# HOSTE'S CONJECTURE FOR GENERALIZED FIBONACCI POLYNOMIALS

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**Abstract.** We prove that for every sequence of polynomials $P_0 = 0$, $P_1 = 1$, $P_i(t) = a_i t P_{i-1}(t) + P_{i-2}(t)$ for integers $a_i$, the complex roots of $P_i(z^{1/2} - z^{-1/2})$ satisfy $\Re e\, z > -1$ . This confirms a conjecture of Hoste on the roots of the Alexander polynomial of 2-bridge knots.

*Keywords:* Fibonacci polynomial, polynomial root, alternating knot, Alexander polynomial, 2-bridge link

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## 1 Introduction

Many important polynomials like Fibonacci, Lucas, Chebyshev, Legendre polynomials and specializations of Dickson polynomials satisfy two-term linear recurrence relations, and roots of these polynomials have been of some interest. See e.g. [Am, Bo, Ma, MZ, WS].

In this paper we will consider a class of polynomials generalizing at least the first three families. (With some effort our study can be applied also to the other classes, but this is not the focus here.)

Define for a sequence of non-zero integers $(a_i)_{i=2}^{\infty}$ polynomials $P_i(t)$ by $P_0 = 0$, $P_1 = 1$, and

$$P_i = a_i t P_{i-1} + P_{i-2}\,. \tag{1}$$

It is very easy to see that the polynomials $P_i$ arising from $(\dots, a_{i-2}, a_{i-1}, 0, a_{i+1}, \dots)$ are the same as those from $(\dots, a_{i-2} + a_{i+1}, \dots)$, thus the non-zero property of $a_i$ is only a technical constraint.

Hoste conjectures for $P_i$ that whenever $t \in \mathbb{C}$ is a root, and $t = z^{1/2} - z^{-1/2}$, then

$$\Re e\, z > -1\,. \tag{2}$$

His conjecture is made in the context of Alexander polynomials, and is considered here only for 2-bridge knots. See [KP, St] for the translation to this context.

We will talk more about knot theory at a separate place, but for now we can content ourselves with this purely algebraic statement, whose proof is the main goal of this paper.

**Theorem 1.1** *Every polynomial $P_i$ satisfies (2).*

This result is motivated by the simplicity of the statement, but its non-trivial (although, as one will see, not necessarily sophisticated) resolution. Knot theoretically, 2-bridge knots and links constitute the most basic class of alternating links, and the lack of decision of such a simple statement on this class was somewhat embarrassing. See [KP, St, LMu] for previous efforts.

In particular we will use the following fact from [KP, St]: if $t$ is a zero of $P_i$, then $|t| < 2$. This implies $\Re e\, z > -\frac{3}{2}$, and the $z$ with $-1 \geqslant \Re e\, z > -\frac{3}{2}$ (to which we will restrict ourselves) have been described. Similarly, we remind that the coefficients of $P_i(z^{1/2} - z^{-1/2})$ alternate, thus there are no negative real zeros.

Our approach will be to work numerically and fundamentally rely on the capacity of MATHEMATICA™ [Wo]. Essentially we reduce the verification to a sequence of verifications of inequalities of 1-variable functions and at a few places 1- or 2-dimensional optimization problems (when there is evidence that the function has no singular behaviour). To support the computation, which is crucial, a MATHEMATICA notebook is available on [St2].

It was already long apparent that the main difficulty in using the recursion (1) lies in subsequences of $a_i$ of the form $(1,1,\dots,1)$ or $(-1,\dots,-1)$. The idea is to treat such subsequences separately using an explicit formula (37). This leads us first to study, in the next section, the location of a particular family of complex numbers $\alpha_i$.

The technical details of this obvious idea have unfortunately grown vaster than originally assumed. One central difficulty is to overcome the limiting process $z \to -1$. Dealing with the singular behaviour there is one (if not the) main reason for the length of the treatise. Thus it is better to go to work quickly. For this same reason, we will move more knot theoretic discussion to a potential further paper generalizing theorem 1.1 using the work in [St]. See §5.

The following abbreviations will be used throughout: 'resp.' will mean 'respectively', 'w.l.o.g.' will stand for 'without loss of generality', and 'r.h.s.' resp. 'l.h.s.' for 'right-hand side' resp. 'left-hand side'.

# 2 Some estimates on arguments and norms

## 2.1 arguments

We will regard the complex plane $\mathbb{C}$ as a real plane $\mathbb{R}^2$, with Cartesian coordinates $x = \Re e\, z$, and $\Im m\, z$, and polar coordinates $r, \theta$. Thus $r = r(z) = |z|$ and $\theta = \theta(z) = \arg z \in \mathbb{R}/2\pi\mathbb{Z}$. In most cases we will identify $\mathbb{R}/2\pi\mathbb{Z}$ with $[0, 2\pi)$, but in some situations, in particular when we write $|\arg z|$ and want it to be small, we will choose $\arg z$ to be in $\mathbb{R}/2\pi\mathbb{Z}$ so that its absolute value is minimal. Similarly, $\measuredangle(z,w)$ will be $|\arg(z/w)|$, taken with this convention (for $z = 0$ or $w = 0$ we can set $\measuredangle(z,w) = 0$). We will *not* use $y$ for $\Im m\, z$, which we will write out explicitly; rather $y$ will be used as in (4). Similarly, we will extensively avoid the use of the complex unit; in the few situations we need it, it will be written as $\sqrt{-1}$.

For this section $z$ will be a complex number such that the number $\hat z = -1/z$ satisfies $|\hat z^{1/2} - \hat z^{-1/2}| \leqslant 2$, $\Re e\, \hat z \leqslant -1$ and $\hat z \neq -1$. We will further assume up to conjugation that $\Im m\, \hat z \geqslant 0$.

The domain of such $z$, call it $\hat D$, is bounded above by the half-circle $\sqrt{x - x^2}$ and below by the graph of the function

$$f(x) = \sqrt{-x^2 - 2x + 7 - 4\sqrt{-2x+3}} \quad \text{for} \quad x \in [1/9, 1] \tag{3}$$

(See the shaded region in the graphics above (93).) This calculation can be found in [KP, St]. For reasons already suggestive here, but much more clearly apparent later, it will be useful to have at hand the substitution

$$y = \sqrt{-2x+3} \in [1, 5/3]\,. \tag{4}$$

Further we will write

$$\alpha = \arg z \in [0, \arccos(1/3)] \quad \text{and} \quad |z| = k \cdot \cos\alpha \tag{5}$$

with $\arccos(1/3) \approx 1.23$ and

$$\frac{1}{3} \leqslant k \leqslant 1\,. \tag{6}$$

The right inequality comes from $\hat{D}$ lying in the half-circle, the left one is a rather crude estimate, which will be sufficient, though.

Let us further stipulate, in general, our treatise in the following of numerical constants given in decimal expansion, *and without ellipsis* at the end. For at most 4 decimals given after the period, this exact number will be meant. An exception occurs for 1.23 which will be used as an approximation of $\arccos(1/3)$. For more than 4 decimals given after the period, a constant rounded on the 6th decimal is represented, or on the last decimal if more than 6 decimals are given.

The goal of this section is to prove some important lemmas on the location of a specific family of numbers

$$\alpha_n = \alpha_n(z) = \frac{1-z^n}{1-z^{n-1}}.$$

**Lemma 2.1**

$$\left|\arg\frac{1-z^{n+1}}{1-z^n}\right| \leqslant \arcsin(1/3). \tag{7}$$

*for* $z \in \hat{D}$ *and* $n \geqslant 1$.

**Proof.** This proof, as well as a core of all following proofs, will use numerical computations with MATHEMATICA™ [Wo]. In certain cases (due to access restrictions) we appealed to the free online portal with restricted (but still rather versatile) functionality. Let us stipulate in general below that all constants given in decimal expansion are rounded on the last decimal.

First, the value on the right of (7) comes from the case $n = 1$, which is an easy exercise. MATHEMATICA™ can also solve the corresponding optimization problems

$$\max_{z\in\hat{D}} \tan\left|\arg\frac{1-z^{n+1}}{1-z^n}\right|$$

with $\tan|\arg w| = \dfrac{|\Im m\, w|}{|\Re e\, w|}$, for $n = 2,3,4$. The values are $\tan|\arg w| \leqslant 0.273396$ for $n = 2$ and $0.227404$ for $n = 3$.

Next, let us for general $n$ exclude $z$ with relatively large $\alpha = \arg z$, say, $\alpha > \pi/4$. For this one can use the form

$$\frac{1-z^{n+1}}{1-z^n} = 1 + \frac{z^n(1-z)}{1-z^n},$$

giving

$$\left|\arg\frac{1-z^{n+1}}{1-z^n}\right| \leqslant \arcsin\frac{|z|^n|1-z|}{|1-z^n|},$$

with the denominator allowing for the estimates

$$|1-z^n| \geqslant 1-|z|^n$$

and more accurately

$$|1-z^n| \geqslant \sqrt{1+|z|^{2n}}$$

if $n\arg z \in [\pi/2, 3\pi/2]$ (which is convenient to use for $2 \leqslant n \leqslant 4$). So assume from now on $\alpha < \pi/4$.

For general $n$, we will look instead on

$$\left|\arg\frac{z^{-n}-1}{z^{-n}-z}\right|$$

with the help of some Euclidean geometry. All notations we introduce will remain valid for the entire section. We will write $\overline{XY}$ for the line segment between $X$ and $Y$, and for its length, while $XY$ will stand for the entire line, or for the ray starting from $X$ in direction $Y$, if so indicated.

We consider the points $A = 0$, $B = 1$ and $C = z$ in $\mathbb{C} \simeq \mathbb{R}^2$, and are interested in the angle $\sphericalangle BPC$ with $P = z^{-n}$:

$$\sphericalangle BPC = \left| \arg \frac{z^{-n} - 1}{z^{-n} - z} \right|$$

It will be important for us that $P$ lies on the boundary of or outside the spiral region

$$S = \{ r \leqslant (k \cos\alpha)^{(2\pi-\theta)/\alpha},\ 0 \leqslant \theta \leqslant 2\pi \} \tag{8}$$

Moreover, for those $P$ close to the positive real axis, we will use that $\arg P = 2\pi - \alpha, 2\pi - 2\alpha$, etc. (Since we allow small $\alpha$, we cannot assume anything about $\arg P$ when it is $\ll 2\pi$.)

Next, we will have to replace the rather clumsy base $(\cos\alpha)^{1/\alpha}$ in the description (8) of $S$ by something more workable. L'Hospital's rule yields

$$\lim_{\alpha \to 0^+} (\cos\alpha)^{1/\alpha} = 1\,.$$

This is one serious source of difficulties, as it means that for small $\alpha$, $S$ collapses closer to the unit disk (and thus $P$ can get arbitrarily close to 1). Further, one finds

$$\lim_{\alpha \to 0^+} ((\cos\alpha)^{1/\alpha})' = -\frac{1}{2}\,,$$

and

$$\lim_{\alpha \to 0^+} ((\cos\alpha)^{1/\alpha})'' = \frac{1}{4}\,.$$

I also obtained some upper estimate on

$$((\cos\alpha)^{1/\alpha})'''$$

for $\alpha \in (0, \arccos(1/3)]$ which together with Taylor's rest term formula allows one to conclude

$$(\cos\alpha)^{1/\alpha} \leqslant 1 - \frac{\alpha}{2} + \frac{\alpha^2}{4}\,. \tag{9}$$

This estimate is backed up also by the graph plot of MATHEMATICA™ and will sometimes by used below.

We need to prove $\sphericalangle BPC \leqslant \arcsin(1/3)$. To do this, we consider the set of $P$ with $\sphericalangle BPC \geqslant \arcsin(1/3)$. It consists of two disk segments $D_{1,2}$ passing through $B = 1$ and $C = z$, with of radius $3/2\overline{BC} = 3/2|1 - z|$ and centers lying on the symmetrizer of $BC$. It will be enough to prove that these disk segments are contained in the spiral $S$. More precisely, for intersections in the quadrant $\{\Re e > 0, \Im m < 0\}$ we it will be enough to show

$$(D_1 \cup D_2) \cap \{\theta = 2\pi - n\alpha\} \subset S \cap \{\theta = 2\pi - n\alpha\}\,.$$

Consider for a moment

$$\delta := \sphericalangle BCA$$

one has

$$\frac{3\pi}{4} \geqslant \delta \geqslant \pi/2\,, \tag{10}$$

the right inequality by Thales' theorem, and the left by checking the inequality

$$f(x) \geqslant \sqrt{\frac{1}{4} + x - x^2} - \frac{1}{2}\,.$$

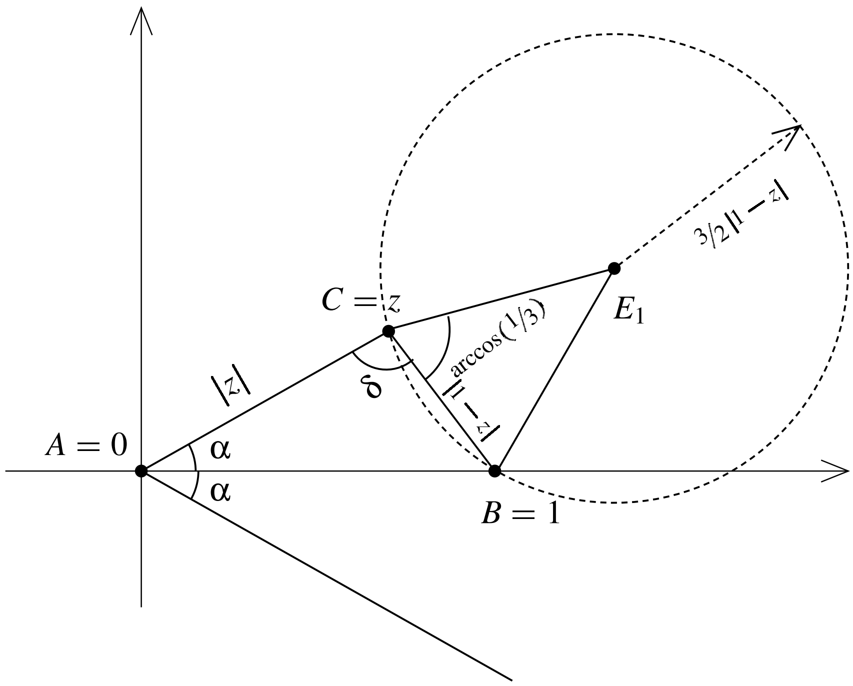
That $\delta \geqslant \pi/2$ easily shows that the symmetrizer of $BC$ and hence the centers of $D_i$ never lie in the quadrant $\{\Re e < 0, \Im m > 0\}$.

We will have to treat the two circle segments separately, and thus it is important to first distinguish them. Let $D_1$, *the upper circle*, be the one whose center lies on the side of the line $BC$ not containing $A = 0$. We call the other circle $D_2$ the *lower circle*.

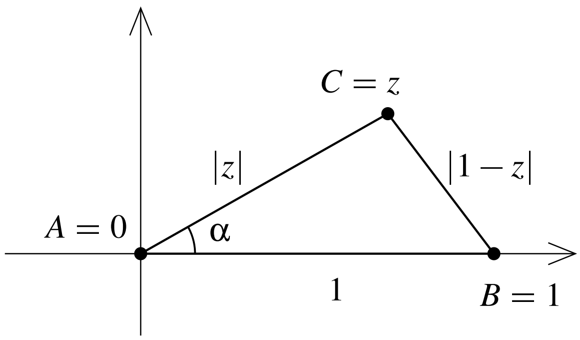
**Case 1.** The upper circle. We will be concerned with checking a few things, which will establish this case.

1. The upper circle does not touch $\{\pi \leqslant \theta \leqslant 2\pi - 2\alpha\}$. The case that it touches $\{\theta = 2\pi - \alpha\}$ will correspond to $n = 1$ which we already discussed.

2. We find intervals $I_\beta$ of $\alpha$ and corresponding angles $\beta \in [0, \pi]$ such that for all $z$ with $\alpha \in I_\beta$ in the given interval the upper circle is contained in
$$\{\theta \leqslant \beta\} \tag{11}$$
and show that the largest possible distance of a point on that circle from the origin is below $((k\cos\alpha)^{1/\alpha})^{\beta - 2\pi}$. When we manage to cover $(0, \pi/4] \ni \alpha$ with such intervals $I_\beta$, we are done.




**Part 1.1.** We start with property 2. The Cosine theorem in $\triangle ABC$ together with (5) and $\sqrt{a^2 + b^2} \leqslant a + b$ gives




$$|1 - z| = \sqrt{1 + k^2\cos^2\alpha - 2k\cos^2\alpha} \leqslant \sin\alpha + (1 - k)\cos\alpha$$

Since the radius of the circle is $\frac{3}{2}|1 - z|$, it is enough to prove the first inequality in the chain

$$|z| + 3|1 - z| \leqslant \frac{1}{k^{2.55}\left(1 - \frac{\alpha}{2} + \frac{\alpha^2}{4}\right)^{2\pi - \beta}} \leqslant \frac{1}{k^{(2\pi - \beta)/\alpha}\left(1 - \frac{\alpha}{2} + \frac{\alpha^2}{4}\right)^{2\pi - \beta}} \leqslant \frac{1}{(k\cos\alpha)^{(2\pi - \beta)/\alpha}},$$

where the second (numeric) estimate uses

$$\frac{2\pi-\beta}{\alpha} \geqslant \frac{\pi}{\alpha} \geqslant \frac{\pi}{\arccos(1/3)} \approx 2.55 > 1\,.$$

Thus it is enough to prove

$$3\sin\alpha + (3-2k)\cos\alpha \leqslant \frac{1}{k^{2.55}\left(1-\frac{\alpha}{2}+\frac{\alpha^2}{4}\right)^{2\pi-\beta}}$$

Comparing partial derivatives in $k$ of both hand-sides and keeping in mind (6) easily shows that it is enough to look at $k=1$:

$$\cos\alpha + 3\sin\alpha \leqslant \frac{1}{\left(1-\frac{\alpha}{2}+\frac{\alpha^2}{4}\right)^{2\pi-\beta}}\,. \tag{12}$$

We begin with $\beta = \pi/12$. This value is good, because (12) holds for all $\alpha$, importantly all close to 0. This is straightforward but tedious to check (using derivatives e.g.) but is mainly owed to the fact that $2\pi - \pi/12 > 6$.

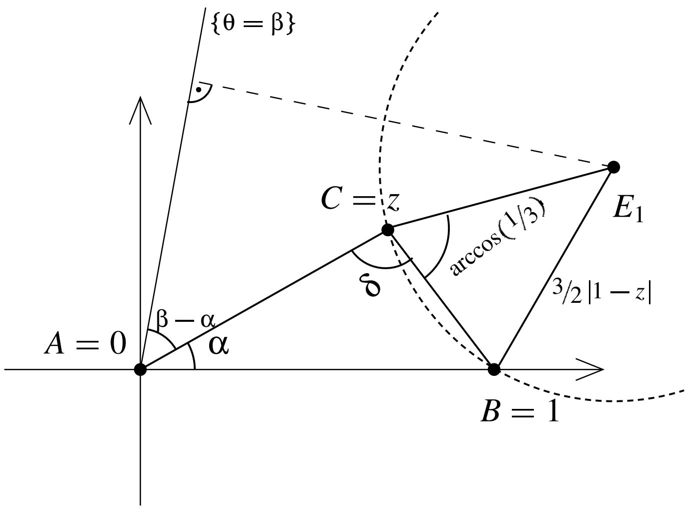
Thus we will be done for all $\alpha \in I_{\pi/12}$, for which we can ascertain that the upper circle in contained in (11) (for $\beta = \pi/12$).

We will need this argument for general $\beta$, so we proceed thus. We will stipulate that for every $\beta$ we choose, we consider as legitimate only values of $\alpha < \beta$.

Now consider the line $L = \{\theta = \beta + m\pi\}$. One can easily calculate the distance of the center of $D_1$ to $L$, which is

$$|z|\sin(\beta-\alpha) + \frac{3}{2}\cos\left(-\frac{\pi}{2} + \arccos(1/3) + \delta - (\beta-\alpha)\right)|1-z|\,.$$

This is explained by the picture below.




We have

$$\eta = 2\pi - \frac{\pi}{2} - (\beta-\alpha) - (2\pi - \delta - \arccos(1/3)) \;=\; (\alpha-\beta) - \frac{\pi}{2} + \delta + \arccos(1/3)\,.$$

This expression has a sign which indicates if the center is in the halfplane containing 1 (then it is positive) or the other halfplane (then it is negative). If the angle

$$\eta = -\frac{\pi}{2} + \arccos(1/3) + \delta - (\beta-\alpha)$$

is negative, this means that the circular arc $A_1$ which bounds the outer circle segment is closest to $L$ in $C = z$, which lies on the desired side of $L$ because we assume $\alpha < \beta$. (Keep in mind that not the entire circle is necessary, but only the part lying on the side of $BC$ not containing 0.)

Thus if $\eta \leqslant 0$, we are done, and we assume $\eta > 0$. Then the minimal distance of a point on the arc $A_1$ to $L$ is

$$|z|\sin(\beta-\alpha) - \frac{3}{2}\left(1-\cos\left(-\frac{\pi}{2}+\arccos(1/3)+\delta-(\beta-\alpha)\right)\right)|1-z|$$

and we wish that this expression is positive, which we will test as

$$|z|\sin(\beta-\alpha) \geqslant \frac{3}{2}\left(1-\cos\left(-\frac{\pi}{2}+\arccos(1/3)+\delta-(\beta-\alpha)\right)\right)|1-z|\,. \tag{13}$$

Using the easy to see $0 < \eta < \pi$, the r.h.s. can be maximized w.r.t. $\delta$ when $\delta$ is maximal, so with (10) set $\delta = \frac{3\pi}{4}$.

Next observe that for fixed $\alpha$, $|z|$ is minimal and $|1-z|$ is maximal when $z$ lies on the boundary part $F$ of $\hat{D}$ which is the graph of $f(x)$ from (3). With the substitution (4), we have

$$\begin{aligned} f(x) &= \tfrac{(y-1)\sqrt{-y^2-2y+7}}{2} \\ |z| = \sqrt{f(x)^2+x^2} &= 2-y \\ |1-z| = \sqrt{f(x)^2+(1-x)^2} &= \sqrt{2}(y-1) \end{aligned} \tag{14}$$

whence also

$$\alpha = \arccos\left(\frac{3-y^2}{2(2-y)}\right) \tag{15}$$

So (13) can be written now

$$(2-y)\sin\left(\beta-\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right) \geqslant \frac{3}{2}\left(1-\cos\left(\frac{\pi}{4}+\arccos(1/3)-\left(\beta-\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right)\right)\right)\cdot\sqrt{2}(y-1) \tag{16}$$

For every relevant $\beta$ is is clear that this inequality will hold when $\alpha$ is small (or equivalently, $y$ is close to 1). So it is relevant to find the equality value of $y$, which can be solved for by MATHEMATICA™ when $\beta$ is specified.

We start with $\beta = \frac{\pi}{12}$. MATHEMATICA™ gives $y \leqslant 1.06718$, from which we get $\alpha \leqslant 0.0695668$ using (15). (We still have to ascertain that $0.0695668 < \beta$, and similarly in the next iterations, but it is not a problem.) Thus $\alpha$ in $I_{\pi/12} = (0, 0.0695668]$ are done.

With $\alpha > 0.0695668$, we test that (12) holds when $\beta = \frac{\pi}{6}$. Solving for equality in (16) with $\beta = \pi/6$ gives $y \leqslant 1.13855$ with $\alpha \leqslant 0.149416$ ($< \pi/6$), so $I_{\pi/6} = [0.0695668, 0.149416]$.

With $\alpha > 0.149416$ we find that (12) holds when $\beta = \frac{\pi}{3}$ giving $y \leqslant 1.28962$ and $\alpha = 0.345344$.

In the next iteration one can use $\beta = 3\pi/5$ yielding $y = 1.52302$ and $\alpha \leqslant 0.776676$. Finally $\beta = 2\pi/3$ pushes $\alpha \geqslant 0.908 > \pi/4$. (Some caution is needed that (12) starts failing when $\beta = 2\pi/3$ for $\alpha \approx 1.2$ if one takes the full range (5) of $\alpha$.)

**Part 1.2.** To show property 1, we consider the line $L = \{\theta = 2\pi - 2\alpha\}$. The property of the arc $A_1$ to lie on the side of $L$ containing $B = 1$ can be written

$$|z|\sin 3\alpha \geqslant \frac{3}{2}(1-\cos\eta)|1-z|\,, \tag{17}$$

where

$$\eta = -\delta - \arccos(1/3) - 3\alpha + \frac{3\pi}{2} \tag{18}$$

Rewrite (17)

$$(\sin 3\alpha)|z| \geqslant \frac{3}{2}\Big(1+\sin\big(\delta+3\alpha+\arccos(1/3)\big)\Big)|1-z| \tag{19}$$

Again let $0 < \alpha < \pi/4$. Thus $\pi/2 < \delta + 3\alpha + \arccos(1/3) < 2\pi$. Thus with (10) the maximal sine in (17) is the one between $\sin(\frac{\pi}{2} + \arccos(1/3) + 3\alpha)$ and $\sin(\frac{3\pi}{4} + \arccos(1/3) + 3\alpha)$.

We will have to discuss two cases.

**Case 1.2.1.** $\arccos(1/3) + 3\alpha < \frac{7\pi}{8}$. Then the maximal sine is $\sin(\frac{\pi}{2} + \arccos(1/3) + 3\alpha)$, so (19) can be simplified to

$$|z| \sin 3\alpha \geqslant \frac{3}{2}(1 + \cos(3\alpha + \arccos(1/3)))|1 - z|$$

which with (14) can again be tested in the form

$$\begin{aligned}(2-y)\sin\left(3\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right) \geqslant \\ \frac{3}{2}\left(1+\cos\left(3\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right)\cdot\frac{1}{3} - \sin\left(3\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right)\cdot\frac{\sqrt{8}}{3}\right)\cdot\sqrt{2}(y-1)\end{aligned} \tag{20}$$

The test of equality using MATHEMATICA™ yields this to be true for $y \leqslant 1.55391$, giving $\alpha = 0.85515 > \frac{\pi}{4}$, so we are done.

**Case 1.2.2.** $\arccos(1/3) + 3\alpha > \frac{7\pi}{8}$. Then the maximal sine is $\sin(\frac{3\pi}{4} + \arccos(1/3) + 3\alpha)$. Then (20) becomes

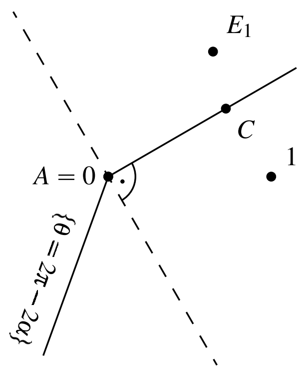
$$\begin{aligned}(2-y)\sin\left(3\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right) \geqslant \\ \frac{3}{2}\left(1+\cos\left(3\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right)\cdot\frac{1/\sqrt{2}-2}{3} - \sin\left(3\arccos\left(\frac{3-y^2}{2(2-y)}\right)\right)\cdot\frac{1/\sqrt{2}+2}{3}\right)\cdot\sqrt{2}(y-1)\end{aligned}$$

MATHEMATICA™ tests it to be true for $y \leqslant 1.48$. This settles $\alpha \leqslant 0.678$.

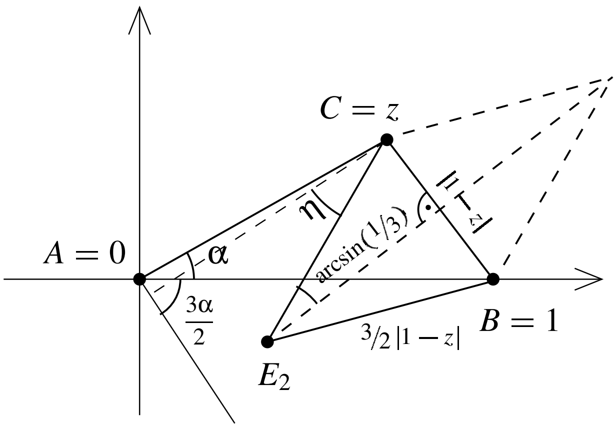
Now assume $\alpha > 0.678$. Then $3\alpha > \pi/2$. Moreover,

$$\frac{3\pi}{2} - \arccos(1/3) - 3\alpha < \frac{\pi}{2},$$

thus in (18) we have $\eta < 0$, which means that the center $E_1$ of the arc $A_1$ lies outside the halfplane of $AC$ ($A = 0, C = z$) containing $B = 1$. But the ray $\{\theta = 2\pi - 2\alpha\}$ lies in that halfplane and makes an angle $\pi > 3\alpha > \pi/2$ with $AC$ (due to our general assumption $\alpha < \pi/4$). Then one can see that the closest point of $\{\pi \leqslant \theta \leqslant 2\pi - 2\alpha\}$ to $E_1$ is the origin $A = 0$. But one also easily sees that $\overline{AE_1} > \overline{CE_1}$, and this finishes this case.



**Case 2.** The lower circle $D_2$, with center $E_2$.




If $\Im m(E_2) > 0$, then $E_2$ lies in the angle segment between the rays $0B$ and $OC$. It is easy to see that the largest polar coordinate radius is attained in the two endpoints of the arc $A_2$, showing it to be 1, clearly within $S$. Thus we are done in this case.

We assume now thus that $\Im m(E_2) < 0$, and thus $2\pi > \theta(E_2) > \pi$.

Every circle in $\mathbb{R}^2$ not containing the origin has two polar radius functions $r = f_{1,2}(\theta)$ (we will used them explicitly later; see (26)). For $D_2$ it is only relevant to consider the larger of the two (which we refer to below as the polar radius).

Note that when $\Im m(E_2) < 0$, the polar radius of $D_2$ will decrease for $\theta > 0$ increasing until $\theta = \theta(E_2) - \pi$ and as long as the polar coordinate is defined. Since the polar radius is 1 for $\theta = 0$, $A_2$ will stay within the spiral $S$ for $0 < \theta \leqslant \theta(E_2) - \pi$. When $\theta > \theta(E_2) - \pi$, the radius of $E_2$ will start increasing, while the radius of $S$ will decrease, and thus only the maximal $\theta$ is needed to test. To see that the radius of $E_2$ is below the radius of $S$ for $\theta = 2\pi - n\alpha$, it will be enough to test this for the smallest relevant values of $n$.

We show that if $\Im m(E_2) < 0$, then $\theta(E_2) > 2\pi - 3\alpha/2$. Then the polar coordinate radius $r$ on $D_2$ will be maximal on $\theta > \pi$ when $\theta = 2\pi - \alpha$. Thus after we checked that the statement holds for $n = 1$, we see it holds for all $n$. Similarly we can replace $3\alpha/2$ by $5\alpha/2$, $7\alpha/2$ and appeal to the previous explicit checks for $n = 2, 3$.

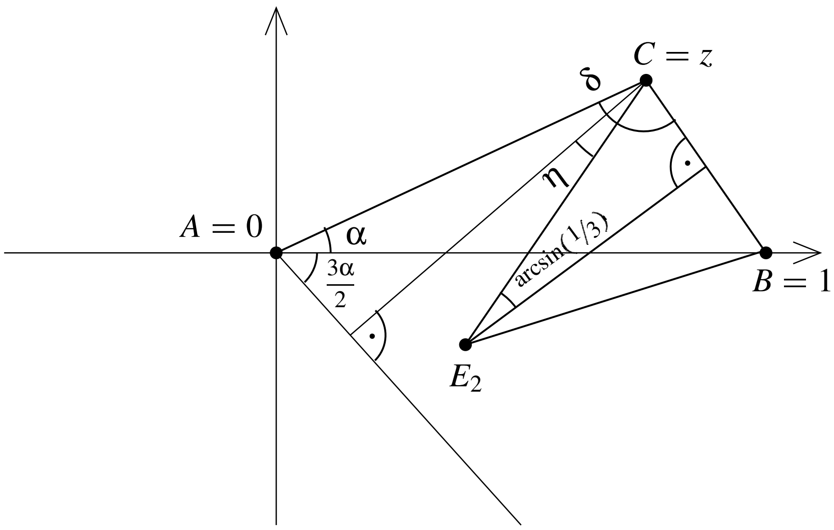
We go on thus now by continuously assuming $\theta(E_2) > \pi$ and trying to show $\theta(E_2) > 2\pi - 3\alpha/2$.

We have to examine when signed distance of $E_2$ to $\{\theta = 2\pi - 3\alpha/2\}$ is negative, where negativity means that $E_2$ lies in the half-plane of $\{\ \theta = m\pi - 3\alpha/2,\ m \in \mathbb{Z}\ \}$ not containing 1. Note: when $3\alpha/2 > \pi/2$, the distance may be negative, but $\pi - 3\alpha/2 < \theta(E_2) < \pi/2$. Thus we test only a necessary condition, not a sufficient one. This is no problem, though, as the extra case was anyway already dealt with. (This scenario is excluded here also because $\alpha < \pi/2$, but the remark will be a little more relevant when we later replace $3\alpha/2$ by $n\alpha + \alpha/2$ for $n > 1$.)

The signed distance is positive iff

$$\sin\frac{5\alpha}{2}|z| \geqslant -\frac{3}{2}\cos\left(\frac{5\alpha}{2} + \arcsin(1/3) + \delta\right)|1 - z| \tag{21}$$

This inequality is explained by the below picture.




We find

$$\eta = \delta - \left(\frac{\pi}{2} - \frac{5\alpha}{2}\right) - \arccos(1/3),$$

thus

$$\cos\eta = -\cos\left(\delta + \frac{5\alpha}{2} + \arcsin(1/3)\right).$$

We consider only the strongest inequality under varying $\delta$ with (10). It can be resolved thus

$$\begin{aligned} \sin\frac{5\alpha}{2}|z| &\geqslant \frac{3}{2}|1-z| \quad \text{for} \quad \frac{5\alpha}{2} + \arcsin(1/3) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \\ \sin\frac{5\alpha}{2}|z| &\geqslant \frac{3}{2}\sin\left(\frac{5\alpha}{2} + \arcsin(1/3)\right)|1-z| \quad \text{for} \quad \frac{5\alpha}{2} + \arcsin(1/3) > \frac{\pi}{2} \\ \sin\frac{5\alpha}{2}|z| &\geqslant -\frac{3}{2}\cos\left(\frac{5\alpha}{2} + \arcsin(1/3) + \frac{3\pi}{4}\right)|1-z| \quad \text{for} \quad \frac{5\alpha}{2} + \arcsin(1/3) < \frac{\pi}{4} \end{aligned}$$

Again we look only on $z$ with $\Im m\, z = f(\Re e\, z)$ with (14). The tests are found successful for $y \leqslant 1.2$ (by plotting graphs, the online portal could not locate the zero), giving $\alpha \leqslant 0.224075$.

The exclusion for $\alpha > \pi/4$ yields also that we need to consider $y \leqslant 1.5266$. (Keeping the exact solution even for a quadratic is bothersome here.)

Replacing $5\alpha/2$ by $7\alpha/2$ in (21) (and appealing to the initial calculation for $n = 2$) for $y \in [1.2, 1.5266]$ shows the inequalities to be satisfied, except for $y \in [1.31, 1.51]$. Finally replacing $7\alpha/2$ by $9\alpha/2$ (and using the check for $n = 3$) settles the inequalities also for these $y$.

This finishes the proof of Lemma 2.1. □

As an afterthought to this lemma, we will remark a way to estimate the quotient of two

$$\alpha_n := \frac{1 - z^n}{1 - z^{n-1}}. \tag{22}$$

**Lemma 2.2** *With $\alpha = \arg z$, and $n \geqslant 2$,*

$$\left|\arg\frac{\alpha_{n+1}}{\alpha_n}\right| \leqslant 2\arcsin(1/3).$$

$$\left|\arg\frac{\alpha_{n+1}}{\alpha_n}\right| \leqslant \alpha + \arcsin(1/3)$$

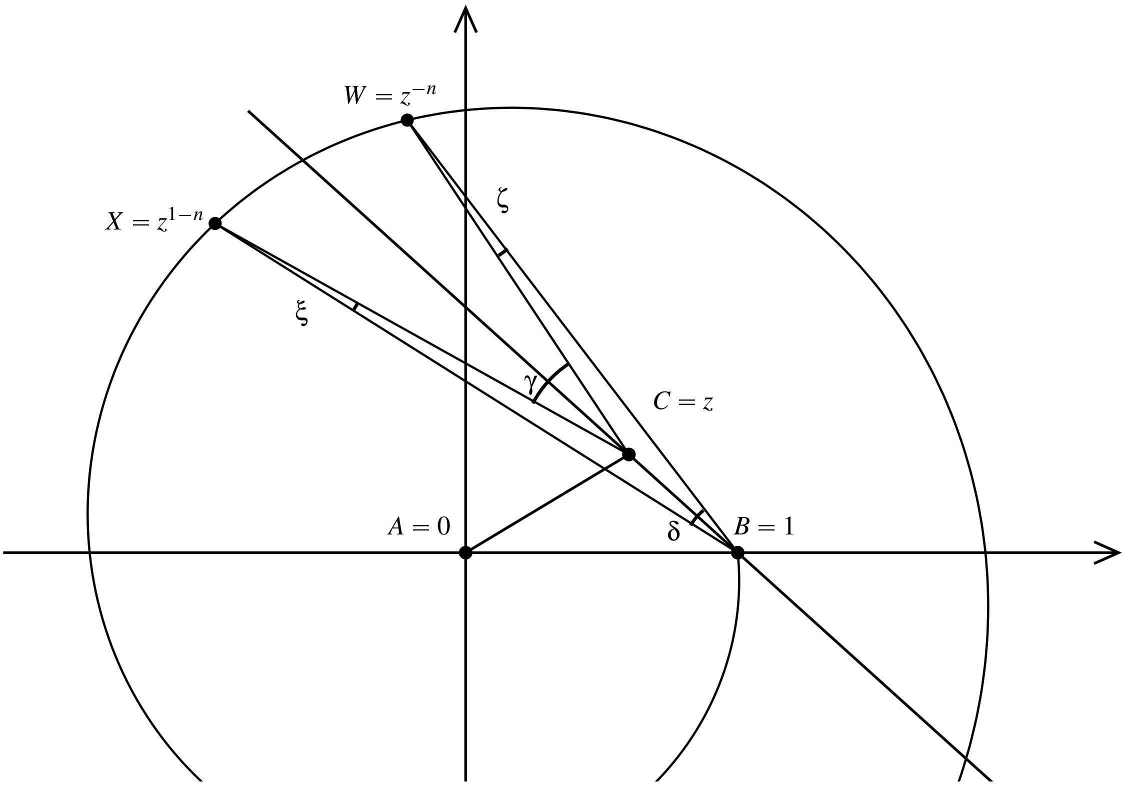
**Proof.** The first inequality is obvious, and for the other one, it is enough to assume that $\arg\alpha_{n+1}$ and $\arg\alpha_n$ have opposite sign.

This means that $X = z^{1-n}$ and $W = z^{-n}$ lie on opposite sides of $BC$ (with $B = 1$ and $C = z$). Let us first argue for the obviously looking fact that $A$, $B$, $C$ line on the same side of $XW$.

If the line segment $\overline{AB}$ is intersected by $XW$ then $n$ must be so that $2\pi - n\alpha > \pi$, $2\pi - (n+1)\alpha < \pi$, but from $\alpha < \pi$ and $|X|, |W| > 1$, one easily sees that $XW$ intersects the negative horizontal axis. Thus $\overline{AB}$ is not intersected by $XW$, and $A$, $B$ lie on the same side of $XW$.

Next, assume $XW$ separates $A$, $C$. Then $XW$ intersects $\overline{AC}$ and contains a point of norm $< |z| < \cos\alpha < 1$. Let $X'$ and $W'$ be the intersections of $XW$ with the unit circle. Let $\beta = \measuredangle X'AW'$. Since the closest point of $\overline{X'W'}$ to $A = 0$ has distance $\cos(\beta/2)$, we must have $\cos(\beta/2) < \cos\alpha$, whence $\beta > \alpha$, and because $S$ in (8) contains the unit disk and $X, W$ are on the boundary of or outside $S$, we have $\measuredangle XAW \geqslant \beta > \alpha$, which is a contradiction to $X$, $W$ being powers of $z$. Thus $A$, $C$ line on the same side of $XW$.

We excluded this obvious absurdity to ascertain the picture is correct:




Be aware that BC and XW may intersect also on the other side of AC; the argument that follows modifies just by a sign change.

Let $\xi = \measuredangle BXC$ and $\zeta = \measuredangle BWC$. Then $\xi + \zeta = \gamma - \delta$ for $\gamma = \measuredangle XCW$ and $\delta = \measuredangle XBW$ (where angles are taken to be positive). Direct calculation shows for $\gamma$ that $\dfrac{z - z^{1-n}}{z - z^{-n}} = \dfrac{z}{\alpha_{n+1}}$ and for $\delta$ that $\dfrac{1 - z^{1-n}}{1 - z^{-n}} = \dfrac{z}{\alpha_n}$. Thus $|\gamma - \delta| \leqslant \max(\gamma, \delta) \leqslant \arcsin(1/3) + \alpha$ (with $\alpha = \arg z$). □

## 2.2 norms

Next we will turn to norms; we use (22). Numerical calculation has *suggested* that $|\alpha_n| \geqslant 0.853324$ (without clear evidence what is this constant), and we will make effort to *prove* some approximation of this, again having to go very close.

**Lemma 2.3** $|\alpha_n| \geqslant 0.85$.

**Proof.** We fix $b = 0.85$. This not only to keep open adapting the argument to other (even a bit better) $b$, but even rather to save a notational mess.

We must prove that the spiral $\{r \leqslant |z|^{\theta/\alpha}\}$ is disjoint from the region

$$\left\{ w \in \mathbb{C} : \left| \frac{1 - zw}{1 - w} \right| \leqslant b \right\}$$

which we will rewrite a bit as

$$\left\{ w \in \mathbb{C} : \left| \frac{1/z - w}{1 - w} \right| \leqslant b|1/z| \right\}$$

The shape of this region depends on $b|1/z|$: if $b/|z| < 1$, it is a circle exterior, if it is $b/|z| = 1$ it is a half-plane, and if it is $b/|z| > 1$ it is a circle interior.

We will consider only the first case here, since it is most relevant. We will thus assume

$$b/|z| < 1$$

which (with (5) and $k \leqslant 1$) is equivalent to

$$b < \cos\alpha \tag{23}$$

We have thus to test whether the circle $A = A_z$ with center $C = C_z$ intersects the spiral $\{r = (k\cos\alpha)^{\theta/\alpha}\}$ (with (5)). The formula for the center is

$$C_z = 1 + (1/z - 1) \cdot \frac{1}{1 - b^2|1/z|^2} \tag{24}$$

and the radius is

$$\rho_z = \left| \frac{1}{z} - 1 \right| \cdot \frac{b|\frac{1}{z}|}{1 - b^2|1/z|^2} \tag{25}$$

The first thing to note is that these circles may not stay below the $x$ axis but will always intersect the $x$ axis above 1 (because the radius is $< |C_z - 1|$). Thus we need to be concerned with intersections below the $x$-axis (i.e., with negative imaginary part).

Note that the polar radius coordinate $r_z(\theta)$ of the circle $A_z$ will decrease for increasing $\theta > \theta(C_z) - \arcsin(\rho_z/|C_z|)$ with $C = C_z$ until $\theta$ reaches $\theta(C)$ and then start increasing, thus it is enough to compare the radius coordinate of the circle with the radius coordinate of the spiral $(k\cos\alpha)^{\theta/\alpha}$ for $\theta \in [\theta(C) - \arcsin(\rho/|C|), \theta(C)]$.

Next we like to argue that we need to consider only $\Re e\, 1/z = 1$. Fix $|z|$, and hence, $1/|z|$, and let $C_0$ and $\rho_0$ be the center and radius of the circle for the $z_0$ of given $|z_0|$ with $\Im m\, 1/z_0 = 1$ and let $\alpha_0 = \arg z_0$.

If $1/z$ moves for fixed $|z|$ away from $\{\Re e\; = 1\}$ (away means so that $\Re e\,(1/z)$ increases while $\Im m\, 1/z$ stays negative), then $\alpha$ gets smaller and $|z| < 1$ constant, so $|z|^{1/\alpha}$ gets smaller and the spiral $\{r \leqslant |z|^{\theta/\alpha}\}$ shrinks. This means that for all these spirals $r \leqslant |z_0|^{\theta/\alpha_0}$ for all $\theta$. We will need this only when $\theta(C_0) - \arcsin(\rho_0/|C_0|) \leqslant \theta < 2\pi$.

But the circle's center $C = C_z$ satisfies

$$|C|^2 = \left| \frac{1}{z} \right|^2 \left( \frac{1}{1 - b^2|1/z|^2} \right)^2 + 1 + 2\Re e \left( \frac{1}{z} - 1 \right) \cdot \frac{1}{1 - b^2|1/z|^2},$$

which will increase, while $\theta(C)$ will decrease, and $\rho$ will also decrease because $\left|\frac{1}{z} - 1\right|$ decreases. This means that all these circles $A_z$ will satisfy in polar coordinates $r_z(\theta) \geqslant |C_0| - \rho_0$ for $\theta(C_0) \leqslant \theta < 2\pi$ and $r_z(\theta) \geqslant r_{z_0}(\theta)$ for $\theta(C_0) - \arcsin(\rho_0/|C_0|) < \theta < \theta(C_0)$. (Note that $r_z(\theta)$ will not exist for many $z$ for these given $\theta$, but this does not cause any problems.)

With this reasoning one can dispose of dealing with the $z$ with $\Re e\, 1/z > 1$. So assume $\Re e\, 1/z = 1$.

With $\Re e\ 1/z = 1$, one can express $1/z$ in terms of $\alpha = \arg z$, and the formulas simplify, and we can write

$$\begin{aligned} |C_z| =: R(\alpha,b) &= \sqrt{1+\left(\frac{\cos\alpha\sin\alpha}{\cos^2\alpha - b^2}\right)^2} \\ \rho_z =: r(\alpha,b) &= \frac{b\sin\alpha}{\cos^2\alpha - b^2} \\ 2\pi - \theta(C_z) =: \gamma(\alpha,b) &= \arctan\frac{\sin\alpha\cos\alpha}{\cos^2\alpha - b^2} \end{aligned}$$

With $\tilde{\theta} = 2\pi - \theta$, the radius formula for the relevant arc of the circle $A_z$ becomes

$$f_{\alpha,b}(\tilde{\theta}) = R(\alpha,b)\cos(\tilde{\theta} - \gamma(\alpha,b)) - \sqrt{r(\alpha,b)^2 - R(\alpha,b)^2\sin^2(\tilde{\theta} - \gamma(\alpha,b))} \tag{26}$$

This function has to be tested to lie above

$$g_\alpha(\tilde{\theta}) = ((\cos\alpha)^{1/\alpha})^{2\pi - \tilde{\theta}}$$

for $\gamma(\alpha,b) \leqslant \tilde{\theta} \leqslant \gamma(\alpha,b) + \arcsin\frac{r(\alpha,b)}{R(\alpha,b)}$.

Since $f,g$ are increasing functions, one can test their inequality by testing that the functions $F_{n,b}(\alpha) := (g_\alpha^{-1} \circ f_{\alpha,b})^{\circ n}(\gamma(\alpha,b))$ satisfy

$$F_{n-1,b} \leqslant F_{n,b} \tag{27}$$

wherever defined.

Here we profit from the fact that $g$ can be explicitly inverted, and

$$g_\alpha^{-1} \circ f_{\alpha,b}(\tilde{\theta}) = 2\pi - \alpha \log_{\cos\alpha} f_{\alpha,b}(\tilde{\theta}). \tag{28}$$

A check with MATHEMATICA™ shows that with $b = 0.85$, (27) holds for $n = 1$, while for $n = 2$, $F_{n,b}$ is no longer defined, i.e.,

$$(g_\alpha^{-1} \circ f_{\alpha,b})^{\circ 2}(\gamma(\alpha,b)) > \gamma(\alpha,b) + \arcsin\frac{r(\alpha,b)}{R(\alpha,b)}. \tag{29}$$

This finished the proof of the lemma for

$$\cos\alpha > 0.85$$

(see (23)). Some similar argument will work also for the other $\alpha$, but we do not get into this here. See lemma 2.9 instead. □

Again we need some improvements and estimates for particular $\alpha_m$.

$$\alpha_m = \frac{1 - z^m}{1 - z^{m-1}}$$

**Lemma 2.4** *1.* $|\alpha_m| \leqslant \frac{3}{2}$ *for* $m \geqslant 3$.

*2.* $|\alpha_m| \leqslant \frac{6}{5}$ *for* $m \geqslant 6$.

*3.* $|\alpha_m| \leqslant 1.1718$ *for* $m \geqslant 7$.

**Proof.** Much of the previous proof can be repeated.

Consider the statement in part 1; the method for the others is the same. The case $m = 3$ can be checked directly (we did not bother for a rigorous argument, but the upper bound can be checked numerically up to any reasonable precision, which will be enough for us). Thus assume $m \geqslant 4$.

Similarly one can check $\alpha > 0.2$; see lemma 2.9.

Let

$$b = 3/2 \quad \text{and} \quad \alpha < 0.2\,. \tag{30}$$

First note the case of small exponents $m$. If $m\alpha < \beta < \pi/2$, then the expression

$$\alpha_m = 1 + \frac{z^{m-1}}{1 + z + \cdots + z^{m-2}} \tag{31}$$

easily shows that

$$|\alpha_m| \leqslant 1 + \frac{1}{(m-1)\cos\beta}\,,$$

thus we are done for

$$m\alpha \,=\, \beta \leqslant \gamma_0 := \arccos\left(\frac{1}{(m-1)(b-1)}\right).$$

With $m \geqslant 4$ and (30) this restricts us to

$$m\alpha \,>\, \arccos(2/3) \,=: \gamma_0 \approx 0.84\,.$$

Since $b/|z| > b > 1$, we have to prove that the spiral part

$$\Sigma_{z,\gamma_0} := \{r \leqslant |z|^{\theta/\alpha},\ \gamma_0 \leqslant \theta \leqslant 2\pi + \gamma_0\} \tag{32}$$

is contained in the complement $\mathbb{C}\backslash A_z$ of the circle $A_z$. The formula for center of $A_z$ is the same as (24), and (25) changes sign

$$\rho_z = \left|\frac{1}{z} - 1\right| \cdot \frac{b|\frac{1}{z}|}{b^2|1/z|^2 - 1}$$

Again one can dispose of $\Re e\ 1/z > 1$, thus. When $|z| = |z'|$ and $|1/z' - 1| > |1/z - 1|$, then we see $\rho_{z'} - \rho_z > |C_{z'} - C_z|$. This means that $A_{z'} \supset A_z$, and since $\theta(z') = \alpha' > \alpha = \theta(z)$, also $\Sigma_{z',\gamma_0} \supset \Sigma_{z,\gamma_0}$. Thus it is clearly enough for $A_z \cap \Sigma_{z,\gamma_0} = \varnothing$ to test $A_{z'} \cap \Sigma_{z',\gamma_0} = \varnothing$, the exclusion for the largest circle $A_z$. This means that for given $|z|$, we need to consider the $z_0$ with largest $|1/z_0 - 1|$, which happens when $\Re e\ 1/z_0 = 1$.

We assume thus $\Re e\ 1/z = 1$. The following formulas are then rather clear.

$$\begin{aligned}
|C_z| \,=:\, R(\alpha,b) \quad &= \quad \sqrt{1 + \left(\frac{\cos\alpha\sin\alpha}{\cos^2\alpha - b^2}\right)^2} \\
\rho_z \,=:\, r(\alpha,b) \quad &= \quad \frac{b\sin\alpha}{b^2 - \cos^2\alpha} \\
\theta(C_z) \,=:\, \gamma(\alpha,b) \quad &= \quad \arctan\frac{\sin\alpha\cos\alpha}{b^2 - \cos^2\alpha}
\end{aligned}$$

Since obviously for $b/|z| > 1$, we have $0 \notin A_z$, we must have $|C_z| > \rho_z$

This means that we are interested in the polar coordinate function of the circle $A_z$,

$$f_{\alpha,b}(\theta) \,=\, R(\alpha,b)\cos(\theta - \gamma(\alpha,b)) - \sqrt{r(\alpha,b)^2 - R(\alpha,b)^2\sin^2(\theta - \gamma(\alpha,b))}$$

defined for all $\gamma(\alpha,b) - \arcsin\frac{r(\alpha,b)}{R(\alpha,b)} < \theta < \gamma(\alpha,b) + \arcsin\frac{r(\alpha,b)}{R(\alpha,b)}$. Note: We regard $\theta$ modulo $2\pi$, so that if $\gamma(\alpha,b) - \arcsin\frac{r(\alpha,b)}{R(\alpha,b)} < 0$, we consider $2\pi > \theta > 2\pi + \gamma(\alpha,b) - \arcsin\frac{r(\alpha,b)}{R(\alpha,b)}$.

This function is geometrically seen increasing between $\gamma < \theta < \gamma + \arcsin\frac{r(\alpha,b)}{R(\alpha,b)}$, and decreasing otherwise (with $\theta$ modulo $2\pi$ when $\gamma(\alpha,b) - \arcsin\frac{r(\alpha,b)}{R(\alpha,b)} < 0$). Thus it is enough with (32) to test with $|z| = \cos\alpha$ that

$$f_{\alpha,b}(\theta) > (\cos\alpha)^{(\theta+2\pi)/\alpha}, \quad \text{for}\ \ \gamma(\alpha,b) > \theta > \gamma(\alpha,b) - \arcsin\frac{r(\alpha,b)}{R(\alpha,b)} \tag{33}$$

and

$$f_{\alpha,b}(\theta) > (\cos\alpha)^{\theta/\alpha}, \quad \text{for } \theta = \gamma_0\,,$$

unless the auxiliary extra value is undefined, i.e.,

$$\arcsin\frac{r(\alpha,b)}{R(\alpha,b)} + \gamma(\alpha,b) < \gamma_0\,. \tag{34}$$

Here, it turns out that this inequality (34) is satisfied when (30).

For the interval test (33), we use the iterated function method of the previous proof. We use the test that with the analogue of (28),

$$g_\alpha^{-1}\circ f_{\alpha,b}(\theta) \;=\; -2\pi + \alpha\log_{\cos\alpha} f_{\alpha,b}(\theta)$$

we have

$$g_\alpha^{-1}\circ f_{\alpha,b}(\gamma(\alpha,b)) \;<\; -\arcsin\frac{r(\alpha,b)}{R(\alpha,b)} + \gamma(\alpha,b)\,.$$

The finishes part 1. For parts 2 and 3 the calculation is similar. Only for the rather delicate bound for $m \geqslant 7$ one has to iterate for $n = 2$ as in (29):

$$(g_\alpha^{-1}\circ f_{\alpha,b})^{\circ 2}(\gamma(\alpha,b)) < \min\left(g_\alpha^{-1}\circ f_{\alpha,b}(\gamma(\alpha,b)),\, -\arcsin\frac{r(\alpha,b)}{R(\alpha,b)} + \gamma(\alpha,b)\right)\,.$$

By testing explicitly $\alpha_m$ for $m \leqslant 10$, one can restrict oneself to $m\alpha \geqslant \gamma_0 = 0.1245$ to have (34) satisfied. □

The following is easy.

**Lemma 2.5** $2 \geqslant |\alpha_2| \geqslant 3/\sqrt{2} = 1.1547\ldots$. *For* $\alpha \leqslant 0.174$ *we have* $|\alpha_2| \geqslant 1.8337\ldots$.

**Proof.** We have

$$\alpha_2 = 1 + z\,.$$

From this the first part is obvious. For the second part, we use the notation of §2. With $\alpha \leqslant 0.174$, and

$$y \;=\; \cos\alpha + \sqrt{\cos^2\alpha - 4\cos\alpha + 3} \tag{35}$$

from (15), we check

$$|\alpha_2| \geqslant \sqrt{f(x)^2 + (x+1)^2} = 2\sqrt{2-y} = 2\sqrt{2 - \cos\alpha - \sqrt{\cos^2\alpha - 4\cos\alpha + 3}} \geqslant 2\sqrt{0.84066} \;=\; 1.8337\ldots\,.$$ □

**Lemma 2.6** $1.5 \geqslant |\alpha_3| \geqslant 0.94744$. *For* $\alpha \leqslant 0.174$*, we have* $|\alpha_3| \geqslant 1.4203\ldots$.

**Proof.** MATHEMATICA easily evaluates the first part. For the second part, again we appeal to rudiment and the calculation in the previous proof using $|z| = 2 - y$ from (14). We have $\alpha_3 = 1 + \frac{z^2}{1+z}$. Thus with $\alpha \leqslant 0.174$,

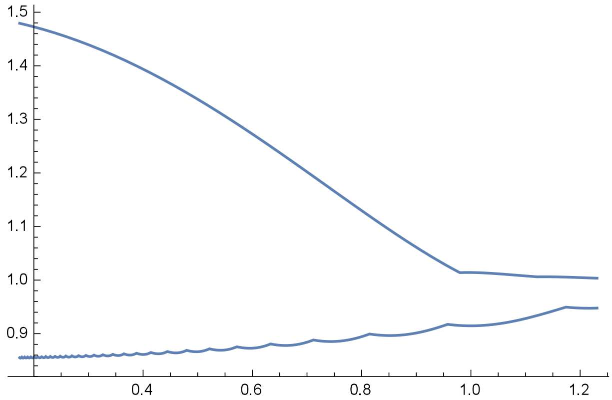
$$|\alpha_3| \geqslant \sqrt{1 + \frac{0.84066^2}{4} + 0.84066\cos(2\cdot 0.174)} \;=\; 1.4203\ldots$$ □

The following bounds were easily calculated by computer.

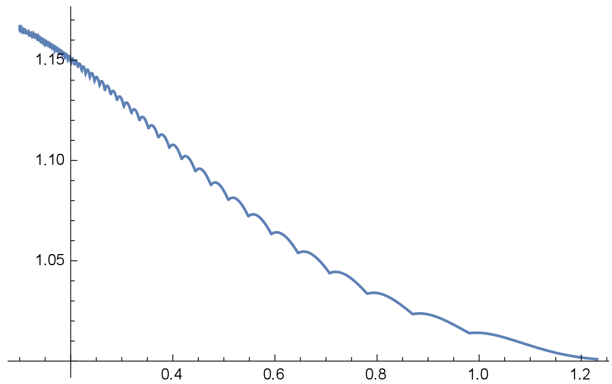
**Lemma 2.7** *For* $\alpha \leqslant 0.174$*, we have* $\frac{4}{3} \geqslant |\alpha_4| \geqslant 1.168$ *and* $\frac{5}{4} \geqslant |\alpha_5| \geqslant 1$.
*For* $\alpha \leqslant 0.35$*, we have* $1.2 \geqslant |\alpha_6| \geqslant 1$.
*For* $\alpha \leqslant 0.3$*, we have* $1.17 \geqslant |\alpha_7| \geqslant 1$. □

**Lemma 2.8** *For all z, we have* $|\alpha_4| \geqslant 0.91465$ *and* $|\alpha_5| \geqslant 0.896456$, $|\alpha_6| \geqslant 0.885285$ *and* $|\alpha_7| \geqslant 0.877903$. □

**Lemma 2.9** *The minimal/maximal values of* $\{|\alpha_m| : m \geqslant 3\}$ *for* $\alpha > 0.174$ *are shown below.*




*For* $\{|\alpha_m| : m \geqslant 7\}$ *and* $\alpha \geqslant 0.1$ *the maximal values are thus:*




**Proof.** We explain the calculation. Fix $\alpha$. Let $v(\alpha) = 2 - \cos\alpha - \sqrt{\cos^2\alpha - 4\cos\alpha + 3}$. Then we can estimate $|\alpha_m|$ for fixed $m$ by a 1-dimensional optimization problem over

$$(\cos m\alpha + \sqrt{-1}\sin m\alpha) \cdot [v(\alpha)^m, \cos^m \alpha] \cup (\cos(m-1)\alpha + \sqrt{-1}\sin(m-1)\alpha) \cdot [v(\alpha)^{m-1}, \cos^{m-1}\alpha]$$

This gives some temporary minimum and maximum values $\mu_m \leqslant |\alpha'_m| \leqslant \nu_m$ for $m' \leqslant m$. Continue increasing $m$ only as long as

$$\frac{1+\cos^m\alpha}{1-\cos^m\alpha} \geqslant \mu_m \text{ or } \frac{1-\cos^m\alpha}{1+\cos^m\alpha} \leqslant \nu_m .$$

(This is a very crude bound, but provides that much extra security.) At that point we know that we can stop and take $\mu_m, \nu_m$.

The gist is that when $\alpha$ is bounded away from 0, not only is the number of iterations per $\alpha$ uniformly bounded, but $\alpha_m$ behave uniformly smoothly. For having an estimate for all $\alpha$, one could solve a certain number of 2-dimensional problems (or formally one could estimate some partial derivative). But given all singular behavior in $\alpha_m$ disappears, there is no reason to distrust numerics, even although it is somewhat time consuming. MATHEMATICA took about 1 day to assemble enough data for this plot and computed 2116 values, which we used to interpolate the function for subsequent work. □

# 3 The conjecture near $-1$

We will now treat only complex numbers $z$ with $|z^{1/2}-z^{-1/2}|<2$ and $\Re e\, z \leqslant -1$. To facilitate further reading, we compile a list of symbols which will be used *consistently* this way *from now on* (though not necessarily used so previously, and with the noted exception).

$z$**:** A complex number with $\Re e\, z \leqslant 1$ and $|z^{1/2}-z^{-1/2}|<2$. We will assume $\Im m\, z>0$. *As an exception*, in §4.1, $z$ will be $-1/z$ of such a number, as in §2.

$\alpha$**:** $|\arg(-z)|$. Clearly $0<\alpha$, and we also know that $\alpha \leqslant \arccos(1/3)$.

$l$**:** A factor we try to estimate a following value of $\Delta(z)$ from the previous value. We have the freedom to choose $l$ to depend on $z$, but in practice we choose it to depend only on $\alpha$.

$k$**:** Up to a factor, $|\Delta(z)|$

$\alpha_m$**:** The expression $\dfrac{1-(-1/z)^m}{1-(-1/z)^{m-1}}$. For $m=1$ we stipulate $\alpha_1=\infty$, which we will use only in the sense that $|1/\alpha_1|=0$.

$a$**:** the number 0.85, serving as a lower bound for $|\alpha_m|$. In some situations, we will replace $a$ in formulas by better lower bounds for special $m$ and $z$.

**Lemma 3.1** *For $v,w\in\mathbb{C}$, when $|v|>|w|$, then* $|\arg(v+w)-\arg(v)| \leqslant \arcsin(|w|/|v|)$. □

**Lemma 3.2** *For $v,w\in\mathbb{C}$, when $|v|>|w|$, and $|\arg(w)|>|\arg(v)|$, then* $|\arg(v+w)| \leqslant |\arg(v)|+|\arg(w)|/2$.

**Proof.** Look at the exterior angle of the triangle with vertices 0, $v$ and $v+w$, and use, say, the Sine law. □

We consider now $\Delta_n = P_n(z^{1/2}-z^{-1/2})$ with $P_n$ as in (1) and $\tilde{\Delta}_n = z^{n/2}\Delta_n$. Further, let us write more precisely $\Delta_n = \Delta(a_2,\dots,a_n)$; and $\tilde{\Delta}_n = \tilde{\Delta}(a_2,\dots,a_n)$.

Then

$$\tilde{\Delta}_n = a_n(z-1)\tilde{\Delta}_{n-1} + z\tilde{\Delta}_{n-2}\,. \tag{36}$$

The following recursion formula is easy to prove by induction and will be important. Let

$$\Delta_{[n]} := \Delta(\dots,a_{l-1},a_l,\underbrace{1,1,\dots,1}_{n-2}),$$

and $\tilde{\Delta}_{[n]} := z^{(l+n)/2-1}\Delta_{[n]}$. Then

$$\Delta_{[n]} = \frac{z^{n/2}}{z^{1/2}+z^{-1/2}}\left[(1-(-1/z)^n)\Delta_{[1]} + (1-(-1/z)^{n-1})\frac{\Delta_{[0]}}{\sqrt{z}}\right] \tag{37}$$

Similarly for

$$\Delta_{[-n]} := \Delta(\dots,a_{l-1},a_l,\underbrace{-1,-1,\dots,-1}_{n-2}),$$

we have

$$\Delta_{[-n]} = -\frac{z^{n/2}}{z^{1/2}+z^{-1/2}}\left[((-1)^n-(1/z)^n)\Delta_{[-1]} + ((-1)^{n-1}-(1/z)^{n-1})\frac{\Delta_{[-0]}}{\sqrt{z}}\right]. \tag{38}$$

We will now study

$$\hat{\Delta}_n = \frac{\tilde{\Delta}_n}{\tilde{\Delta}_{n-1}} = \sqrt{z}\frac{\Delta_n}{\Delta_{n-1}}\,.$$

Let

$$\alpha_n := \frac{1-(-1/z)^n}{1-(-1/z)^{n-1}} \tag{39}$$

and compare with (22). The change from $z$ to $-1/z$ makes the results in §2 applicable for the $z$ we consider here. Also, $\alpha$ of §2 is equal to $|\arg(-z)|$ here.

Note that (with the obvious definition of $\hat{\Delta}_{[n]}$), (37) implies

$$\hat{\Delta}_{[n]} = z\frac{\alpha_n\tilde{\Delta}_{[1]}+\tilde{\Delta}_{[0]}}{\tilde{\Delta}_{[1]}+\tilde{\Delta}_{[0]}/\alpha_{n-1}} = z\alpha_n\frac{\hat{\Delta}_{[1]}+1/\alpha_n}{\hat{\Delta}_{[1]}+1/\alpha_{n-1}}\,. \tag{40}$$

We will be done by proving $|\hat{\Delta}_n| \geqslant l$ for some constant $l > 0$.

The choice of $l$ will be a central problem, as will be the design of a condition that one can put through induction over $n$. We will generally assume

$$\frac{1}{2} \leqslant l < \frac{a}{\cos\alpha} = \frac{0.85}{\cos\alpha}\,. \tag{41}$$

Furthermore, we can let $l$ depend on $z$, but we will let it depend on $\alpha$ only. Let

$$v = |z-1| \geqslant 2\,.$$

With $\alpha = |\arg(-z)|$, and

$$\alpha' = \arccos\left(2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}\right), \tag{42}$$

we need the expression

$$\begin{aligned}\delta &= \arcsin\frac{\sin\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right)+2\alpha'\right)}{\cos\alpha'\sqrt{4(4+\tan^2\alpha')l^2+\frac{1}{\cos^2\alpha'}-4\cdot\frac{l}{\cos\alpha'}\cdot\sqrt{4+\tan^2\alpha'}\cos\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right)+2\alpha'\right)}} \\ &\quad +\arctan\frac{\tan\alpha'}{2} \\ &\leqslant 2\arcsin(1/3)+2\alpha\,.\end{aligned} \tag{43}$$

The stated inequality will be proved below.

We claim now the following property.

**Theorem 3.1** *Let $0 < \alpha = |\arg(-z)| < 0.1091$. Then there is an $l$ such that for all $n \geqslant 1$, the following inequalities hold.*

$$|\hat{\Delta}_n| \geqslant l \quad \textit{and} \quad |\arg(-\hat{\Delta}_n)| \leqslant 2\arcsin(1/3)+2|\arg(-z)| \quad \textit{if} \quad a_n = 1 \tag{44}$$

$$|\arg(-\hat{\Delta}_n)-\arg(1-z)| \leqslant \arcsin(\frac{|z|}{lv}) \quad \textit{if} \quad a_n = 1 \tag{45}$$

$$|\hat{\Delta}_n| \geqslant l \quad \textit{and} \quad |\arg(\hat{\Delta}_n)| \leqslant 2\arcsin(1/3)+2|\arg(-z)| \quad \textit{if} \quad a_n = -1 \tag{46}$$

$$|\arg(\hat{\Delta}_n)-\arg(1-z)| \leqslant \arcsin(\frac{|z|}{lv}) \quad \textit{if} \quad a_n = -1 \tag{47}$$

$$|\hat{\Delta}_n| \geqslant 2v-\frac{|z|}{l} \quad \textit{and} \quad |\arg(-\hat{\Delta}_n)| \leqslant \delta \quad \textit{if} \quad a_n > 1 \tag{48}$$

$$|\arg(-\hat{\Delta}_n)-\arg(1-z)| \leqslant \arcsin(\frac{|z|}{2lv}) \quad \textit{if} \quad a_n > 1 \tag{49}$$

$$|\hat{\Delta}_n| \geqslant 2v-\frac{|z|}{l} \quad \textit{and} \quad |\arg(\hat{\Delta}_n)| \leqslant \delta \quad \textit{if} \quad a_n < -1 \tag{50}$$

$$|\arg(\hat{\Delta}_n)-\arg(1-z)| \leqslant \arcsin(\frac{|z|}{2lv}) \quad \textit{if} \quad a_n < -1 \tag{51}$$

*(See below for the meaning of these inequalities when $n = 1$.)*

**Proof.** The induction start is no problem once one stipulates that when $w \neq 0$, then $|w/0| = \infty$ and taking $\arg(w/0)$ to be $\arg(w)$. The arguments below go through without serious modifications. Alternatively, one starts with $n = 2$. In the case $(a_i)_{i=2}^{\infty}$ starts with $(1,1,\dots,1)$ or $(-1,\dots,-1)$, one should let $\hat{\Delta}_{[1]} = \infty$ in (40) (obtaining $z\alpha_n$) and use (41).

From now on, we dwell only upon the induction step. First note that with (43), all alternatives imply

$$|\hat{\Delta}_n| \geqslant l \quad \textit{and} \quad |\arg(\pm\hat{\Delta}_n)| \leqslant 2\arcsin(1/3) + 2\alpha, \tag{52}$$

if we ascertain

$$2|1-z| - \frac{|z|}{l} \geqslant l. \tag{53}$$

One can check that for given (fixed) $\alpha = |\arg(-z)|$, the value $2|1-z| - \frac{|z|}{l}$ is minimal when $|z|$ is the argument of $\Gamma$ in (64), and that

$$\Phi(\alpha, l) \geqslant l. \tag{54}$$

We assume now all inequalities (44) – (51), and in particular (52), hold for index $\leqslant n$. Using $|\hat{\Delta}_n| \geqslant l$ and (36), one immediately obtains (45), (47), (49) and (51). Use lemma 3.1. This simple inequality in the lemma will be often used implicitly below. Similarly (and very extensively) will be done with the Sine and Cosine laws.

Similarly easily one obtains the norm estimates in (48) and (50). For the rest, we need to discuss a few cases for $a_{n+1}$. Since mirroring only changes signs in $\Delta$, it will be enough, up to mirroring to assume $a_{n+1} > 0$.

**Case 1.** $a_{n+1} = 1$. So $\hat{\Delta}_{n+1} = \hat{\Delta}_{[m+1]}$ for some $m \geqslant 1$, and we use the recursion (40). We can assume $a_{n-m+1} \neq 1$.

**Part 1.1.** We discuss first norms. We have to prove

$$|z| \cdot |\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant l|\hat{\Delta}_{[1]} + 1/\alpha_m| \tag{55}$$

with

$$\alpha_m = \frac{1-(-1/z)^m}{1-(-1/z)^{m-1}}$$

and stipulating

$$1/\alpha_1 = 0. \tag{56}$$

**Case 1.1.1.** $a_{n-m+1} = -1$. The angle between $\alpha_{m+1}\hat{\Delta}_{[1]}$ and 1 is at most $|\arg(\alpha_{m+1})| + |\arg(\hat{\Delta}_{[1]})|$, which is below $\arcsin(1/3) + 2\arcsin(1/3) + 2|\arg(-z)|$, by (44) and (7).

Let $\Delta = \hat{\Delta}_{[1]}$ and $k = |\Delta| \geqslant l$ and $|\alpha_n| \geqslant a$ for all $n$. Also $|z| \geqslant 1/\cos\alpha$.

Let $\gamma = \measuredangle(\Delta, 1/\alpha_m) \leqslant 3\arcsin(1/3) + 2\alpha$. Then by lemma 2.2,

$$\measuredangle(\Delta, 1/\alpha_{m+1}) \leqslant \min(\gamma + \arcsin(1/3) + \alpha, 3\arcsin(1/3) + 2\alpha).$$

Let $p = |1/\alpha_m| \in [0, 1/a]$ (keep in mind (56)) and $q = |\alpha_{m+1}| \in [a, \infty)$.

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant \min_{q \geqslant a} \sqrt{1 + q^2k^2 + 2q \cdot k\cos\, \min(3\arcsin(1/3) + 2\alpha, \gamma + \arcsin(1/3) + \alpha)}. \tag{57}$$

Now

$$\text{when } \alpha < 0.185 \text{ then } 3\arcsin(1/3) + 2\alpha < \tfrac{\pi}{2}, \tag{58}$$

thus

$$ak \geqslant al \geqslant -\cos(3\arcsin(1/3) + 2\alpha),$$

which means that the radicand in (57) is minimized over $q \geqslant a$ for $q = a$. This yields

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant \sqrt{1 + a^2k^2 + 2a \cdot k\cos\, \min(3\arcsin(1/3) + 2\alpha, \gamma + \arcsin(1/3) + \alpha)}.$$

Similarly

$$|\hat{\Delta}_{[1]} + 1/\alpha_m| \leqslant \max_{0 \leqslant p \leqslant 1/a} \sqrt{k^2 + p^2 + 2kp\cos\gamma},$$

giving

$$|\hat{\Delta}_{[1]}+1/\alpha_m| \leqslant \max\left(\sqrt{k^2+\frac{1}{a^2}+2\frac{k}{a}\cos\gamma}, k\right).$$

We need to prove

$$1+a^2k^2+2ak\cos(\gamma+\arcsin(1/3)+\alpha) \geqslant l^2\left(k^2+\frac{1}{a^2}+2\frac{k}{a}\cos(\gamma)\right)\cdot\cos^2\alpha, \tag{59}$$

and

$$1+a^2k^2+2ak\cos(\gamma+\arcsin(1/3)+\alpha) \geqslant l^2k^2\cdot\cos^2\alpha, \tag{60}$$

for $0\leqslant\gamma\leqslant 2\arcsin(1/3)+\alpha$.

The second alternative (60) is easily dealt from (41) and (58). Thus consider the first alternative (59). The linear coefficient in $k$, when all terms are put left, is

$$\cos\gamma\left(\cos(\arcsin(1/3)+\alpha)\cdot 2ka-2\frac{k}{a}l^2\cos^2\alpha\right)-\sin\gamma\cdot 2ka\cdot\sin(\arcsin(1/3)+\alpha). \tag{61}$$

We test that with $l$ as in (81)

$$\cos(\arcsin(1/3)+\alpha)-\frac{l^2\cos^2\alpha}{a^2} > 0.$$

Then the expression (61) as a function of $\gamma\geqslant 0$, is decreasing for small $\gamma$ until a local minimum at

$$\gamma=\pi+\arctan\left(-\frac{\sin(\alpha+\arcsin(1/3))}{\cos(\arcsin(1/3)+\alpha)-l^2\cos^2\alpha/a^2}\right).$$

Now, one tests that with $l$ as in (81)

$$\pi+\arctan\left(-\frac{\sin(\arcsin(1/3)+\alpha)}{\cos(\arcsin(1/3)+\alpha)-l^2\cos^2\alpha/a^2}\right) > 2\arcsin(1/3)+\alpha,$$

which shows that we need to evaluate (61) for the maximal $\gamma = 2\arcsin(1/3)+\alpha$. This leads then to testing the following version of inequality (59),

$$\cos(2\arcsin(1/3)+\alpha)2ka\left(\cos(\arcsin(1/3)+\alpha)-\frac{l^2\cos^2\alpha}{a^2}\right)-\sin(2\arcsin(1/3)+\alpha)\cdot 2ka\sin(\arcsin(1/3)+\alpha) \geqslant k^2\left(l^2\cos^2\alpha-a^2\right)+\left(\frac{l^2\cos^2\alpha}{a^2}-1\right),$$

or

$$k^2\left(l^2\cos^2\alpha-a^2\right)+ 2ka\left[-\cos(2\arcsin(1/3)+\alpha)\left(\cos(\arcsin(1/3)+\alpha)-\frac{l^2\cos^2\alpha}{a^2}\right)+\sin(2\arcsin(1/3)+\alpha)\sin(\arcsin(1/3)+\alpha)\right]+ \left(\frac{l^2\cos^2\alpha}{a^2}-1\right) \leqslant 0 \text{ for all } k\geqslant l. \tag{62}$$

We will collect this inequality as the first of a list of inequalities to test with (81).

**Case 1.1.2.** $a_{n-m+1}<-1$. Similar to previous case but easier because (51) implies (47) and (50) implies (46) (using (43)).

**Case 1.1.3.** $a_{n-m+1}>1$. Let $\Delta=\hat{\Delta}_{[1]}$ and $k=|\Delta|$. We have

$$k\geqslant 2|1-z|-\frac{|z|}{l}. \tag{63}$$

We will work with $l$ depending on $\alpha$ only; see (81) below. One has now to check when for given $\alpha = |\arg(-z)|$, the norm $2|1-z| - \frac{|z|}{l}$ is minimal. Let $\kappa = |z|$. Then the formulas (14) and (15) show that

$$\frac{1}{\cos\alpha} \leqslant \kappa \leqslant 2 - \cos\alpha + \sqrt{\cos^2\alpha - 4\cos\alpha + 3}\,.$$

The function

$$\Gamma(\kappa) = 2\sqrt{\kappa^2 + 2\cos\alpha\kappa + 1} - \kappa/l$$

has a local minimum at $\kappa = -\cos\alpha + \frac{\sin\alpha}{\sqrt{4l^2-1}}$ (keep in mind (41)). Thus we are led to the value

$$\Phi(\alpha,l) := \Gamma\left( \max\left(\min\left(2-\cos\alpha+\sqrt{\cos^2\alpha - 4\cos\alpha+3}, -\cos\alpha + \frac{\sin\alpha}{\sqrt{4l^2-1}}\right), \frac{1}{\cos\alpha}\right)\right). \tag{64}$$

In our case with (81) one can test that

$$-\cos\alpha + \frac{\sin\alpha}{\sqrt{4l^2-1}} \leqslant \frac{1}{\cos\alpha}, \quad \text{for } 0 \leqslant \alpha \leqslant 1.23\,. \tag{65}$$

Therefore,

$$\Phi(\alpha,l) = \Gamma\left(\frac{1}{\cos\alpha}\right) = 2\sqrt{4+\tan^2\alpha} - \frac{1}{l\cos\alpha}\,. \tag{66}$$

Thus one can eliminate $z$ in (63) and test for

$$k \geqslant \Phi(\alpha,l)\,. \tag{67}$$

The angle between $-\hat{\Delta}_{[1]}$ and $1/\alpha_{m+1}$ is at most $|\arg(\alpha_{m+1})| + |\arg(-\hat{\Delta}_{[1]})|$ which is below $\arcsin(1/3)+\delta$ by (48) and (7) (keeping in mind the change from $z$ to $-1/z$):

$$\measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_{m+1}) \leqslant \arcsin(1/3) + \delta\,. \tag{68}$$

Let $\gamma = \measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_{m+1})$. Then by lemma 2.2,

$$\measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_m) \leqslant \gamma + \arcsin(1/3) + \alpha,$$

and similarly to (68)

$$\measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_m) \leqslant \arcsin(1/3) + \delta\,.$$

Then by testing

$$a\Phi(\alpha,l) \geqslant 1\,, \tag{69}$$

we have

$$ak > \cos\gamma \tag{70}$$

and can conclude with $k = |\hat{\Delta}_{[1]}|$

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant \sqrt{|\alpha_{m+1}|^2k^2 - 2|\alpha_{m+1}|k\cos\gamma + 1} \geqslant a\sqrt{k^2 + \left(\frac{1}{a}\right)^2 - 2k\frac{1}{a}\cos(\gamma)}\,,$$

with the second inequality using (70). Here we must work harder. If $|\alpha_{m+1}| \geqslant \nu_{m+1} > a$, then similarly

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant \nu_{m+1}\sqrt{k^2 + \left(\frac{1}{\nu_{m+1}}\right)^2 - 2k\frac{1}{\nu_{m+1}}\cos(\gamma)}\,.$$

The reverse estimate for $|\hat{\Delta}_{[1]} + 1/\alpha_m|$ is far more painful.

Assume $\nu'_m \geqslant |\alpha_m| \geqslant \nu_m$ and let

$$\Omega(\nu) = \sqrt{k^2 + \left(\frac{1}{\nu}\right)^2 - 2\frac{k}{\nu}\cos\min\left(\arcsin(1/3) + \gamma + \alpha, \arcsin(1/3) + \delta\right)}\,.$$

Then

$$|\hat{\Delta}_{[1]} + 1/\alpha_m| \leqslant \max(\Omega(\nu'_m), \Omega(\nu_m))\,. \tag{71}$$

Thus for (55) with $|z| \geqslant 1/\cos\alpha$ we need to ascertain

$$\nu_{m+1}^2\left(k^2 + \left(\frac{1}{\nu_{m+1}}\right)^2 - 2k\frac{1}{\nu_{m+1}}\cos(\gamma)\right) \geqslant l^2\cdot\cos^2\alpha\cdot\left(\max\Omega(\nu'_m), \Omega(\nu_m)\right)^2 \tag{72}$$

for $\gamma \leqslant \arcsin(1/3) + \delta$.

We will write this inequality parametrically as $(72)(\nu_{m+1}, \nu_m, \nu'_m)$.

**Case 1.1.3.1.** $m = 1$. We have $1/\alpha_m = 0$ by convention and $2 \geqslant |\alpha_{m+1}| \geqslant 1.1$ by lemma 2.5. Thus we need to ascertain $(72)(1.1, \infty, \infty)$, which using $\cos(\gamma) \leqslant 1$ simplifies to

$$1.1k - 1 \geqslant lk\cos\alpha$$

or

$$1.1\Phi(\alpha, l) - 1 \geqslant l\Phi(\alpha, l)\cos\alpha \tag{73}$$

**Case 1.1.3.2.** $m = 2$. By lemma 2.5 and lemma 2.6, we need to ascertain $(72)(1.4, 1.1, 2)$. Let

$$a' = 1.4, b = 1.1, b' = 2\,.$$

Thus we need to ascertain

$$a'^2\left(k^2 + \left(\frac{1}{a'}\right)^2 - 2k\frac{1}{a'}\cos(\gamma)\right) \geqslant l^2\left(k^2 + \left(\frac{1}{b}\right)^2 - 2k\frac{1}{b}\cos\min\left(\arcsin(1/3) + \gamma + \alpha, \arcsin(1/3) + \delta\right)\right)\cdot\cos^2\alpha\,. \tag{74}$$

and

$$a'^2\left(k^2 + \left(\frac{1}{a'}\right)^2 - 2k\frac{1}{a'}\cos(\gamma)\right) \geqslant l^2\left(k^2 + \left(\frac{1}{b'}\right)^2 - 2k\frac{1}{b'}\cos\min\left(\arcsin(1/3) + \gamma + \alpha, \arcsin(1/3) + \delta\right)\right)\cdot\cos^2\alpha\,. \tag{75}$$

One easily sees that it is enough to take $\gamma \leqslant \delta - \alpha$ and discard the second minimum alternative.

**Case 1.1.3.2.1.** (74) We maximize first the linear coefficient in $k$. Regarding this inequality as depending on $\gamma$, one has to examine when

$$-a'\cos\gamma + \frac{1}{b}l^2\cos^2\alpha\cos(\gamma + \arcsin(1/3) + \alpha) = \left(l^2\cos^2\alpha\frac{\sqrt{8}\cos\alpha - \sin\alpha}{3b} - a'\right)\cos\gamma - \frac{l^2\cos^2\alpha}{3b}\cdot(\cos\alpha + \sqrt{8}\sin\alpha)\sin\gamma$$

is minimal when $0 \leqslant \gamma \leqslant \delta - \alpha$. If

$$\frac{l^2\cos^2\alpha}{b} < a'\,,$$

in particular if $b, a' \geqslant a$ because of (41), this expression is easily seen to be decreasing in $\gamma \geqslant 0$ until

$$\tan\gamma = \Xi_3(\alpha, l, a', b) = \frac{l^2\cos^2\alpha(\cos\alpha + \sqrt{8}\sin\alpha)/b}{3a' - (\sqrt{8}\cos\alpha - \sin\alpha)l^2\cos^2\alpha/b}$$

This gives the minimal value

$$-\left[a'\cos(\min(\delta-\alpha,\arctan\Xi_3(\alpha,l,a',b)))-\frac{1}{b}l^2\cos^2\alpha\frac{\sqrt{8}}{3}\cos\min\left(\delta,\alpha+\arctan\Xi_3(\alpha,l,a',b)\right)+\right.$$
$$\left.\frac{l^2\cos^2\alpha}{3b}\sin\min\left(\delta,\alpha+\arctan\Xi_3(\alpha,l,a',b)\right)\right]$$

Thus (74) can be rewritten

$$\begin{aligned}&k^2(l^2\cos^2\alpha-a'^2)+\\ &2k\cdot\left[a'\cos(\min(\delta-\alpha,\arctan\Xi_3(\alpha,l,a',b)))-\right.\\ &\left.l^2\cos^2\alpha\frac{\sqrt{8}}{3b}\cos\min\left(\delta,\alpha+\arctan\Xi_3(\alpha,l,a',b)\right)+\frac{l^2\cos^2\alpha}{3b}\sin\min\left(\delta,\alpha+\arctan\Xi_3(\alpha,l,a',b)\right)\right]\\ &+\left(\frac{l^2\cos^2\alpha}{b^2}-1\right)\quad\leqslant\quad 0\end{aligned}\tag{76}$$

$$\text{for all } k\geqslant\Phi(\alpha,l) \text{ from (66)}.$$

Call this inequality $(76)(a',b)$. So we need to test $(76)(1.4,1.1)$.

**Case 1.1.3.2.2.** (75) By repetition of the previous calculation, we need to test $(76)(1.4,2)$.

**Case 1.1.3.3.** $m=3,4,5,6$. We need to ascertain $(72)(1,1,1.5)$ by lemmas 2.6 and 2.7. We need to test $(76)(1,1)$ and $(76)(1,1.5)$.

**Case 1.1.3.4.** $m>6$. We need to ascertain $(72)(a,a,1.1718)$ by lemma 2.4.

Thus we need to ascertain

$$a^2\left(k^2+\left(\frac{1}{a}\right)^2-2k\frac{1}{a}\cos(\gamma)\right)\geqslant l^2\left(k^2+\left(\frac{1}{a}\right)^2-2k\frac{1}{a}\cos\min\left(\arcsin(1/3)+\gamma+\alpha,\arcsin(1/3)+\delta\right)\right)\cdot\cos^2\alpha,\tag{77}$$

and

$$a^2\left(k^2+\left(\frac{1}{a}\right)^2-2k\frac{1}{a}\cos(\gamma)\right)\geqslant l^2\left(k^2+\left(\frac{1}{1.1718}\right)^2-2k\frac{1}{1.1718}\cos\min\left(\arcsin(1/3)+\gamma+\alpha,\arcsin(1/3)+\delta\right)\right)\cdot\cos^2\alpha.\tag{78}$$

**Case 1.1.3.4.1.** (77). We maximize first the linear coefficient. Regarding this inequality as depending on $\gamma$, one has to examine when

$$-a^2\cos\gamma+l^2\cos^2\alpha\cos(\gamma+\arcsin(1/3)+\alpha)=\left(l^2\cos^2\alpha\frac{\sqrt{8}\cos\alpha-\sin\alpha}{3}-a^2\right)\cos\gamma-\frac{l^2\cos^2\alpha}{3}\cdot(\cos\alpha+\sqrt{8}\sin\alpha)\sin\gamma$$

is minimal when $0\leqslant\gamma\leqslant\delta-\alpha$. This expression is decreasing until

$$\tan\gamma=\Xi(\alpha,l)=\frac{l^2\cos^2\alpha(\cos\alpha+\sqrt{8}\sin\alpha)}{3a^2-(\sqrt{8}\cos\alpha-\sin\alpha)l^2\cos^2\alpha}$$

This gives the minimal value

$$-\left[a^2\cos\min\left(\delta-\alpha,\arctan\Xi(\alpha,l)\right)-\right.$$
$$\left.l^2\cos^2\alpha\frac{\sqrt{8}}{3}\cos\min\left(\delta,\alpha+\arctan\Xi(\alpha,l)\right)+\frac{l^2\cos^2\alpha}{3}\sin\min\left(\delta,\alpha+\arctan\Xi(\alpha,l)\right)\right]$$

Thus (77) can be rewritten

$$k^2(l^2\cos^2\alpha - a^2) + k\cdot\frac{2}{a}\cdot\Big[a^2\cos(\min(\delta-\alpha,\arctan\Xi(\alpha,l))) - l^2\cos^2\alpha\frac{\sqrt{8}}{3}\cos\min(\delta,\alpha+\arctan\Xi(\alpha,l)) + \frac{l^2\cos^2\alpha}{3}\sin\min(\delta,\alpha+\arctan\Xi(\alpha,l))\Big] + \left(\frac{l^2\cos^2\alpha}{a^2}-1\right) \leqslant 0 \tag{79}$$

$$\text{for all } k \geqslant \Phi(\alpha,l) \text{ from (66)}.$$

**Case 1.1.3.4.2.** (78). We maximize first the linear coefficient. Regarding this inequality as depending on $\gamma$, one has to examine when

$$-a\cos\gamma + \frac{1}{1.1718}l^2\cos^2\alpha\cos(\gamma+\arcsin(1/3)+\alpha) = \left(l^2\cos^2\alpha\frac{(\sqrt{8}\cos\alpha-\sin\alpha)}{3\cdot 1.1718} - a\right)\cos\gamma - \frac{l^2\cos^2\alpha}{3\cdot 1.1718}\cdot(\cos\alpha+\sqrt{8}\sin\alpha)\sin\gamma$$

is minimal when $0\leqslant\gamma\leqslant\delta-\alpha$. This expression is decreasing until

$$\tan\gamma = \Xi_2(\alpha,l) = \frac{l^2\cos^2\alpha(\cos\alpha+\sqrt{8}\sin\alpha)}{3a\cdot 1.1718 - (\sqrt{8}\cos\alpha-\sin\alpha)l^2\cos^2\alpha}$$

This gives the minimal value

$$-\Big[a\cos(\min(\delta-\alpha,\arctan\Xi_2(\alpha,l))) - \frac{1}{1.1718}l^2\cos^2\alpha\frac{\sqrt{8}}{3}\cos\min(\delta,\alpha+\arctan\Xi_2(\alpha,l)) + \frac{l^2\cos^2\alpha}{3\cdot 1.1718}\sin\min(\delta,\alpha+\arctan\Xi_2(\alpha,l))\Big]$$

Thus (78) can be rewritten

$$k^2(l^2\cos^2\alpha - a^2) + 2k\cdot\Big[a\cos(\min(\delta-\alpha,\arctan\Xi_2(\alpha,l))) - l^2\cos^2\alpha\frac{\sqrt{8}}{3\cdot 1.1718}\cos\min(\delta,\alpha+\arctan\Xi_2(\alpha,l)) + \frac{l^2\cos^2\alpha}{3\cdot 1.1718}\sin\min(\delta,\alpha+\arctan\Xi_2(\alpha,l))\Big] + \left(\frac{l^2\cos^2\alpha}{1.1718^2}-1\right) \leqslant 0 \tag{80}$$

$$\text{for all } k \geqslant \Phi(\alpha,l) \text{ from (66)}.$$

Maximizing a quadratic expression in $k$ over an interval is no problem, and MATHEMATICA shows that with $a = 0.85$ one can choose

$$l = 0.595 - \alpha/15 \tag{81}$$

to satisfy the inequalities in all cases simultaneously when $\alpha < 0.1091$. The inequality (80) (or (76)$(a, 1.1718)$) in case 1.1.3.4.2 is by far the hardest to get stand, and leads to this severe restraint for $\alpha$. This is the reason we had to dig so deep in §2.

**Part 1.2.** Next, we need to care about angles. Let again $\Delta = \hat{\Delta}_{[1]}$.

**Case 1.2.1.** If $a_{n-m+1} < 0$, then we have some difficulty.

We assume $|\Delta| \geqslant l$ and $\arg(\Delta) \leqslant 2\arcsin(1/3) + 2|\arg(-z)|$ and like to show

$$\left|\arg\frac{\Delta + 1/\alpha_{m+1}}{\Delta + 1/\alpha_m}\right| = \measuredangle(\Delta + 1/\alpha_{m+1}, \Delta + 1/\alpha_m) \leqslant \arcsin(1/3) + |\arg(-z)|\,.$$

This argument is longer and moved out to lemma 3.3. With (40) we have

$$\left|\arg -\hat{\Delta}_{n+1}\right| \leqslant 2\arcsin(1/3) + 2|\arg(-z)|\,,$$

as we wanted in (44).

**Case 1.2.2.** $a_{n-m+1} > 1$. Again using (40), it is enough to see

$$\left|\arg\frac{\Delta + 1/\alpha_{m+1}}{\Delta + 1/\alpha_m}\right| \leqslant \arcsin(1/3) + |\arg(-z)|\,.$$

Now

$$|\Delta| \geqslant 2|1-z| - \frac{|z|}{l}, \quad \measuredangle(1/\alpha_{m+1}, 1/\alpha_m) \leqslant \arcsin(1/3) + |\arg(-z)| \quad \text{and} \quad \left|\frac{1}{\alpha_{m[+1]}}\right| \leqslant \frac{1}{a},$$

with the agreement that '$m[+1]$' means 'both for $m$ and $m+1$'.

We have $\left|\frac{1}{\alpha_{m[+1]}}\right| \leqslant |\Delta|$ by (69). Trigonometry in the triangles with vertices $0, \Delta, \Delta + 1/\alpha_m$ and $0, \Delta, \Delta + 1/\alpha_{m+1}$ and an easy argument shows that

$$\measuredangle(\Delta + 1/\alpha_{m+1}, \Delta + 1/\alpha_m)$$

is maximal when $|\Delta| = 2|1-z| - |z|/l$, $|\alpha_m| = |\alpha_{m+1}| = a$, $\measuredangle(1/\alpha_{m+1}, 1/\alpha_m) = \arcsin(1/3) + \alpha$ and $\arg(-\Delta\alpha_m) = -\arg(-\Delta\alpha_{m+1})$. This gives

$$\sin\frac{\measuredangle(\Delta + 1/\alpha_{m+1}, \Delta + 1/\alpha_m)}{2} \leqslant \frac{\sin\frac{\arcsin(1/3) + |\arg(-z)|}{2}}{a\sqrt{\left(2|1-z| - \frac{|z|}{l}\right)^2 + \left(\frac{1}{a}\right)^2 - 2\frac{1}{a}\left(2|1-z| - \frac{|z|}{l}\right)\cos\frac{\arcsin(1/3) + |\arg(-z)|}{2}}}$$

First, we checked when $\alpha = |\arg(-z)|$ is fixed the minimal value of

$$2|1-z| - |z|/l$$

to be in (64). Thus we can set $|\Delta| = k = \Phi(\alpha, l)$. Now to see

$$\measuredangle(\Delta + 1/\alpha_{m+1}, \Delta + 1/\alpha_m)/2 < (\arcsin(1/3) + \alpha)/2$$

it is enough, with (69) in mind, to see for $\alpha = |\arg(-z)|$ that

$$\frac{\sin\frac{\arcsin(1/3)+\alpha}{2}}{a\sqrt{\Phi(\alpha, l)^2 + \left(\frac{1}{a}\right)^2 - \frac{2}{a}\cdot\Phi(\alpha, l)\cos\frac{\arcsin(1/3)+\alpha}{2}}} \leqslant \sin\frac{\arcsin(1/3) + \alpha}{2}$$

which is true for $l$ in (81) and $a = 0.85$.

This finishes $a_{n+1} = 1$. The case $a_{n+1} = -1$ is analogous.

**Case 2.** $a_{n+1} > 1$. We need to derive the estimate (48) of the angle. The norm estimate is quite clear (recall remark below (54)).

We will use (36) in the form

$$\hat{\Delta}_{n+1} = a_{n+1}(z-1) + \frac{z}{\hat{\Delta}_n}\,.$$

Consider $\triangle ABC$ with $\eta, \beta, \gamma$ angles at $A, B, C$. Let $\overline{AB} = 2|1-z|$, $\overline{AC} = |z|/l$ and

$$\eta = 2\arcsin(1/3) + |\arg(1 - 1/z)| + 2|\arg(-z)|\,.$$

We have

$$\sin|\arg(1-1/z)| = \frac{\cos\alpha\sin\alpha}{\sqrt{1+3\cos^2\alpha}} \qquad \cos|\arg(1-1/z)| = \frac{1+\cos^2\alpha}{\sqrt{1+3\cos^2\alpha}}$$

We need an upper bound on $\beta$. First, from $l \geqslant 0.5$ (see (41)) we have $\overline{AC} < \overline{AB}$, thus $\beta \leqslant \pi/2$.

The Sine and Cosine theorems give

$$\sin\beta \quad = \quad \frac{\sin\eta\cdot|z|}{\sqrt{4|1-z|^2l^2+|z|^2-4\cdot|1-z|\cdot l\cdot|z|\cdot\cos\eta}}$$

This estimates

$$\measuredangle(\tilde{\Delta}_{n+1},a_{n+1}(z-1)\tilde{\Delta}_n) = \measuredangle(\tilde{\Delta}_{n+1},(z-1)\tilde{\Delta}_n) = \measuredangle(-\tilde{\Delta}_{n+1},(1-z)\tilde{\Delta}_n) \leqslant \beta$$

and

$$|\arg(-\hat{\Delta}_{n+1})| = \measuredangle(-\tilde{\Delta}_{n+1},\tilde{\Delta}_n) \leqslant \beta+|\arg(1-z)|. \tag{82}$$

Next, for given fixed $|z|$, it is easy to see that $|1-1/z|$ (or equivalently, $|1-z|$) is smallest, and all of $|\arg(-z)|$, $|\arg(1-z)|$, $|\arg(1-1/z)|$ largest when $\Re e\, z' = -1$. Thus consider only this case. Then the formulas in §2 show that

$$|\arg(-z')| =: \alpha' \leqslant \arccos\left(2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}\right),$$

leading to (42). Now assuming $z' = z$, we have

$$|z| = \frac{1}{\cos\alpha} \qquad |1-z| = \sqrt{4+\tan^2\alpha} \qquad |\arg(1-z)| = \arctan\left(\frac{\tan\alpha}{2}\right) \qquad |\arg(1-1/z)| = \arctan\left(\frac{\cos\alpha\sin\alpha}{1+\cos^2\alpha}\right). \tag{83}$$

Thus

$$\sin\beta \quad = \quad \frac{\sin\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha\sin\alpha}{1+\cos^2\alpha}\right)+2\alpha\right)}{\cos\alpha\sqrt{4(4+\tan^2\alpha)l^2+\frac{1}{\cos^2\alpha}-4\cdot\frac{l}{\cos\alpha}\cdot\sqrt{4+\tan^2\alpha}\cos\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha\sin\alpha}{1+\cos^2\alpha}\right)+2\alpha\right)}},$$

which with (82) and (83) yields the expression (43). The inequality there is to be tested:

$$\arcsin\frac{\sin\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right)+2\alpha'\right)}{\cos\alpha'\sqrt{4(4+\tan^2\alpha')l^2+\frac{1}{\cos^2\alpha'}-4\cdot\frac{l}{\cos\alpha'}\cdot\sqrt{4+\tan^2\alpha'}\cos\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right)+2\alpha'\right)}} + \arctan\frac{\tan\alpha'}{2} \quad \leqslant \quad 2\arcsin(1/3)+2\alpha.$$

It is true for $\alpha \leqslant 1.23$.

The case $a_{n+1} < -1$ is analogous.

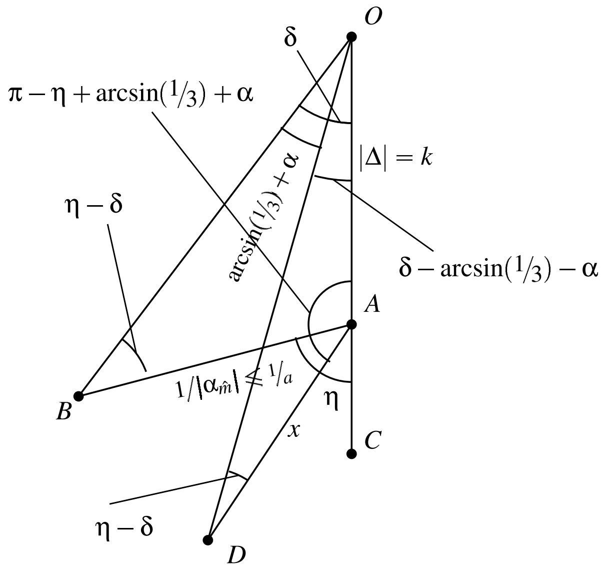
With this the induction is complete. □

**Lemma 3.3** *Let* $|\Delta| \geqslant l$ *and* $\arg(\Delta) \leqslant 2\arcsin(1/3)+2|\arg(-z)|$ *and* $\alpha = |\arg(-z)| < 0.11$. *Then*

$$\measuredangle(\Delta+1/\alpha_{m+1},\Delta+1/\alpha_m) \leqslant \arcsin(1/3)+|\arg(-z)|.$$

**Proof.**

**Case 1.** We assume first none of $\alpha_m$, $\alpha_{m+1}$ is $\alpha_2$, i.e., $m \geqslant 3$. See figure




We set $\overline{OA} = |\Delta| = k$. Let $\overline{AB} = 1/|\alpha_m| \leqslant 1/a$, where we assume w.l.o.g. that $\measuredangle(\Delta, 1/\alpha_m) > \measuredangle(\Delta, 1/\alpha_{m+1})$. (Otherwise, exchange $\alpha_m$ and $\alpha_{m+1}$ in the following.) We set next

$$\eta = \measuredangle BAC \leqslant |\arg(\Delta)| + |\arg(\alpha_m)| \leqslant 3\arcsin(1/3) + 2|\arg(-z)|\,. \tag{84}$$

Let $\overline{AD} = 1/|\alpha_{m+1}|$ and $\measuredangle BAD = \measuredangle(1/\alpha_m, 1/\alpha_{m+1}) \leqslant \arcsin(1/3) + |\arg(-z)|$, by lemma 2.2.

Note that if $B, D$ are on different side of $AC$, then

$$\left|\arg \frac{\Delta + 1/\alpha_{m+1}}{\Delta + 1/\alpha_m}\right| = \measuredangle(\Delta + 1/\alpha_{m+1}, \Delta + 1/\alpha_m) \leqslant \measuredangle(1/\alpha_{m+1}, 1/\alpha_m) \leqslant \arcsin(1/3) + |\arg(-z)|\,,$$

as we wanted. We thus assume them on the same side of (or on) $AC$, and w.l.o.g.

$$\arg\left(\frac{1}{\alpha_m \Delta}\right), \arg\left(\frac{1}{\alpha_{m+1}\Delta}\right) \geqslant 0\,. \tag{85}$$

We are interested in proving a lower bound $|1/\alpha_{m+1}| = \overline{AD} \geqslant x$, so that $\measuredangle BOD \leqslant \alpha + \arcsin(1/3)$.

For this it is enough to assume that $\overline{AB}$ is maximal, $1/a$, and so is $\measuredangle BAD = \arcsin(1/3) + \alpha$. This assumption is justified because for $\alpha \leqslant 0.11$ we have $\eta < \pi/2$ by (58) and (84).

We also set $\measuredangle BOD = \alpha + \arcsin(1/3)$ and calculate $\overline{AD} = x$.

Let

$$\tilde{\delta}(\eta, k, a) = \arcsin \frac{\sin\eta}{\sqrt{a^2k^2 + 1 + 2ak\cos\eta}}\,,$$

and we have

$$\delta(\eta, k, a) = \measuredangle AOB = \begin{cases} \tilde{\delta}(\eta, k, a) & \text{if } \cos\eta > -ka \\ \pi - \tilde{\delta}(\eta, k, a) & \text{otherwise} \end{cases} \tag{86}$$

It is geometrically obvious that $\delta$ increases with $\eta$, until $\eta = \frac{\pi}{2} + \arcsin\left(\frac{1}{ka}\right) > \frac{\pi}{2}$, which does not occur for $\alpha < 0.174$ by (58) and (84).

It helps that $\measuredangle OBA = \measuredangle ODA = \eta - \delta$. The Sine law gives

$$x = \frac{1}{a} \cdot \frac{\sin(\delta - \arcsin(1/3) - \alpha)}{\sin(\delta)}$$

The derivative immediately shows this to be increasing in $\delta$ (and hence $\eta$), and to get maximal value, set $\eta = 3\arcsin(1/3)+2\alpha$ by (84). Moreover, $\delta(\eta,k,a)$ is also decreasing in $k \geqslant l$ (using $\cos\eta > 0$), so set $k = l$. Then we have

$$\frac{1}{|\alpha_{m+1}|} = x \leqslant \frac{1}{a}\cdot\frac{\sin(\delta(3\arcsin(1/3)+2\alpha,l,a)-\arcsin(1/3)-\alpha)}{\sin(\delta(3\arcsin(1/3)+2\alpha,l,a))}.$$

To recall, this is an upper bound for $x$, for which the angle $\measuredangle DOB$ is too large, as we do not desire.

Thus if we use lemma 2.4 and show that

$$\frac{2}{3} \geqslant \frac{1}{a}\cdot\frac{\sin(\delta(3\arcsin(1/3)+2\alpha,l,a)-\arcsin(1/3)-\alpha)}{\sin(\delta(3\arcsin(1/3)+2\alpha,l,a))}, \tag{87}$$

we are done.

A MATHEMATICA plot of the r.h.s. for $\alpha \leqslant 0.2$ shows that the maximal value for $x$ is for $\alpha \to 0$, with the value being below $2/3$.

This motivated lemma 2.4, which finishes this case.

**Case 2.** $m = 1$. With $\alpha \leqslant 0.174$ and following (35) in the proof of lemma 2.5, we have

$$\frac{1}{|\alpha_2|} \leqslant \frac{1}{2\sqrt{2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}}}.$$

We have to show (with the convention $1/|\alpha_m| = 0$)

$$\delta\left(3\arcsin(1/3)+2\alpha,l,2\sqrt{2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}}\right) \leqslant \arcsin(1/3)+\alpha.$$

But this test fails.

We have to estimate $|\arg(\alpha_2)| = |\arg(1-1/z)|$ better. We have

$$|\arg(\alpha_2)| = \left|\arg\left(1-\frac{1}{z}\right)\right| \leqslant |\arg(-z)|$$

and by using $|1/z| < 1$ and lemma 3.2, we have even $|\arg(\alpha_2)| \leqslant |\arg(-z)|/2$. Thus in (84) we can replace the r.h.s. $3\arcsin(1/3)+2\alpha$ by $2\arcsin(1/3)+2.5\alpha$, and thus we test

$$\delta\left(2\arcsin(1/3)+2.5\alpha,l,2\sqrt{2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}}\right) \leqslant \arcsin(1/3)+\alpha,$$

and this test succeeds for $\alpha \leqslant 0.110115$.

**Case 3.** $m = 2$. With $\alpha \leqslant 0.174$ and lemma 2.6, we have $1/|\alpha_3| \leqslant 0.7040644$ and $1/|\alpha_2| \geqslant 0.5453$ by lemma 2.5.

We need to ascertain (if $\measuredangle(\Delta,1/\alpha_3) \leqslant \measuredangle(\Delta,1/\alpha_2)$)

$$\frac{2}{3} \geqslant 0.5453\cdot\frac{\sin(\delta(3\arcsin(1/3)+2\alpha,l,\frac{1}{0.5453})-\arcsin(1/3)-\alpha)}{\sin(\delta(3\arcsin(1/3)+2\alpha,l,\frac{1}{0.5453}))},$$

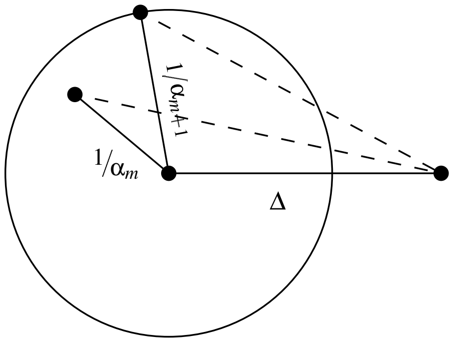
but it is subsumed by (87), and (if $\measuredangle(\Delta,1/\alpha_2) \leqslant \measuredangle(\Delta,1/\alpha_3)$)

$$\frac{1}{2} \geqslant 0.7040644\cdot\frac{\sin(\delta(3\arcsin(1/3)+2\alpha,l,\frac{1}{0.7040644})-\arcsin(1/3)-\alpha)}{\sin(\delta(3\arcsin(1/3)+2\alpha,l,\frac{1}{0.7040644}))},$$

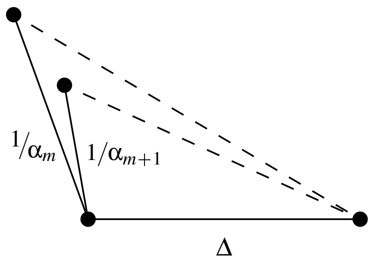
and then we are done.

This discussion deals with the situation that if $\arg\left(\frac{1}{\alpha_{m+1}\Delta}\right) \leqslant \arg\left(\frac{1}{\alpha_m\Delta}\right)$ then $\arg(\Delta+1/\alpha_{m+1}) < \arg(\Delta+1/\alpha_m) - \arcsin(1/3)-\alpha$, respectively if $\arg\left(\frac{1}{\alpha_{m+1}\Delta}\right) \geqslant \arg\left(\frac{1}{\alpha_m\Delta}\right)$ then $\arg(\Delta+1/\alpha_m) < \arg(\Delta+1/\alpha_{m+1}) - \arcsin(1/3)-\alpha$

(keep in mind (85)).




We need to consider also $\arg\left(\frac{1}{\alpha_{m+1}\Delta}\right) \leqslant \arg\left(\frac{1}{\alpha_m\Delta}\right)$ and $\arg(\Delta + 1/\alpha_{m+1}) > \arg(\Delta + 1/\alpha_m) + \arcsin(1/3) + \alpha$ (similarly with indices $m$ and $m+1$ exchanged, as drawn below).




We handle the situation thus. Since $\eta < \pi/2$, we have $\eta - \delta < \pi/2$. Then, when $\arg\left(\frac{1}{\alpha_{m+1}\Delta}\right) \leqslant \arg\left(\frac{1}{\alpha_m\Delta}\right)$, the largest difference $\arg(\Delta + 1/\alpha_{m+1}) - \arg(\Delta + 1/\alpha_m)$ is seen to be achieved when $\arg(1/\alpha_{m+1}) = \arg(1/\alpha_m)$, where the previous argument estimated the angle. This is why we are done. □

# 4 The conjecture far from $-1$

## 4.1 back to $\alpha_i$

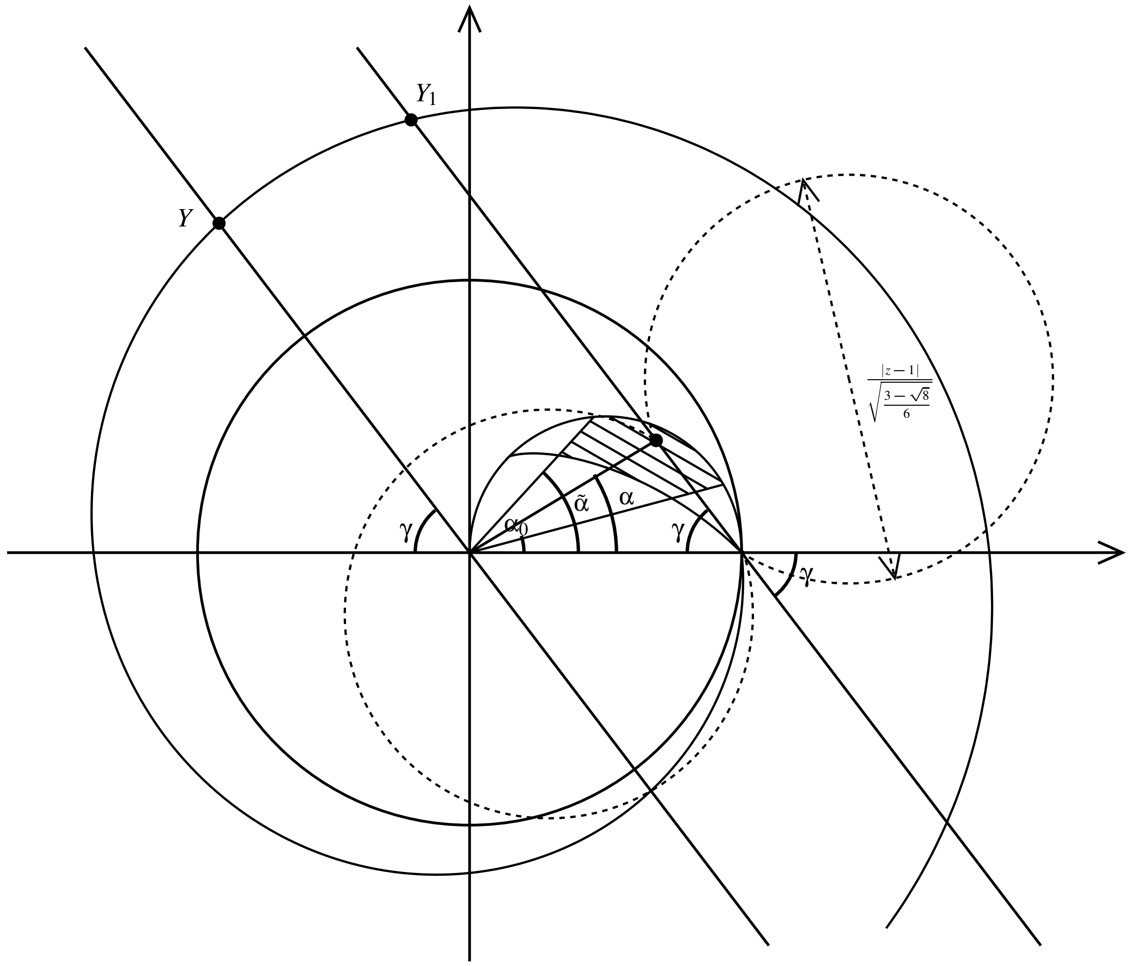
*In this subsection we use the notation of* §2 *for z.*

**Lemma 4.1** *If* $\alpha \geqslant 0.1091$, *then*

$$\left|\arg\frac{\alpha_n}{\alpha_{n-1}}\right| \leqslant \arcsin(1/3).$$

Again, the fact that $\alpha$ is bounded away from 0 will be used in a decisive way. It is the technical difficulty of proving this lemma for small $\alpha$ that leaves to two separate parts in §3 and §4.

**Proof.** As for lemma 2.2, using lemma 2.1, we are done unless $\arg(\alpha_n)$ and $\arg(\alpha_{n-1})$ have opposite sign. So assume this now.




We will test that when $\arg(\alpha_n)$ and $\arg(\alpha_{n-1})$ have opposite sign, then $|\arg(\alpha_{n[-1]})| \leqslant \arcsin(1/3)/2$, with the agreement that '$n[-1]$' means 'both for $n$ and $n-1$'.

Again we consider two circle arcs ending on $B = 1$ and $C = z$. But this time we measure the angle $\arcsin(1/3)/2$, so the diameter of the circle arcs is

$$\frac{|z-1|}{\sin(\arcsin(1/3)/2)} = \frac{|z-1|}{\sqrt{\frac{3-\sqrt{8}}{6}}}.$$

In particular, since 1 is on these arcs, their points have norm at most

$$1 + \frac{|z-1|}{\sqrt{\frac{3-\sqrt{8}}{6}}}.$$

We compare these arcs with the spiral

$$\left\{ r = \left(\frac{1}{\cos\alpha}\right)^{(2\pi-\theta)/\alpha} \right\}, \tag{88}$$

which we consider here not only for $0 \leqslant \theta \leqslant 2\pi$ but also for $\theta < 0$. Let for any $w \neq 0$,

$$\tilde{\theta}(w) = 2\pi - \theta(w).$$

We will use again that

$$\left(\frac{1}{\cos\alpha}\right)^{1/\alpha} \geqslant \frac{1}{\left(1-\frac{\alpha}{2}+\frac{\alpha^2}{4}\right)}. \tag{89}$$

We are interested in the angles $\measuredangle BX_{1,2}C$ for two points $X_{1,2}$ on this spiral lying on opposite sides of the line $BC$, and let the angle $\tilde\theta(X_1) < \tilde\theta(X_2)$ thus $\tilde\theta(X_1) + \arg z = \tilde\theta(x_2)$.

Now we would like to test all $z$ with

$$\alpha_0 \leqslant \alpha = \arg z \leqslant \tilde\alpha.$$

(We will later see how to choose $\alpha_0, \tilde\alpha$ so that the test works.)

An easy thought using (14) and (35) shows that the maximal $|1-z|$ is

$$\Sigma(\tilde\alpha) = \sqrt{2}\left(\cos\tilde\alpha - 1 + \sqrt{\cos^2\tilde\alpha - 4\cos\tilde\alpha + 3}\right).$$

Let $\gamma = \measuredangle ABC$. Using (10), now $\gamma \geqslant \frac{\pi}{4} - \alpha \geqslant \frac{\pi}{4} - \tilde\alpha$.

Let $Y_1$ be the first intersection of $BC$ with the spiral. "First" means for smallest $\tilde\theta$. Note that we did not investigate whether there is no intersection for small $\tilde\theta$. However, we know that $\tilde\theta$ must be a multiple of $\alpha$, we assumed $\alpha \geqslant 0.1$, and we can test directly enough small values of $n$ (as done at the end of the proof) to ascertain that $\tilde\theta \geqslant \pi/2$. This saves us from bothering about the possibility of an intersection close to $B = 1$. (We did check that indeed such an intersection does not occur, but the details are too painful and distracting here.) We assume thus $Y_1$ has the smallest $\tilde\theta(Y_1) \geqslant \pi/2$.

Consider a parallel to $BC$ through $A = 0$. Let it intersect the spiral in $Y$. This shows that

$$\tilde\theta(X_2) \geqslant \tilde\theta(Y_1) \geqslant \tilde\theta(Y) \geqslant \pi + \gamma,$$

and thus that

$$\tilde\theta(X_1) \geqslant \pi + \gamma - \alpha \geqslant \frac{5\pi}{4} - 2\alpha \geqslant \frac{5\pi}{4} - 2\tilde\alpha. \tag{90}$$

Thus if we have

$$\frac{1}{\left(1-\frac{\alpha_0}{2}+\frac{\alpha_0^2}{4}\right)^{5\pi/4-2\tilde\alpha}} \geqslant 1 + \frac{\Sigma(\tilde\alpha)}{\sqrt{\frac{3-\sqrt{8}}{6}}},$$

we would be done. This can be reorganized as

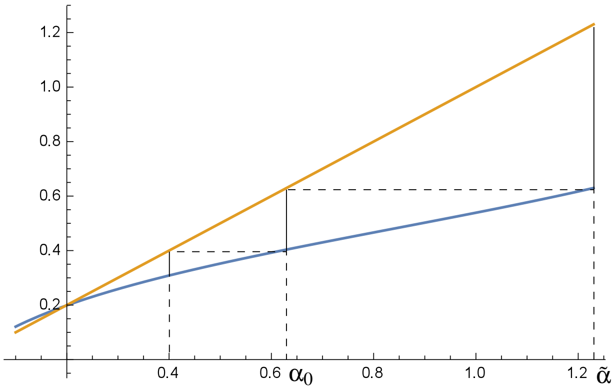
$$1-\sqrt{4\left(\frac{\sqrt{3-\sqrt{8}}}{\sqrt{6}\Sigma(\tilde\alpha)+\sqrt{3-\sqrt{8}}}\right)^{1/\beta} - 3} \leqslant \alpha_0 \leqslant \min\left(\tilde\alpha, 1+\sqrt{4\left(\frac{\sqrt{3-\sqrt{8}}}{\sqrt{6}\Sigma(\tilde\alpha)+\sqrt{3-\sqrt{8}}}\right)^{1/\beta} - 3}\right). \tag{91}$$

for

$$\beta = \frac{5\pi}{4} - 2\tilde\alpha. \tag{92}$$

This test does not work well.

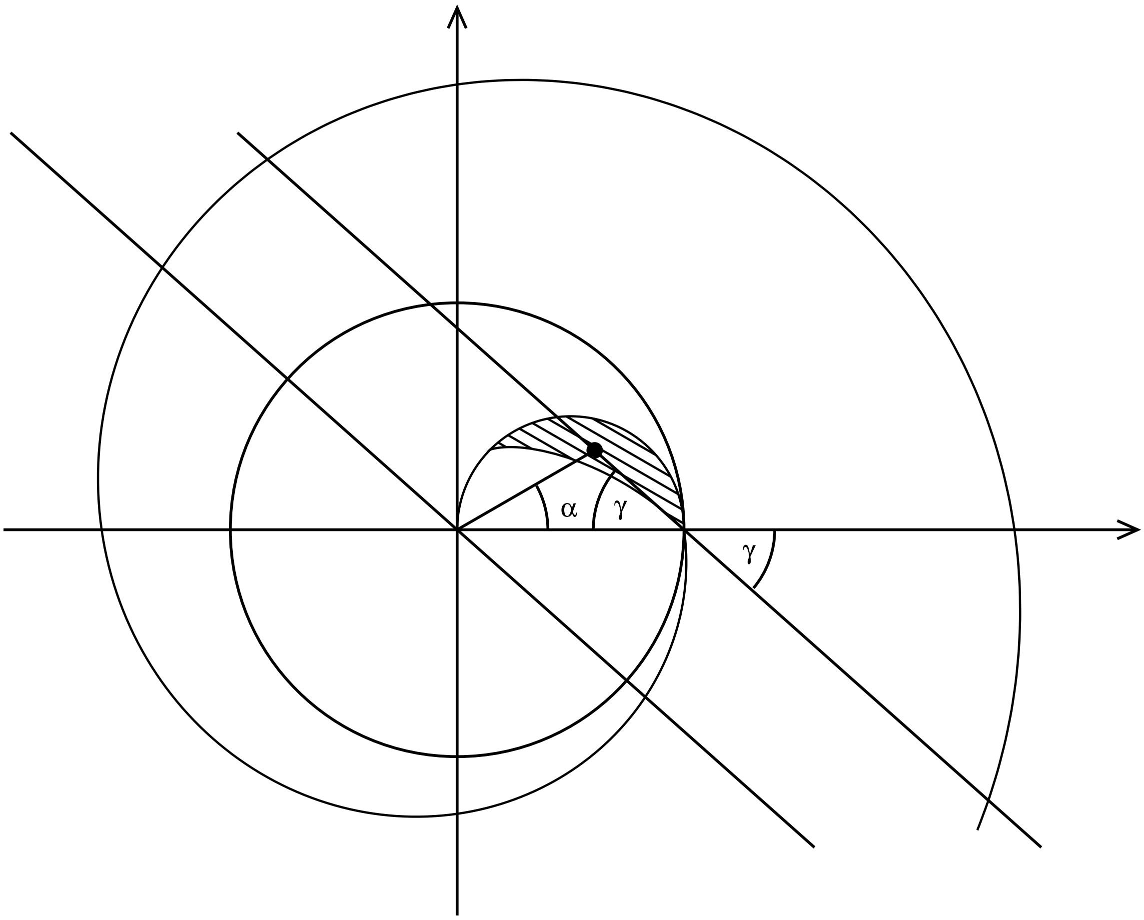
**Case 1.** Assume $\tilde\alpha \geqslant 0.2$. However, (91) does work well when we replace $\beta$ in (92) by $\beta = 13\pi/4 - 2\tilde\alpha$. Then, the right alternative in (91) is $\tilde\alpha$, and the left hand-side is defined for all $0.2 \leqslant \tilde\alpha \leqslant 1.23$ and remains visibly below the left hand-side.




Thus by successively replacing $\tilde{\alpha}$ by $\alpha_0$, one can cover $[0.2, 1.23]$ by intervals $[\alpha_0, \tilde{\alpha}]$. This argument leaves us to deal only with points $X_{1,2}$ near the first two intersections of $BC$ with the spiral.

Since $\gamma \leqslant \pi/2 - \alpha$, one easily sees that then $\tilde{\theta}(X_1) \leqslant 2\pi + \gamma$ and hence

$$\tilde{\theta}(X_2) = (n+1)\arg z \leqslant 5\pi/2\,.$$




Also using the argument for (90),

$$\tilde{\theta}(X_2) \geqslant 5\pi/4 - \alpha$$

Then we can limit $n$ and have a finite number of 2-parameter tests. For $0.2 \leqslant \alpha \leqslant 0.3$,

$$12.08 < \frac{5\pi}{4 \cdot 0.3} - 1 \leqslant n+1 \leqslant \frac{5\pi}{2 \cdot 0.2} \leqslant 39.27\,, \tag{93}$$

thus

$$12 \leqslant n \leqslant 38\,.$$

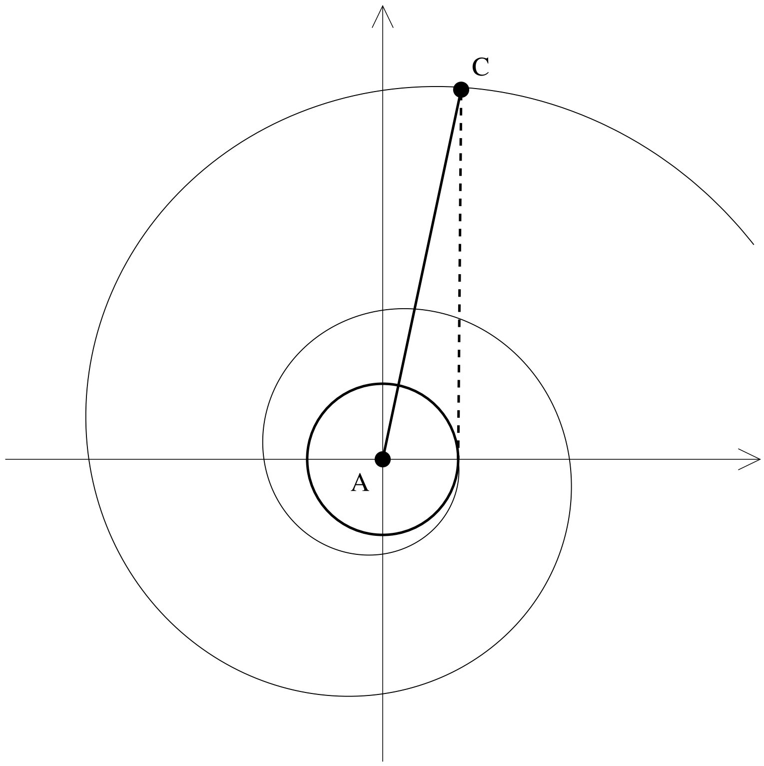
For $1.23 \geqslant \alpha > 0.3$, we test $2 \leqslant n \leqslant 25$. We were able to test in fact $n \leqslant 99$ for $\alpha \leqslant 0.3$ and $n \leqslant 49$ for $\alpha > 0.3$.

**Case 2.** Assume $0.1091 \leqslant \tilde{\alpha} \leqslant 0.2$. In that case we have to test one more intersection of $BC$ and the spiral (88).

We can estimate for the fourth intersection $Y_4$ that $\tilde{\theta}(Y_4) \geqslant 4\pi$, thus $\tilde{\theta}(X_1) \geqslant 4\pi - \alpha \geqslant 4\pi - \tilde{\alpha}$. We test then (91) for $\beta = 4\pi - \tilde{\alpha}$, and find it successful (in the sense of case 1).

We are left now with testing the first 3 intersections, and find first an estimate for $\tilde{\theta}(Y_3)$ for the third intersection. The following diagram shows that if $\tilde{\theta}(Y_3) \geqslant \frac{7\pi}{2}$, then

$$\tilde{\theta}(X_1) \leqslant \tilde{\theta}(Y_3) \leqslant \frac{7\pi}{2} + \arcsin\left(\left(1 - \frac{0.1091}{2} + \frac{0.1091^2}{4}\right)^{7\pi/2}\right).$$




Then (93) modifies to

$$\frac{5\pi}{4\cdot 0.2} - 2 \leqslant n \leqslant \frac{1}{0.1091}\left[\frac{7\pi}{2} + \arcsin\left(\left(1 - \frac{0.1091}{2} + \frac{0.1091^2}{4}\right)^{7\pi/2}\right)\right] \leqslant 106.22$$

This justifies us to test further explicitly for $0.1 \leqslant \alpha \leqslant 0.2$ the values $n = 100, \ldots, 119$, in addition to case 1. With this check we are done. □

## 4.2 The estimate for large α

Now we repeat theorem 3.1 using lemma 4.1.

With $\alpha = |\arg(-z)|$, and (42) we need the expression

$$\delta = \arcsin \frac{\sin\left(2\arcsin(1/3) + \arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right) + \alpha'\right)}{\cos\alpha'\sqrt{4(4+\tan^2\alpha')l^2 + \frac{1}{\cos^2\alpha'} - 4\cdot\frac{l}{\cos\alpha'}\cdot\sqrt{4+\tan^2\alpha'}\cos\left(2\arcsin(1/3) + \arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right) + \alpha'\right)}} \tag{94}$$

$$+\arctan\frac{\tan\alpha'}{2}$$
$$\leqslant\quad 2\arcsin(1/3)+\alpha\,.$$

The claimed inequality will be proved below.

**Theorem 4.1** *Let* $0.1091 \leqslant \alpha = |\arg(-z)| \leqslant \arccos(1/3) \approx 1.23$. *Then there is an* $l = l(\alpha)$ *such that for all* $n \geqslant 1$, *the following inequalities hold.*

$$|\hat\Delta_n| \geqslant l \quad \textit{and} \quad |\arg(-\hat\Delta_n)| \leqslant 2\arcsin(1/3) + |\arg(-z)| \quad \textit{if} \quad a_n = 1 \tag{95}$$

$$|\arg(-\hat\Delta_n) - \arg(1-z)| \leqslant \arcsin(\frac{|z|}{lv}) \quad \textit{if} \quad a_n = 1 \tag{96}$$

$$|\hat\Delta_n| \geqslant l \quad \textit{and} \quad |\arg(\hat\Delta_n)| \leqslant 2\arcsin(1/3) + |\arg(-z)| \quad \textit{if} \quad a_n = -1 \tag{97}$$

$$|\arg(\hat\Delta_n) - \arg(1-z)| \leqslant \arcsin(\frac{|z|}{lv}) \quad \textit{if} \quad a_n = -1 \tag{98}$$

$$|\hat\Delta_n| \geqslant 2v - \frac{|z|}{l} \quad \textit{and} \quad |\arg(-\hat\Delta_n)| \leqslant \delta \quad \textit{if} \quad a_n > 1 \tag{99}$$

$$|\arg(-\hat\Delta_n) - \arg(1-z)| \leqslant \arcsin(\frac{|z|}{2lv}) \quad \textit{if} \quad a_n > 1 \tag{100}$$

$$|\hat\Delta_n| \geqslant 2v - \frac{|z|}{l} \quad \textit{and} \quad |\arg(\hat\Delta_n)| \leqslant \delta \quad \textit{if} \quad a_n < -1 \tag{101}$$

$$|\arg(\hat\Delta_n) - \arg(1-z)| \leqslant \arcsin(\frac{|z|}{2lv}) \quad \textit{if} \quad a_n < -1 \tag{102}$$

*(See below for the meaning of these inequalities when* $n = 1$.*)*

**Proof.** The induction start is no problem once one stipulates that when $w \neq 0$, then $|w/0| = \infty$ and taking $\arg(w/0)$ to be $\arg(w)$. The arguments below go through without serious modifications. Alternatively, one starts with $n = 2$.

From now on, we dwell only upon the induction step. First note that with (94), all alternatives imply

$$|\hat\Delta_n| \geqslant l \quad \textit{and} \quad |\arg(\pm\hat\Delta_n)| \leqslant 2\arcsin(1/3) + \alpha, \tag{103}$$

after we ascertained (54).

We assume now all inequalities (95) – (102), and in particular (103), hold for index $\leqslant n$. Using $|\hat\Delta_n| \geqslant l$ and (36), and lemma 3.1, one immediately obtains (96), (98), (100) and (102), as well as the norm estimates in (99) and (101). For the rest, we need to discuss a few cases for $a_{n+1}$, and again it is enough to treat $a_{n+1} > 0$.

**Case 1.** $a_{n+1} = 1$. So $\hat\Delta_{n+1} = \hat\Delta_{[m+1]}$ for some $m \geqslant 1$, and we use the recursion (40). We can assume $a_{n-m+1} \neq 1$.

**Part 1.1.** We discuss first norms. We have to prove

$$|z| \cdot |\alpha_{m+1}\hat\Delta_{[1]} + 1| \geqslant l|\hat\Delta_{[1]} + 1/\alpha_m| \tag{104}$$

with

$$\alpha_m = \frac{1-(-1/z)^m}{1-(-1/z)^{m-1}}$$

and stipulating (56).

**Case 1.1.1.** $a_{n-m+1} = -1$. The angle between $\alpha_{m+1}\hat\Delta_{[1]}$ and 1 is at most $|\arg(\alpha_{m+1})| + |\arg(\hat\Delta_{[1]})|$, which is below $\arcsin(1/3) + 2\arcsin(1/3) + |\arg(-z)|$. Similarly for $\alpha_m$ instead of $\alpha_{m+1}$.

Let $\Delta = \hat\Delta_{[1]}$ and $k = |\Delta| \geqslant l$ and $|\alpha_n| \geqslant a$ for all $n$. Also $|z| \geqslant 1/\cos\alpha$.

Let $\gamma = \measuredangle(\Delta, 1/\alpha_m) \leqslant 3\arcsin(1/3) + \alpha$. Then by lemma 4.1,

$$\measuredangle(\Delta, 1/\alpha_{m+1}) \leqslant \min(\gamma + \arcsin(1/3), 3\arcsin(1/3) + \alpha)\,.$$

Let $p = |1/\alpha_m| \in [0, 1/a]$ (keep in mind (56)) and $q = |\alpha_{m+1}| \in [a, \infty)$.

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant \min_{q \geqslant a} \sqrt{1 + q^2k^2 + 2q \cdot k \cos\, \min\left(3\arcsin(1/3) + \alpha, \gamma + \arcsin(1/3)\right)}\,. \tag{105}$$

Now we have to test thus

$$a \cdot k \geqslant \underline{a \cdot l \geqslant -\cos(3\arcsin(1/3) + \alpha)} \geqslant -\cos\min(\gamma + \arcsin(1/3), 3\arcsin(1/3) + \alpha)$$

to ascertain that the radicand in (105) is minimized over $q \geqslant a$ for $q = a$. This yields

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant \sqrt{1 + a^2k^2 + 2a \cdot k \cos\, \min\left(3\arcsin(1/3) + \alpha, \gamma + \arcsin(1/3)\right)}\,.$$

Similarly

$$|\hat{\Delta}_{[1]} + 1/\alpha_m| \leqslant \max_{0 \leqslant p \leqslant 1/a} \sqrt{k^2 + p^2 + 2kp\cos\gamma},$$

giving

$$|\hat{\Delta}_{[1]} + 1/\alpha_m| \leqslant \max\left(\sqrt{k^2 + \frac{1}{a^2} + 2\frac{k}{a}\cos\gamma}, k\right).$$

We need to prove for (104)

$$1 + a^2k^2 + 2ak\cos(\gamma + \arcsin(1/3)) \geqslant l^2\left(k^2 + \frac{1}{a^2} + 2\frac{k}{a}\cos(\gamma)\right) \cdot \cos^2\alpha\,, \tag{106}$$

and

$$1 + a^2k^2 + 2ak\cos(\gamma + \arcsin(1/3)) \geqslant l^2k^2 \cdot \cos^2\alpha \tag{107}$$

for $0 \leqslant \gamma \leqslant 2\arcsin(1/3) + \alpha$.

The second alternative (107) is easier dealt with to test.

Thus consider the first alternative (106). The linear coefficient in $k$, when all terms are put left, is

$$\cos\gamma\left(\frac{\sqrt{8}}{3} \cdot 2ka - 2\frac{k}{a}l^2\cos^2\alpha\right) - \sin\gamma \cdot \frac{2ka}{3}\,. \tag{108}$$

We test that with $l$ as in (123)

$$\sqrt{8} - \frac{3l^2\cos^2\alpha}{a^2} > 0\,.$$

Then the expression (108) as a function of $\gamma$, is decreasing for small $\gamma \geqslant 0$ until a local minimum at

$$\gamma = \pi + \arctan\left(-\frac{1}{\sqrt{8} - 3l^2\cos^2\alpha/a^2}\right)$$

Now, one tests that with $l$ as in (123)

$$\pi + \arctan\left(-\frac{1}{\sqrt{8} - 3l^2\cos^2\alpha/a^2}\right) > 2\arcsin(1/3) + \alpha$$

which shows that we need to evaluate (108) for the maximal $\gamma = 2\arcsin(1/3) + \alpha$. This leads then to testing the following version of inequality (106),

$$\cos(2\arcsin(1/3) + \alpha)2ka\left(\frac{\sqrt{8}}{3} - \frac{l^2\cos^2\alpha}{a^2}\right) - \sin(2\arcsin(1/3) + \alpha) \cdot \frac{2ka}{3} \geqslant k^2\left(l^2\cos^2\alpha - a^2\right) + \left(\frac{l^2\cos^2\alpha}{a^2} - 1\right),$$

or

$$
\begin{aligned}
&k^2\left(l^2\cos^2\alpha - a^2\right) + \\
2ka\Bigg[-\cos(2\arcsin(1/3)+\alpha)\left(\frac{\sqrt{8}}{3} - \frac{l^2\cos^2\alpha}{a^2}\right) + &\frac{\sin(2\arcsin(1/3)+\alpha)}{3}\Bigg] + \\
&\left(\frac{l^2\cos^2\alpha}{a^2} - 1\right) \leqslant 0 \text{ for all } k \geqslant l\,.
\end{aligned} \tag{109}
$$

Again we collect (109) with some inequalities to follow to be tested when we fix (123).

**Case 1.1.2.** $a_{n-m+1} < -1$. Similar to previous case but easier because (102) implies (98) and (101) implies (97) (with (53) and (94) checked).

**Case 1.1.3.** $a_{n-m+1} > 1$. Let $\Delta = \hat{\Delta}_{[1]}$ and $k = |\Delta|$. We have

$$k \geqslant 2|1-z| - \frac{|z|}{l}\,.$$

In our case with (123) one can again test that

$$-\cos\alpha + \frac{\sin\alpha}{\sqrt{4l^2-1}} \leqslant \frac{1}{\cos\alpha}\,, \quad \text{for } 0 \leqslant \alpha \leqslant 1.23\,, \tag{110}$$

to eliminate $z$ in the range condition for $z$ and test for (67).

The angle between $-\hat{\Delta}_{[1]}$ and $1/\alpha_{m+1}$ is at most $|\arg(\alpha_{m+1})| + |\arg(-\hat{\Delta}_{[1]})|$ which is below $\arcsin(1/3)+\delta$ by (48) and (50):

$$\measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_{m+1}) \leqslant \arcsin(1/3) + \delta\,. \tag{111}$$

Let $\gamma = \measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_{m+1})$. Then by lemma 4.1,

$$\measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_m) \leqslant \gamma + \arcsin(1/3)\,,$$

and similarly to (111)

$$\measuredangle(-\hat{\Delta}_{[1]}, 1/\alpha_m) \leqslant \arcsin(1/3) + \delta\,.$$

Then by testing

$$a\Phi(\alpha, l) \geqslant 1 \tag{112}$$

we have

$$ak > \cos\gamma$$

and can conclude

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant a\sqrt{k^2 + \left(\frac{1}{a}\right)^2 - 2k\frac{1}{a}\cos(\gamma)}\,.$$

Here we must work harder. If $|\alpha_{m+1}| \geqslant \nu_{m+1} \geqslant a$, then

$$|\alpha_{m+1}\hat{\Delta}_{[1]} + 1| \geqslant \nu_{m+1}\sqrt{k^2 + \left(\frac{1}{\nu_{m+1}}\right)^2 - 2k\frac{1}{\nu_{m+1}}\cos(\gamma)}\,.$$

The reverse estimate for $|\hat{\Delta}_{[1]} + 1/\alpha_m|$ is far more painful.

Assume $\nu'_m \geqslant |\alpha_m| \geqslant \nu_m$ and let

$$\Omega(\nu) = \sqrt{k^2 + \left(\frac{1}{\nu}\right)^2 - 2\frac{k}{\nu}\cos\min\left(\arcsin(1/3)+\gamma, \arcsin(1/3)+\delta\right)}\,,$$

so that

$$|\hat{\Delta}_{[1]} + 1/\alpha_m| \leqslant \max(\Omega(\nu'_m), \Omega(\nu_m)). \tag{113}$$

Thus we need to ascertain

$$\nu_{m+1}^2 \left( k^2 + \left( \frac{1}{\nu_{m+1}} \right)^2 - 2k \frac{1}{\nu_{m+1}} \cos(\gamma) \right) \geqslant l^2 \cdot \cos^2\alpha \cdot \left( \max \Omega(\nu'_m), \Omega(\nu_m) \right)^2 \tag{114}$$

for $\gamma \leqslant \arcsin(1/3) + \delta$.

We will write this inequality parametrically as $(114)(\nu_{m+1}, \nu_m, \nu'_m)$.

**Case 1.1.3.1.** $m = 1$. There is no difference to §3, and we are done by testing again (73).

**Case 1.1.3.2.** $m = 2$. By lemma 2.5 and lemma 2.6, we need to ascertain $(114)(0.947, 1.1, 2)$. Let

$$a' = 0.947, b = 1.1, b' = 2.$$

Thus we need to ascertain

$$a'^2 \left( k^2 + \left( \frac{1}{a'} \right)^2 - 2k \frac{1}{a'} \cos(\gamma) \right) \geqslant l^2 \left( k^2 + \left( \frac{1}{b} \right)^2 - 2k \frac{1}{b} \cos\min\left(\arcsin(1/3) + \gamma, \arcsin(1/3) + \delta\right) \right) \cdot \cos^2\alpha, \tag{115}$$

and

$$a'^2 \left( k^2 + \left( \frac{1}{a'} \right)^2 - 2k \frac{1}{a'} \cos(\gamma) \right) \geqslant l^2 \left( k^2 + \left( \frac{1}{b'} \right)^2 - 2k \frac{1}{b'} \cos\min\left(\arcsin(1/3) + \gamma, \arcsin(1/3) + \delta\right) \right) \cdot \cos^2\alpha. \tag{116}$$

Again one can discard the second maximum alternative, and restrict to $0 \leqslant \gamma \leqslant \delta$.

**Case 1.1.3.2.1.** (115) We maximize first the linear coefficient. Regarding this inequality as depending on $\gamma$, one has to examine when

$$-a' \cos\gamma + \frac{1}{b} l^2 \cos^2\alpha \cos(\gamma + \arcsin(1/3)) = \left( l^2 \cos^2\alpha \frac{\sqrt{8}}{3b} - a' \right) \cos\gamma - \frac{l^2 \cos^2\alpha}{3b} \cdot \sin\gamma$$

is minimal when $0 \leqslant \gamma \leqslant \delta$. If

$$\frac{l^2 \cos^2\alpha}{b} < a',$$

in particular if $b, a' \geqslant a$ (again because of (41)), this expression is easily seen to be decreasing in $\gamma \geqslant 0$ until

$$\tan\gamma = \Xi_3(\alpha, l, a', b) = \frac{l^2 \cos^2\alpha / b}{3a' - \sqrt{8} l^2 \cos^2\alpha / b}$$

This gives the minimal value

$$-\left[ \left( a' - \frac{1}{b} l^2 \cos^2\alpha \frac{\sqrt{8}}{3} \right) \cos\min\left(\delta, \arctan \Xi_3(\alpha, l, a', b)\right) + \frac{l^2 \cos^2\alpha}{3b} \sin\min\left(\delta, \arctan \Xi_3(\alpha, l, a', b)\right) \right]$$

Thus (115) can be rewritten

$$\begin{aligned} & k^2 (l^2 \cos^2\alpha - a'^2) + \\ 2k \cdot \Bigg[ \left( a' - l^2 \cos^2\alpha \frac{\sqrt{8}}{3b} \right) & \cos\min\left(\delta, \arctan \Xi_3(\alpha, l, a', b)\right) + \\ & \frac{l^2 \cos^2\alpha}{3b} \sin\min\left(\delta, \arctan \Xi_3(\alpha, l, a', b)\right) \Bigg] \\ & + \left( \frac{l^2 \cos^2\alpha}{b^2} - 1 \right) \leqslant 0 \end{aligned} \tag{117}$$

$$\text{for all } k \geqslant \Phi(\alpha, l) \text{ from (66)}.$$

Call this inequality $(117)(a', b)$. So we need to test $(117)(0.947, 1.1)$.

**Case 1.1.3.2.2.** (116) By repetition of the previous calculation, we need to test $(117)(0.947, 2)$.

**Case 1.1.3.3.** $m = 3, 4, 5$. We need to ascertain $(114)(0.885285, 0.896456, 1.5)$.

**Case 1.1.3.4.** $m > 5$. We need to ascertain $(114)(a, a, 1.2)$.

Thus we need to ascertain

$$a^2\left(k^2 + \left(\frac{1}{a}\right)^2 - 2k\frac{1}{a}\cos(\gamma)\right) \geqslant l^2\left(k^2 + \left(\frac{1}{a}\right)^2 - 2k\frac{1}{a}\cos\min\left(\arcsin(1/3) + \gamma, \arcsin(1/3) + \delta\right)\right) \cdot \cos^2\alpha. \quad (118)$$

and

$$a^2\left(k^2 + \left(\frac{1}{a}\right)^2 - 2k\frac{1}{a}\cos(\gamma)\right) \geqslant l^2\left(k^2 + \left(\frac{5}{6}\right)^2 - 2k\frac{5}{6}\cos\min\left(\arcsin(1/3) + \gamma, \arcsin(1/3) + \delta\right)\right) \cdot \cos^2\alpha. \quad (119)$$

**Case 1.1.3.4.1.** (118). We maximize first the linear coefficient. Regarding this inequality as depending on $\gamma$, one has to examine when

$$-a^2\cos\gamma + l^2\cos^2\alpha\cos(\gamma + \arcsin(1/3)) = \left(l^2\cos^2\alpha\frac{\sqrt{8}}{3} - a^2\right)\cos\gamma - \frac{l^2\cos^2\alpha}{3}\sin\gamma$$

is minimal when $0 \leqslant \gamma \leqslant \delta$. This expression is decreasing until

$$\tan\gamma = \frac{l^2\cos^2\alpha}{3a^2 - \sqrt{8}l^2\cos^2\alpha}$$

This gives the maximal value

$$-\left[\left(a^2 - l^2\cos^2\alpha\frac{\sqrt{8}}{3}\right)\cos\min\left(\delta, \arctan\left(\frac{l^2\cos^2\alpha}{3a^2 - \sqrt{8}l^2\cos^2\alpha}\right)\right) + \frac{l^2\cos^2\alpha}{3}\sin\min\left(\delta, \arctan\left(\frac{l^2\cos^2\alpha}{3a^2 - \sqrt{8}l^2\cos^2\alpha}\right)\right)\right]$$

Thus (118) can be rewritten

$$\begin{aligned} & k^2(l^2\cos^2\alpha - a^2) + \\ k \cdot \frac{2}{a} \cdot \Bigg[ & \left(a^2 - l^2\cos^2\alpha\frac{\sqrt{8}}{3}\right)\cos\min\left(\delta, \arctan\left(\frac{l^2\cos^2\alpha}{3a^2 - \sqrt{8}l^2\cos^2\alpha}\right)\right) + \\ & \frac{l^2\cos^2\alpha}{3}\sin\min\left(\delta, \arctan\left(\frac{l^2\cos^2\alpha}{3a^2 - \sqrt{8}l^2\cos^2\alpha}\right)\right)\Bigg] \\ & + \left(\frac{l^2\cos^2\alpha}{a^2} - 1\right) \leqslant 0 \end{aligned} \quad (120)$$

$$\text{for all } k \geqslant 2|1 - z| - \frac{|z|}{l}. \quad (121)$$

Again, one can eliminate $z$ in (121) and test for $k$ in (67).

**Case 1.1.3.4.2.** (119). We maximize first the linear coefficient. Regarding this inequality as depending on $\gamma$, one has to examine when

$$-a\cos\gamma + \frac{5}{6}l^2\cos^2\alpha\cos(\gamma + \arcsin(1/3)) = \left(l^2\cos^2\alpha\frac{5\sqrt{8}}{18} - a\right)\cos\gamma - \frac{5l^2\cos^2\alpha}{18} \cdot \sin\gamma$$

is minimal when $0 \leqslant \gamma \leqslant \delta$. This expression is decreasing until

$$\tan\gamma = \Xi_2(\alpha,l) \;=\; \frac{5l^2\cos^2\alpha}{18a - 5\sqrt{8}l^2\cos^2\alpha}$$

This gives the maximal value

$$-\left[\left(a - \frac{5}{6}l^2\cos^2\alpha\frac{\sqrt{8}}{3}\right)\cos\min\left(\delta, \arctan\Xi_2(\alpha,l)\right) + \frac{5l^2\cos^2\alpha}{18}\sin\min\left(\delta, \arctan\Xi_2(\alpha,l)\right)\right]$$

Thus (119) can be rewritten

$$\begin{aligned} & k^2(l^2\cos^2\alpha - a^2) + \\ 2k\cdot\left[\left(a - l^2\cos^2\alpha\frac{5\sqrt{8}}{18}\right)\cos\min\left(\delta, \arctan\Xi_2(\alpha,l)\right) + \frac{5l^2\cos^2\alpha}{18}\right. & \left.\sin\min\left(\delta, \arctan\Xi_2(\alpha,l)\right)\right] \\ & + \left(\frac{25l^2\cos^2\alpha}{36} - 1\right) \;\leqslant\; 0 \end{aligned} \tag{122}$$

$$\text{for all } k \geqslant \Phi(\alpha,l) \text{ from (66)}.$$

Maximizing a quadratic expression in $k$ over an interval is no problem, and MATHEMATICA shows that with $a = 0.85$ one can choose

$$\begin{aligned} l &= (0.61 - \alpha/30)/(\cos\alpha)^{11/10-\alpha/60} \text{ for } \alpha \geqslant 0.77 \\ l &= (0.61 - \alpha/30)/(\cos\alpha)^{13/10-\alpha/60} \text{ for } \alpha < 0.77 \end{aligned} \tag{123}$$

to satisfy all inequalities in the subcases of Part 1.1 simultaneously when $\alpha \leqslant 1.23$. The inequality (122) in case 1.1.3.4.2 is the hardest to satisfy, and is one further reason we had to dig so deep in §2.

**Part 1.2.** Next, we need to care about angles. Let $\Delta = \hat{\Delta}_{[1]}$.

**Case 1.2.1.** Continue assuming $a_{n+1} = 1$ and $\hat{\Delta}_{n+1} = \hat{\Delta}_{[m+1]}$. If $a_{n-m+1} < 0$, then we have some difficulty. For $m > 1$, much of this is moved to lemma 4.2.

Using lemma 4.2 for $m > 2$,

$$\left|\arg\frac{\Delta + 1/\alpha_{m+1}}{\Delta + 1/\alpha_m}\right| \leqslant \arcsin(1/3).$$

With (40) we have

$$\left|\arg -\hat{\Delta}_{n+1}\right| \leqslant |\arg(-z)| + |\arg(\alpha_{m+1})| + \left|\arg\frac{\Delta + 1/\alpha_{m+1}}{\Delta + 1/\alpha_m}\right| \leqslant 2\arcsin(1/3) + |\arg(-z)|,$$

as we wanted in (95).

For $m = 2$, we observe that (by comparing (31) after the variable change)

$$z\alpha_3 \;=\; z\cdot\frac{1-\left(-\frac{1}{z}\right)^3}{1-\left(-\frac{1}{z}\right)^2} \;=\; z\cdot\left(1 + \frac{\left(-\frac{1}{z}\right)^2}{1-\frac{1}{z}}\right) \;=\; z + \frac{1}{z-1},$$

which easily shows that

$$|\arg(-z\alpha_3)| \leqslant |\arg(-z)|.$$

This is how we can grant ourselves an extra $\arcsin(1/3)$ on the right in (126) for $m = 2$.

We need to treat $m = 1$ extra here. We have with $\alpha_2 = 1 - 1/z$ and $\Delta = \hat{\Delta}_{[1]}$ from (40),

$$-\hat{\Delta}_{[2]} \;=\; -z\alpha_2\frac{\hat{\Delta}_{[1]} + 1/\alpha_2}{\hat{\Delta}_{[1]}} \;=\; -z\left(\alpha_2 + \frac{1}{\Delta}\right) \;=\; 1 - z + \frac{-z}{1-z}\cdot\frac{1-z}{\Delta}.$$

We like to use lemma 3.2 and check

$$|1-z| > \left|\frac{-z}{\Delta}\right|,$$

which (see (124) below) is equivalent to

$$2\sqrt{2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}} > \frac{1}{l(\alpha)}.$$

Thus using (98) and (102), and lemma 3.2,

$$|\arg(-\hat{\Delta}_{[2]})| \leqslant \max\left(|\arg(1-z)|,\ \frac{1}{2}\left(\left|\arg\frac{-z}{1-z}\right|+\arcsin\left(\frac{|z|}{l|1-z|}\right)+|\arg(1-z)|\right)\right).$$

Thus to show is

$$\frac{1}{2}\left(\alpha+\arcsin\left(\frac{|z|}{l|1-z|}\right)\right) = \frac{1}{2}\left(|\arg(1-z)|+\left|\arg\frac{-z}{1-z}\right|+\arcsin\left(\frac{|z|}{l|1-z|}\right)\right) \leqslant |\arg(-z)|+2\arcsin(1/3).$$

As long as $l$ in (81) depends only on $\alpha = |\arg(-z)|$, for given $\alpha$, one can see that $1/|\alpha_2| = |z|/|1-z| = 1/\left|1-\frac{1}{z}\right|$ is maximal when $-1/z = x+f(x)\sqrt{-1}$, with the notation in §2. Then from (35),

$$\left|1-\frac{1}{z}\right| = \sqrt{f(x)^2+(1+x)^2} = 2\sqrt{2-y} = 2\sqrt{2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}} \tag{124}$$

We have to show

$$\alpha+\arcsin\left(\frac{1}{l|\alpha_2|}\right) \leqslant 2\alpha+4\arcsin\left(\frac{1}{3}\right),$$

or

$$\arcsin\left(\frac{1}{2\sqrt{2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}}\,l(\alpha)}\right) \leqslant \alpha+4\arcsin\left(\frac{1}{3}\right).$$

This test succeeds for $\alpha \geqslant 0.1$.

**Case 1.2.2.** $a_{n-m+1} > 1$. Again using (40), it is enough to see

$$\left|\arg\frac{\Delta+1/\alpha_{m+1}}{\Delta+1/\alpha_m}\right| \leqslant \arcsin(1/3).$$

Now

$$|\Delta| \geqslant 2|1-z|-\frac{|z|}{l},\quad \sphericalangle(1/\alpha_{m+1},1/\alpha_m) \leqslant \arcsin(1/3) \quad\text{and}\quad \left|\frac{1}{\alpha_{m[+1]}}\right| \leqslant \frac{1}{a}.$$

We have $\left|\frac{1}{\alpha_{m[+1]}}\right| \leqslant |\Delta|$ by (112). Trigonometry in the triangles with vertices $0,\Delta,\Delta+1/\alpha_m$ and $0,\Delta,\Delta+1/\alpha_{m+1}$ and an easy argument shows that

$$\sphericalangle(\Delta+1/\alpha_{m+1},\Delta+1/\alpha_m)$$

is maximal when $|\Delta| = 2|1-z|-|z|/l$, $|\alpha_{m[+1]}| = a$, $\sphericalangle(1/\alpha_{m+1},1/\alpha_m) = \arcsin(1/3)$ and $\arg(-\Delta\alpha_m) = -\arg(-\Delta\alpha_{m+1})$. This gives

$$\sin\frac{\sphericalangle(\Delta+1/\alpha_{m+1},\Delta+1/\alpha_m)}{2} \leqslant \frac{\sin\frac{\arcsin(1/3)}{2}}{a\sqrt{\left(2|1-z|-\frac{|z|}{l}\right)^2+\left(\frac{1}{a}\right)^2-2\frac{1}{a}\left(2|1-z|-\frac{|z|}{l}\right)\cos\frac{\arcsin(1/3)}{2}}}$$

First, we check again with (110) that

$$2|1-z|-|z|/l$$

decreases with $|z|$ when $\alpha$ is fixed. Thus we can set $\Re e\, z = -1$, whence $|z| = 1/\cos\alpha$ and $|1-z| = \sqrt{4+\tan^2\alpha}$ (see (83)). Now to see

$$\sphericalangle(\Delta+1/\alpha_{m+1},\Delta+1/\alpha_m)/2 \leqslant \arcsin(1/3)/2$$

it is enough to see for $\alpha = |\arg(-z)|$, with $\Phi(\alpha,l)$ as in (54), and testing (69), that

$$\frac{\sin\frac{\arcsin(1/3)}{2}}{a\sqrt{\Phi(\alpha,l)^2+\left(\frac{1}{a}\right)^2-\frac{2}{a}\Phi(\alpha,l)\cos\frac{\arcsin(1/3)}{2}}} \leqslant \sin\frac{\arcsin(1/3)}{2},$$

which is true for $l$ in (123) and $a = 0.85$.

This finishes $a_{n+1} = 1$. The case $a_{n+1} = -1$ is analogous.

**Case 2.** $a_{n+1} > 1$. We need to derive the estimate (99) of the angle. The norm estimate is quite clear (recall the remark below (103)).

We will use (36) in the form

$$\hat{\Delta}_{n+1} = a_{n+1}(z-1) + \frac{z}{\hat{\Delta}_n}.$$

Consider $\triangle ABC$ with $\eta,\beta,\gamma$ angles at $A,B,C$. Let $\overline{AB} = 2|1-z|$, $\overline{AC} = |z|/l$ and

$$\eta = 2\arcsin(1/3) + |\arg(1-1/z)| + |\arg(-z)|.$$

We have (compare (83))

$$\sin|\arg(1-1/z)| = \frac{\cos\alpha\sin\alpha}{\sqrt{1+3\cos^2\alpha}}, \quad \cos|\arg(1-1/z)| = \frac{1+\cos^2\alpha}{\sqrt{1+3\cos^2\alpha}}.$$

We need an upper bound on $\beta$. First, from $l \geqslant 0.5$ we have $\overline{AC} < \overline{AB}$, thus $\beta \leqslant \pi/2$.

The Sine and Cosine theorems give

$$\sin\beta = \frac{\sin\eta\cdot|z|}{\sqrt{4|1-z|^2l^2+|z|^2-4\cdot|1-z|\cdot l\cdot|z|\cdot\cos\eta}}$$

This estimates

$$\measuredangle(\tilde{\Delta}_{n+1},(z-1)\tilde{\Delta}_n) = \measuredangle(-\tilde{\Delta}_{n+1},(1-z)\tilde{\Delta}_n) \leqslant \beta$$

and

$$|\arg(-\hat{\Delta}_{n+1})| = \measuredangle(-\tilde{\Delta}_{n+1},\tilde{\Delta}_n) \leqslant \beta + |\arg(1-z)|. \tag{125}$$

Next, for given fixed $|z|$, it is easy to see that $|1-1/z|$ (or equivalently, $|1-z|$) is smallest, and all of $|\arg(-z)|$, $|\arg(1-z)|$, $|\arg(1-1/z)|$ largest when $\Re e\, z' = -1$. Thus consider only this case. Then the formulas in §2 show that

$$|\arg(-z')| =: \alpha' \leqslant \arccos\left(2-\cos\alpha-\sqrt{\cos^2\alpha-4\cos\alpha+3}\right),$$

leading to (42). Now assuming $z' = z$, we have (83). Thus

$$\sin\beta = \frac{\sin\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha\sin\alpha}{1+\cos^2\alpha}\right)+\alpha\right)}{\cos\alpha\sqrt{4(4+\tan^2\alpha)l^2+\frac{1}{\cos^2\alpha}-4\cdot\frac{l}{\cos\alpha}\cdot\sqrt{4+\tan^2\alpha}\cos\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha\sin\alpha}{1+\cos^2\alpha}\right)+\alpha\right)}},$$

which with (125) and (83) yields the expression (94). The inequality there is to be tested:

$$\arcsin\frac{\sin\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right)+\alpha'\right)}{\cos\alpha'\sqrt{4(4+\tan^2\alpha')l^2+\frac{1}{\cos^2\alpha'}-4\cdot\frac{l}{\cos\alpha'}\cdot\sqrt{4+\tan^2\alpha'}\cos\left(2\arcsin(1/3)+\arctan\left(\frac{\cos\alpha'\sin\alpha'}{1+\cos^2\alpha'}\right)+\alpha'\right)}}$$
$$+\arctan\frac{\tan\alpha'}{2} \leqslant 2\arcsin(1/3)+\alpha.$$

It is true for $\alpha \leqslant 1.23$.

The case $a_{n+1} < -1$ is analogous.

With this the induction is complete. □

**Lemma 4.2** *Let* $|\Delta| \geqslant l$ *and* $\arg(\Delta) \leqslant 2\arcsin(1/3) + |\arg(-z)|$ *with* $\alpha = |\arg(-z)| > 0.1$. *Then for* $m > 2$,

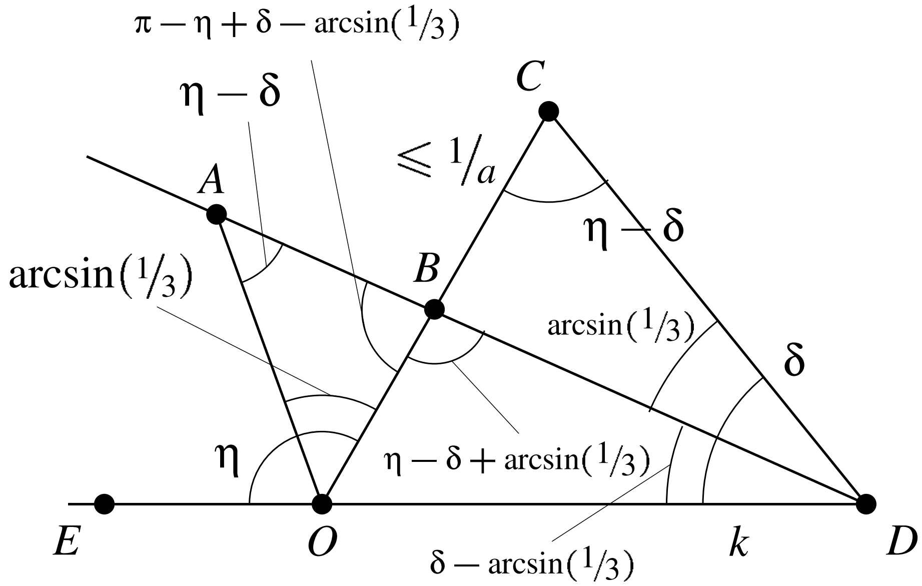
$$\measuredangle(\Delta + 1/\alpha_{m+1}, \Delta + 1/\alpha_m) \leqslant \arcsin(1/3)\,.$$

*For* $m = 2$,

$$\measuredangle(\Delta + 1/\alpha_3, \Delta + 1/\alpha_2) \leqslant 2\arcsin(1/3)\,. \tag{126}$$

**Proof. Part 1.** Inner ray. To some extent we use the proof of lemma 3.3.

Consider this picture




Let $\hat{m} \in \{m, m+1\}$ be so that $\measuredangle(\Delta, 1/\alpha_{\hat{m}}) \geqslant \measuredangle(\Delta, 1/\alpha_{\tilde{m}})$ for $\tilde{m} = 2m + 1 - \hat{m}$.

We set $D = 0$. Now $O$ is $\Delta$ and $\overline{OC} = 1/|\alpha_{\hat{m}}|$. We fix $\measuredangle BCD = \arcsin(1/3)$ and must consider the largest distance between $O$ and a point in $\triangle AOB$, which is

$$\max\ \overline{OA}, \overline{OB}\,. \tag{127}$$

We must prove that $|1/\alpha_{\tilde{m}}|$ is larger than (127), which we rewrite below as (128). Let $k = \overline{OD} = |\Delta| \geqslant l$. Here

$$\eta \;=\; \measuredangle EOC \;\leqslant\; 3\arcsin(1/3) + \alpha\,.$$

**Case 1.1.** $m > 2$. Our goal is

$$\max(\overline{OA}, \overline{OB}) \leqslant \frac{1}{M_{\alpha,\tilde{m}}}\,, \tag{128}$$

where

$$M_{\alpha,\tilde{m}} \;=\; \max\{\,|\alpha_m| : m \geqslant \tilde{m}, |\arg(-z)| = \alpha\,\}\,. \tag{129}$$

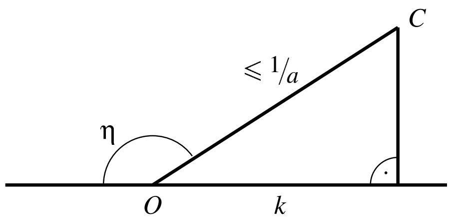
These numbers were determined in lemma 2.9.

Our first goal is to show that $\overline{OB}$ can be often removed from that alternative.

First, one tests that

$$\cos\eta \;\geqslant\; \cos(3\arcsin(1/3) + \alpha) > -la \geqslant -\frac{k}{\overline{OC}}\,, \tag{130}$$

thus $\delta < \pi/2$. This will be important.




The problem so far depends on three parameters, $\eta$, $\overline{OC}$ and $k$. We have to remove first the dependence on $\overline{OC}$ and $k$.

**Case 1.1.1.** Assume

$$\eta - \delta < \pi/2 - \arcsin(1/3)/2\,. \tag{131}$$

We have again by Sine law

$$\overline{OA} = x = \overline{OC} \cdot \frac{\sin(\delta - \arcsin(1/3))}{\sin\delta} \tag{132}$$

with $\delta$ as in (86). (If the expression is negative, we are done.)

Observe that

$$\overline{OB} = \frac{\overline{OA}\sin(\eta-\delta)}{\sin(\eta-\delta+\arcsin(1/3))}\,. \tag{133}$$

Because of the case assumption, we have $\overline{OB} < \overline{OA}$, and thus can discard $\overline{OB}$ in (127).

First, fix $k$, $\eta$. We observe again from (132) and calculus that $\overline{AO}$ is largest when $\delta$ is largest. Increasing $\overline{OC} \leqslant 1/a$ at fixed $k$, $\eta$ does not lead out of the case assumption (131) and increases $\delta$ and hence by (132) also $\overline{OA}$. Thus it is enough for the rest of the argument to set $\overline{OC} = 1/a$. Then (132) becomes

$$\overline{OA} = x = \frac{1}{a} \cdot \frac{\sin(\delta - \arcsin(1/3))}{\sin\delta}\,. \tag{134}$$

By testing (130), one ascertains that $\delta(\eta,k,a)$ is decreasing in increasing $k \geqslant l$ for fixed $\eta, a$. Thus it is enough to decrease $k$ until $k = l$ (and this does not lead out of the case assumption).

We observed the expression for $\overline{OA}$ is increasing in $\delta$. Under the assumption of the case, $\delta$ is increasing in $\eta$, for fixed $k$ (now $k = l(\alpha)$).

Thus put only the maximal

$$\eta = \min\left(3\arcsin(1/3) + \alpha, \frac{\pi}{2} + \arcsin\frac{1}{la}\right). \tag{135}$$

(Note that $\eta = \dfrac{\pi}{2} + \arcsin\dfrac{1}{la}$ is where $\delta$ is maximal over $\eta$ for $k = l = l(\alpha)$ fixed, and then $\eta - \delta = \dfrac{\pi}{2}$, which is out of our case. Thus the stated value for $\eta$ is in fact more generous than we need. It turns out, though, that for $a = 0.85$, the first alternative is always smaller.)

We have to evaluate (134) for $k = l = l(\alpha)$ and $\eta$ in (135) and test $\leqslant 1/M_{\alpha,\tilde m}$ to get (128).

This test succeeds for $\tilde m \geqslant 5$. For $\tilde m = 4$ we can succeed by setting $a_4 = 0.896456 = \min\{|\alpha_3|, |\alpha_5|\}$ by lemmas 2.8 and 2.6. For $\tilde m = 3$ we must be more careful, and replace $a$ by

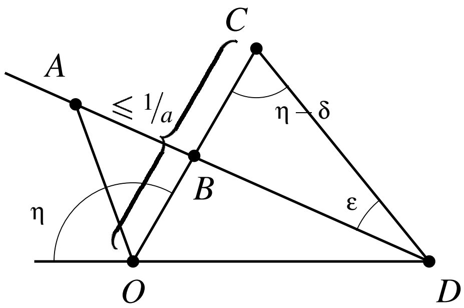
$$a_3(\alpha) = \min\{\,|\alpha_2|, |\alpha_4| \,:\, |\arg(-z)| = \alpha\,\}\,,$$

and let $a_3$ depend on $\alpha$.

**Case 1.1.2.** We assume

$$\eta - \delta \geqslant \pi/2 - \arcsin(1/3)/2\,. \tag{136}$$

So we have the picture

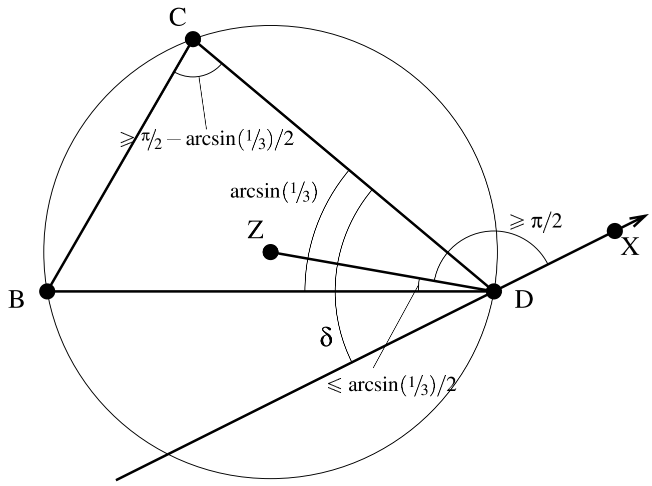



(137)

Let $M_\alpha = M_{\alpha,3}$ from (129). To remove the dependence on α, we use that since $\eta \leqslant 3\arcsin(1/3) + \alpha$,

$$M_\alpha \leqslant \tilde{M}_\eta := \max\{\, M_\alpha : \arccos(1/3) \geqslant \alpha \geqslant \max(0.1, \eta - 3\arcsin(1/3))\,\}.$$

We show that it's impossible that both $\overline{OB} > \overline{OA}$ and $\overline{OB} \geqslant 1/\tilde{M}_\eta$. Assume that it is so. We will show the contradiction that $\sphericalangle BDC < \arcsin(1/3)$. The picture




shows that when $D$ moves right, then $\sphericalangle BDC$ decreases. This can be seen thus.

Let $X$ be a point on the ray indicated by the arrow.

Consider the circumcenter $Z$ of $\triangle BCD$. Assume first $\sphericalangle BCD < \pi/2$, so $Z$ lies on the same side of $BD$ as $C$. From (136) we have that

$$\sphericalangle ZDB \leqslant \arcsin(1/3)/2. \tag{138}$$

Now, it is easy to see that $\sphericalangle BDO$ decreases when $\overline{OA}$ is fixed and $\overline{OB} \geqslant \overline{OA}$ increases. Moreover, for $\overline{OB} = \overline{OA}$ we have $\sphericalangle BDO = \eta - \arcsin(1/3)/2 - \pi/2$ and

$$\eta \leqslant 3\arcsin(1/3) + \alpha \leqslant \pi/2 + 2\arcsin(1/3). \tag{139}$$

Thus $\sphericalangle BDO \leqslant 3\arcsin(1/3)/2$. Hence with (138)

$$\sphericalangle ZDO \leqslant 2\arcsin(1/3) < \pi/2.$$

This conclusion obviously works also (even more easily) when $\sphericalangle BCD \geqslant \pi/2$, so $Z$ lies on the opposite side of $BD$ to $C$ (then $\sphericalangle ZDO < \sphericalangle BDO$).

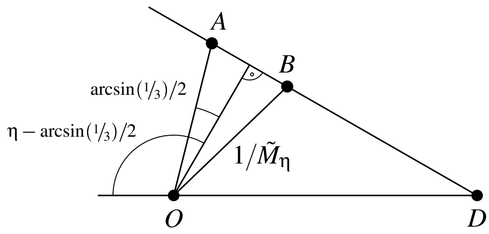
And so

$$\sphericalangle ZDX = \pi - \sphericalangle ZDO > \pi/2.$$

Thus the ray $DX$ lies outside the circumcircle of $\triangle BCD$ and $\sphericalangle BDC$ decreases locally when $D$ moves right in direction $X$. Then $\sphericalangle BDC$ must also decrease globally when $D$ moves right (and $\overline{OB} \geqslant \overline{OA}$ remains true under this move; the condition (136) will also be preserved when $D$ moves right, because $C$ is fixed and $B$ moves on $\overline{OC}$ so that $\sphericalangle BCD$ increases).

Moreover, $D$ is easily seen to move right when $\overline{OB}$ increases at fixed $\overline{AO}$, or both $\overline{OB}$ and $\overline{AO}$ increase by the same amount. This shows that we need to consider only $\overline{AO} = \overline{OB} = 1/\tilde{M}_\eta$ in the picture (137).

Using the picture




one easily finds

$$k = \overline{OD} = -\frac{\cos(\arcsin(1/3)/2)}{\tilde{M}_\eta \cos(\eta - \arcsin(1/3)/2)} = -\frac{\sqrt{3+\sqrt{8}}}{\sqrt{6}\tilde{M}_\eta \cos(\eta - \arcsin(1/3)/2)}. \tag{140}$$

We consider only $k \leqslant 3\overline{OC} \leqslant 3/a$, because otherwise $\sphericalangle BDC < \sphericalangle ODC < \arcsin(1/3)$ and we are done. This gives

$$\eta \geqslant \frac{\pi}{2} + \frac{\arcsin(1/3)}{2} + \arcsin\left(\frac{a\sqrt{3+\sqrt{8}}}{3\sqrt{6}\tilde{M}_\eta}\right) \geqslant \frac{\pi}{2} + \frac{\arcsin(1/3)}{2}.$$

(Note that from (136) and assuming $\delta > \arcsin(1/3)$, we have $\cos(\eta - \arcsin(1/3)/2) < 0$.) Thus

$$\tilde{M}_\eta \leqslant \tilde{M}_{\pi/2+\arcsin(1/3)/2},$$

and

$$\eta \geqslant \frac{\pi}{2} + \frac{\arcsin(1/3)}{2} + \arcsin\left(\frac{a\sqrt{3+\sqrt{8}}}{3\sqrt{6}\tilde{M}_{\pi/2+\arcsin(1/3)/2}}\right).$$

Let $\hat{a} = \frac{1}{\overline{OC}} \geqslant a$. Thus with (140) we have to examine

$$\sphericalangle BDC =: \varepsilon = \arcsin\frac{\sin\eta}{\sqrt{\hat{a}^2k^2+1+2\hat{a}k\cos\eta}} - \arcsin\frac{\sin\eta}{\sqrt{\tilde{M}_\eta^2k^2+1+2\tilde{M}_\eta k\cos\eta}} < \arcsin(1/3), \tag{141}$$

for (recall (139))

$$\frac{\pi}{2} + \frac{\arcsin(1/3)}{2} + \arcsin\left(\frac{a\sqrt{3+\sqrt{8}}}{3\sqrt{6}\tilde{M}_{\pi/2+\arcsin(1/3)/2}}\right) \leqslant \eta \leqslant \frac{\pi}{2} + 2\arcsin(1/3).$$

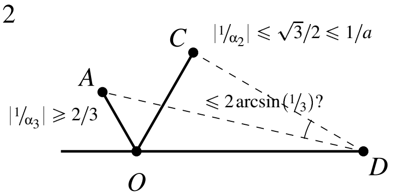
Again because of (130) we can set $\hat{a} = a$ in (141).

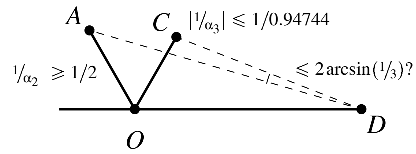
We calculated $\varepsilon = \sphericalangle BDC = \varepsilon(\eta)$ explicitly and showed by plot $\varepsilon < \arcsin(1/3)$. To simplify the calculation, we used from lemma 2.9

$$\tilde{M}_\eta \leqslant \tilde{M}_{\pi/2+\arcsin(1/3)/2} \approx M_{\pi/2-2.5\arcsin(1/3)} \approx 1.18666 < 1.2.$$

Thus not both $\overline{OB} > \overline{OA}$ and $\overline{OB} \geqslant 1/\tilde{M}_\eta$. But $\overline{OB} > \overline{OA}$ by (133) and case assumption. Thus $\overline{OA} < \overline{OB} \leqslant 1/\tilde{M}_\eta \leqslant 1/M_\alpha$ and we are done.

**Case 1.2.** $m = 2$




The left case picture is in fact handled in case 1.1. Consider the right picture.

Our goal is

$$\max(\overline{OA}, \overline{OB}) \leqslant \frac{1}{\hat{M}_\alpha}, \tag{142}$$

where

$$\hat{M}_\alpha = \max\left\{ |\alpha_2| = \left|1 - \frac{1}{z}\right| : |\arg(-z)| = \alpha \right\} = \sqrt{1 + 3\cos^2\alpha}. \tag{143}$$

Our first goal is to show that $\overline{OB}$ can be often removed from that alternative.

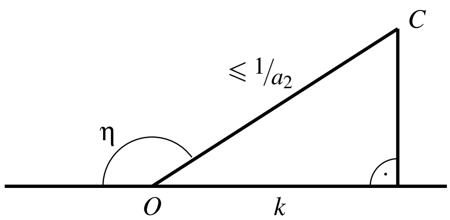
First, we tested from (130) with

$$a_2 = \min|\alpha_3| = 0.94744 > a \tag{144}$$

that

$$\cos\eta > \cos(3\arcsin(1/3) + \alpha) > -la_2 > -ka_2,$$

thus $\delta < \pi/2$. This will be important. (As in case 1.1, one can set $\overline{OC} \leqslant 1/a_2$ to $\overline{OC} = 1/a_2$.)




The problem so far depends on two parameters, $\eta$ and $k$. We have to remove somehow the dependence on $k$.

**Case 1.2.1.** $\eta - \delta < \pi/2 - 3\arcsin(1/3)/2$. We have again

$$\overline{OA} = x = \frac{1}{a_2} \cdot \frac{\sin(\delta - 2\arcsin(1/3))}{\sin\delta} \tag{145}$$

with $\delta$ as in (86). (If the expression is negative, we are done.)

First, fix $k$. Observe that

$$\overline{OB} = \frac{\overline{OA}\sin(\eta - \delta + \arcsin(1/3))}{\sin(\eta - \delta + 2\arcsin(1/3))}. \tag{146}$$

Because of the case assumption, we have $\overline{OB} < \overline{OA}$, and thus can discard $\overline{OB}$.

We observe again that $\overline{AO}$ is largest when $\delta$ is largest. By testing (130), one ascertains that $\delta(\eta, k, a_2)$ is decreasing in $k \geqslant l$ for fixed $\eta, a_2$. Thus it is enough to decrease $k$ until $k = l$ (and this does not lead out of the case assumption).

We observed the expression for $\overline{OA}$ is increasing in $\delta$. Under the assumption of the case, $\delta$ is increasing in $\eta$, for fixed $k$ (now $k = l(\alpha)$).

Thus put only the maximal

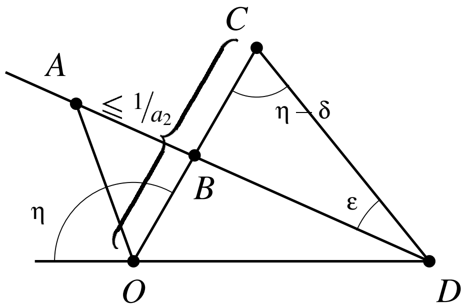
$$\eta = \min\left(3\arcsin(1/3) + \alpha, \frac{\pi}{2} + \arcsin\frac{1}{la_2}\right). \tag{147}$$

We have to evaluate (145) for $k = l = l(\alpha)$ and $\eta$ in (147) and test $\leqslant 1/\hat{M}_\alpha$ to get (142).

**Case 1.2.2.** We assume

$$\eta - \delta \geqslant \pi/2 - 3\arcsin(1/3)/2\,. \tag{148}$$

So we have the picture




(149)

Note that with (148) and assuming $\delta > 2\arcsin(1/3)$ we have

$$\eta > \frac{\pi}{2} + \frac{1}{2}\arcsin(1/3), \quad \text{so} \quad \eta - 3\arcsin(1/3) > \frac{\pi}{2} - 2.5\arcsin(1/3) > 0.1\,. \tag{150}$$

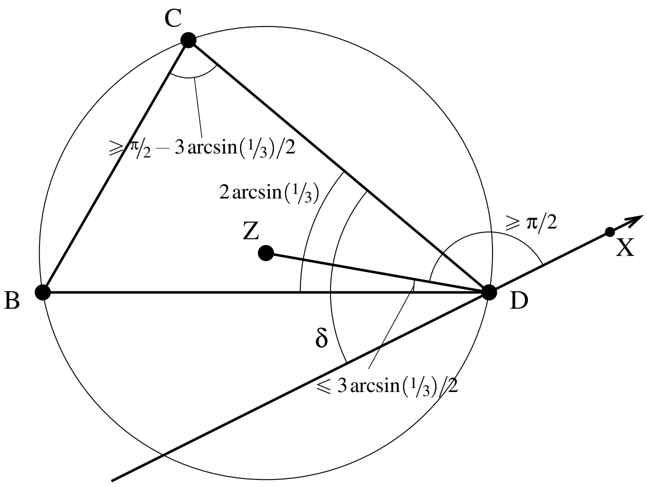
To remove the dependence on $\alpha$, we again set using $\eta \leqslant 3\arcsin(1/3) + \alpha$,

$$\hat{M}_\alpha \leqslant \bar{M}_\eta := \max\{\, \hat{M}_\alpha : \arccos(1/3) \geqslant \alpha \geqslant \eta - 3\arcsin(1/3)\,\}\,.$$

We have from (143)

$$\bar{M}_\eta = \sqrt{1 + 3\cos^2(\eta - 3\arcsin(1/3))}\,. \tag{151}$$

We show that it's impossible that both $\overline{OB} \geqslant \overline{OA}$ and $\overline{OB} \geqslant 1/\bar{M}_\eta$. Assume that it is so. We will show the contradiction that $\sphericalangle BDC < 2\arcsin(1/3)$. The picture




shows that when $D$ moves right, then $\sphericalangle BDC$ decreases. This can be seen thus.

Let $X$ be a point on the ray indicated by the arrow.

Consider the circumcenter $Z$ of $\triangle BCD$. Assume first $\sphericalangle BCD < \pi/2$, so $Z$ lies on the same side of $BD$ as $C$. From (148) we have that $\sphericalangle ZDB \leqslant 3\arcsin(1/3)/2$.

Now, it is easy to see that $\sphericalangle BDO$ decreases when $\overline{OA}$ is fixed and $\overline{OB} \geqslant \overline{OA}$ increases. Moreover, for $\overline{OB} = \overline{OA}$ we have $\sphericalangle BDO = \eta - \arcsin(1/3)/2 - \pi/2$ and $\eta \leqslant 3\arcsin(1/3) + \alpha \leqslant \pi/2 + 2\arcsin(1/3)$. Thus $\sphericalangle BDO \leqslant 3\arcsin(1/3)/2$. Hence

$$\sphericalangle ZDO \leqslant 3\arcsin(1/3) < \pi/2\,.$$

This conclusion obviously works also (even more easily) when $\sphericalangle BCD \geqslant \pi/2$, so $Z$ lies on the opposite side of $BD$ to $C$.

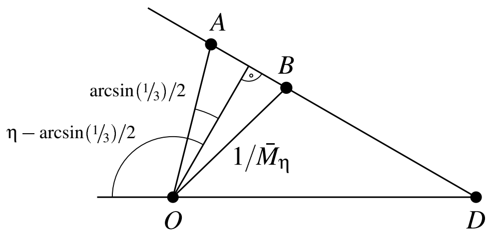
And so

$$\sphericalangle ZDX = \pi - \sphericalangle ZDO > \pi/2\,.$$

Thus the ray $DX$ lies outside the circumcircle of $\triangle BCD$ and $\sphericalangle BDC$ decreases locally when $D$ moves right in direction $X$. Then $\sphericalangle BDC$ must also decrease globally when $D$ moves right (and $\overline{OB} \geqslant \overline{OA}$ remains true under this move; the condition (148) will also be preserved when $D$ moves right, because $C$ is fixed and $B$ moves on $\overline{OC}$ so that $\sphericalangle BCD$ increases).

Moreover, $D$ is easily seen to move right when $\overline{OB}$ increases at fixed $\overline{AO}$, or both $\overline{OB}$ and $\overline{AO}$ increase by the same amount. This shows that we need to consider only $\overline{AO} = \overline{OB} = 1/\bar{M}_\eta$ in the picture (149).

Using the picture




one easily finds

$$k = \overline{OD} = -\frac{\cos(\arcsin(1/3)/2)}{\bar{M}_\eta \cos(\eta - \arcsin(1/3)/2)} = -\frac{\sqrt{3+\sqrt{8}}}{\sqrt{6}\bar{M}_\eta \cos(\eta - \arcsin(1/3)/2)}\,. \tag{152}$$

We consider only $k \leqslant \dfrac{9}{2\sqrt{8}a_2}$, because otherwise $\sphericalangle BDC < \sphericalangle ODC < 2\arcsin(1/3)$ and we are done. This gives (again with $\cos(\eta - \arcsin(1/3)/2) < 0$ from (150))

$$\eta \geqslant \frac{\pi}{2} + \frac{\arcsin(1/3)}{2} + \arcsin\left(\frac{2\sqrt{8}a_2\sqrt{3+\sqrt{8}}}{9\sqrt{6}\bar{M}_\eta}\right) \geqslant \frac{\pi}{2} + \frac{\arcsin(1/3)}{2}\,, \tag{153}$$

and again from (151) and the rightmost inequality in (153),

$$\bar{M}_\eta \leqslant \bar{M}_{\pi/2+\arcsin(1/3)/2} = \sqrt{1+3\cos^2(\pi/2 - 2.5\arcsin(1/3))} = \sqrt{1+3\sin^2(2.5\arcsin(1/3))}\,.$$

Thus with (152) and (151) we have to examine

$$\varepsilon = \arcsin\frac{\sin\eta}{\sqrt{a_2^2k^2 + 1 + 2a_2k\cos\eta}} - \arcsin\frac{\sin\eta}{\sqrt{\bar{M}_\eta^2k^2 + 1 + 2\bar{M}_\eta k\cos\eta}} < 2\arcsin(1/3)$$

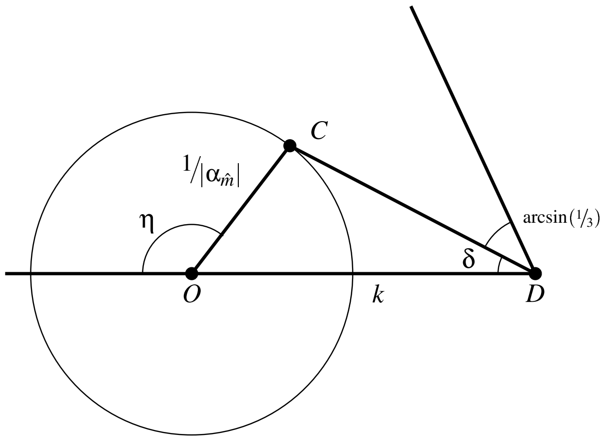
for (recall (139))

$$2.106539 \approx \frac{\pi}{2} + \frac{\arcsin(1/3)}{2} + \arcsin\left(\frac{2\sqrt{8}a_2\sqrt{3+\sqrt{8}}}{9\sqrt{6}\sqrt{1+3\sin^2(2.5\arcsin(1/3))}}\right) \leqslant \eta \leqslant \frac{\pi}{2} + 2\arcsin(1/3) \approx 2.25047\,.$$

We calculate $\varepsilon = \sphericalangle BDC = \varepsilon(\eta)$ explicitly and show by plot $\varepsilon < 2\arcsin(1/3)$.

Thus not both $\overline{OB} \geqslant \overline{OA}$ and $\overline{OB} \geqslant 1/\bar{M}_\eta$. But $\overline{OB} \geqslant \overline{OA}$ by (146) and case assumption (148). Thus $\overline{OA} \leqslant \overline{OB} \leqslant 1/\bar{M}_\eta \leqslant 1/\hat{M}_\alpha$ and we are done also with $m = 2$.

**Part 2.** Outer ray. However, we are not yet done with the proof. Because of the possibility $\eta - \delta > \pi/2$, there is a further option (which does not occur in the proof of lemma 3.3).




We need to consider also $\arg(\Delta + \alpha_{\tilde{m}}) > \arg(\Delta + \alpha_{\hat{m}}) + \arcsin(1/3)$, with the meaning of $\tilde{m}, \hat{m} \in \{m, m+1\}$ from the beginning of the proof.

We have to exclude that the upper ray intersects or touches the circle.

We assume thus that

$$\eta - \delta > \pi/2. \tag{154}$$

This occurs when $3\arcsin(1/3) + \alpha > \eta > \pi/2 + \arcsin\left(\frac{1}{ka}\right)$, thus assume in particular that

$$3\arcsin(1/3) + \alpha \geqslant \frac{\pi}{2} + \arcsin\left(\frac{1}{ka}\right) \tag{155}$$

in particular $\delta$ decreases with $\eta$ when $\eta > \pi/2 + \arcsin\left(\frac{1}{ka}\right)$. The ray is closest when $\delta$ is smallest, thus assume $\eta = 3\arcsin(1/3) + \alpha$.

This means we have to test

$$k \cdot \sin(\delta(3\arcsin(1/3) + \alpha, k, 3/2) + \arcsin(1/3)) > \frac{1}{a}, \tag{156}$$

for $\hat{m} \geqslant 3$ and

$$k \cdot \sin(\delta(3\arcsin(1/3) + \alpha, k, 2) + 2\arcsin(1/3)) > \frac{1}{a_2} \tag{157}$$

for $\hat{m} = 2$ (with $a_2$ as in (144)). We have to replace for the third argument $a$ of $\delta$ upper bounds for $|\alpha_{\hat{m}}|$, because we want to show that the angle is too large even for the smallest $\overline{OC} = |1/\alpha_{\hat{m}}|$. In the case $m = 2$ we will handle (157) by assuming $\hat{m} = 2$, otherwise ($\hat{m} = 3$) the treatment of (156) will handle the situation.

We will use the formula (86) for $\delta$. Observe that because of (154), we must have $\delta < \pi/2$, and thus the first alternative in (86) will apply.

**Case 2.1.** $\hat{m} \geqslant 3$. Note that we can assume $k \leqslant 3/a$, because for $k > 3/a$ we have (156) even if we set $\delta = 0$.

Thus combining with (155), we can assume

$$\frac{3}{a} \geqslant k \geqslant \frac{1}{a \cdot \sin\left(3\arcsin(1/3) + \alpha - \frac{\pi}{2}\right)} \tag{158}$$

Since we checked (130), we observed that $\delta$ decreases with $k$. We will first test (156) by evaluating the first factor on the left for the minimal $k$ in (158) and the second factor for the maximal $k$.

$$\frac{1}{a\cdot\sin(3\arcsin(1/3)+\alpha-\pi/2)}\cdot\sin\left(\arcsin(1/3)+\arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1+\frac{81}{4a^2}+\frac{9}{a}\cos(3\arcsin(1/3)+\alpha)}}\right)>\frac{1}{a}. \tag{159}$$

Now $\cos(3\arcsin(1/3)+\alpha)\geqslant -1$, and

$$\sqrt{1+\frac{81}{4a^2}-\frac{9}{a}}=\frac{9}{2a}-1>2, \tag{160}$$

thus

$$\arcsin(1/3)+\arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1+\frac{81}{4a^2}+\frac{9}{a}\cos(3\arcsin(1/3)+\alpha)}}<\arcsin(1/3)+\arcsin(1/2)<\pi/2$$

and $3\arcsin(1/3)+\alpha-\pi/2\leqslant 2\arcsin(1/3)<\pi/2$, thus in

$$\sin(3\arcsin(1/3)+\alpha-\pi/2)<\sin\left(\arcsin(1/3)+\arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1+\frac{81}{4a^2}+\frac{9}{a}\cos(3\arcsin(1/3)+\alpha)}}\right)$$

we can drop the outer sines, and it is implied by

$$2\arcsin(1/3)+\alpha\leqslant\frac{\pi}{2}+\arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1+\frac{81}{4a^2}+\frac{9}{a}\cos(3\arcsin(1/3)+\alpha)}}. \tag{161}$$

MATHEMATICA tests this to be true for (and finishes off) $\alpha\leqslant 1.07$, but for $\alpha\in[1.07,1.23]$, one more twist is needed. Let $\alpha\in[1.07,1.23]$. Then we split (158) into two parts

$$k\in\left[\frac{1}{a\cdot\sin(3\arcsin(1/3)+\alpha-\pi/2)},\frac{1.6}{a}\right]\qquad\text{and}\qquad k\in\left[\frac{1.6}{a},\frac{3}{a}\right]$$

The first range can be tested thus. First fix (160)

$$\left(\frac{3}{2}\cdot\frac{1.6}{a}\right)^2+1-2\cdot\frac{3}{2}\cdot\frac{1.6}{a}\cdot\frac{2\sqrt{8}}{9}>4,$$

using that because of $2\arcsin(1/3)+\pi/2\geqslant 3\arcsin(1/3)+\alpha\geqslant\pi/2$, we have

$$-\frac{2\sqrt{8}}{9}\leqslant\cos(3\arcsin(1/3)+\alpha)\leqslant 0. \tag{162}$$

Then test in the style of (161)

$$2\arcsin(1/3)+\alpha<\frac{\pi}{2}+\arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1+\frac{9\cdot 1.6^2}{4a^2}+\frac{3\cdot 1.6}{a}\cos(3\arcsin(1/3)+\alpha)}},$$

and the second by splitting further at $1.6/a, 1.609/a, 1.635/a, 1.69/a, 1.81/a, 2.1/a, 3/a$: for each two consecutive numbers $b,c$ in the sequence $1.6, 1.609, 1.635, 1.69, 1.81, 2.1, 3$, we examine

$$b\sin\left(\arcsin(1/3)+\arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1+\frac{9c^2}{4a^2}+\frac{3c}{a}\cos(3\arcsin(1/3)+\alpha)}}\right)>1,$$

and all these tests are successful for $\alpha\in[1.07,\arccos(1/3)]$.

**Case 2.2.** $\hat{m} = 2$. We consider (157). Note that we can assume $k \leqslant \frac{9}{2\sqrt{8}a_2}$, because for larger $k$ we have (157) even if we set $\delta = 0$.

Thus combining with (155), we can assume

$$\frac{9}{2\sqrt{8}a_2} \geqslant k \geqslant \frac{1}{a_2 \cdot \sin\left(3\arcsin(1/3) + \alpha - \frac{\pi}{2}\right)} \tag{163}$$

Since we checked (130), we observed that $\delta$ decreases with $k$. We will first test (157) by evaluating the first factor on the left for the minimal $k$ in (163) and the second factor for the maximal $k$.

$$\frac{1}{a_2 \cdot \sin(3\arcsin(1/3) + \alpha - \pi/2)} \cdot \sin\left(2\arcsin(1/3) + \arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1 + \frac{81}{8a_2^2} + \frac{9}{\sqrt{2}a_2}\cos(3\arcsin(1/3)+\alpha)}}\right) > \frac{1}{a_2}. \tag{164}$$

Now because of (162)

$$2\arcsin(1/3) + \arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1 + \frac{81}{8a_2^2} + \frac{9}{\sqrt{2}a_2}\cos(3\arcsin(1/3)+\alpha)}} \leqslant 2\arcsin(1/3) + \arcsin\left(\frac{1}{\sqrt{1 + \frac{81}{8a_2^2} - \frac{9}{\sqrt{2}a_2} \cdot \frac{2\sqrt{8}}{9}}}\right) < \frac{\pi}{2}$$

and $3\arcsin(1/3) + \alpha - \pi/2 < \pi/2$, thus in

$$\sin(3\arcsin(1/3) + \alpha - \pi/2) < \sin\left(2\arcsin(1/3) + \arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1 + \frac{81}{8a_2^2} + \frac{9}{\sqrt{2}a_2}\cos(3\arcsin(1/3)+\alpha)}}\right)$$

we can drop the outer sines, and it is implied by

$$\arcsin(1/3) + \alpha \leqslant \frac{\pi}{2} + \arcsin\frac{\sin(3\arcsin(1/3)+\alpha)}{\sqrt{1 + \frac{81}{8a_2^2} + \frac{9}{\sqrt{2}a_2}\cos(3\arcsin(1/3)+\alpha)}}.$$

But this is now evident even with the rightmost arc sine term dropped.

This finishes also $m = 2$. □

# 5 Afterword

It is clear from the proof that the method will apply to such linear recurrent polynomials $P_i$ also for other starting values $P_{0,1}$. Of course, as long as $P_0 = 0$, the outcome says nothing new. But for $P_0 \neq 0$ one can still gain similar statements if $|P_1(t)/P_0(t)|$ is large enough for the relevant $z$ (with $t = z^{1/2} - z^{-1/2}$). In fact, the Alexander polynomials of alternating Montesinos links (a generalization of 2-bridge links) satisfy such a recurrence (as exploited in [St]), but the conditions on $P_{0,1}$ seem not strong enough to make the induction estimates here start properly. Thus the adaptation of our method remains a future (and quite challenging) undertaking.

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