 2020) (S. Cabello, D.Z. Chen, eds.), Leibniz Int. Proc. Inform., vol. **164**, Schloss Dachstuhl-Leibniz-Zentrum fuer Informatik, 2020, 25:1–25:17, `https://regina-normal.github.io/data.html`

[Bu] J. v. Buskirk, *Positive links have positive Conway polynomial*, Springer Lecture Notes in Math. **1144** (1983), 146–159.

[CC] A. Cavallo and C. Collari, *Slice-torus concordance invariants and Whitehead doubles of links,* Canad. J. Math. **72(6)** (2020), 1423–1462.

[CG] T. Cochran and R. Gompf, *Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and property P.* Topology **27(4)** (1988), 495–512.

[Cr] P. Cromwell, *Homogeneous links,* J. London Math. Soc. (2) **39(3)** (1989), 535–552.

[Ga1] D. Gabai, *Foliations and the topology of 3-manifolds.* J. Differential Geom. **18(3)** (1983), 445–503.

[Ga2] D. Gabai, *Detecting fibred links in S^3*. Comment. Math. Helv. **61(4)** (1986), 519–555.

[Ga3] D. Gabai, *Foliations and the topology of 3-manifolds. II.* J. Differential Geom. **26(3)** (1987), 461–478.

[GiLv] P. M. Gilmer and C. Livingston, *Signature Jumps and Alexander Polynomials for Links*, Proc. Amer. Math. Soc. **144(12)** (2016), 5407–5417.

[GoLi] C. Gordon and C. A. Litherland, *On the signature of a link,* Invent. Math. **47(1)** (1978), 53–69.

[Hs] F. Hosokawa, *On ∇-polynomials of links*, Osaka Math. J. **10** (1958), 273–282.

[Ho] J. Hoste, *The first coefficient of the Conway polynomial,* Proc. Amer. Math. Soc. **95(2)** (1985), 299–302.

[HT] J. Hoste and M. Thistlethwaite, *KnotScape*, a knot polynomial calculation program, available at `http://www.math.utk.edu/~morwen`.

[IMT] T. Ito, K. Motegi, and M. Teragaito, *Generalized torsion and Dehn filling,* Topol. Appl. **301** (2021) 107515.
[It1] T. Ito, *Alexander polynomial obstruction of bi-orderability for rationally homologically fibered knot groups.* New York J. Math. **23** (2017), 497–503.
[It2] T. Ito, *A note on HOMFLY polynomial of positive braid links,* Internat. J. Math. **33**(4) (2022), Paper No. 2250031, 18 pp.
[It3] T. Ito, *Successively almost positive links*, arXiv.2111.14361.
[It4] T. Ito, *A slice Crowmell inequality for homogeneous links,* Bull. Austral. Math. Soc. **114(1)** (2026), 182–190.
[IS] T. Ito and A. Stoimenow, *Weakly successively almost positive links II: Concordance and Class inclusions*, in preparation.
[Le] L. Lewark, *Rasmussen's spectral sequences and the $\mathfrak{sl}_N$-concordance invariants,* Adv. Math. **260** (2014), 59–83.
[LMo] C. Livingston and A. Moore, *KnotInfo unknotting number reference page*, `https://knotinfo.math.indiana.edu/descriptions/unknotting_number.html#specificknots`
[Mo] H. R. Morton, *Seifert circles and knot polynomials*, Proc. Camb. Phil. Soc. **99** (1986), 107–109.
[MS] K. Murasugi and A. Stoimenov, *The Alexander polynomial of planar even valence graphs*, Adv. Appl. Math. **31(2)** (2003), 440–462.
[PT] J. Przytycki and K. Taniyama, *Almost positive links have negative signature,* J. Knot Theory Ramifications **19(2)** (2010), 187–289.
[Rud] L. Rudolph, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. (N.S.) **29(1)** (1993), 51–59.
[ST] M. Scharlemann and A. Thompson, *Link genus and Conway moves*, Comment. Math. Helvetici **64** (1989), 527–535.
[Shi] Y. Shinohara, *On the signature of knots and links,* Trans. Amer. Math. Soc. **156** (1971), 273–285.
[St2] A. Stoimenow, *Positive knots, closed braids and the Jones polynomial*, `math/9805078`, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **2(2)** (2003), 237–285.
[St3] A. Stoimenow, *Gauß diagram sums on almost positive knots*, Compos. Math. **140(1)** (2004), 228–254.
[St4] A. Stoimenow, *On polynomials and surfaces of variously positive links,* J. Eur. Math. Soc. (JEMS) **7(4)** (2005), 477–509.
[St5] A. Stoimenow, *Realizing Alexander Polynomials by Hyperbolic Links*, Expo. Math. **28(2)** (2010), 133–178.
[St6] A. Stoimenow, *On the crossing number of semiadequate links*, Forum Math. **26(4)** (2014), 1187–1246.
[St7] A. Stoimenow, *Minimal genus of links and fibering of canonical surfaces,* Illinois J. Math. **59(2)** (2015), 399–448.
[St8] A. Stoimenow, *Coefficients and non-triviality of the Jones polynomial*, J. Reine Angew. Math. (Crelle's J.) **657** (2011), 1–55.
[St9] A. Stoimenow, *Application of braiding sequences IV: link polynomials and geometric invariants*, Annales de l'Institut Fourier **70(4)** (2020), 1431–1475,
[St10] A. Stoimenow, *On the crossing number of positive knots and braids and braid index criteria of Jones and Morton-Williams-Franks*, Trans. Amer. Math. Soc. **354(10)** (2002), 3927–3954.
[St11] A. Stoimenow, *Knot data tables*, `http://www.stoimenov.net/stoimeno/homepage/ptab/`
[Tr] P. Traczyk, *Nontrivial negative links have positive signature.* Manuscripta Math. 61 (1988), no. 3, 279–284.

Department of Mathematics, Kyoto University, Kyoto 606-8502, JAPAN
Email address: `tetitoh@math.kyoto-u.ac.jp`

Gyeongju-si, Gyeongsangbuk-do 38067, Republic of Korea
Email address: `stoimeno@stoimenov.net`