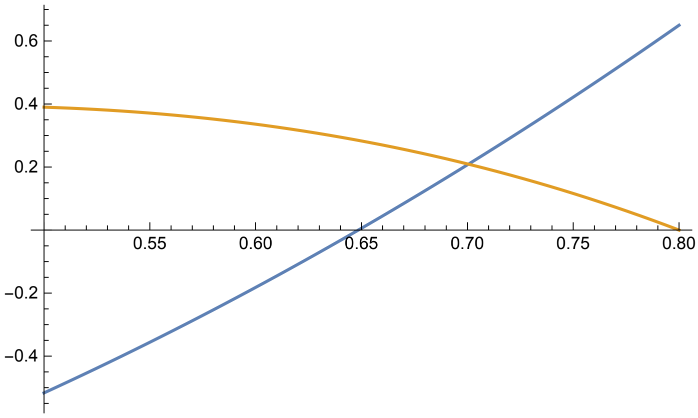
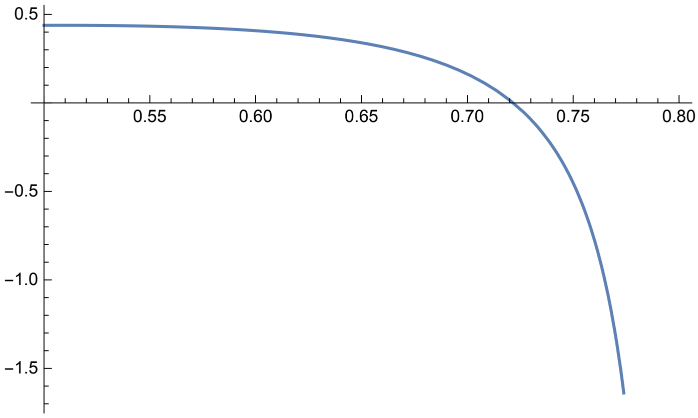
```
Plot[{2.5 l^2 - 1.6 Sin[Pi / 3 - ArcSin[1 / 2 l]], 2 l (0.64 - l^2)}, {l, 0.5, 0.8}]
```




```
NSolve[2.5 l^2 - 1.6 Sin[Pi / 3 - ArcSin[1 / 2 l]] == 2 l (0.64 - l^2)]
{{l → -1.26054}, {l → -0.792605}, {l → 0.700515}}
```

```
NSolve[
  (2.5 l^2 - 1.6 Sin[Pi / 3 - ArcSin[1 / 2 l]])^2 / 4 / (l^2 - 0.64) + (1 - 25 l^2 / 16) == 0]
{{l → -1.}, {l → -1.}, {l → -0.798729}, {l → 0.721617}}
```

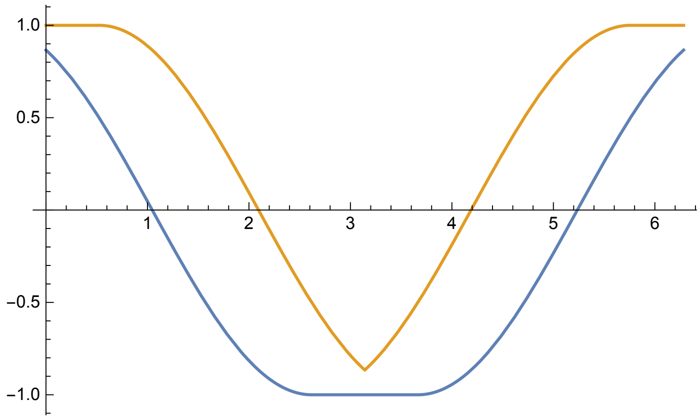
```
Plpot[(2.5 l^2 - 1.6 Sin[Pi / 3 - ArcSin[1 / 2 l]])^2 / 4 / (l^2 - 0.64) + (1 - 25 l^2 / 16),
  {l, 0.5, 0.8}]
```



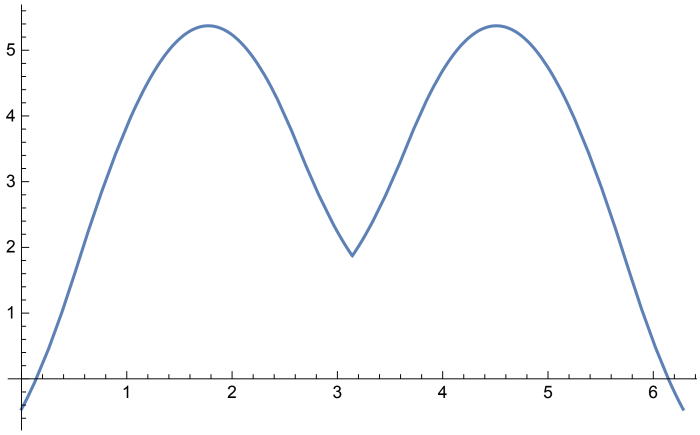

```
(2.75 * 0.8 + 1) / 4
0.8
```

```
MinCos[gm_] :=
  If[gm > 5 Pi / 6 && gm < 7 Pi / 6, -1, Min[Cos[gm - Pi / 6], Cos[gm + Pi / 6]]]

MaxCos[gm_] := If[gm < Pi / 6 || gm > 11 Pi / 6, 1, Max[Cos[gm - Pi / 6], Cos[gm + Pi / 6]]]
```

```
Plot[{MinCos[gm], MaxCos[gm]}, {gm, 0, 2 Pi}]
```



```
Plot[(l^2 / 0.8 * MaxCos[gm] - 0.8 MinCos[gm]) / (0.8^2 - l^2) /. {l → 0.71},
  {gm, 0, 2 Pi}]
```



```
NSolve[(l^2 / 0.8 * MaxCos[gm] - 0.8 MinCos[gm]) / (0.8^2 - l^2) == 2.75 /. {l → 0.71}]
```

NSolve::ratnz : NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result. >>

```
{{gm → 0.740331}, {gm → 2.83738}, {gm → 3.4458},
  {gm → 5.54285}, {gm → ConditionalExpression[-0.72094 + 6.28319 C[1],
     (C[1] ∈ Integers && C[1] ≥ 2.) || (C[1] ∈ Integers && C[1] ≤ 0)]},
  {gm → ConditionalExpression[0.72094 + 6.28319 C[1],
     (C[1] ∈ Integers && C[1] ≥ 1.) || (C[1] ∈ Integers && C[1] ≤ -1.)]}}
```

```
3.445800416419704 + 2.8373848907598824 - 2 Pi
```

```
0.
```

```
((1 - l^2 / 0.8^2) + (0.8^2 - l^2) * k^2 -
   2 k (l^2 / 0.8 * MaxCos[gm] - 0.8 MinCos[gm])) /. {l → 0.71, gm → 0.51, k → 2.75}
```

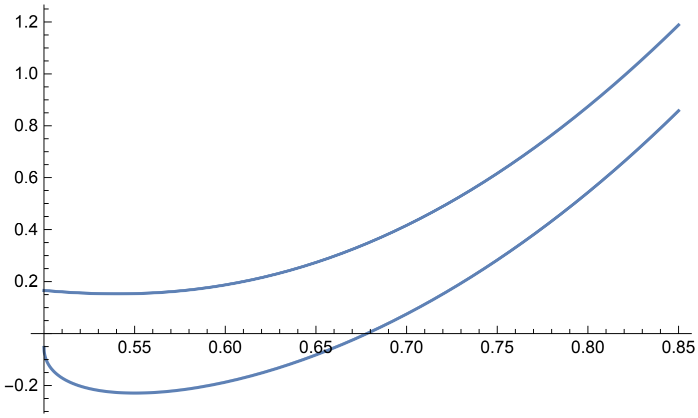
```
0.0260133
```

```
(* NORM ESTIMATE |\Dl| ge l old redundant *)
```

```
MaxQ[a_, b_, c_, d_] := If[-b / (2 a) < d, a d^2 + b d + c, -b^2 / (4 a) + c] /; a < 0
```

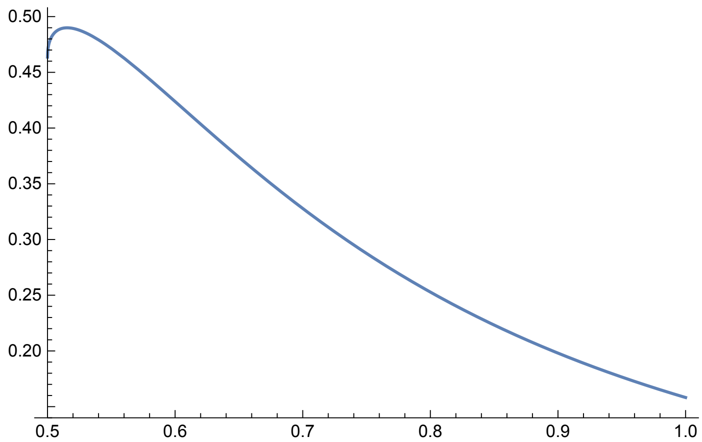
```
Fct[a_, l_] :=
 {MaxQ[l^2 - a^2, 2 l^2 / a - a (Sqrt[3 (4 l^2 - 1)] - 1) / (2 l), (l / a)^2 - 1, l],
  MaxQ[l^2 - a^2, 2 a - l (Sqrt[10 (16 l^2 - 1) / 3] - Sqrt[2 / 3]) / (4 a),
   (l / a)^2 - 1, 4 - 1 / l]}
```

```
Plot[Fct[0.85, l], {l, 0.5, 0.85}]
```



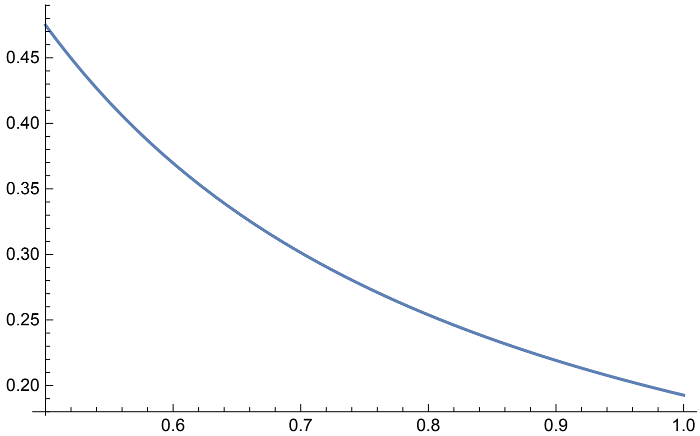

```
beta[l_] := ArcSin[1 / (2 l * Sqrt[16 l^2 + 1 - 4 Sqrt[4 l^2 - 1]])]
```

```
Plot[beta[l], {l, 0.5, 1}]
```



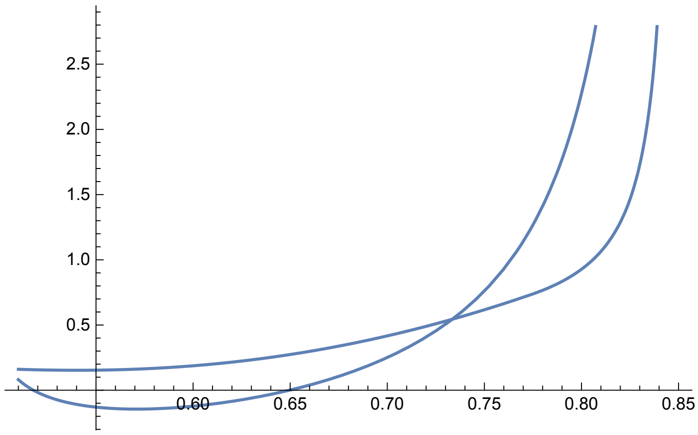
```
beta[l_] := ArcSin[2 * Sqrt[8 / 81] / Sqrt[16 l^2 + 1 - 8 l * 7 / 9]]
```

```
Plot[beta[l], {l, 0.5, 1}]
```

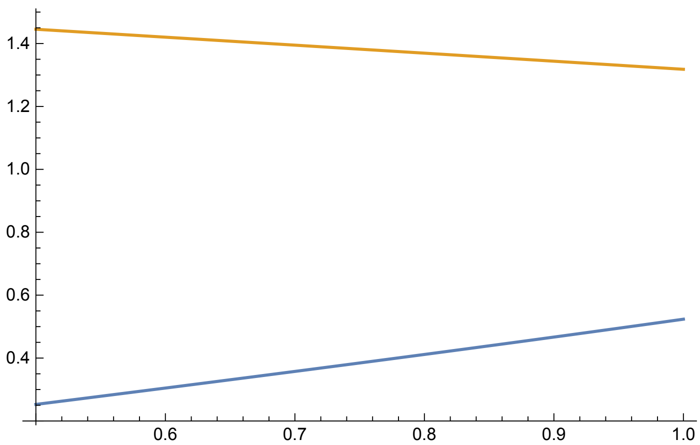



```
Fct[a_, l_] :=
 {MaxQ[l^2 - a^2, 2 l^2 / a - a (Sqrt[8 (4 l^2 - 1)] - 1) / (3 l), (l / a) ^2 - 1, l],
  MaxQ[l^2 - a^2,
   2 a - 2 l^2 (Sqrt[8 / 9] * Cos[beta[l]] - Sqrt[1 / 9] * Sin[beta[l]]) / (a),
   (l / a) ^2 - 1, 4 - 1 / l]}
```

```
Plot[Fct[0.85, l], {l, 0.51, 0.85}]
```




```
Plot[{ArcSin[1 / 2 l], ArcCos[1 / 4 l]}, {l, 0.5, 1}]
```



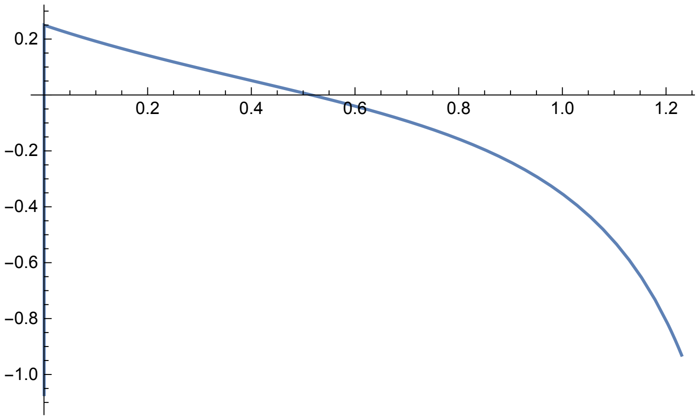
```
Fct[0.85, 0.61]
```

```
{-0.634736, -0.00353456}
```

```
(* END NORM ESTIMATE |\Dl|\ge 1 *)
```

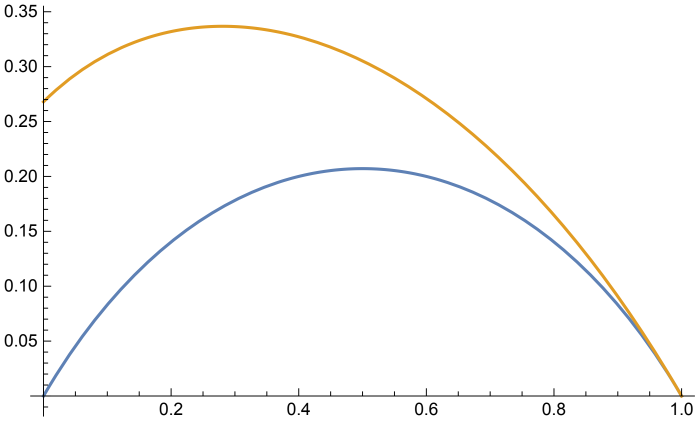
```
(* stuff for Lemma 2.1 *)
g := D[Cos[ap]^(1/ap), {ap, 2}];
Plot[g /. {ap → a}, {a, 0, 1.23}]
```

```
(* test (cos ap)^1/ap ≥1-ap/2+ap^2/4 *)
```




```
(* delta of section 2 ≤ 3Pi/4 *)
```

```
Plot[{Sqrt[1/4 + x - x^2] - 1/2, Sqrt[-x^2 - 2 x + 7 - 4 Sqrt[-2 x + 3]]}, {x, 0, 1}]
```




```
Simplify[Sqrt[1/4 + x - x^2] - 1/2 - Sqrt[-x^2 - 2 x + 7 - 4 Sqrt[-2 x + 3]] ≥ 0, 0 ≤ x ≤ 1]
```

$\sqrt{1+4x-4x^2} \geq 1+2\sqrt{7-4\sqrt{3-2x}-2x-x^2}$

```
(Pi / 4 + ArcSin[1/Sqrt[6]]) * 2./5
```

```
0.482373
```

```
Simplify[
 1/4 + x - x^2 + 1/4 - Sqrt[1/4 + x - x^2] >= -x^2 - 2 x + 7 - 4 Sqrt[-2 x + 3], 0 ≤ x ≤ 1]
```

$8\sqrt{3-2x}+6x \geq 13+\sqrt{1+4x-4x^2}$

```
Simplify[64 (3 - 2 x) ≥ 1 + 4 x - 4 x^2, 0 ≤ x ≤ 1]
```

```
True
```

```
Simplify[64 (3 - 2 x) + 1 + 4 x - 4 x^2 - 16 Sqrt[(3 - 2 x) * (1 + 4 x - 4 x^2)] ≥ (13 - 6 x)^2]
```

$5\, x^2 + 2 \sqrt{3 + 10\, x - 20\, x^2 + 8\, x^3} \leq 3 + 4\, x$

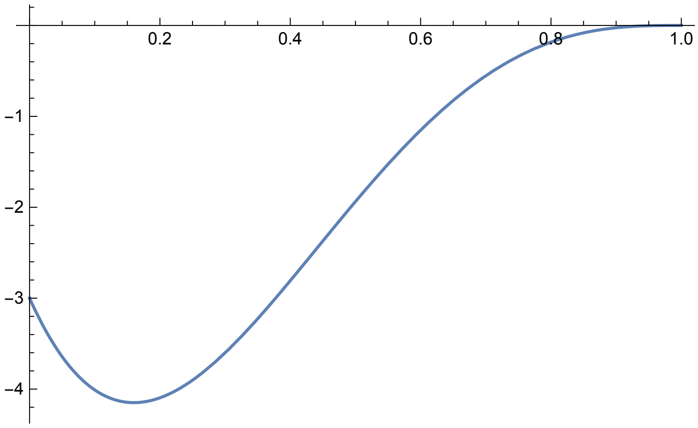
```
Simplify[-5 x^2 + 4 x + 3 ≥ 0, 0 ≤ x ≤ 1]
```

True

```
Simplify[Expand[(-5 x^2 + 4 x + 3)^2 - 4 (3 + 10 x - 20 x^2 + 8 x^3)] ≥ 0, 0 ≤ x ≤ 1]
```

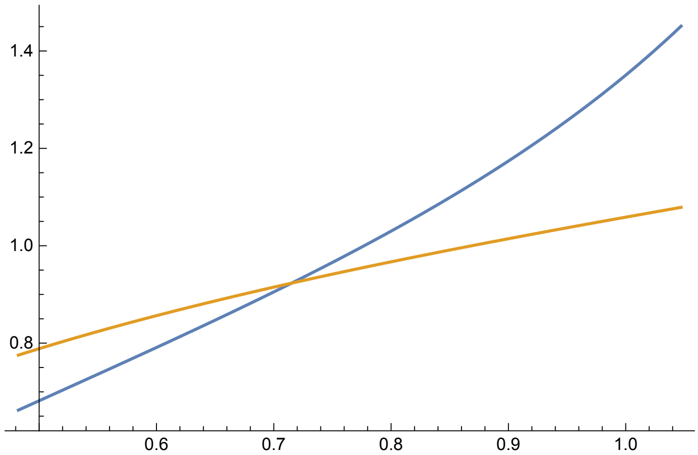
$x \geq 1$

```
Plot[Expand[(-5 x^2 + 4 x + 3)^2 - 4 (3 + 10 x - 20 x^2 + 8 x^3)], {x, 0, 1}]
```

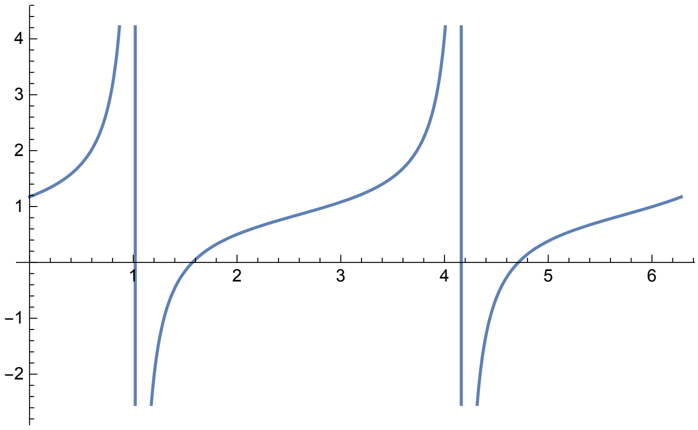



```
(* end delta of section 2 ≤ 3Pi/4 *)
```

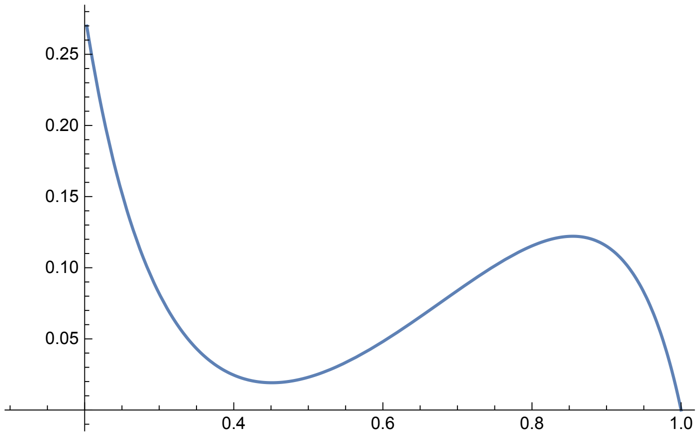
```
Plot[{Sin[3 al / 2] / Sin[3 al / 2 + ArcCos[Sqrt[4 / 24]] - ArcSin[1 / 3]],
  Sqrt[6 / 4] Sin[al] / Sin[al + ArcSin[1 / 3]]}, {al, 0.4823, Pi / 3}]
```



```
Plot[Cos[x] / Cos[x + 0.55], {x, 0, 2 Pi}]
```

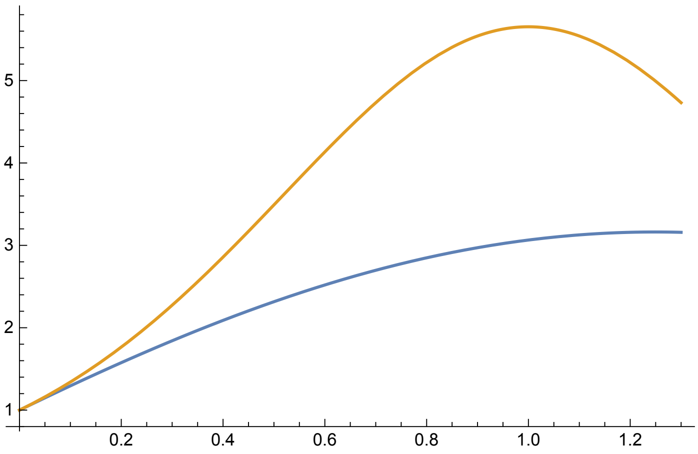


```
f[x_] := Sqrt[-x^2 - 2 x + 7 - 4 Sqrt[-2 x + 3]]

al[x_] := ArcTan[f[x] / x]

bt1[x_] := ArcSin[Sqrt[5 / 6]] - ArcTan[f[x] / (1 - x)]

Plot[Sin[3 al[x] / 2] / Sin[bt1[x] + 3 al[x] / 2] -
  Sqrt[3 / 2] * Sin[al[x]] / Sin[ArcSin[Sqrt[5 / 6]] - bt1[x] + al[x]], {x, 1 / 9, 1}]
```

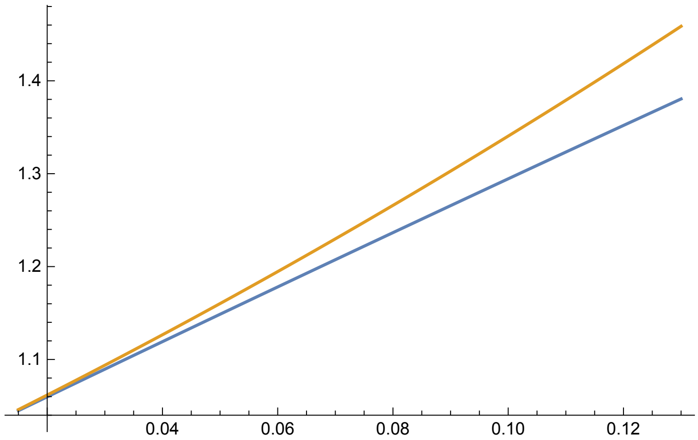


```
(* upper arc *)
```

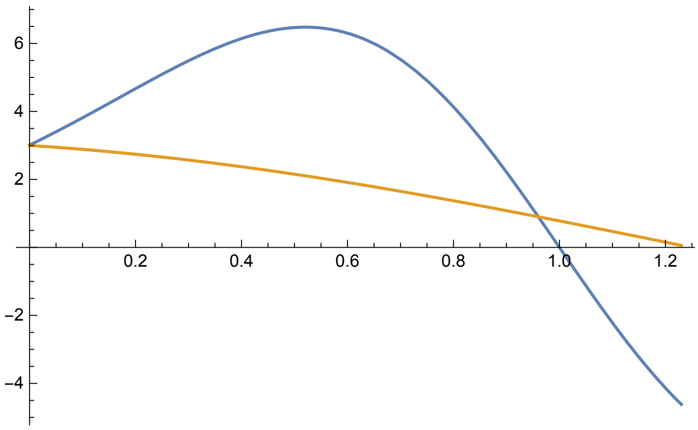
```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4)^(23 Pi / 12)}, {al, 0, 1.3}]
```



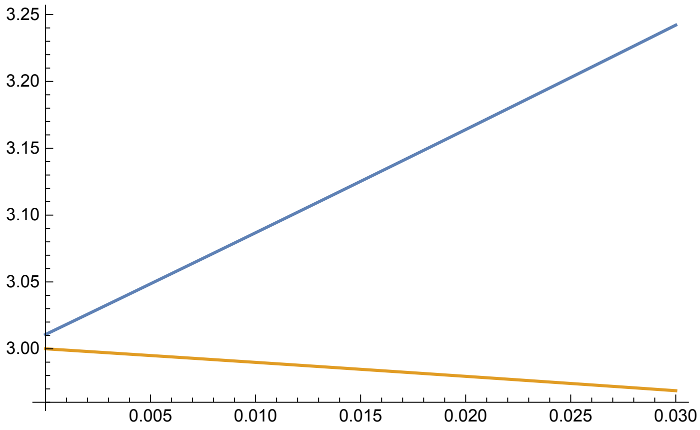

```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4)^(23 Pi / 12)}, {al, 0.015, 0.13}]
```




```
g := D[1 / (1 - al / 2 + al^2 / 4)^(23 Pi / 12), al];
Plot[{g /. {al → a}, -Sin[a] + 3 Cos[a]}, {a, 0, 1.23}]
```



```
Plot[{g /. {al → a}, -Sin[a] + 3 Cos[a]}, {a, 0, 0.03}]
```




```
(* NSolve[Cos[al]+3Sin[al]==1/(1-al/2+al^2/4)^(23Pi/12)] - don't try that! *)
```

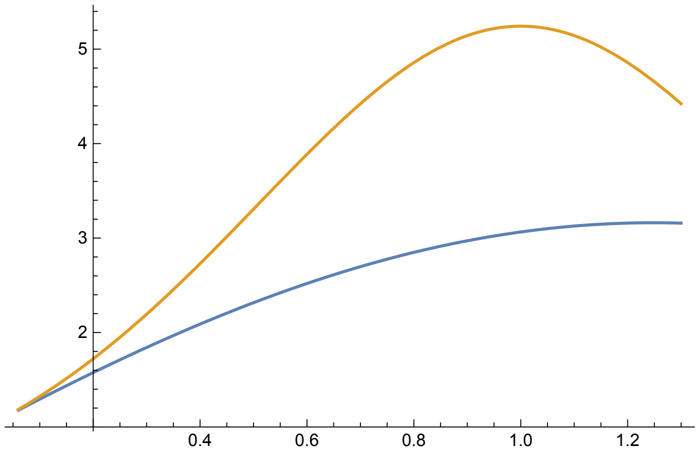
```
NSolve[(2 - y) Sin[bt - ArcCos[(3 - y^2) / 2 / (2 - y)]] ==
    1.5 (1 - Cos[ArcCos[1 / 3] + Pi / 4 - bt + ArcCos[(3 - y^2) / 2 / (2 - y)]])
     (y - 1) Sqrt[2] /. {bt → Pi / 12}]
```

```
{{y → 1.06718}}
```

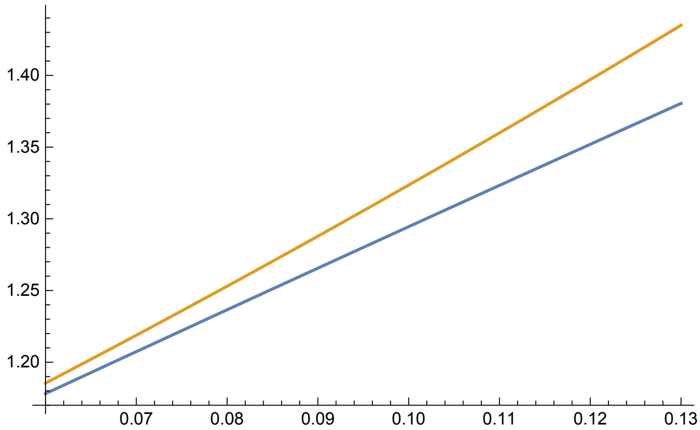
```
ArcCos[(3 - y^2) / 2 / (2 - y)] /. {y -> 1.0671760258350662`}
```

```
0.0695668
```

```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4)^(11 Pi / 6)}, {al, 0.06, 1.3}]
```



```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4) ^ (11 Pi / 6)}, {al, 0.06, 0.13}]
```



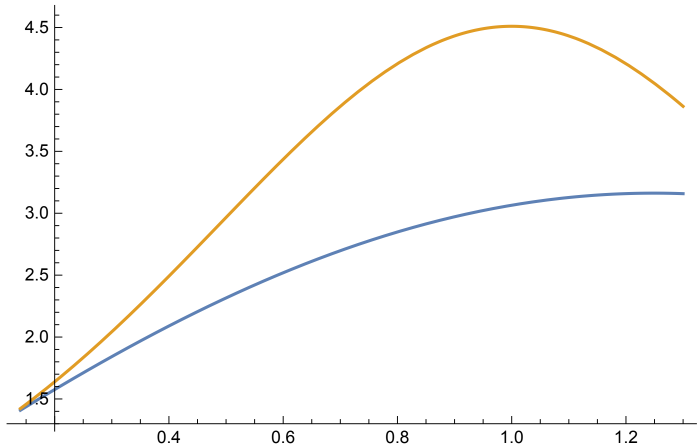

```
NSolve[(2 - y) Sin[bt - ArcCos[(3 - y^2) / 2 / (2 - y)]] ==
   1.5 (1 - Cos[ArcCos[1 / 3] + Pi / 4 - bt + ArcCos[(3 - y^2) / 2 / (2 - y)]])
    (y - 1) Sqrt[2] /. {bt → Pi / 6}]
```

```
{{y → 1.13855}}
```

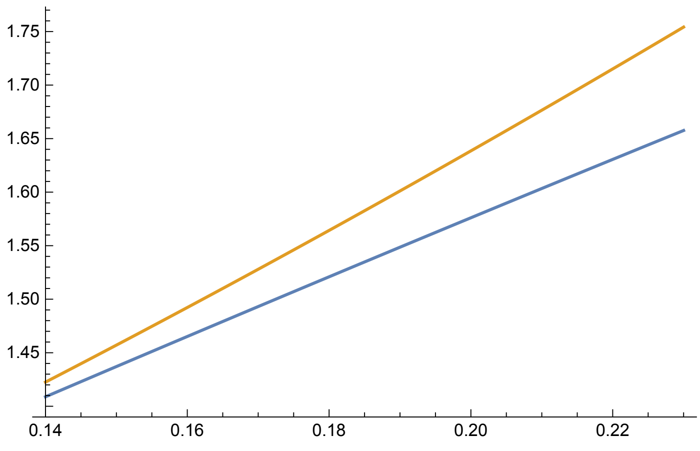
```
ArcCos[(3 - y^2) / 2 / (2 - y)] /. {y -> 1.138550249824842`}
```

```
0.149416
```

```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4) ^ (5 Pi / 3)}, {al, 0.14, 1.3}]
```




```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4) ^ (5 Pi / 3)}, {al, 0.14, .23}]
```



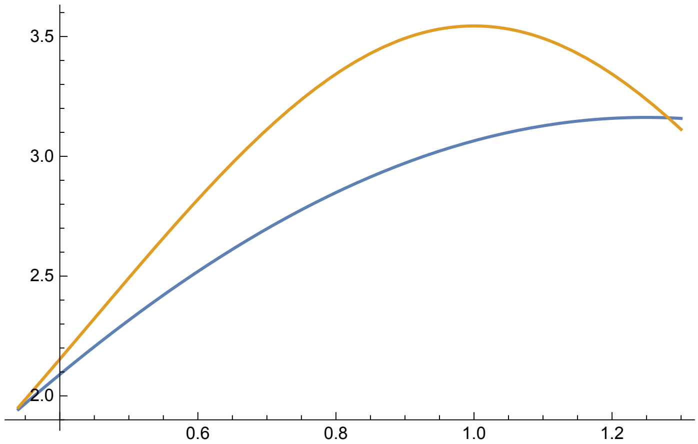
```
NSolve[(2 - y) Sin[bt - ArcCos[(3 - y^2) / 2 / (2 - y)]] ==
   1.5 (1 - Cos[ArcCos[1 / 3] + Pi / 4 - bt + ArcCos[(3 - y^2) / 2 / (2 - y)]])
    (y - 1) Sqrt[2] /. {bt → Pi / 3}]
```

{{y → 1.28962}}

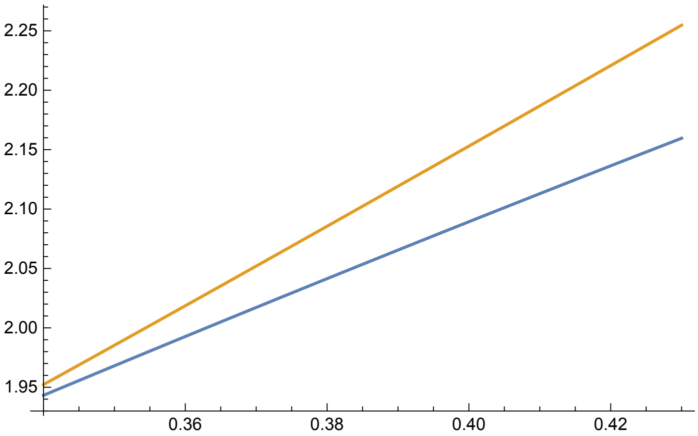
```
ArcCos[(3 - y^2) / 2 / (2 - y)] /. {y -> 1.2896247613700633`}
```

0.345344

```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4) ^ (7 Pi / 5)}, {al, 0.34, 1.3}]
```




```
Plot[{Cos[al] + 3 Sin[al], 1 / (1 - al / 2 + al^2 / 4) ^ (7 Pi / 5)}, {al, 0.34, .43}]
```



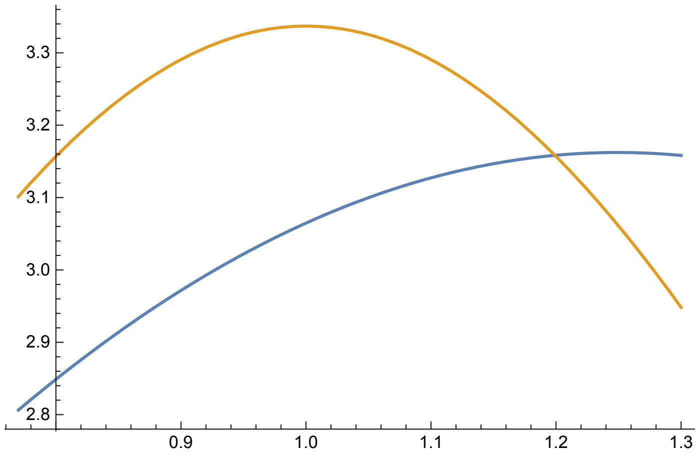

```
NSolve[(2 - y) Sin[bt - ArcCos[(3 - y^2) / 2 / (2 - y)]] ==
   1.5 (1 - Cos[ArcCos[1 / 3] + Pi / 4 - bt + ArcCos[(3 - y^2) / 2 / (2 - y)]])
    (y - 1) Sqrt[2] /. {bt → 3 Pi / 5}]
```

{{y → 1.52302}, {y → 1.58014 - 1.29155 i}, {y → 1.58014 + 1.29155 i}}

```
ArcCos[(3 - y^2) / 2 / (2 - y)] /. {y -> 1.5230204354638972`}
```

0.776676

```
Plot[{Cos[al] + 3 Sin[al], 1/(1 - al/2 + al^2/4)^(4 Pi/3)}, {al, 0.77, 1.3}]
```



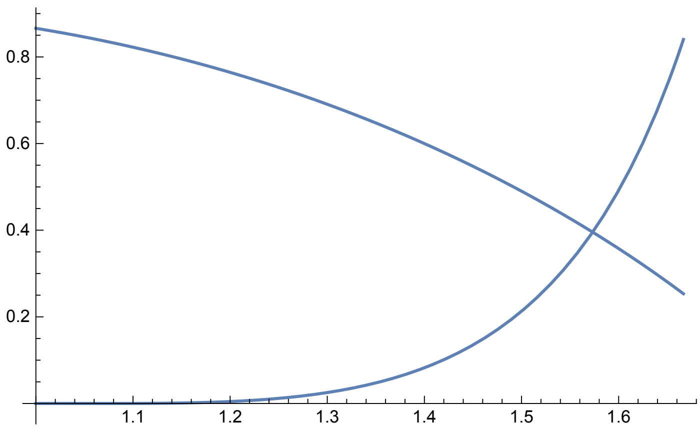

```
NSolve[(2 - y) Sin[bt - ArcCos[(3 - y^2)/2/(2 - y)]] ==
   1.5 (1 - Cos[ArcCos[1/3] + Pi / 4 - bt + ArcCos[(3 - y^2)/2/(2 - y)]])
    (y - 1) Sqrt[2] /. {bt → 2 Pi/3}]
```

```
{{y → 1.55998 - 1.54698 i}, {y → 1.55998 + 1.54698 i}, {y → 1.57321}}
```

```
ArcCos[(3 - y^2)/2/(2 - y)] /. {y -> 1.573209405718236`}
```

```
0.908322
```

```
Plot[{(2 - y) Sin[bt - ArcCos[(3 - y^2)/2/(2 - y)]],
   1.5 (1 - Cos[ArcCos[1/3] + Pi / 4 - bt + ArcCos[(3 - y^2)/2/(2 - y)]])
    (y - 1) Sqrt[2]} /. {bt → 2 Pi/3}, {y, 1, 5/3}]
```

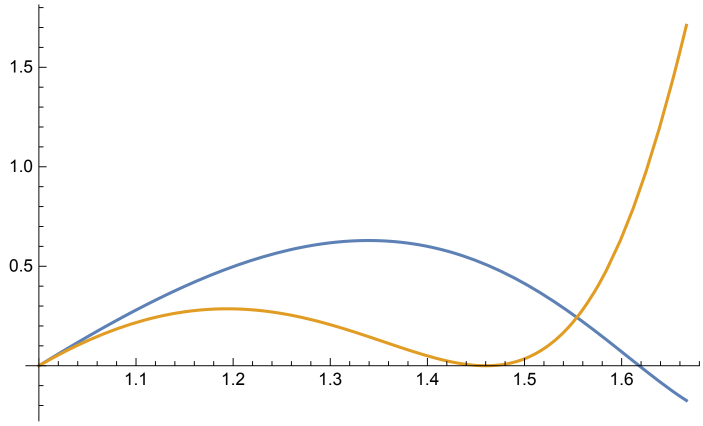



```
N[Pi / 4]
```

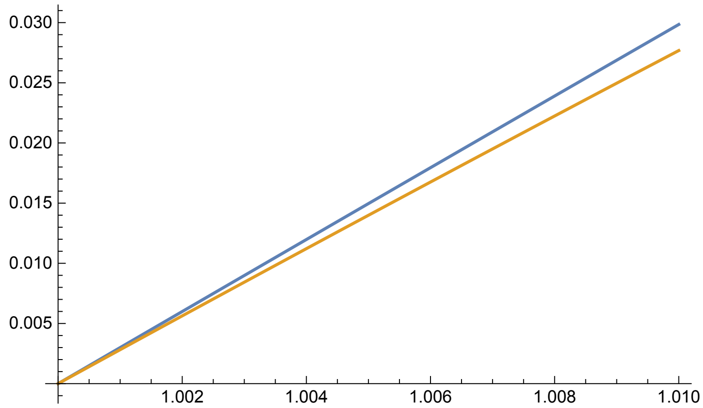
```
0.785398
```

```
(* ADDED STUFF OF CASE 1.2.1 (20) AND 1.2.2  *)
```

```
Plot[{(2 - y) Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]],
  1.5 (1 + Cos[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3 - Sqrt[8]
      Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3) * Sqrt[2] * (y - 1)}, {y, 1, 5 / 3}]
```




```
Plot[{(2 - y) Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]],
  1.5 (1 + Cos[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3 - Sqrt[8]
      Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3) * Sqrt[2] * (y - 1)}, {y, 1, 1.01}]
```



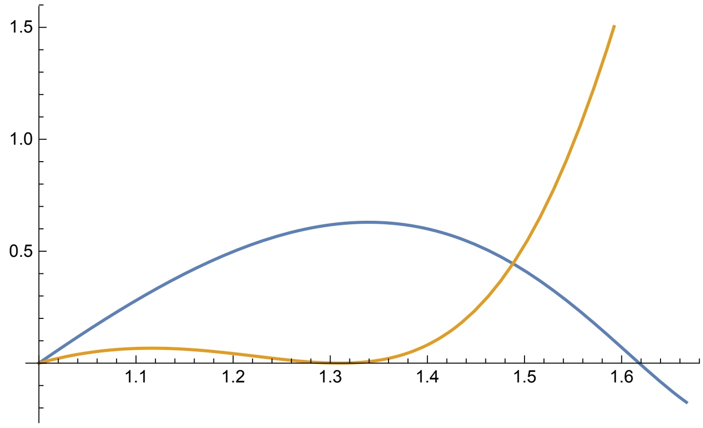

```
FindRoot[(2 - y) Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] -
  1.5 (1 + Cos[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3 -
     Sqrt[8] Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3) * Sqrt[2] * (y - 1), {y, 1.55}]
```

```
{y → 1.55391}
```

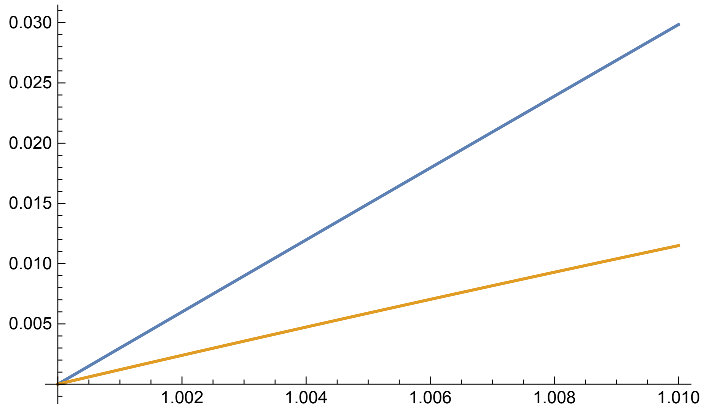
```
ArcCos[(3 - y^2) / 2 / (2 - y)] /. %
```

```
0.855146
```

```
Plot[{(2 - y) Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]],
  1.5 (1 + Cos[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] * (Sqrt[.5] - 2) / 3 - (Sqrt[.5] + 2)
      Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3) * Sqrt[2] * (y - 1)}, {y, 1, 5 / 3}]
```




```
Plot[{(2 - y) Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]],
  1.5 (1 + Cos[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] * (Sqrt[.5] - 2) / 3 - (Sqrt[.5] + 2)
      Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3) * Sqrt[2] * (y - 1)}, {y, 1, 1.01}]
```




```
FindRoot[(2 - y) Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] -
  1.5 (1 + Cos[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] * (Sqrt[.5] - 2) / 3 - (Sqrt[.5] + 2)
      Sin[3 ArcCos[(3 - y^2) / 2 / (2 - y)]] / 3) * Sqrt[2] * (y - 1), {y, 1.55}]
```

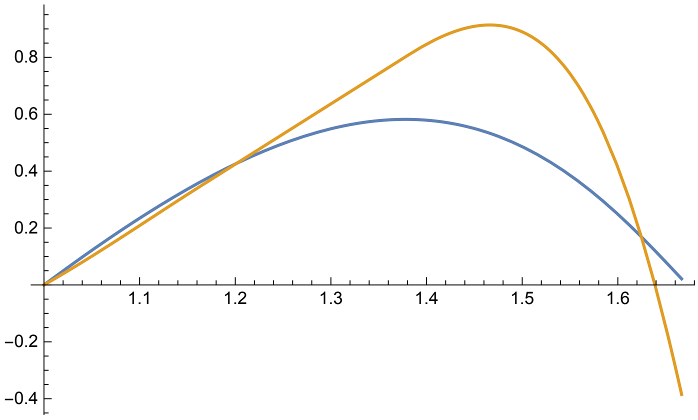
```
{y → 1.48815}
```

```
ArcCos[(3 - y^2) / 2 / (2 - y)] /. {y → 1.48}
```

```
0.678585
```

```
(* end upper arc *)

(* lower arc does not touch {theta= -3ap/2, -5ap/2,...} *)

MinCos[a_] := If[-Sin[a + Pi / 2] <= 0 && -Sin[a + 3 Pi / 4] ≥ 0,
  -1, Min[Cos[a + Pi / 2], Cos[a + 3 Pi / 4]]]
```

```
Plot[{Sin[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2] * (2 - y) ,
  -3 / 2 * MinCos[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2 + ArcSin[1 / 3]] *
   Sqrt[2] * (y - 1)}, {y, 1, 5 / 3}]
```




```
FindRoot[Sin[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2] * (2 - y) +
  3 / 2 * MinCos[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2 + ArcSin[1 / 3]] *
   Sqrt[2] * (y - 1), {y, 1.2}]
```

```
{y → 1.20158}
```

```
NSolve[ArcCos[(3 - y^2) / 2 / (2 - y)] == Pi / 4]
```

```
{{y → -0.112389}, {y → 1.5266}}
```

```
{(Pi / 4 - ArcSin[1 / 3]) * 0.4, (Pi / 2 ArcSin[1 / 3]) * 0.4}
```

```
{0.178225, 0.213526}
```

```
ArcCos[(3 - y^2) / 2 / (2 - y)] /. {y → 1.2}
```

```
0.224075
```

```
NSolve[Sin[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2] * (2 - y) ==
  3 / 2 * Sin[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2 + ArcSin[1 / 3]] * Sqrt[2] * (y - 1)]
```

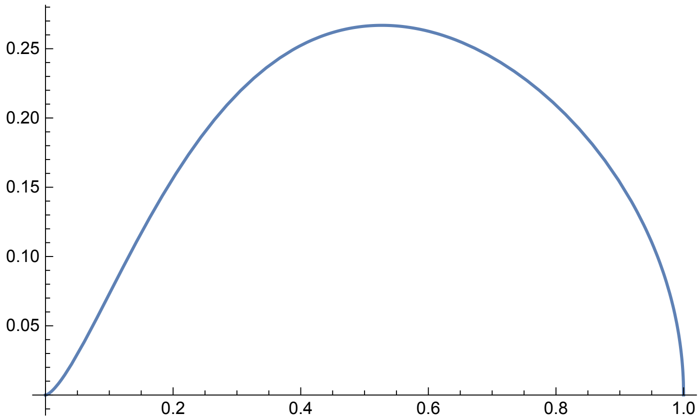
```
{Sin[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2] * (2 - y) ,
  -3 / 2 * MinCos[5 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2 + ArcSin[1 / 3]] *
   Sqrt[2] * (y - 1)} /. {y → 1.2}
```

```
{0.425077, 0.424264}
```

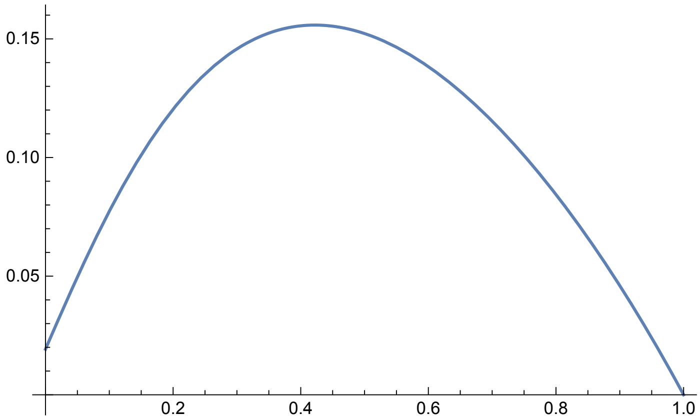
```
? Arg
```

Arg[*z*] gives the argument of the complex number *z*. >>

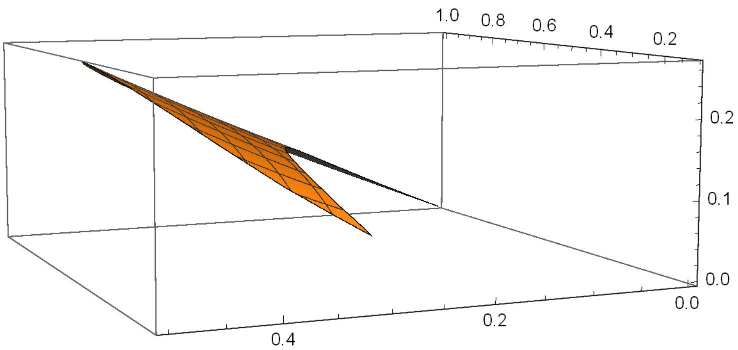
```
Plot[ComplexExpand[Arg[(1 - z^3) / (1 - z^2) /. {z → x + I Sqrt[x - x^2]}]], {x, 0, 1}]
```



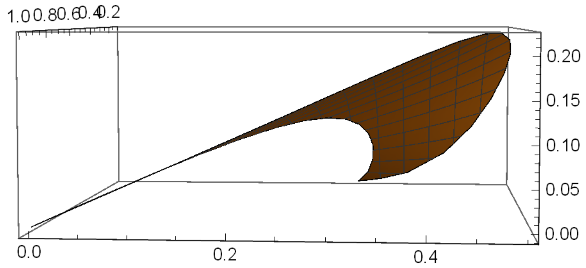

```
f[x_] := Sqrt[-x^2 - 2 x + 7 - 4 Sqrt[-2 x + 3]];
Plot[ComplexExpand[Arg[(1 - z^3) / (1 - z^2) /. {z → x + I f[x]}]], {x, 0, 1}]
```




```
Plot3D[ComplexExpand[Arg[(1 - z^3) / (1 - z^2) /. {z → x + I y}]],
 {x, 1 / 9, 1}, {y, f[x], Sqrt[x - x^2]}]
```



```
Plot3D[ComplexExpand[Arg[(1 - z^4) / (1 - z^3) /. {z → x + I y}]],
 {x, 1/9, 1}, {y, f[x], Sqrt[x - x^2]}]
```




```
? Plot
```

Plot[f, {x, x_{min}, x_{max}}] generates a plot of f as a function of x from x_{min} to x_{max}.

Plot[{f_1, f_2, …}, {x, x_{min}, x_{max}}] plots several functions f_i. >>

```
MyArg[a_ + I b_] := Abs[b / a];
Maximize[MyArg[ComplexExpand[(1 - z^3) / (1 - z^2) /. {z → x + I y}]],
 {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]
```

```
{-Root[-729 + 9600 ⌗1^2 + 2048 ⌗1^4 &, 1],
 {x → Root[{-729 + 9600 ⌗1^2 + 2048 ⌗1^4 &, ⌗1^2 + 4 ⌗1^2 ⌗2 + 10 ⌗1^2 ⌗2^2 - 9 ⌗2^3 + 12 ⌗1^2 ⌗2^3 +
       9 ⌗2^4 + 9 ⌗1^2 ⌗2^4 &}, {1, 2}], y → Root[{-729 + 9600 ⌗1^2 + 2048 ⌗1^4 &,
     ⌗1^2 + 4 ⌗1^2 ⌗2 + 10 ⌗1^2 ⌗2^2 - 9 ⌗2^3 + 12 ⌗1^2 ⌗2^3 + 9 ⌗2^4 + 9 ⌗1^2 ⌗2^4 &,
     ⌗1 + 2 ⌗1 ⌗2 + 2 ⌗1 ⌗2^2 + ⌗1 ⌗2^3 + 2 ⌗2 ⌗3 + ⌗2^2 ⌗3 + ⌗1 ⌗2 ⌗3^2 + ⌗3^3 &}, {1, 2, 1}]}}
```

```
N[Out[27]]
```

```
{0.273396, {x → 0.527202, y → 0.499259}}
```

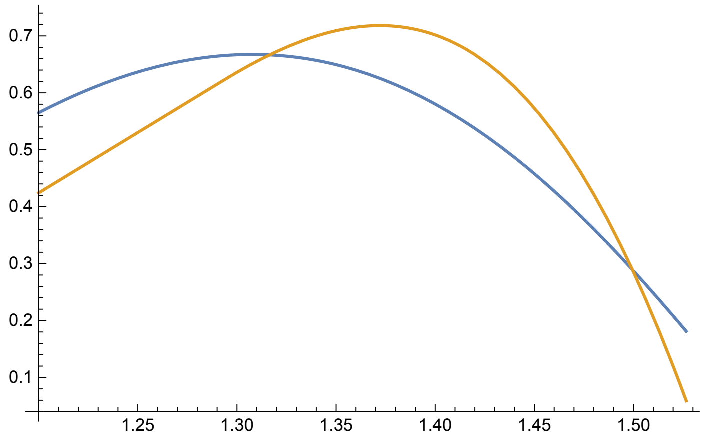
```
N[Maximize[MyArg[ComplexExpand[(1 - z^4) / (1 - z^3) /. {z → x + I y}]],
  {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]
```

```
{0.227404, {x → 0.650517, y → 0.476807}}
```

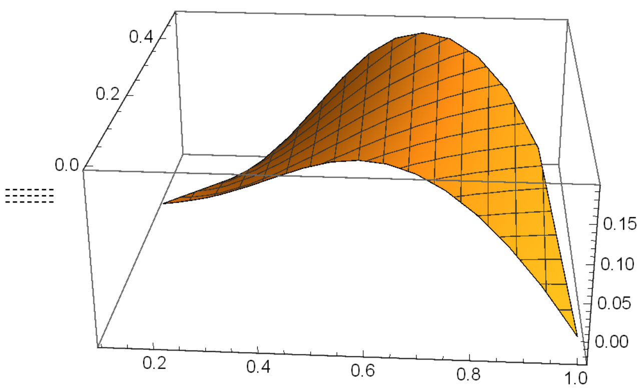
```
N[Maximize[MyArg[ComplexExpand[(1 - z^5) / (1 - z^4) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]] // Timing
```

```
{20.935334, {0.196696, {x → 0.731362, y → 0.443251}}}
```

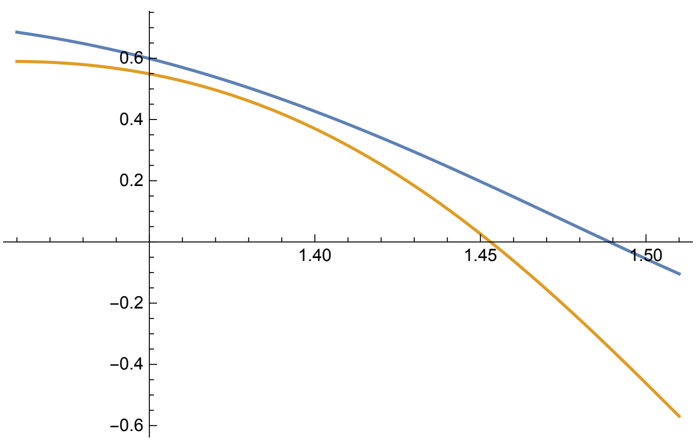
```
Plot[{Sin[7 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2] * (2 - y) ,
  -3 / 2 * MinCos[7 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2 + ArcSin[1 / 3]] *
   Sqrt[2] * (y - 1)}, {y, 1.2, 1.5266}]
```




```
Plot3D[ComplexExpand[Arg[(1 - z^5) / (1 - z^4) /. {z → x + I y}]],
 {x, 1 / 9, 1}, {y, f[x], Sqrt[x - x^2]}]
```

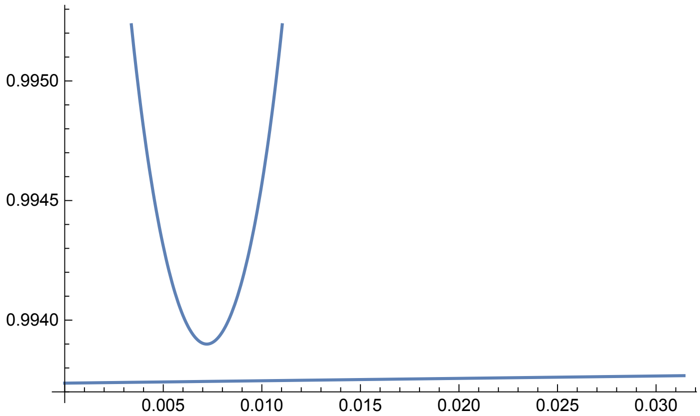



```
Plot[{Sin[9 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2] * (2 - y) ,
  -3 / 2 * MinCos[9 ArcCos[(3 - y^2) / 2 / (2 - y)] / 2 + ArcSin[1 / 3]] *
   Sqrt[2] * (y - 1)}, {y, 1.31, 1.51}]
```

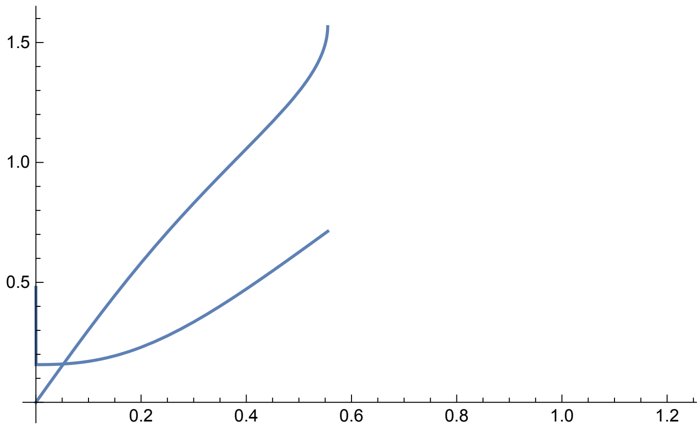



```
(* end lower arc does not touch {theta= -3ap/2, -5ap/2,...} *)
```

```
(* |\ap_n|\ge 0.85 *)
R[al_, b_] := Sqrt[1 + (Cos[al] Sin[al] / (Cos[al]^2 - b^2))^2] /; b < Cos[al]
r[al_, b_] := b Sin[al] / (Cos[al]^2 - b^2)
Gm[al_, b_] := ArcTan[Cos[al] Sin[al] / (Cos[al]^2 - b^2)]
F[al_, b_, th_] :=
 R[al, b] Cos[th - Gm[al, b]] - Sqrt[r[al, b]^2 - R[al, b]^2 * Sin[th - Gm[al, b]]^2]
Plot[{Cos[al]^((2 Pi - th) / al), F[al, 0.85, th]} /. {al → 0.002}, {th, 0, Pi / 100}]
```

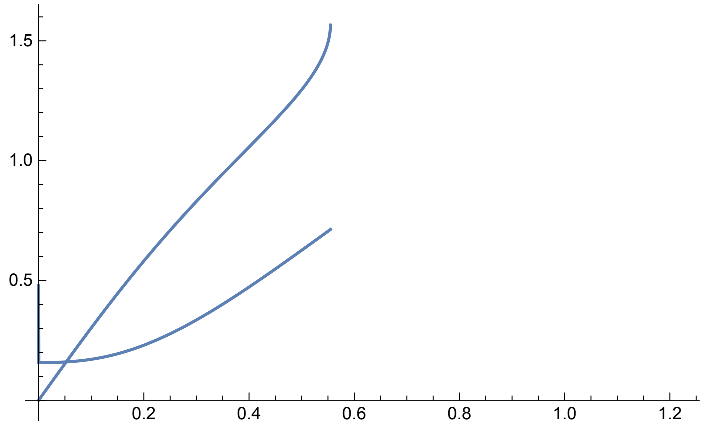



```
Plot[{2 Pi - al Log[Cos[al], F[al, b, Gm[al, b]]] - Gm[al, b],
   ArcSin[r[al, b] / R[al, b]]} /. {b → 0.85}, {al, 0, ArcTan[Sqrt[8]]}]
```

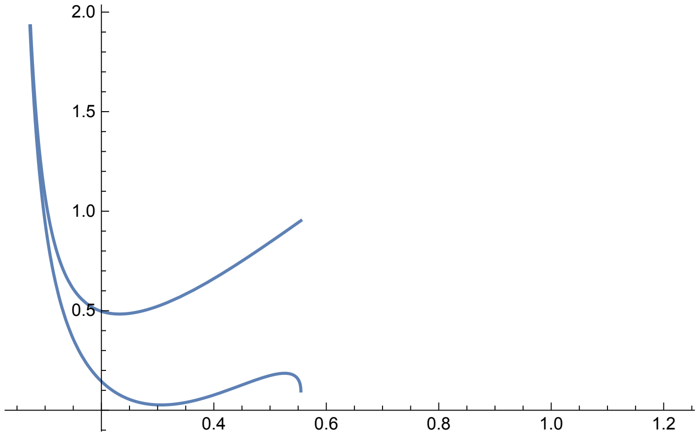



```
FindRoot[2 Pi - al Log[Cos[al], F[al, b, Gm[al, b]]] - Gm[al, b] ==
   ArcSin[r[al, b] / R[al, b]] /. {b → 0.85}, {al, 0.034}]
{al → 0.0524235}
```

```
Plot[{Nest[2 Pi - al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 1] -
    Nest[2 Pi - al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 0],
  ArcSin[r[al, b] / R[al, b]]} /. {b → 0.85}, {al, 0, ArcTan[Sqrt[8]]}]
```




```
Plot[{Nest[2 Pi - al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 2] -
    Nest[2 Pi - al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 1],
  Nest[2 Pi - al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 2] -
    Gm[al, b] - ArcSin[r[al, b] / R[al, b]]} /. {b → 0.85},
 {al, 0.052423546149187816, ArcTan[Sqrt[8]]}]
```




```
(* END |\ap_m|\ge 0.85 *)

(* |\ap_n|\le 1.5 n\ge 3 *)


(* test that A_z\supset A_Z' when Re 1/z=1 *)


(* P 76,77 of block - hasn't proved useful *)


nw[k_] := Sqrt[1 + k^2]; nwp[k_, th_] := Sqrt[1 + k^2 + 2 k Sin[th]];
Lw[b_, k_] := b * nw[k] / (b^2 nw[k]^2 - 1);
Lwp[b_, k_, th_] := b * nwp[k, th] / (b^2 nwp[k, th]^2 - 1);
lw[b_, k_] := 1 / (b^2 nw[k]^2 - 1); lwp[b_, k_, th_] := 1 / (b^2 nwp[k, th]^2 - 1);
Df[b_, k_, th_] := (Lw[b, k] - Lwp[b, k / 2, th]) ^2 -
  lw[b, k] ^2 - lwp[b, k / 2, th] ^2 + 2 Cos[th] lw[b, k] lwp[b, k / 2, th]
```

Df[b, k, th]

$$-\frac{1}{\left(-1+b^2\left(1+k^2\right)\right)^2}-\frac{1}{\left(-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)\right)^2}+\frac{2\,\mathrm{Cos}[th]}{\left(-1+b^2\left(1+k^2\right)\right)\left(-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)\right)}+\left(\frac{b\sqrt{1+k^2}}{-1+b^2\left(1+k^2\right)}-\frac{b\sqrt{1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]}}{-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)}\right)^2$$

D[Df[b, k, th], {th, 2}];

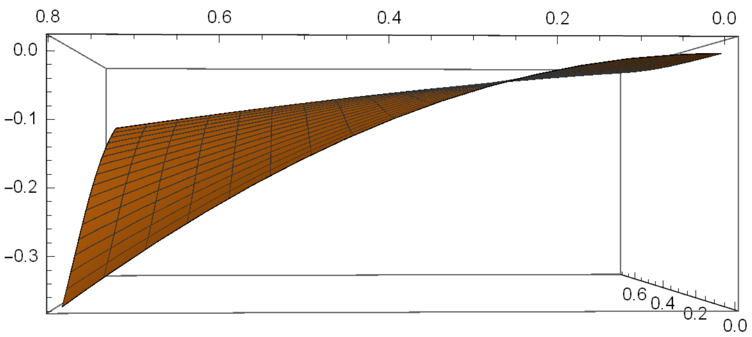
D[Df[b, k, th], th]

$$\frac{2\,b^2\,k\,\mathrm{Cos}[th]}{\left(-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)\right)^3}-\frac{2\,b^2\,k\,\mathrm{Cos}[th]^2}{\left(-1+b^2\left(1+k^2\right)\right)\left(-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)\right)^2}-\frac{2\,\mathrm{Sin}[th]}{\left(-1+b^2\left(1+k^2\right)\right)\left(-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)\right)}+2\left(\frac{b^3\,k\,\mathrm{Cos}[th]\sqrt{1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]}}{\left(-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)\right)^2}-\frac{b\,k\,\mathrm{Cos}[th]}{2\sqrt{1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]}\left(-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)\right)}\right)\left(\frac{b\sqrt{1+k^2}}{-1+b^2\left(1+k^2\right)}-\frac{b\sqrt{1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]}}{-1+b^2\left(1+\frac{k^2}{4}+k\,\mathrm{Sin}[th]\right)}\right)$$

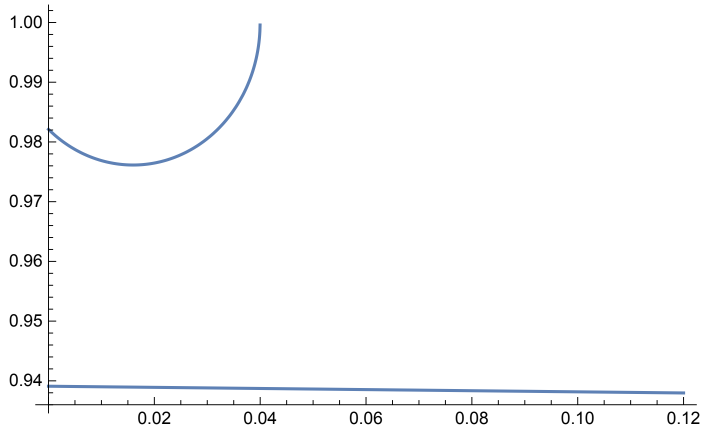
% /. {th → 0}

$$\frac{2\,b^2\,k}{\left(-1+b^2\left(1+\frac{k^2}{4}\right)\right)^3}-\frac{2\,b^2\,k}{\left(-1+b^2\left(1+\frac{k^2}{4}\right)\right)^2\left(-1+b^2\left(1+k^2\right)\right)}+2\left(\frac{b^3\,k\sqrt{1+\frac{k^2}{4}}}{\left(-1+b^2\left(1+\frac{k^2}{4}\right)\right)^2}-\frac{b\,k}{2\sqrt{1+\frac{k^2}{4}}\left(-1+b^2\left(1+\frac{k^2}{4}\right)\right)}\right)\left(-\frac{b\sqrt{1+\frac{k^2}{4}}}{-1+b^2\left(1+\frac{k^2}{4}\right)}+\frac{b\sqrt{1+k^2}}{-1+b^2\left(1+k^2\right)}\right)$$

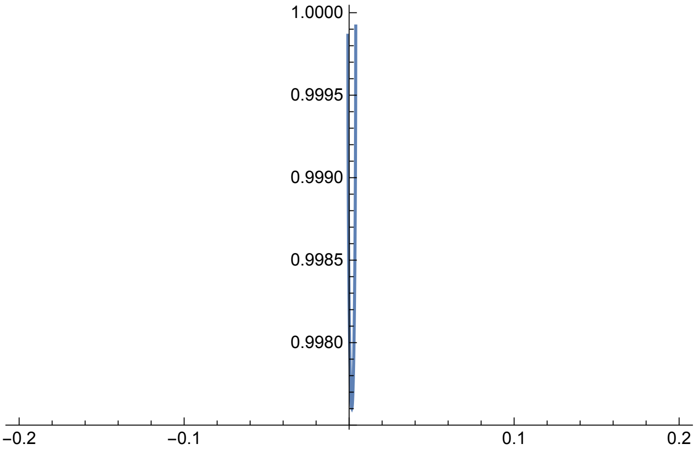
```
Plot3D[Df[b, k, th] /. {b → 1.5}, {k, 0, 0.7}, {th, 0, Pi / 4}]
```



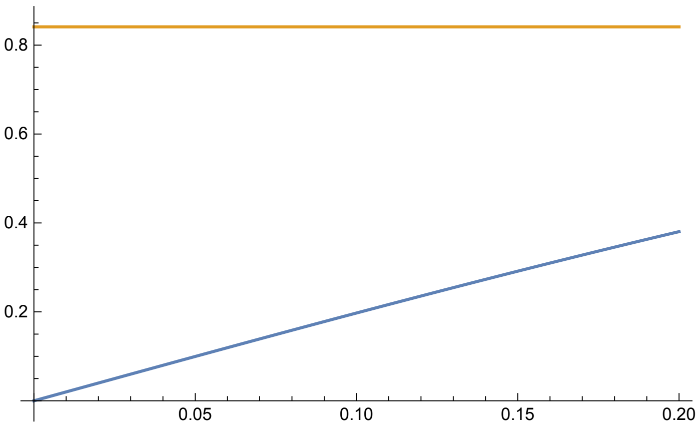

```
(* Lemma 2.4 *)
R[al_, b_] := Sqrt[1 + (Cos[al] Sin[al] / (Cos[al]^2 - b^2))^2] /; b > Cos[al]
r[al_, b_] := -b Sin[al] / (Cos[al]^2 - b^2) /; b > Cos[al]
Gm[al_, b_] := ArcTan[-Cos[al] Sin[al] / (Cos[al]^2 - b^2)] /; b > Cos[al]
F[al_, b_, th_] :=
 R[al, b] Cos[th - Gm[al, b]] - Sqrt[r[al, b]^2 - R[al, b]^2 * Sin[th - Gm[al, b]]^2]
Plot[{Cos[al]^((2 Pi + th) / al), F[al, 1.5, th]} //. {al → 0.02}, {th, 0, 0.12}]
```



```
Plot[F[al, 1.5, th] //. {al → 0.002}, {th, -.2, 0.2}]
```



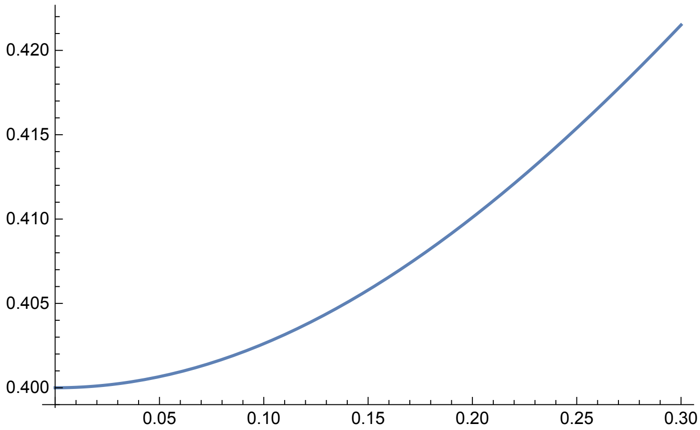

```
(* equality 34 *)

gmzero[b_, m_] := ArcCos[1 / (m - 1) / (b - 1)];

Plot[{ArcSin[r[al, 1.5] / R[al, 1.5]] + Gm[al, 1.5], gmzero[1.5, 4]}, {al, 0, 0.2}]
```



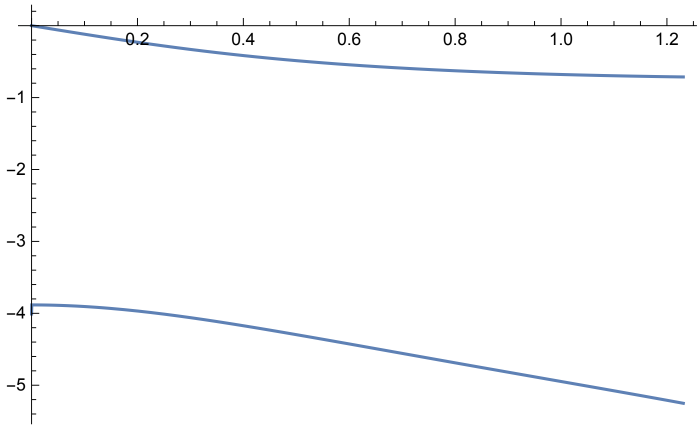

```
(* derivatives have proved unneeded now *)

D[ArcSin[r[al, 1.5] / R[al, 1.5]] - Gm[al, 1.5], al]
```

$$-\mathrm{Gm}^{(1,0)}[\mathrm{al}, 1.5] + \frac{\frac{r^{(1,0)}[\mathrm{al},1.5]}{R[\mathrm{al},1.5]} - \frac{r[\mathrm{al},1.5]\, R^{(1,0)}[\mathrm{al},1.5]}{R[\mathrm{al},1.5]^2}}{\sqrt{1 - \frac{r[\mathrm{al},1.5]^2}{R[\mathrm{al},1.5]^2}}}$$

```
Plot[Out[9], {al, 0.0, 0.3}] (* must be less than 3 *)
```



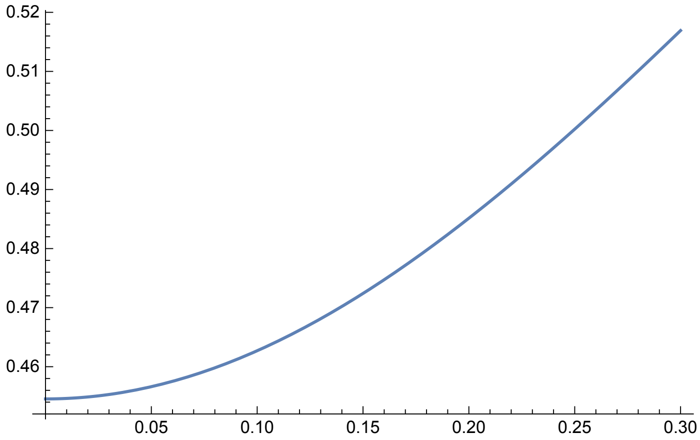
```
(* this finished |ap_m|le 1.5 *)

Plot[{-2 Pi + al Log[Cos[al], F[al, b, Gm[al, b]]] - Gm[al, b],
   -ArcSin[r[al, b] / R[al, b]]} /. {b → 1.5}, {al, 0, ArcTan[Sqrt[8]]}]
```



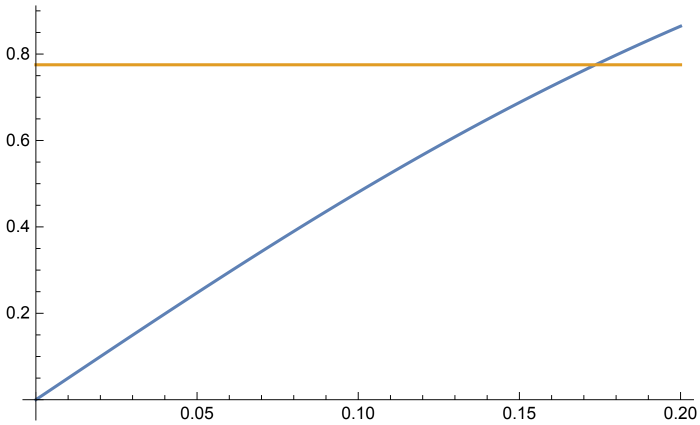
```
(* again, derivatives have proved unneeded now *)

D[ArcSin[r[al, 1.2] / R[al, 1.2]] - Gm[al, 1.2], al]
```

$$-\mathrm{Gm}^{(1,0)}[\mathrm{al},\,1.2] + \frac{\frac{\mathrm{r}^{(1,0)}[\mathrm{al},1.2]}{\mathrm{R}[\mathrm{al},1.2]} - \frac{\mathrm{r}[\mathrm{al},1.2]\,\mathrm{R}^{(1,0)}[\mathrm{al},1.2]}{\mathrm{R}[\mathrm{al},1.2]^2}}{\sqrt{1 - \frac{\mathrm{r}[\mathrm{al},1.2]^2}{\mathrm{R}[\mathrm{al},1.2]^2}}}$$

```
Plot[Out[16], {al, 0.0, 0.3}](* must be less than 6 *)
```

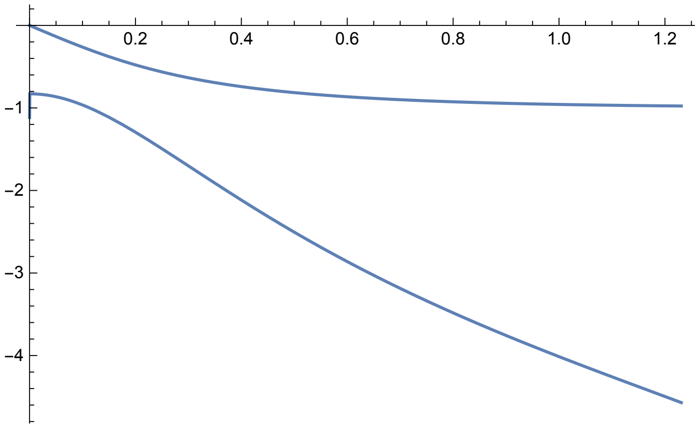


```
(* now the bound 1.2 for m\ge 6 *)

gmzero[1.2, 6.001]

0.0199983

Plot[{ArcSin[r[al, 1.2] / R[al, 1.2]] + Gm[al, 1.2], gmzero[1.2, 8]}, {al, 0, 0.2}]
```

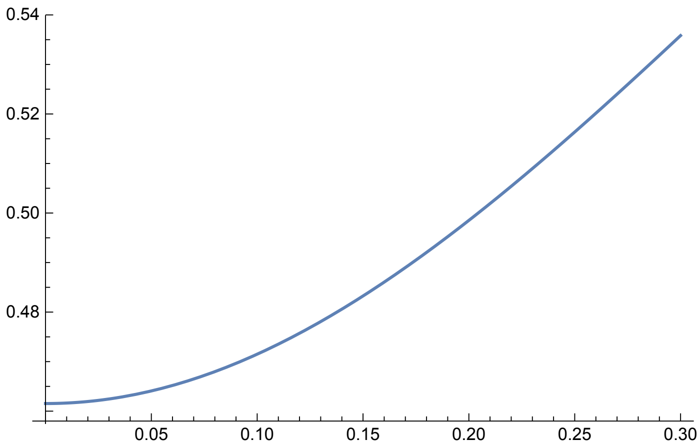


```
FindRoot[
 ArcSin[r[al, b] / R[al, b]] + Gm[al, b] - gmzero[b, 8] /. {b → 1.2}, {al, 0.15}]

{al → 0.173611}
```

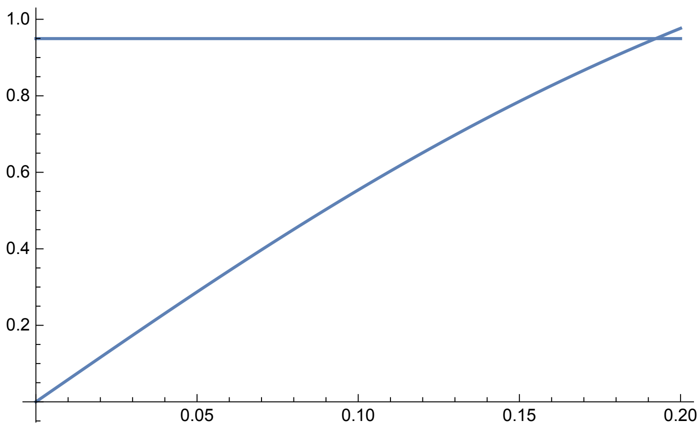
```
Plot[{-2 Pi + al Log[Cos[al], F[al, b, Gm[al, b]]] - Gm[al, b],
   -ArcSin[r[al, b] / R[al, b]]} /. {b → 1.2}, {al, 0, ArcTan[Sqrt[8]]}]
```



```
(* ok, so works for |\ap_n|le 1.2 n≥7 *)

(* now we see if we can carve out a bit more : 1.1718 *)

D[ArcSin[r[al, 7/6] / R[al, 7/6]] - Gm[al, 7/6], al];

Plot[Out[20], {al, 0.0, 0.3}]
```



```
Plot[{ArcSin[r[al, b] / R[al, b]] + Gm[al, b], gmzero[b, 11]} /. {b → 1.1718},
 {al, 0, 0.2}]
```

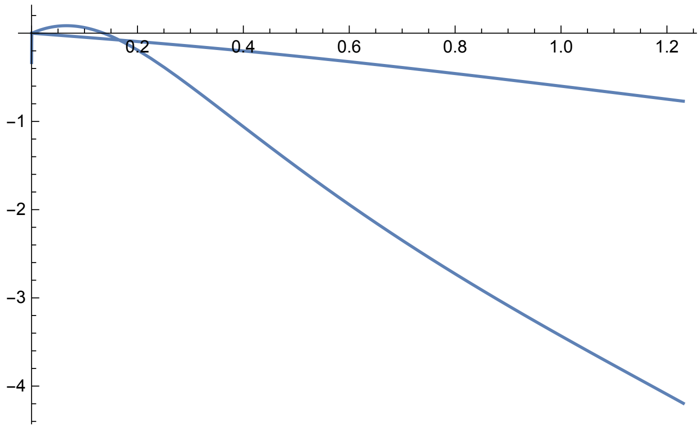


```
(* the value of gm_0 when we test explicitly |ap_m|le 10 *)
```

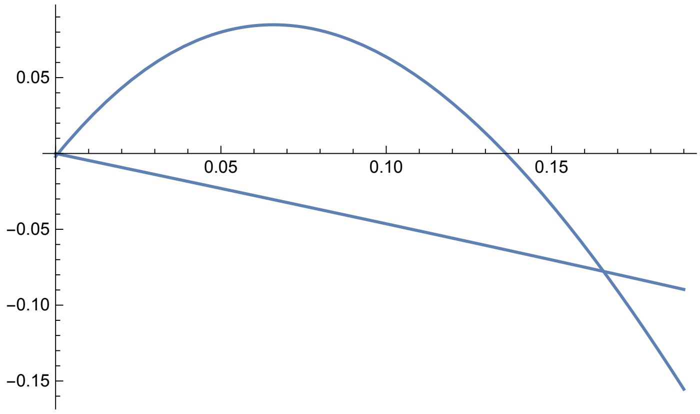
```
FindRoot[
 ArcSin[r[al, b] / R[al, b]] + Gm[al, b] - gmzero[b, 11] /. {b → 1.1718}, {al, 0.15}]
```

```
{al → 0.192264}
```

```
Plot[{-2 Pi + al Log[Cos[al], F[al, b, Gm[al, b]]],
   Gm[al, b] - ArcSin[r[al, b] / R[al, b]]} /. {b → 1.1718}, {al, 0, ArcTan[Sqrt[8]]}]
```



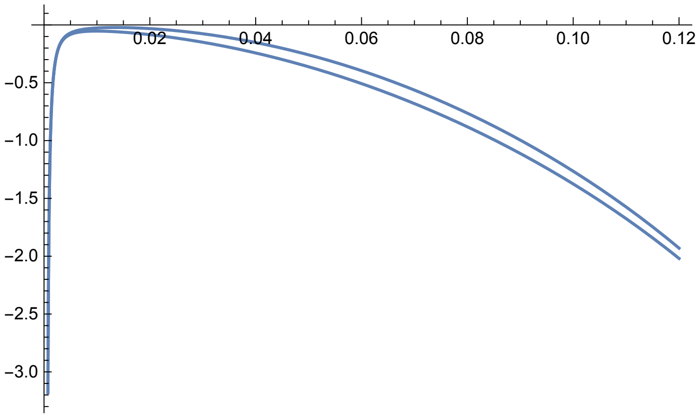

```
Plot[{-2 Pi + al Log[Cos[al], F[al, b, Gm[al, b]]],
   Gm[al, b] - ArcSin[r[al, b] / R[al, b]]} /. {b → 1.1718}, {al, 0.0, 0.19}]
```




```
FindRoot[-2 Pi + al Log[Cos[al], F[al, b, Gm[al, b]]] -
   Gm[al, b] + ArcSin[r[al, b] / R[al, b]] /. {b → 1.1718}, {al, 0.15}]
```

```
{al → 0.16572}
```

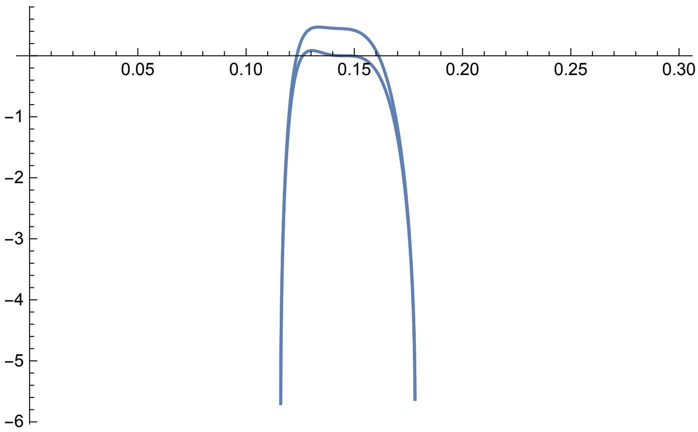
```
Plot[{Nest[-2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 2] -
    Nest[-2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 1],
   Nest[-2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 2] - Gm[al, b] +
    ArcSin[r[al, b] / R[al, b]]} /. {b → 1.1718}, {al, 0.0001, 0.12}]
```




```
(* evidence that with less than 1.1718 does not work *)

FindRoot[Nest[-2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 2] - Nest[
    -2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 1] /. {b → 1.17}, {al, 0.02893}]
```

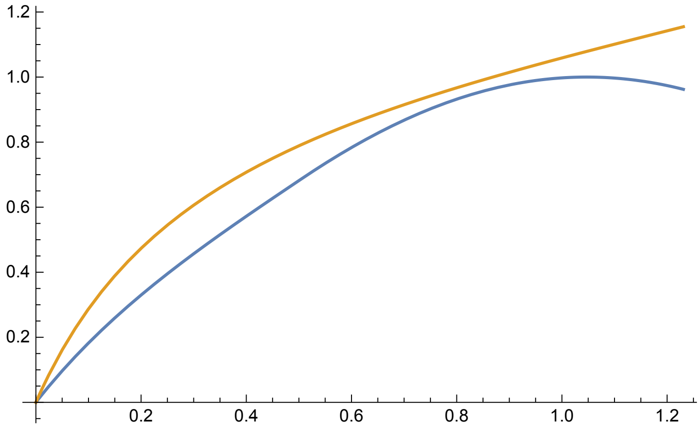
```
{al → 0.0541909}
```

```
Plot[{Nest[-2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 3] -
    Nest[-2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 2],
   Nest[-2 Pi + al Log[Cos[al], F[al, b, #]] &, Gm[al, b], 3] - Gm[al, b] +
    ArcSin[r[al, b] / R[al, b]]} /. {b → 1.16}, {al, 0.00, 0.3}]
```

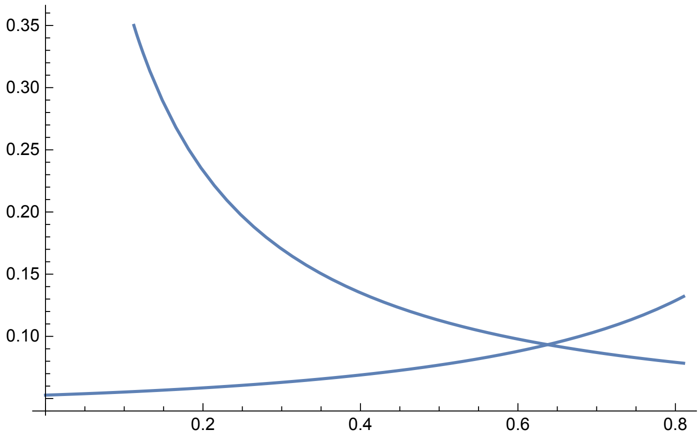



```
(* this is some old exp stuff likely not needed *)
```

```
Plot[{Sin[3 al / 2] / If[3 al / 2 + ArcSin[Sqrt[5 / 6]] - ArcSin[1 / 3] > Pi / 2,
     1, Sin[3 al / 2 + ArcSin[Sqrt[5 / 6]] - ArcSin[1 / 3]]],
  Sqrt[3 / 2] Sin[al] / Sin[ArcSin[1 / 3] + al]}, {al, 0, ArcCos[1 / 3]}]
```




```
Plot[{Sin[3 al / 2] / Sin[bt1 + 3 al / 2],
    Sqrt[3 / 2] * Sin[al] / Sin[ArcSin[Sqrt[5 / 6]] - bt1 + al]} /. {al → 0.04},
 {bt1, 0, ArcSin[Sqrt[5 / 6]] - ArcSin[1 / 3]}]
```




```
Sin[Pi / 6]
```

$\frac{1}{2}$

```
(* one small piece for maximum |\ap_4| *)

MyAbs[a_ + b_ I] := Sqrt[a^2 + b^2]

f[x_] := Sqrt[-x^2 - 2 x + 7 - 4 Sqrt[-2 x + 3]];

N[Maximize[MyAbs[ComplexExpand[(1 - z^5) / (1 - z^4) /. {z → x + I y}]],
  {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]

{1.25, {x → 1., y → 0.}}
```

```
N[Maximize[MyAbs[ComplexExpand[(1 - z^6) / (1 - z^5) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]
{1.2, {x → 1., y → 0.}}

N[Minimize[MyAbs[ComplexExpand[(1 - z^6) / (1 - z^5) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]
{0.885285, {x → 0.541873, y → 0.498244}}

N[Minimize[MyAbs[ComplexExpand[(1 - z^7) / (1 - z^6) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]
{0.877903, {x → 0.623465, y → 0.484517}}

N[Minimize[MyAbs[ComplexExpand[(1 - z^5) / (1 - z^4) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]
{0.896456, {x → 0.433933, y → 0.495616}}

N[Minimize[MyAbs[ComplexExpand[(1 - z^6) / (1 - z^5) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Min[Tan[0.4] * x, Sqrt[x - x^2]]}, {x, y}]]
{1.00523, {x → 0.620643, y → 0.262404}}

N[Minimize[MyAbs[ComplexExpand[(1 - z^3) / (1 - z^2) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Min[Tan[0.45] * x, Sqrt[x - x^2]]}, {x, y}]]
{1.20486, {x → 0.57842, y → 0.279409}}

N[Minimize[MyAbs[ComplexExpand[(1 - z^2) / (1 - z^1) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]

{1.1547, {x → 0.111111, y → 0.31427}}

Sqrt[108] / 9 // N
1.1547

N[Maximize[MyAbs[ComplexExpand[(1 - z^4) / (1 - z^3) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]
{1.33333, {x → 1., y → 0.}}

N[Minimize[MyAbs[ComplexExpand[(1 - z^4) / (1 - z^3) /. {z → x + I y}]],
   {0 ≤ x ≤ 1, f[x] ≤ y ≤ Sqrt[x - x^2]}, {x, y}]]
{0.914653, {x → 0.292293, y → 0.454816}}
```

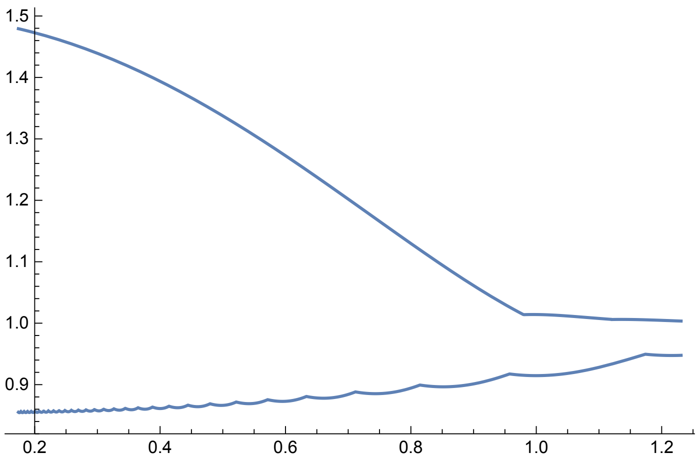
```
AlMax[ap_, n_] := (AlMax[ap, n] =
     NMaximize[{Sqrt[(1 - Cos[n * ap] * r^n)^2 + Sin[n * ap]^2 * r^(2 n)] /
         Sqrt[(1 - Cos[(n - 1) * ap] * r^(n - 1))^2 + Sin[(n - 1) * ap]^2 * r^(2 n - 2)],
        r ≥ 2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3] && r ≤ Cos[ap]},
       r, MaxIterations -> 1000][[1]]) /; ap > 0.0 && ap < Pi / 2;
AlMin[ap_, n_] := (AlMin[ap, n] = NMinimize[
       {Sqrt[(1 - Cos[n * ap] * r^n)^2 + Sin[n * ap]^2 * r^(2 n)] /
         Sqrt[(1 - Cos[(n - 1) * ap] * r^(n - 1))^2 + Sin[(n - 1) * ap]^2 * r^(2 n - 2)],
        r ≥ 2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3] && r ≤ Cos[ap]},
       r, MaxIterations -> 1000][[1]]) /; ap > 0.0 && ap < Pi / 2;

AMax[ap_, n_: 2] := (AMax[ap, n] =
     Block[{a = AlMax[ap, n], b = AlMin[ap, n], c = n, d, f1},
      While[++c; (d = (1 + Cos[ap]^c) / (1 - Cos[ap]^c)) > a || 1 / d < b ,
        a = Max[a, AlMax[ap, c]]; b = Min[b, AlMin[ap, c]]];
      f1 = OpenAppend[AMaxLogFile]; WriteString[f1, "AMax[" <> ToString[ap] <>
        "," <> ToString[n] <> "]={" <> ToString[a] <> "," <> ToString[b] <> "}
"]; Close[f1]; {a, b}
     ]) /; ap > 0.1 && ap < Pi / 2;

$IterationLimit = Infinity
```

∞

```
AMaxLogFile = "test.txt";

AMax[0.24, 3]
```

{1.46087, 0.856832}

```
Plot[AMax[ap, 3], {ap, 0.174, ArcCos[1 / 3]}]
```



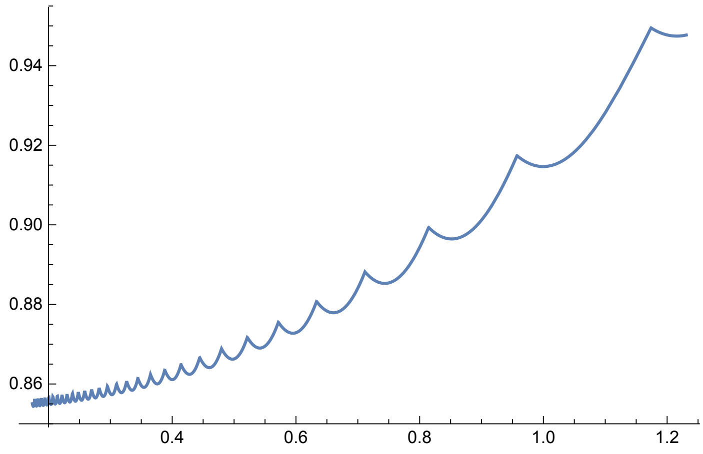

```
SetDirectory["C:\\Users\\admin\\Documents\\math"]
```

C:\Users\admin\Documents\math

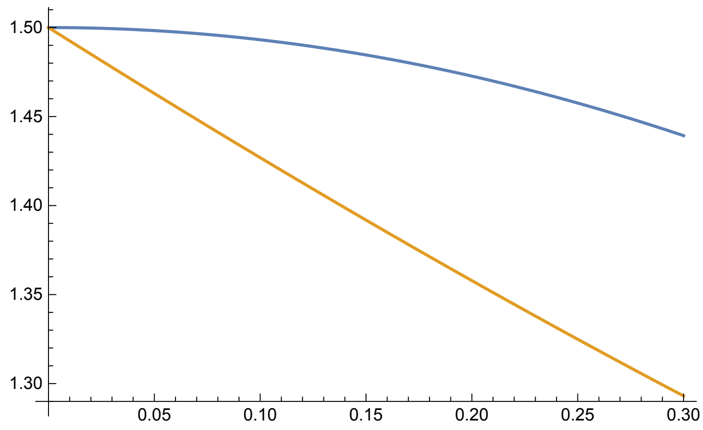
```
<< alpha_bd.m
```

```
almax3 = Interpolation[Union[{#[[1]], #[[3]]} & /@ data]];
almin3 = Interpolation[Union[{#[[1]], #[[4]]} & /@ data]];
```

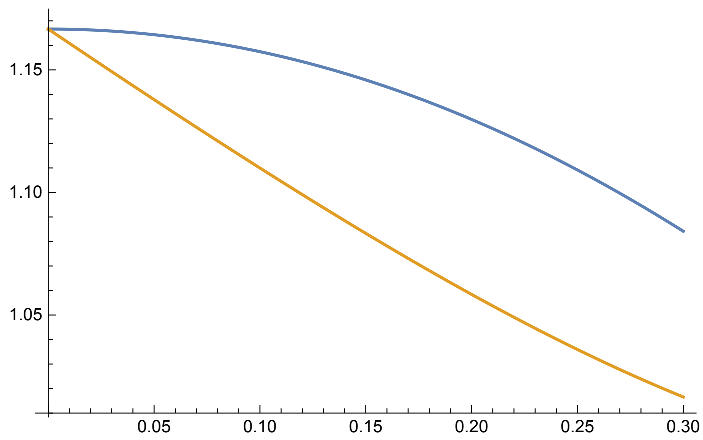
```
Plot[almin3[ap], {ap, 0.174, ArcCos[1/3]}]
```




```
Plot[{AlMax[ap, 3], AlMin[ap, 3]}, {ap, 0.0, 0.3}]
```




```
Plot[{AlMax[ap, 7], AlMin[ap, 7]}, {ap, 0.0, 0.3}]
```




```
PrintTime[t_] := Block[{a = Timing[t]}, Print[a[[1]]]; a[[2]]]
```

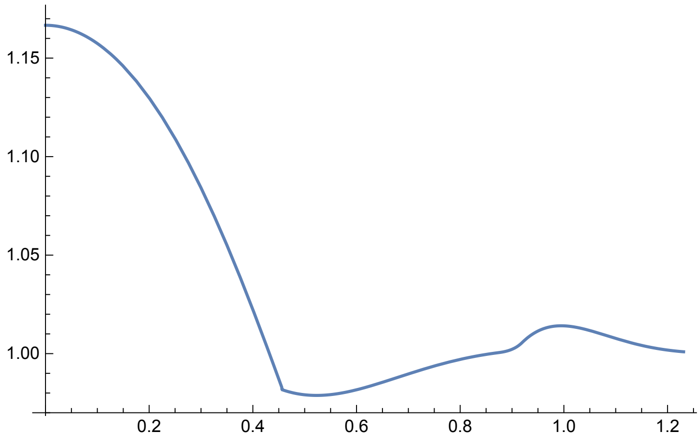
```
Attributes[Plot]
```

```
{HoldAll, Protected, ReadProtected}
```

```
SetAttributes[PrintTime, {HoldAll}]
```

```
Plot[AlMax[ap, 7], {ap, 0.0, ArcCos[1/3]}] // PrintTime
```

```
210.445
```

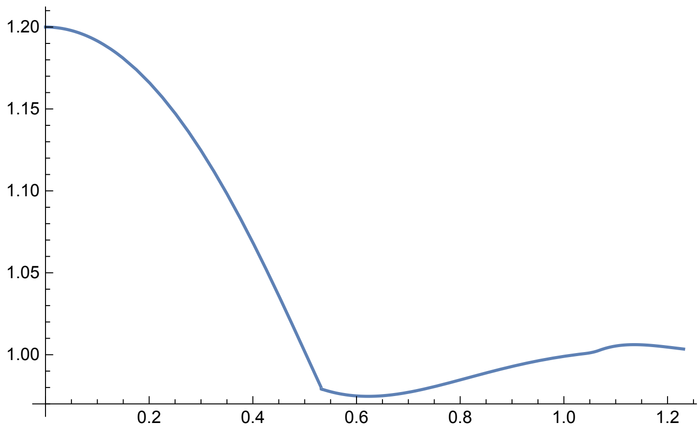



```
NMaximize[{AlMax[al, 7], 0 ≤ al ≤ 0.3}, al]
```

$\{1.16667, \{al \to 4.75233 \times 10^{-6}\}\}$

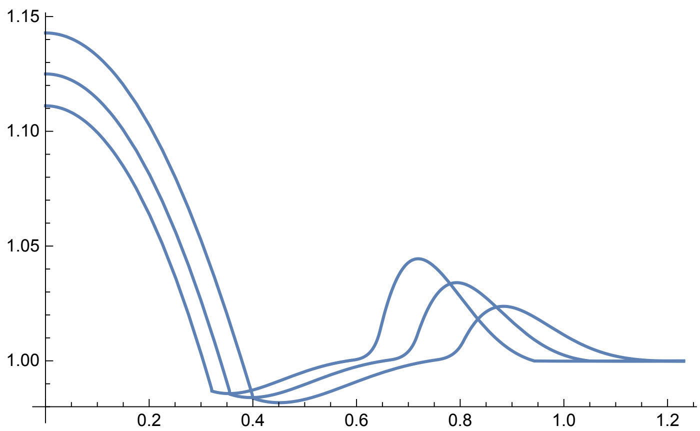
```
Plot[AlMax[ap, 6], {ap, 0.0, ArcCos[1/3]}] // PrintTime
```

```
192.396
```



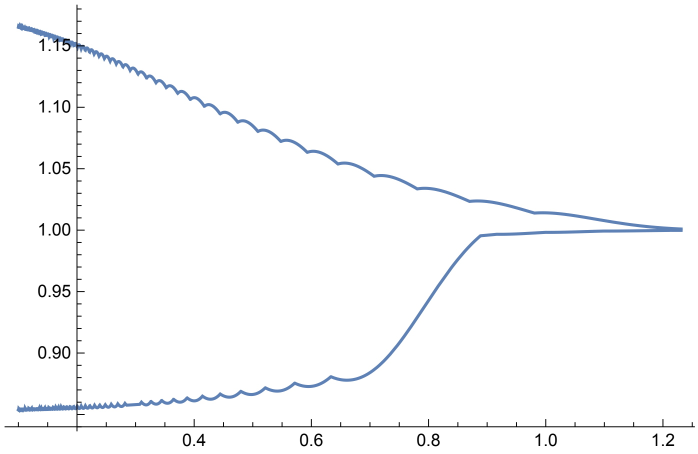

```
Plot[AlMax[ap, #] & /@ Range[8, 10], {ap, 0.0, ArcCos[1/3]}] // PrintTime
```

```
2435.57
```



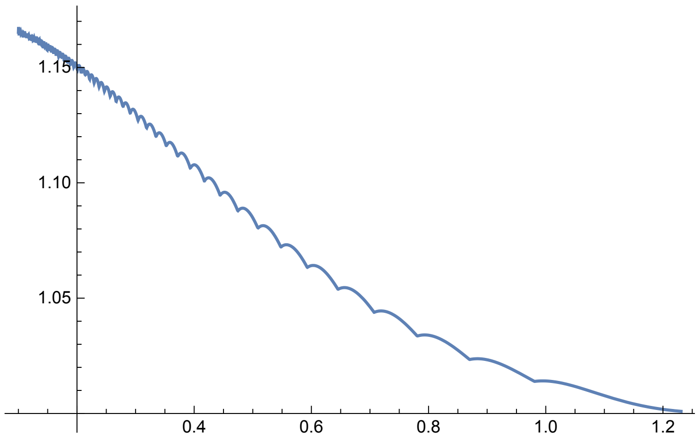
```
<< "almax_bd_7.txt"; AMaxLogFile = "almax_bd_7a.txt";
Plot[AMax[ap, 7], {ap, 0.1, ArcCos[1/3]}] // PrintTime
```

```
0.
```




```
<< almax_bd_7.m;
(* make interpolation file with almax7[] - done! *)
almax7 = Interpolation[Union[{#[[1]], #[[3]]} & /@ data7]];
Plot[almax7[ap], {ap, 0.1, ArcCos[1/3]}] // PrintTime
```

```
0.
```



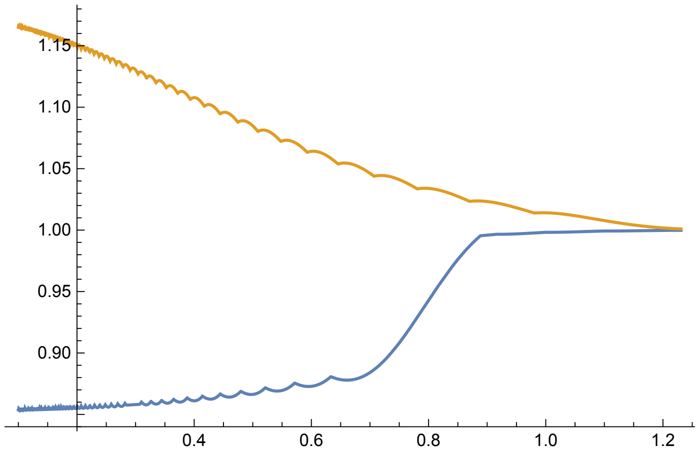

```
SetDirectory["C:\\Users\\admin\\Documents\\math"]
```

```
C:\Users\admin\Documents\math
```

```
<< almax_bd_7a.m;
almax7a = Interpolation[Union[{#[[1]], #[[3]]} & /@ data7a]];
almin7a = Interpolation[Union[{#[[1]], #[[4]]} & /@ data7a]];
```

```
Plot[{almin7a[ap], almax7a[ap]}, {ap, 0.1, ArcCos[1/3]}] // PrintTime
```

0.




```
AlMax[as_, aq_, n_] :=
  NMaximize[{Sqrt[(1 - Cos[n * ap] * r^n)^2 + Sin[n * ap]^2 * r^(2 n)] /
       Sqrt[(1 - Cos[(n - 1) * ap] * r^(n - 1))^2 + Sin[(n - 1) * ap]^2 * r^(2 n - 2)],
      r ≥ 2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3] - 0.1 &&
       r ≤ Cos[ap] && as ≤ ap ≤ aq}, {r, ap},
     MaxIterations -> 1000][[1]] /; as >= 0.0 && aq < Pi/2;
AlMin[as_, aq_, n_] := NMinimize[
     {Sqrt[(1 - Cos[n * ap] * r^n)^2 + Sin[n * ap]^2 * r^(2 n)] /
       Sqrt[(1 - Cos[(n - 1) * ap] * r^(n - 1))^2 + Sin[(n - 1) * ap]^2 * r^(2 n - 2)],
      r ≥ 2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3] - 0.1 &&
       r ≤ Cos[ap] && as ≤ ap ≤ aq}, {r, ap},
     MaxIterations -> 1000][[1]] /; as >= 0.0 && aq < Pi/2;
```

```
AlMax[0., 0.3, 7]
```

1.16667

```
AlMax[0, ArcCos[1/3], 8]
```

1.14286

```
AlMax[0, ArcCos[1/3], 9]
```

1.125

```
AlMax[0.1, ArcCos[1/3], 10]
```

1.09948

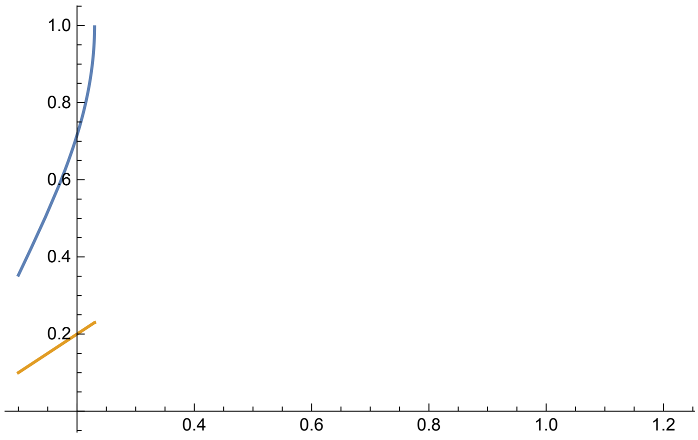
```
{AlMax[0, 0.174, 4], AlMin[0, 0.174, 4]}
```

{1.33333, 1.16818}

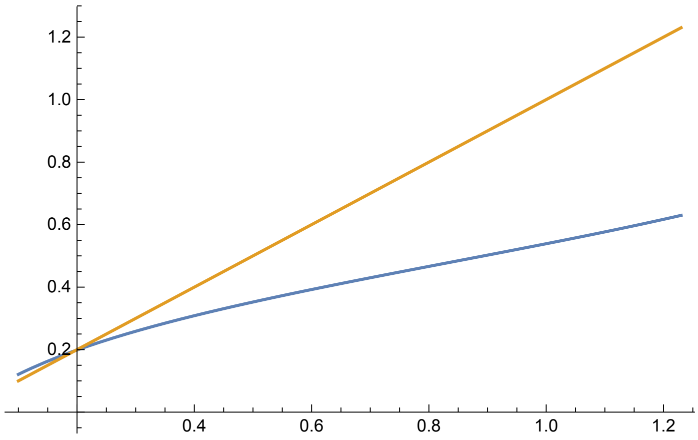
```
(* for Lemma 4.1 *)
```

```
Sig[ap_] = Sqrt[2] * (-1 + Cos[ap] + Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]);
```

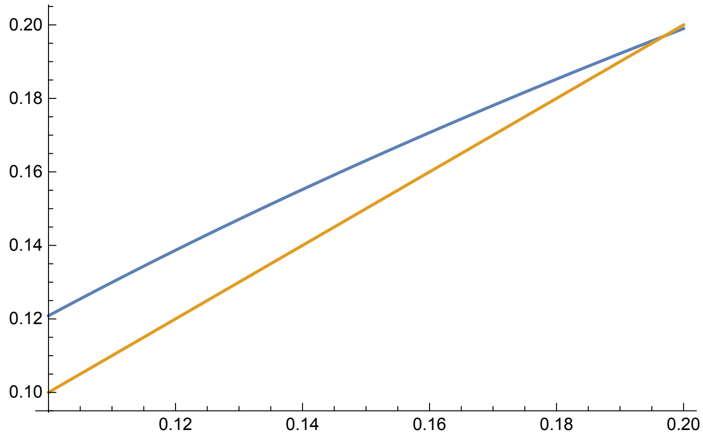
```
Plot[{1 - Sqrt[4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^
      (1 / (5 Pi / 4 - 2 ap)) - 3],
  Min[ap, 1 + Sqrt[4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^
      (1 / (5 Pi / 4 - 2 ap)) - 3]]}, {ap, 0.1, 1.23}]
```



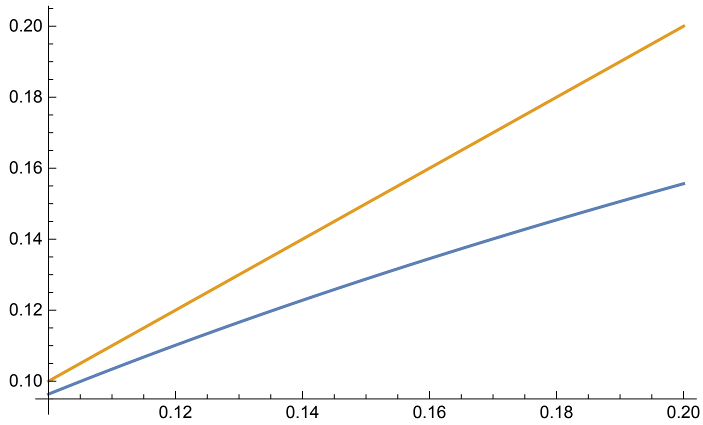

```
(* this fails totally , so try 5Pi/4→ 13Pi/4 *)
Plot[{1 - Sqrt[4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^
      (1 / (13 Pi / 4 - 2 ap)) - 3],
  Min[ap, 1 + Sqrt[4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^
      (1 / (13 Pi / 4 - 2 ap)) - 3]]}, {ap, 0.1, 1.23}]
```



```
Plot[{1 - Sqrt[4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^
      (1 / (13 Pi / 4 - 2 ap)) - 3],
  Min[ap, 1 + Sqrt[4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^
      (1 / (13 Pi / 4 - 2 ap)) - 3]]}, {ap, 0.1, 0.2}]
```




```
Plot[{1 - Sqrt[
    4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^ (1 / (4 Pi - ap)) - 3],
  Min[ap, 1 + Sqrt[4 (Sqrt[3 - Sqrt[8]] / (Sqrt[6] Sig[ap] + Sqrt[3 - Sqrt[8]])) ^
      (1 / (4 Pi - ap)) - 3]]}, {ap, 0.1, 0.2}]
```




```
(* maximal arg of ap_n/ap_{n-1} for ap betw as and aq
      needed for |arg(ap_n/ap_n-1)}\le \as Lemma 4.1 *)

alpha[ap_, n_] := ComplexExpand[((1 - Cos[n * ap] * r^n) + I Sin[n * ap] * r^(n)) *
    ((1 - Cos[(n - 1) * ap] * r^(n - 1)) - I Sin[(n - 1) * ap] * r^(n - 1))];

ArMax[as_, aq_, n_] :=
  NMaximize[Abs[Arg[alpha[ap, n] * Conjugate[alpha[ap, n - 1]]]],
     {r ≥ 2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3] - 0.00001 &&
       r ≤ Cos[ap] && as ≤ ap ≤ aq}, {r, ap},
     MaxIterations -> 1000][[1]] /; as > 0.0 && aq < Pi/2;

ArMax[0.1, 0.3, 2]

0.
```

```
ArMax[0.1, 0.3, #] & /@ Range[3, 100]
```

GreaterEqual::nord : Invalid comparison with 1.21666 – 0.389633 ⅈ attempted. ≫

NMaximize::bcons :

The following constraints are not valid: $\left\{r \geq 1.99999 - \text{Cos}[ap] - \sqrt{3 - 4\,\text{Cos}[ap] + \text{Cos}[ap]^2},\ 0.1 \leq ap,\ ap \leq 0.3,\ r \leq \text{Cos}[ap]\right\}$.
Constraints should be equalities, inequalities, or domain specifications involving the variables. ≫

First::normal : Nonatomic expression expected at position 1 in First[Optimization`NMinimizeDump`convNM]. ≫

GreaterEqual::nord : Invalid comparison with 1.21666 – 0.389633 ⅈ attempted. ≫

NMaximize::bcons :

The following constraints are not valid: $\left\{r \geq 1.99999 - \text{Cos}[ap] - \sqrt{3 - 4\,\text{Cos}[ap] + \text{Cos}[ap]^2},\ 0.1 \leq ap,\ ap \leq 0.3,\ r \leq \text{Cos}[ap]\right\}$.
Constraints should be equalities, inequalities, or domain specifications involving the variables. ≫

First::normal : Nonatomic expression expected at position 1 in First[Optimization`NMinimizeDump`convNM]. ≫

GreaterEqual::nord : Invalid comparison with 1.21666 – 0.389633 ⅈ attempted. ≫

General::stop : Further output of GreaterEqual::nord will be suppressed during this calculation. ≫

NMaximize::bcons :

The following constraints are not valid: $\left\{r \geq 1.99999 - \text{Cos}[ap] - \sqrt{3 - 4\,\text{Cos}[ap] + \text{Cos}[ap]^2},\ 0.1 \leq ap,\ ap \leq 0.3,\ r \leq \text{Cos}[ap]\right\}$.
Constraints should be equalities, inequalities, or domain specifications involving the variables. ≫

General::stop : Further output of NMaximize::bcons will be suppressed during this calculation. ≫

First::normal : Nonatomic expression expected at position 1 in First[Optimization`NMinimizeDump`convNM]. ≫

General::stop : Further output of First::normal will be suppressed during this calculation. ≫

```
{0.0148787, 0.0148227, 0.0146668, 0.0143387, 0.0137587, 0.0129065,
 0.0123621, 0.0122432, 0.0125264, 0.013249, 0.0145175, 0.016543,
 0.0197237, 0.0248414, 0.0335697, 0.0475489, 0.0623476, 0.0660671,
 Abs[Arg[(1 - ⅈ Conjugate[-r^19 Sin[19 ap] + r^39 Cos[20 ap] Sin[19 ap] +
        r^20 Sin[20 ap] - r^39 Cos[19 ap] Sin[20 ap]] + Conjugate[-r^19 Cos[19 ap] -
       r^20 Cos[20 ap] + r^39 Cos[19 ap] Cos[20 ap] + r^39 Sin[19 ap] Sin[20 ap]])
     (1 - r^20 Cos[20 ap] - r^21 Cos[21 ap] + r^41 Cos[20 ap] Cos[21 ap] +
      r^41 Sin[20 ap] Sin[21 ap] + ⅈ (-r^20 Sin[20 ap] + r^41 Cos[21 ap] Sin[20 ap] +
        r^21 Sin[21 ap] - r^41 Cos[20 ap] Sin[21 ap]))]],
 Abs[Arg[(1 - ⅈ Conjugate[-r^20 Sin[20 ap] + r^41 Cos[21 ap] Sin[20 ap] +
        r^21 Sin[21 ap] - r^41 Cos[20 ap] Sin[21 ap]] + Conjugate[-r^20 Cos[20 ap] -
       r^21 Cos[21 ap] + r^41 Cos[20 ap] Cos[21 ap] + r^41 Sin[20 ap] Sin[21 ap]])
     (1 - r^21 Cos[21 ap] - r^22 Cos[22 ap] + r^43 Cos[21 ap] Cos[22 ap] +
      r^43 Sin[21 ap] Sin[22 ap] + ⅈ (-r^21 Sin[21 ap] + r^43 Cos[22 ap] Sin[21 ap] +
        r^22 Sin[22 ap] - r^43 Cos[21 ap] Sin[22 ap]))]],
 0.0653299, 0.0651481, 0.0622527, 0.0624242,
 0.0625676,
 0.0646209,
 0.0627903,
 0.0644399,
 0.0629507,
 0.0642953,
 0.0642338,
 0.0631148,
```

0.064128,
0.0631904,
0.0632209,
0.0632476,
0.0639683,
0.0632913,
0.0633093,
0.0638806,
0.0633391,
0.0638331,
0.0633623,
0.0637923,
0.0633807,
0.0633883,
0.0633952,
0.0637268,
0.0637131,
0.0634115,
0.0636882,
0.0634197,
0.0634231,
0.0634262,
0.0150368,
0.00655235,
0.0167459,
0.0636221,
0.0636148,
0.063439,
0.0634404,
0.00684406,
0.00688182,
0.0613457,
0.0477452,
0.00698025, 0.0230865,
0.00703511, 0.0166086,
0.00708262, 0.00710393,
0.00712375, 0.0157901,
0.0165671, 0.0165596,
0.0165525, 0.00210511,
0.0158909, 0.0165324,
0.0165261, 0.0072496,
0.0165143, 0.0159678,
0.0165033, 0.00728383,
0.016493, 0.0160166,
0.0164833, 0.00730887,
0.0164743, 0.01647, 0.0164659,

```
 0.0164619, 0.016458,
 0.0164542, 0.00207718,
 0.00335097, 0.0063389}
```

```
Max @@ Select[Out[3], NumberQ]
```

```
0.0660671
```

```
Select[Range[3, 100], Not[NumberQ[Out[3][[# - 2]]]] &]
```

```
{21, 22}
```

```
{ArMax[0.1, 0.28, #], ArMax[0.28, 0.32, #]} & /@ %
```

```
{{0.0655941, 0.0431408}, {0.0655396, 0.0614958}}
```

```
Max @@ Select[% // Flatten, NumberQ]
```

```
0.0655941
```

```
ArMax[0.1, 0.2, #] & /@ Range[101, 120]
```

```
{0.0161177, 0.0164371, 0.0161298, 0.0164309, 0.016428, 0.00735409, 0.00735538,
 0.0161559, 0.0161605, 0.0164145, 0.00735926, 0.00661533, 0.00736053,
 0.016181, 0.0164029, 0.0164008, 0.0163987, 0.0073622, 0.0163947, 0.00374413}
```

```
Max @@ %
```

```
0.0164371
```

```
ArMax[0.3, ArcCos[1 / 3], #] & /@ Range[2, 50]
```

GreaterEqual::nord : Invalid comparison with 1.43542 − 0.546564 *i* attempted. ≫

NMaximize::bcons : The following constraints are not valid:

$\left\{r \geq 1.99999 - \mathrm{Cos}[ap] - \sqrt{3 - 4\,\mathrm{Cos}[ap] + \mathrm{Cos}[ap]^2},\ 0.3 \leq ap,\ ap \leq \mathrm{ArcCos}\left[\frac{1}{3}\right],\ r \leq \mathrm{Cos}[ap]\right\}$.

Constraints should be equalities, inequalities, or domain specifications involving the variables. ≫

First::normal : Nonatomic expression expected at position 1 in First[Optimization`NMinimizeDump`convNM]. ≫

GreaterEqual::nord : Invalid comparison with 1.43542 − 0.546564 *i* attempted. ≫

NMaximize::bcons : The following constraints are not valid:

$\left\{r \geq 1.99999 - \mathrm{Cos}[ap] - \sqrt{3 - 4\,\mathrm{Cos}[ap] + \mathrm{Cos}[ap]^2},\ 0.3 \leq ap,\ ap \leq \mathrm{ArcCos}\left[\frac{1}{3}\right],\ r \leq \mathrm{Cos}[ap]\right\}$.

Constraints should be equalities, inequalities, or domain specifications involving the variables. ≫

First::normal : Nonatomic expression expected at position 1 in First[Optimization`NMinimizeDump`convNM]. ≫

GreaterEqual::nord : Invalid comparison with 1.43542 − 0.546564 *i* attempted. ≫

General::stop : Further output of GreaterEqual::nord will be suppressed during this calculation. ≫

NMaximize::bcons : The following constraints are not valid:

$\left\{r \geq 1.99999 - \mathrm{Cos}[ap] - \sqrt{3 - 4\,\mathrm{Cos}[ap] + \mathrm{Cos}[ap]^2},\ 0.3 \leq ap,\ ap \leq \mathrm{ArcCos}\left[\frac{1}{3}\right],\ r \leq \mathrm{Cos}[ap]\right\}$.

Constraints should be equalities, inequalities, or domain specifications involving the variables. ≫

General::stop : Further output of NMaximize::bcons will be suppressed during this calculation. ≫

First::normal : Nonatomic expression expected at position 1 in First[Optimization`NMinimizeDump`convNM]. ≫

General::stop : Further output of First::normal will be suppressed during this calculation. ≫

```
{0., 0.192645, 0.137818, 0.11212, 0.0977676,
 0.08885, 0.0829114, 0.0787583, 0.0757455, 0.0734965,
 Abs[Arg[(1 - i Conjugate[-r^10 Sin[10 ap] + r^21 Cos[11 ap] Sin[10 ap] +
          r^11 Sin[11 ap] - r^21 Cos[10 ap] Sin[11 ap]] + Conjugate[-r^10 Cos[10 ap] -
         r^11 Cos[11 ap] + r^21 Cos[10 ap] Cos[11 ap] + r^21 Sin[10 ap] Sin[11 ap]])
     (1 - r^11 Cos[11 ap] - r^12 Cos[12 ap] + r^23 Cos[11 ap] Cos[12 ap] +
       r^23 Sin[11 ap] Sin[12 ap] + i (-r^11 Sin[11 ap] + r^23 Cos[12 ap] Sin[11 ap] +
          r^12 Sin[12 ap] - r^23 Cos[11 ap] Sin[12 ap]))]],
 0.0695872, 0.0693803, 0.0685294, 0.0678376,
 0.0672688,
 0.0594137,
 Abs[
  Arg[(1 - i Conjugate[-r^17 Sin[17 ap] + r^35 Cos[18 ap] Sin[17 ap] +
          r^18 Sin[18 ap] - r^35 Cos[17 ap] Sin[18 ap]] + Conjugate[-r^17 Cos[17 ap] -
         r^18 Cos[18 ap] + r^35 Cos[17 ap] Cos[18 ap] + r^35 Sin[17 ap] Sin[18 ap]])
     (1 - r^18 Cos[18 ap] - r^19 Cos[19 ap] + r^37 Cos[18 ap] Cos[19 ap] +
       r^37 Sin[18 ap] Sin[19 ap] + i (-r^18 Sin[18 ap] + r^37 Cos[19 ap] Sin[18 ap] +
          r^19 Sin[19 ap] - r^37 Cos[18 ap] Sin[19 ap]))]],
 0.0606729, 0.0611267, 0.0614959, 0.0617981,
 0.0572094,
 0.0420037,
 0.0279388,
 Abs[
  Arg[(1 - i Conjugate[-r^25 Sin[25 ap] + r^51 Cos[26 ap] Sin[25 ap] +
          r^26 Sin[26 ap] - r^51 Cos[25 ap] Sin[26 ap]] + Conjugate[-r^25 Cos[25 ap] -
         r^26 Cos[26 ap] + r^51 Cos[25 ap] Cos[26 ap] + r^51 Sin[25 ap] Sin[26 ap]])
     (1 - r^26 Cos[26 ap] - r^27 Cos[27 ap] + r^53 Cos[26 ap] Cos[27 ap] +
       r^53 Sin[26 ap] Sin[27 ap] + i (-r^26 Sin[26 ap] + r^53 Cos[27 ap] Sin[26 ap] +
          r^27 Sin[27 ap] - r^53 Cos[26 ap] Sin[27 ap]))]], 0.00836949, 0.0170312,
 0.01708, Abs[Arg[(1 - i Conjugate[-r^29 Sin[29 ap] + r^59 Cos[30 ap] Sin[29 ap] +
          r^30 Sin[30 ap] - r^59 Cos[29 ap] Sin[30 ap]] + Conjugate[-r^29 Cos[29 ap] -
         r^30 Cos[30 ap] + r^59 Cos[29 ap] Cos[30 ap] + r^59 Sin[29 ap] Sin[30 ap]])
     (1 - r^30 Cos[30 ap] - r^31 Cos[31 ap] + r^61 Cos[30 ap] Cos[31 ap] +
       r^61 Sin[30 ap] Sin[31 ap] + i (-r^30 Sin[30 ap] + r^61 Cos[31 ap] Sin[30 ap] +
          r^31 Sin[31 ap] - r^61 Cos[30 ap] Sin[31 ap]))]],
 0.0171358, 0.017148, 0.0171525, 0.0171509,
 Abs[
  Arg[(1 - i Conjugate[-r^34 Sin[34 ap] + r^69 Cos[35 ap] Sin[34 ap] +
          r^35 Sin[35 ap] - r^69 Cos[34 ap] Sin[35 ap]] + Conjugate[-r^34 Cos[34 ap] -
         r^35 Cos[35 ap] + r^69 Cos[34 ap] Cos[35 ap] + r^69 Sin[34 ap] Sin[35 ap]])
     (1 - r^35 Cos[35 ap] - r^36 Cos[36 ap] + r^71 Cos[35 ap] Cos[36 ap] +
       r^71 Sin[35 ap] Sin[36 ap] + i (-r^35 Sin[35 ap] + r^71 Cos[36 ap] Sin[35 ap] +
          r^36 Sin[36 ap] - r^71 Cos[35 ap] Sin[36 ap]))]],
 Abs[Arg[(1 - i Conjugate[-r^35 Sin[35 ap] + r^71 Cos[36 ap] Sin[35 ap] +
          r^36 Sin[36 ap] - r^71 Cos[35 ap] Sin[36 ap]] + Conjugate[-r^35 Cos[35 ap] -
         r^36 Cos[36 ap] + r^71 Cos[35 ap] Cos[36 ap] + r^71 Sin[35 ap] Sin[36 ap]])
```

```
    (1 - r^36 Cos[36 ap] - r^37 Cos[37 ap] + r^73 Cos[36 ap] Cos[37 ap] +
      r^73 Sin[36 ap] Sin[37 ap] + ⅈ (-r^36 Sin[36 ap] + r^73 Cos[37 ap] Sin[36 ap] +
         r^37 Sin[37 ap] - r^73 Cos[36 ap] Sin[37 ap]))]], 0.0171211,
 Abs[Arg[(1 - ⅈ Conjugate[-r^37 Sin[37 ap] + r^75 Cos[38 ap] Sin[37 ap] +
         r^38 Sin[38 ap] - r^75 Cos[37 ap] Sin[38 ap]] + Conjugate[-r^37 Cos[37 ap] -
        r^38 Cos[38 ap] + r^75 Cos[37 ap] Cos[38 ap] + r^75 Sin[37 ap] Sin[38 ap]])
    (1 - r^38 Cos[38 ap] - r^39 Cos[39 ap] + r^77 Cos[38 ap] Cos[39 ap] +
      r^77 Sin[38 ap] Sin[39 ap] + ⅈ (-r^38 Sin[38 ap] + r^77 Cos[39 ap] Sin[38 ap] +
         r^39 Sin[39 ap] - r^77 Cos[38 ap] Sin[39 ap]))]], 0.0141154,
 Abs[Arg[(1 - ⅈ Conjugate[-r^39 Sin[39 ap] + r^79 Cos[40 ap] Sin[39 ap] +
         r^40 Sin[40 ap] - r^79 Cos[39 ap] Sin[40 ap]] + Conjugate[-r^39 Cos[39 ap] -
        r^40 Cos[40 ap] + r^79 Cos[39 ap] Cos[40 ap] + r^79 Sin[39 ap] Sin[40 ap]])
    (1 - r^40 Cos[40 ap] - r^41 Cos[41 ap] + r^81 Cos[40 ap] Cos[41 ap] +
      r^81 Sin[40 ap] Sin[41 ap] + ⅈ (-r^40 Sin[40 ap] + r^81 Cos[41 ap] Sin[40 ap] +
         r^41 Sin[41 ap] - r^81 Cos[40 ap] Sin[41 ap]))]],
 0.013153, 0.0133509, 0.013535, 0.0137063,
 0.00524113,
 0.011747,
 Abs[
  Arg[
   (1 - ⅈ Conjugate[-r^46 Sin[46 ap] + r^93 Cos[47 ap] Sin[46 ap] + r^47 Sin[47 ap] -
         r^93 Cos[46 ap] Sin[47 ap]] + Conjugate[-r^46 Cos[46 ap] -
        r^47 Cos[47 ap] + r^93 Cos[46 ap] Cos[47 ap] + r^93 Sin[46 ap] Sin[47 ap]])
    (1 - r^47 Cos[47 ap] - r^48 Cos[48 ap] + r^95 Cos[47 ap] Cos[48 ap] +
      r^95 Sin[47 ap] Sin[48 ap] + ⅈ (-r^47 Sin[47 ap] + r^95 Cos[48 ap] Sin[47 ap] +
         r^48 Sin[48 ap] - r^95 Cos[47 ap] Sin[48 ap]))]], 0.00265065,
 Abs[Arg[(1 - ⅈ Conjugate[-r^48 Sin[48 ap] + r^97 Cos[49 ap] Sin[48 ap] +
         r^49 Sin[49 ap] - r^97 Cos[48 ap] Sin[49 ap]] + Conjugate[-r^48 Cos[48 ap] -
        r^49 Cos[49 ap] + r^97 Cos[48 ap] Cos[49 ap] + r^97 Sin[48 ap] Sin[49 ap]])
    (1 - r^49 Cos[49 ap] - r^50 Cos[50 ap] + r^99 Cos[49 ap] Cos[50 ap] +
      r^99 Sin[49 ap] Sin[50 ap] + ⅈ (-r^49 Sin[49 ap] + r^99 Cos[50 ap] Sin[49 ap] +
         r^50 Sin[50 ap] - r^99 Cos[49 ap] Sin[50 ap]))]]}
```

```
Max @@ Select[Out[8], NumberQ]
```

```
0.192645
```

```
Select[Range[2, 50], Not[NumberQ[Out[8][[# - 1]]]] &]
```

```
{12, 19, 27, 31, 36, 37, 39, 41, 48, 50}
```

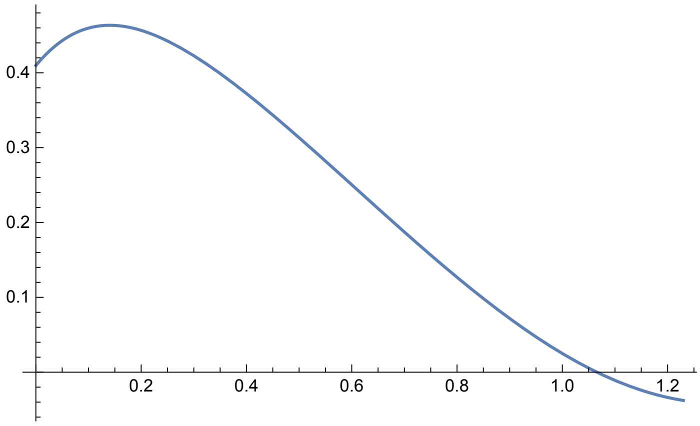
```
{ArMax[0.24, 0.72, #], ArMax[0.72, ArcCos[1/3], #]} & /@
 {12, 19, 27, 31, 36, 37, 39, 41, 48, 50}
```

```
{{0.0717782, 0.016317}, {0.0664011, 0.0030907}, {0.0625677, 0.000216905},
 {0.0171139, 0.0000562068}, {0.0171445, 0.0000191567},
 {0.0171335, 0.0000200041}, {0.0124667, 3.7024×10^-6}, {0.0170703, 6.06762×10^-6},
 {0.0169317, 3.29248×10^-7}, {0.0143996, 4.87262×10^-7}}
```

```
Max @@ Select[Union[Out[19]] // Flatten, NumberQ]
```

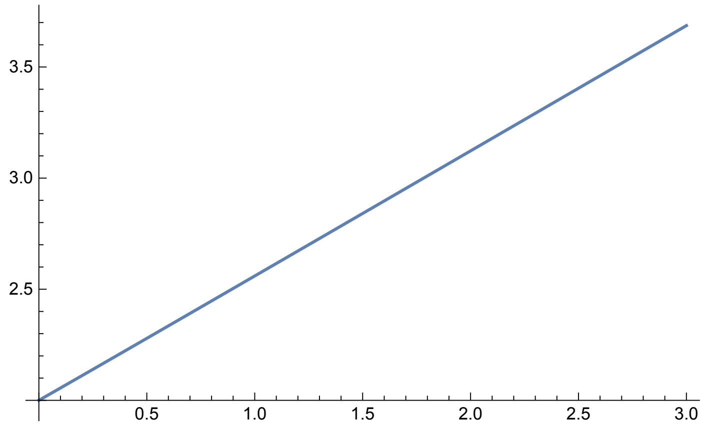
0.0717782

```
(* end of proof of arg( ap_n/ap_n-1) lemma *)


(* deforming the calc of delta *)

(* THIS IS sin delta old and wrong *)

sind[ap_, l_] :=
 Sin[2 ap + 2 ArcSin[1 / 3] + ArcTan[Sin[ap] Cos[ap] / (1 + Cos[ap]^2)]] /
  Sqrt[4 (4 + Tan[ap]^2) l^2 + 1 / Cos[ap]^2 - 4 Sqrt[4 + Tan[ap]^2] l
     Cos[2 ap + ArcSin[1 / 3] + 2 ArcTan[Sin[ap] Cos[ap] / (1 + Cos[ap]^2)]] / Cos[ap]]


Plot[sind[ap, 0.61], {ap, 0, 1.23}]
```



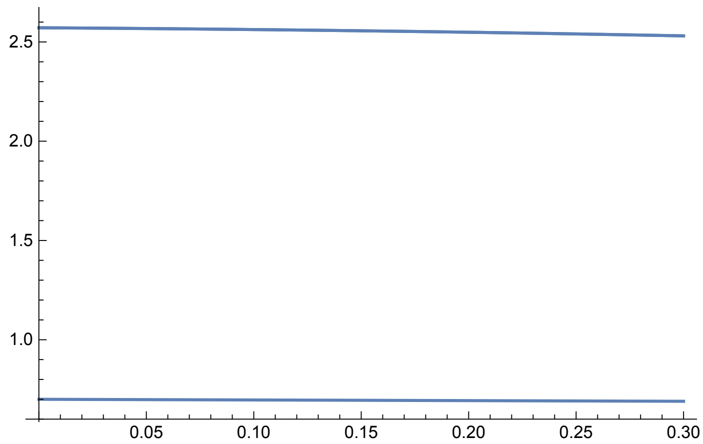

```
(* max of ax^2+bx+c over x\ge d *)

MaxQ[a_, b_, c_, d_] := If[-b / (2 a) < d, a d^2 + b d + c, -b^2 / (4 a) + c] /; a < 0

Primeap[ap_] := ArcCos[2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]]

(* this is delta in paper *)

beta[l_, ap_] := ArcSin[sind[Primeap[ap], l]] + ArcTan[Tan[Primeap[ap]] / 2]


Gm[ap_, l_, k_] := 2 Sqrt[k^2 + 2 Cos[ap] k + 1] - k / l;
```

```
Plot[Gm[ap, l, k] //. {l -> 0.7 - ap / 30, ap → 0.1}, {k, 0, 3}]
```



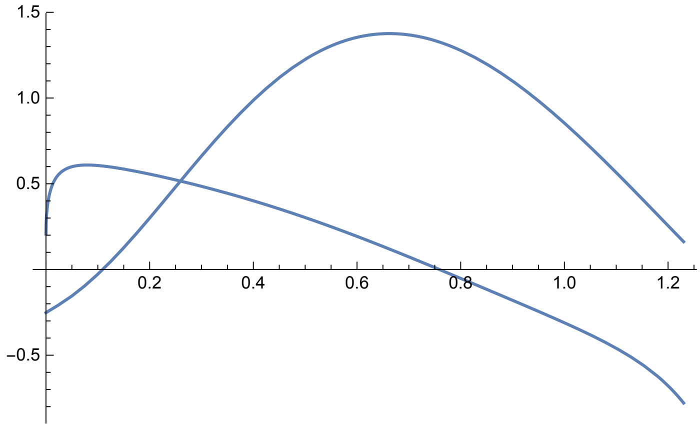

```
Phi[ap_, l_] := Gm[ap, l, Max[Min[2 - Cos[ap] + Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3],
    -Cos[ap] + Sin[ap] / Sqrt[4 l^2 - 1]], 1 / Cos[ap]]]
```

```
Plot[{2 Sqrt[4 + Tan[ap]^2] - 1 / l / Cos[ap], 1 * Phi[ap, l], l} /. {l -> 0.7 - ap / 30},
 {ap, 0, 0.3}]
```

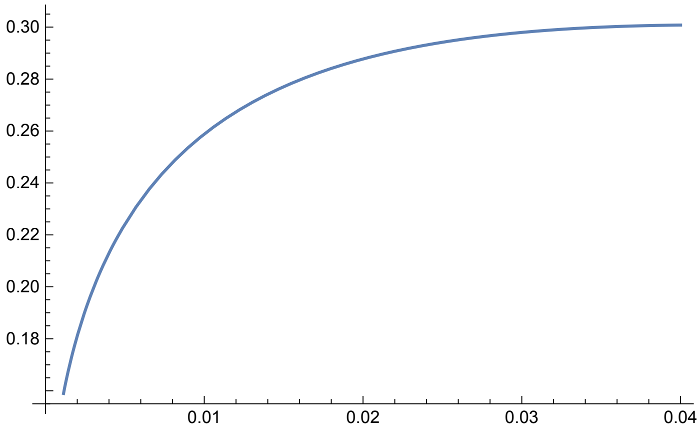



```
Fct[a_, l_, ap_] := {MaxQ[l^2 Cos[ap]^2 - a^2,
    2 l^2 Cos[ap]^2 / a - 2 a Cos[3 * ArcSin[1 / 3] + 2 ap] , (l Cos[ap] / a)^2 - 1, l],
  MaxQ[l^2 Cos[ap]^2 - a^2, 2 a - 2 l^2 Cos[ap]^2
      (Sqrt[8 / 9] * Cos[beta[l, ap]] - Sqrt[1 / 9] * Sin[beta[l, ap]]) / (a),
   (l Cos[ap] / a)^2 - 1, Phi[ap, l]]}
```

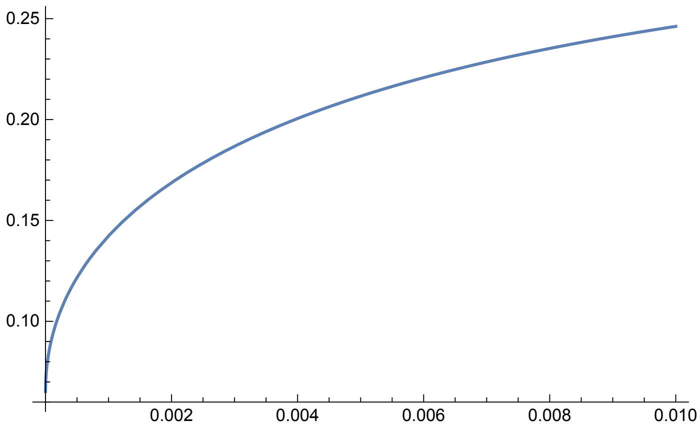
```
Plot[Fct[0.85, 0.7 - ap / 30, ap], {ap, 0.00, 1.23}]
```



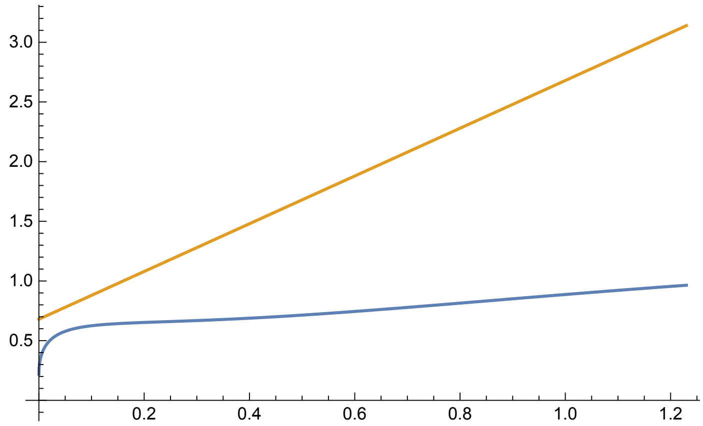

```
Plot[Fct[0.85, 0.59 - ap / 30, ap][[2]], {ap, 0.00, 0.04}]
```




```
Plot[Fct[0.85, 0.59 - ap / 30, ap][[2]], {ap, 0.00, 0.01}]
```

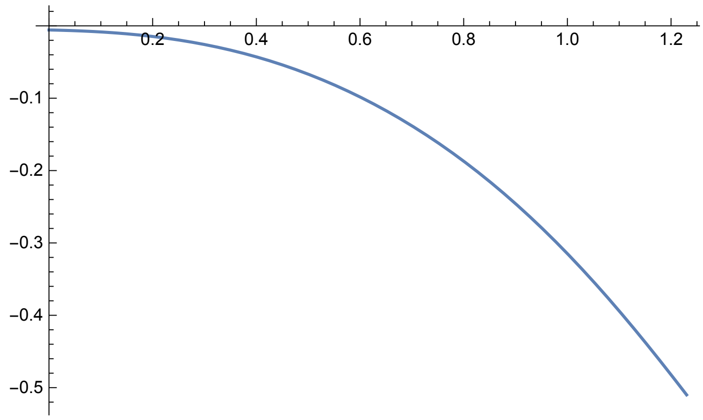


```
Plot[{beta[0.61 - ap / 30, ap], 2 ArcSin[1 / 3] + 2 ap}, {ap, 0, 1.23}]
```




```
H[a_, l_, al_] :=
 Sin[(ArcSin[1 / 3] + al) / 2] / a / (Sqrt[(2 Sqrt[4 + Tan[al] ^2] - 1 / (l )) ^2 + 1 / a^2 -
        2 / a * (2 Sqrt[4 + Tan[al] ^2] - 1 / (l )) Cos[(ArcSin[1 / 3] + al) / 2]]) -
  Sin[(ArcSin[1 / 3] + al) / 2]
```

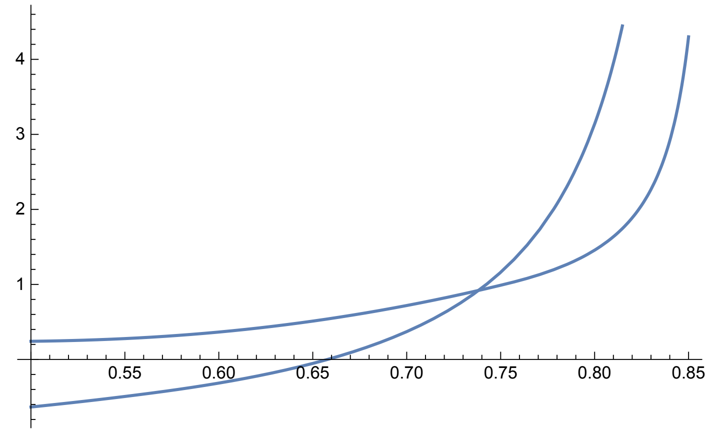
```
Plot[H[0.85, 0.61 - ap / 13, ap], {ap, 0, 1.23}]
```



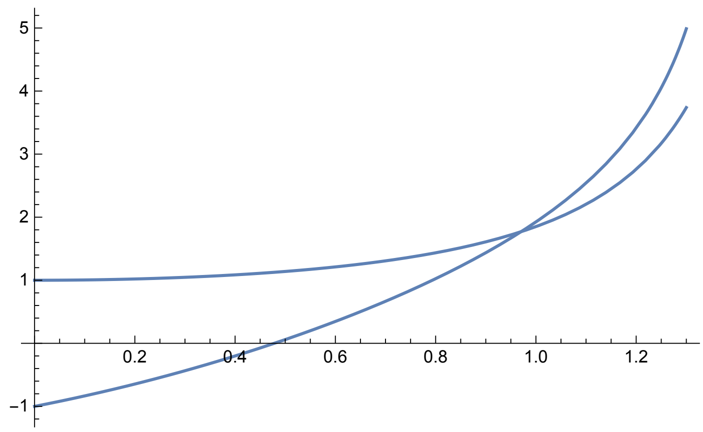

```
Fct[0.85, 0.61, 0]
```

```
{-0.624456, -0.00353456}
```

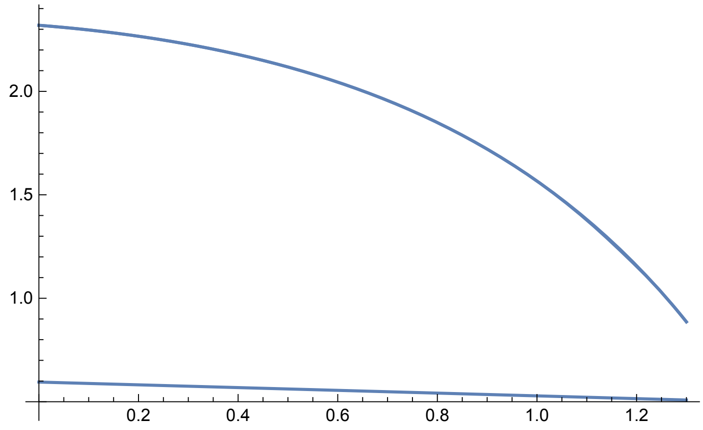
```
Plot[Fct[0.85, l, 0.2], {l, 0.5, 0.85}]
```



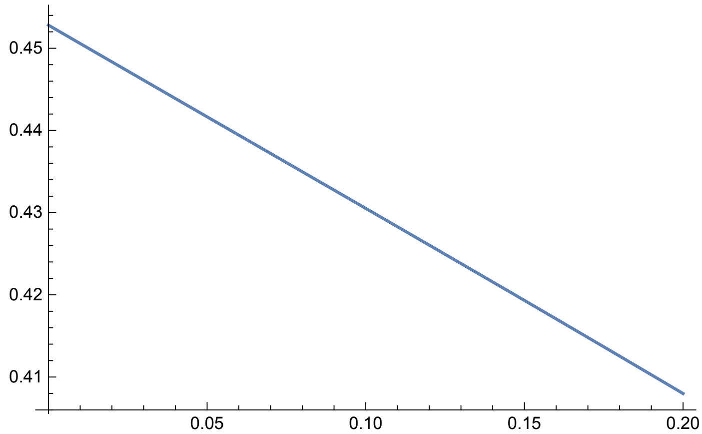

```
(* end of old and wrong stuff *)

(* this is the current and most reasonable approach *)

(* the test for small alpha w/t lemma 2.3 *)
(* for the general proof Phi \ge l *)
(* below is ok for ap≤0.1 *)
Plot[{-Cos[ap] + Sin[ap] / Sqrt[4 l^2 - 1], 1 / Cos[ap]} /. {l -> 0.595 - ap / 15},
 {ap, 0, 1.3}]
```



```
Plot[{2 Sqrt[4 + Tan[ap]^2] - 1 / l / Cos[ap], Phi[ap, l], l} /. {l → 0.595 - ap / 15},
 {ap, 0, 1.3}]
```

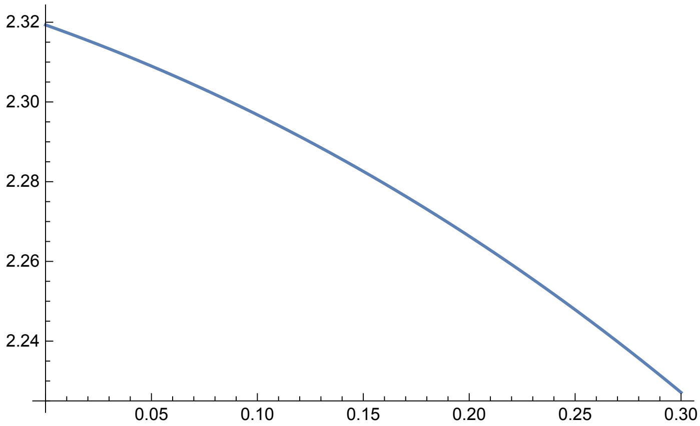



```
(* this is stuff added 8/27 and must be tested Case 1.1.1 *)
Plot[Cos[ArcSin[1 / 3] + ap] - l^2 Cos[ap]^2 / a^2 //.
  {a → 0.85, l → 0.595 - ap / 15}, {ap, 0, 0.2}]
```

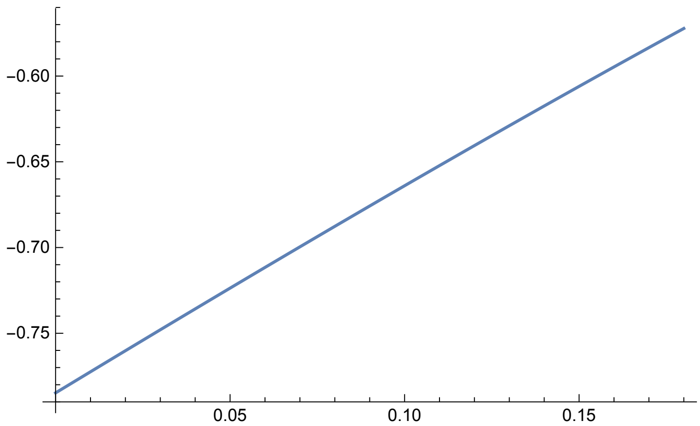



```
(* continue old suff *)

sind[ap_, l_] :=
 Sin[ 2 ArcSin[1 / 3] + ArcTan[Sin[ap] Cos[ap] / (1 + Cos[ap]^2)] + 2 ap] /
   Sqrt[4 (4 + Tan[ap]^2) l^2 + 1 / Cos[ap]^2 -
     4 Sqrt[4 + Tan[ap]^2] l Cos[2 ArcSin[1 / 3] +
         ArcTan[Sin[ap] Cos[ap] / (1 + Cos[ap]^2)] + 2 ap] / Cos[ap]] / Cos[ap]

Primeap[ap_] := ArcCos[2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]]

(* delta of paper start of section 3 *)

beta[l_, ap_] := ArcSin[sind[Primeap[ap], l]] + ArcTan[Tan[Primeap[ap]] / 2]

MaxQ[a_, b_, c_, d_] := If[-b / (2 a) < d, a d^2 + b d + c, -b^2 / (4 a) + c] /; a < 0

Gm[ap_, l_, k_] := 2 Sqrt[k^2 + 2 Cos[ap] k + 1] - k / l;
```

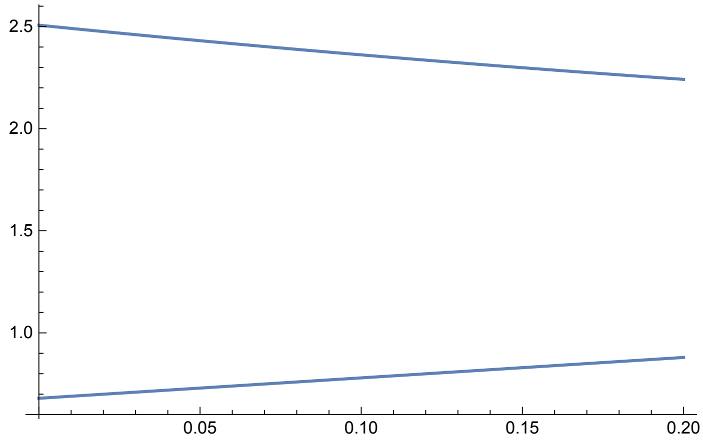
```
Phi[ap_, l_] := Gm[ap, l, Max[Min[2 - Cos[ap] + Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3],
    -Cos[ap] + Sin[ap] / Sqrt[4 l^2 - 1]], 1 / Cos[ap]]]

Plot[Phi[ap, 0.595 - ap / 15], {ap, 0, 0.3}]
```



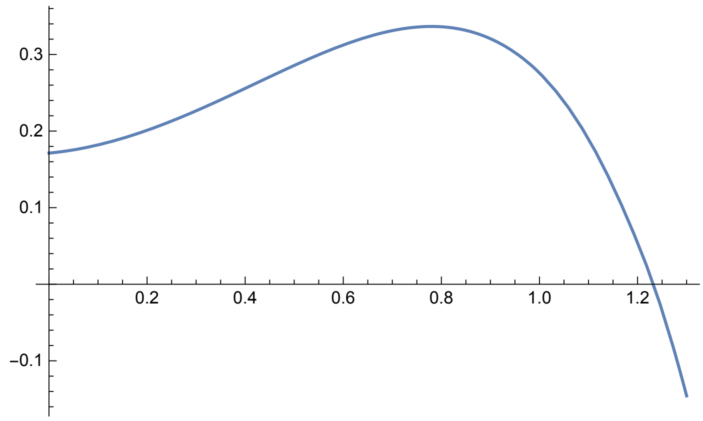

```
(* more new stuff to test *)

(* Test in Case 1.1.1 what was before first part *)

Test111[ap_, l_, a_] := MaxQ[l^2 Cos[ap]^2 - a^2,
   2 a (-Cos[2 ArcSin[1 / 3] + ap] * (Cos[ArcSin[1 / 3] + ap] - l^2 Cos[ap]^2 / a^2) +
      Sin[2 ArcSin[1 / 3] + ap] * Sin[ArcSin[1 / 3] + ap]), (l Cos[ap] / a)^2 - 1, l];

(* MaxQ[l^2Cos[ap]^2-a^2,2l^2Cos[ap]^2/a-2a Cos[3*ArcSin[1/3]+2ap] ,
 (l Cos[ap]/a)^2-1,l], old 1st part Case 1.1.1 *)

Plot[Test111[ap, 0.595 - ap / 15, 0.85], {ap, 0, 0.18}]
```



```
Plot[{Pi + ArcTan[
     - Sin[ArcSin[1/3] + ap] / (Cos[ArcSin[1/3] + ap] - l^2 Cos[ap]^2/a^2)],
   2 ArcSin[1/3] + ap} //. {a → 0.85, l → 0.595 - ap/15}, {ap, 0, 0.2}]
```

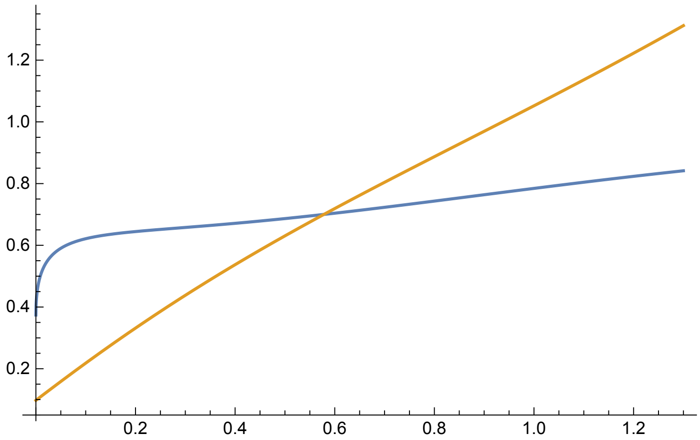



```
(* Case 1.1.3.1 *)

(* test this - ok *)

Plot[1.1 Phi[ap, l] - 1 - l Phi[ap, l] * Cos[ap] /. {l → 0.595 - ap/15}, {ap, 0, 1.3}]
```

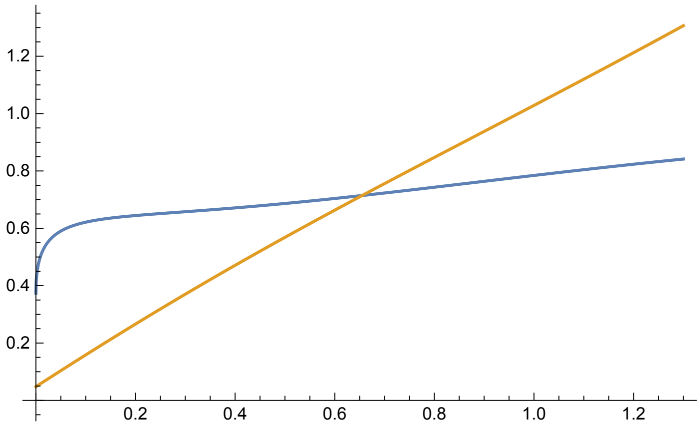



```
Eta3[ap_, l_, a_, b_] :=
 ArcTan[l^2 Cos[ap]^2
   (Cos[ap] + Sqrt[8] Sin[ap]) / (3 a b - (Sqrt[8] Cos[ap] - Sin[ap]) l^2 Cos[ap]^2)]

(* test in case 1.1.3.2.1, 1.1.3.2.2 *)

Test58[ap_, l_, a_, b_] := MaxQ[l^2 Cos[ap]^2 - a^2,
   2 * (a Cos[Min[beta[l, ap] - ap, Eta3[ap, l, a, b]]] -
      Sqrt[8] l^2 Cos[ap]^2/(3 b) * Cos[Min[beta[l, ap], Eta3[ap, l, a, b] + ap]] +
      l^2 Cos[ap]^2/(3 b) * Sin[Min[beta[l, ap], Eta3[ap, l, a, b] + ap]]),
   (l Cos[ap] / b)^2 - 1, Phi[ap, l]];
```

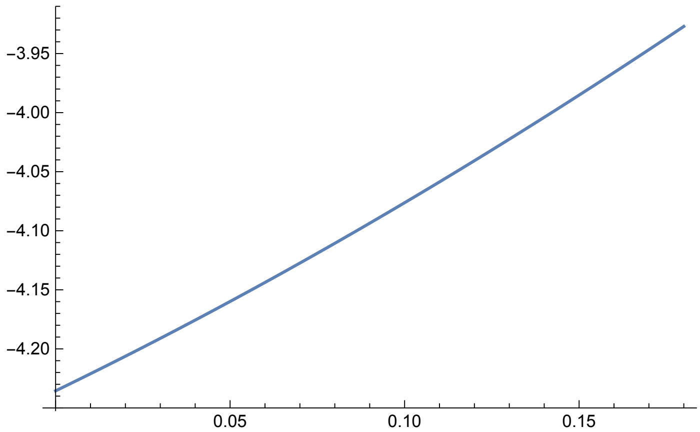
```
Plot[{beta[0.595 - ap/15, ap], Eta3[ap, 0.595 - ap/15, 1.4, 1.1] + ap}, {ap, 0, 1.3}]
```




```
Tst11321[ap_, l_] := Test58[ap, l, 1.4, 1.1];
Plot[{beta[0.595 - ap/15, ap], Eta3[ap, 0.595 - ap/15, 1.4, 2] + ap}, {ap, 0, 1.3}]
```

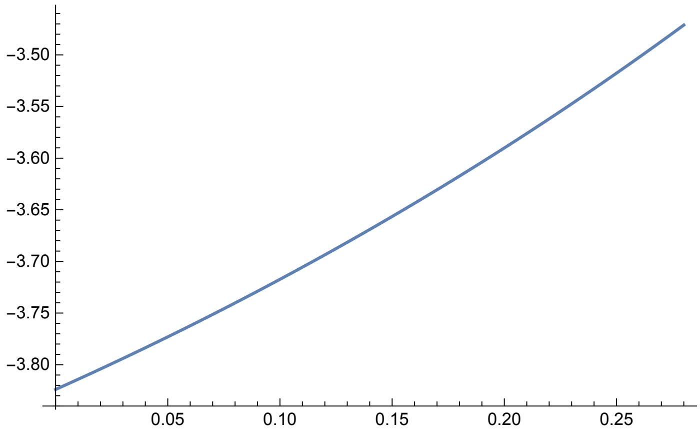



```
Plot[Tst11321[ap, 0.595 - ap/15], {ap, 0, 0.18}]
```

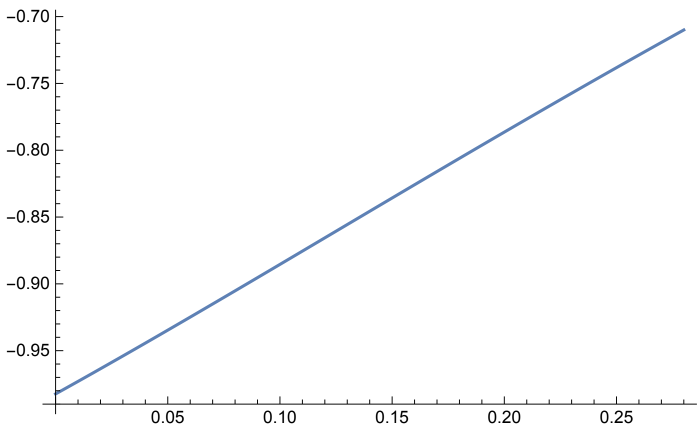



```
Tst11322[ap_, l_] := Test58[ap, l, 1.4, 2];
```

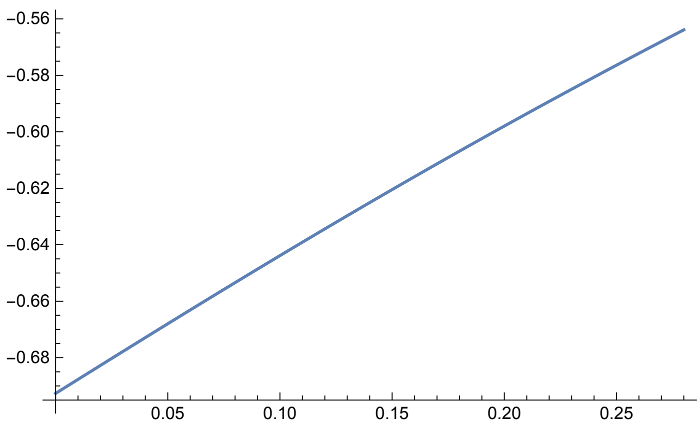
```
Plot[Tst11322[ap, 0.595 - ap/15], {ap, 0, 0.28}]
```




```
(* Case 1.1.3.3.1 *)

Tst11331[ap_, l_] := Test58[ap, l, 1, 1];

Plot[Tst11331[ap, 0.595 - ap/15], {ap, 0, 0.28}]
```

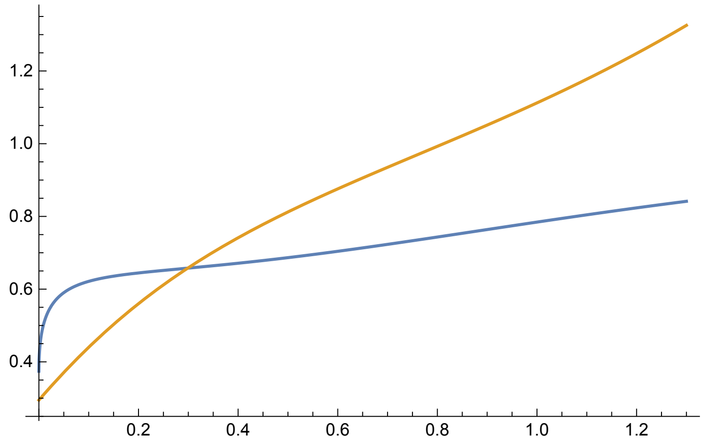



```
(* Case 1.1.3.3.2 *)

Tst11332[ap_, l_] := Test58[ap, l, 1, 1.5];

Plot[Tst11332[ap, 0.595 - ap/15], {ap, 0, 0.28}]
```

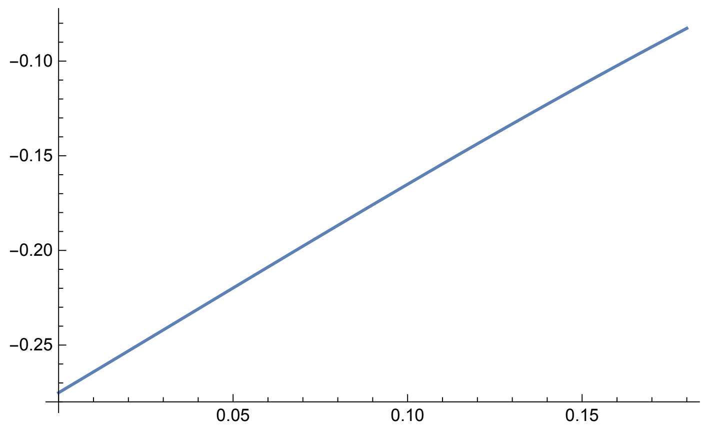



```
(* Case 1.1.3.4.1 *)
```

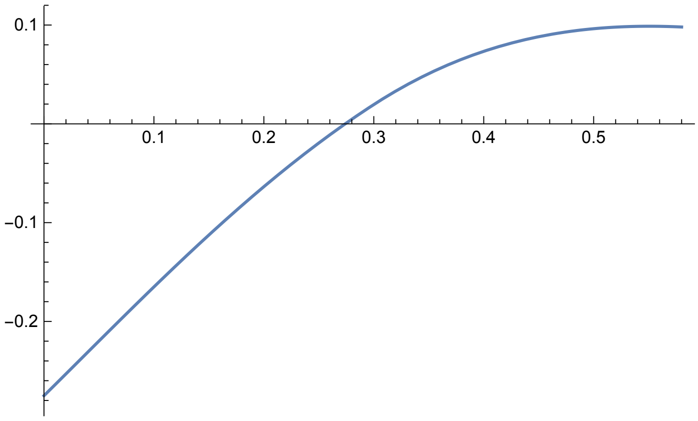
```
(* in paper Eta is the tangent of that !*)

Eta[ap_, l_, a_] := ArcTan[l^2 Cos[ap]^2
    (Cos[ap] + Sqrt[8] Sin[ap]) / (3 a^2 - (Sqrt[8] Cos[ap] - Sin[ap]) l^2 Cos[ap]^2)]

Plot[{beta[0.595 - ap / 15, ap], Eta[ap, 0.595 - ap / 15, 0.85] + ap}, {ap, 0, 1.3}]
```



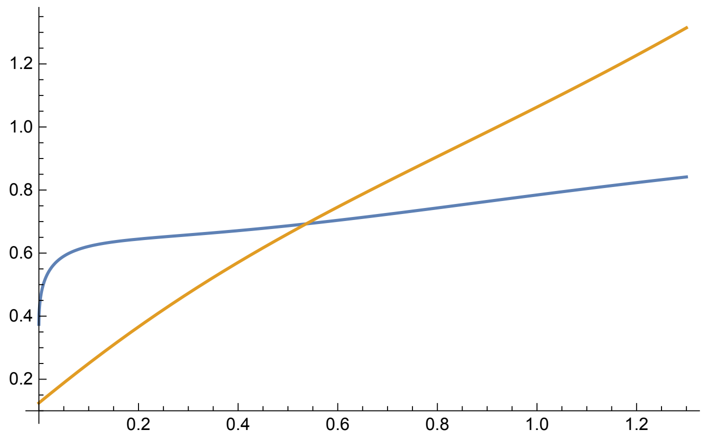

```
Tst11341[ap_, l_, a_] := MaxQ[l^2 Cos[ap]^2 - a^2,
  2 / a * (a^2 Cos[Min[beta[l, ap] - ap, Eta[ap, l, a]]] -
      Sqrt[8] l^2 Cos[ap]^2 / 3 * Cos[Min[beta[l, ap], Eta[ap, l, a] + ap]] +
      l^2 Cos[ap]^2 / 3 * Sin[Min[beta[l, ap], Eta[ap, l, a] + ap]]),
  (l Cos[ap] / a)^2 - 1, Phi[ap, l]]

Plot[Tst11341[ap, 0.595 - ap / 15, 0.85], {ap, 0, 0.18}]
(* Case 1.1.3.4.2 *)
```



```
Plot[Test58[ap, 0.595 - ap / 15, 0.85, 0.85], {ap, 0, 0.58}]
```

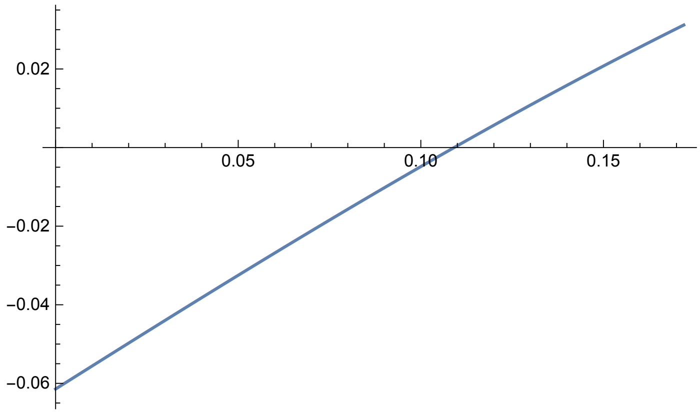



```
Eta2[ap_, l_, a_] := ArcTan[2 l^2 Cos[ap]^2 (Cos[ap] + Sqrt[8] Sin[ap]) /
    (9 a - 2 (Sqrt[8] Cos[ap] - Sin[ap]) l^2 Cos[ap]^2)]
Plot[{beta[0.595 - ap / 15, ap], Eta2[ap, 0.595 - ap / 15, 0.85] + ap}, {ap, 0, 1.3}]
```

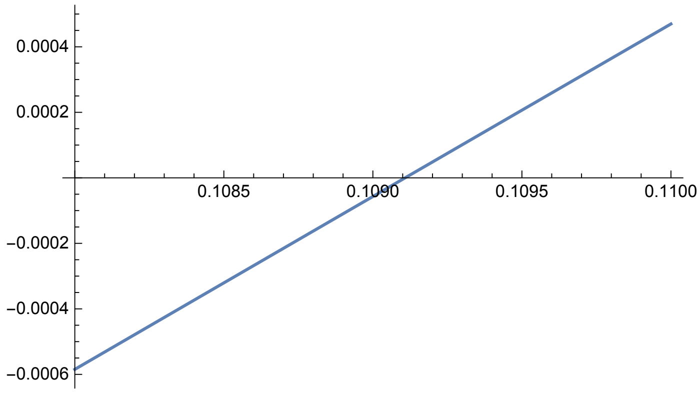



```
(* Test11342[ap_,l_,a_]:=MaxQ[l^2Cos[ap]^2-a^2,
   2*(a Cos[Min[beta[l,ap]-ap,Eta2[ap,l,a]]]-
      4Sqrt[8]l^2Cos[ap]^2/15*Cos[Min[beta[l,ap],Eta2[ap,l,a]+ap]]+
      4l^2Cos[ap]^2/15*Sin[Min[beta[l,ap],Eta2[ap,l,a]+ap]]),
   (l Cos[ap])^2*(4/5)^2-1,Phi[ap,l]]

  old version *)
Test11342[ap_, l_, a_] := Test58[ap, l, a, 1.1718];
(* 3rd arg a - lower bound for |ap n+1|, 4th arg= upper bound for |ap n| *)
```

```
Plot[Test11342[ap, 0.595 - ap / 15, 0.85], {ap, 0, 0.172}]
```

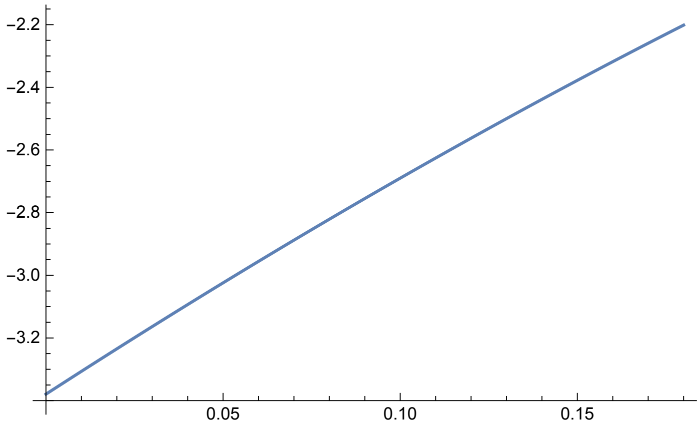



```
Plot[Test11342[ap, 0.595 - ap / 15, 0.85], {ap, 0.108, 0.11}]
```

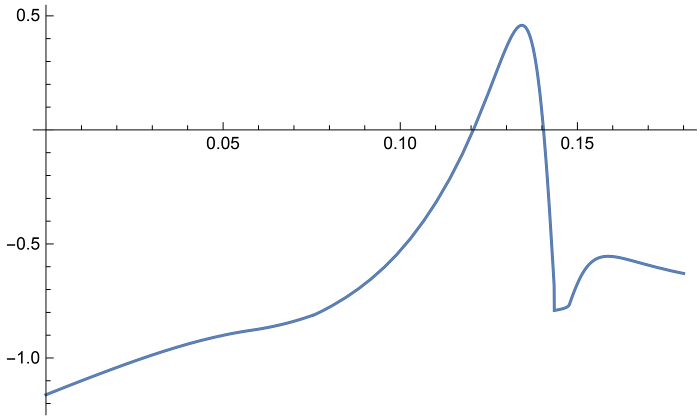



```
(* some old stuff *)
```

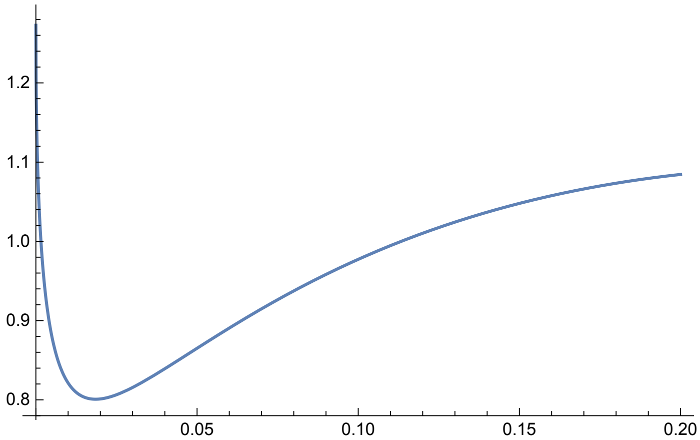
```
Plot[Test58[ap, 0.6 - ap / 30, AlMin[ap, 4], AlMax[ap, 3]], {ap, 0, 0.18}]
```



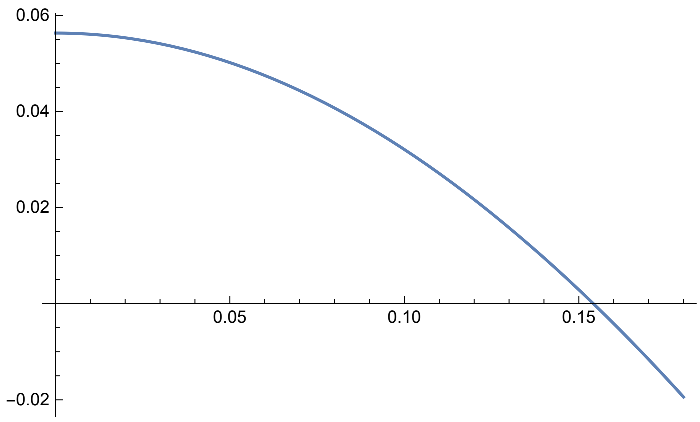
```
Plot[Test58[ap, 0.7 - ap / 30, AlMin[ap, 44], AlMax[ap, 43]], {ap, 0, 0.18}]
```



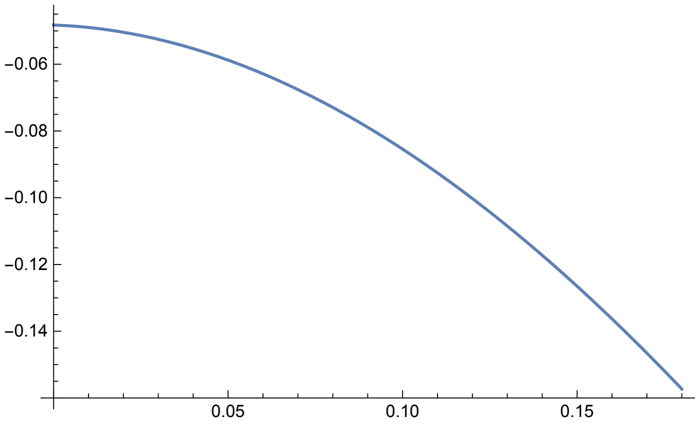

```
Plot[2 * Phi[ap, 0.7 - ap / 30] * Cos[ArcSin[1 / 3] + beta[0.51 - ap / 30, ap]] -
   (2 / 3 + 1 / 0.85), {ap, 0, 0.2}]
(* this is the reason the 11342 test is worse than 11341 *)
```



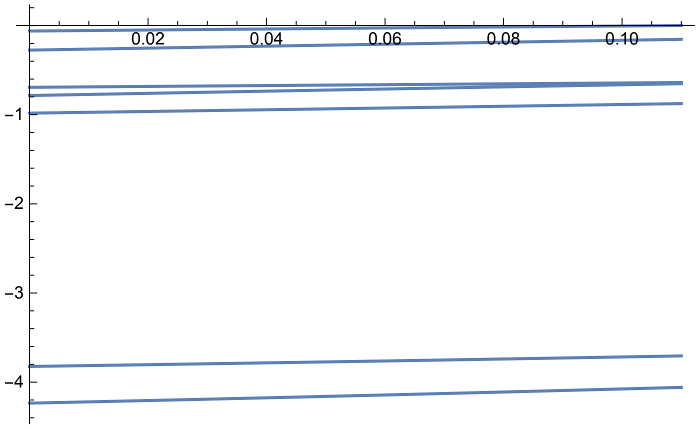

```
Plot[Test58[ap, 0.501, 0.853, 1 / 0.853], {ap, 0, 0.18}]
```



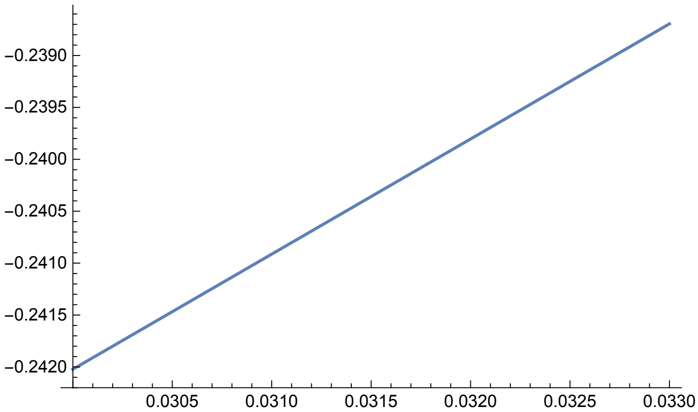
```
Plot[Test58[ap, 0.58 - ap / 30, 0.851, 1.01], {ap, 0, 0.18}]
```



```
(* end old stuff *)

Fct[a_, l_, ap_] := {Test111[ap, l, a], Tst11321[ap, l], Tst11322[ap, l],
  Tst11331[ap, l], Tst11332[ap, l], Tst11341[ap, l, a], Test11342[ap, l, a]}

Plot[Fct[0.85, 0.595 - ap / 15, ap], {ap, 0, 0.11}]
```

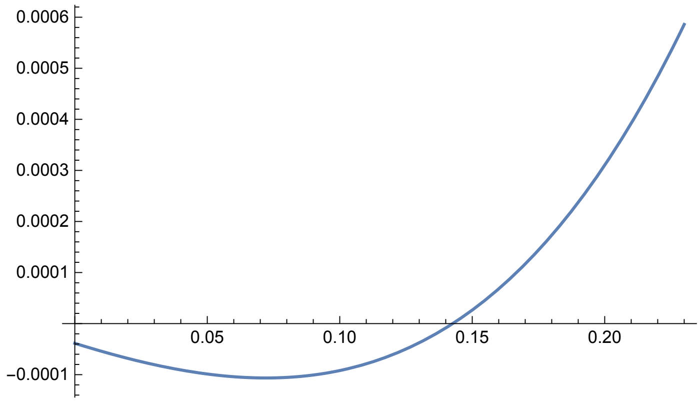


```
Plot[Fct[0.85, 0.595 - ap / 15, ap][[6]], {ap, 0.03, 0.033}]
```

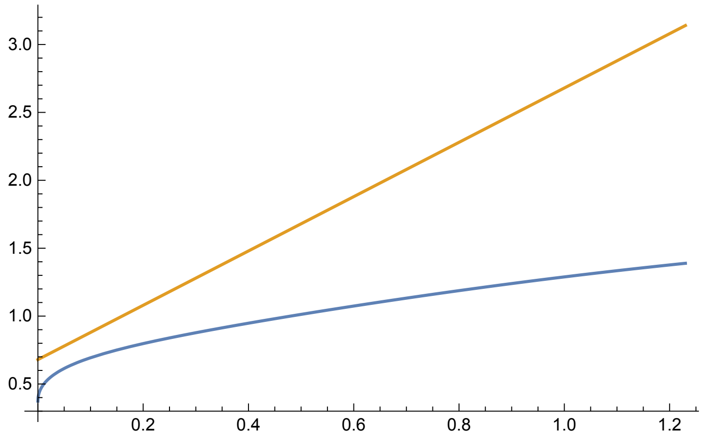


```
H[a_, l_, al_] := Sin[(ArcSin[1 / 3] + al) / 2] / a / Sqrt[Phi[al, l]^2 + 1 / a^2 -
      2 / a * Phi[al, l] * Cos[(ArcSin[1 / 3] + al) / 2]] - Sin[(ArcSin[1 / 3] + al) / 2]
```

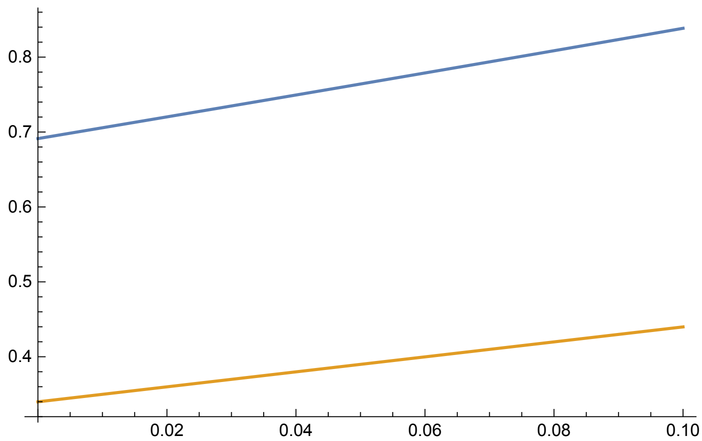
```
Plot[H[0.85, 0.595 - ap / 15, ap], {ap, 0, 0.23}]
```



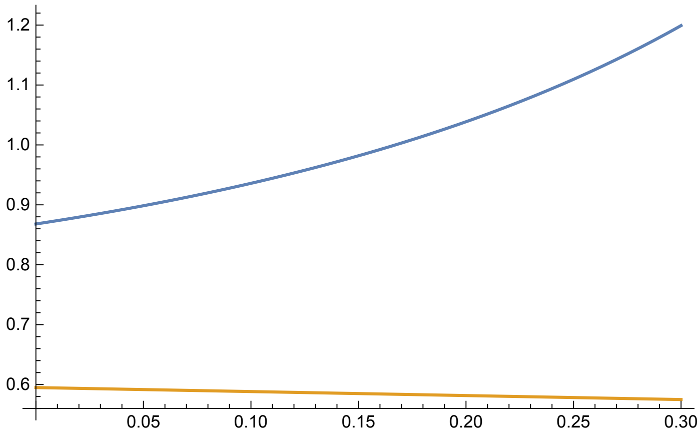

```
Plot[{beta[0.595 - ap / 15, ap], 2 ArcSin[1 / 3] + 2 ap}, {ap, 0, 1.23}]
```



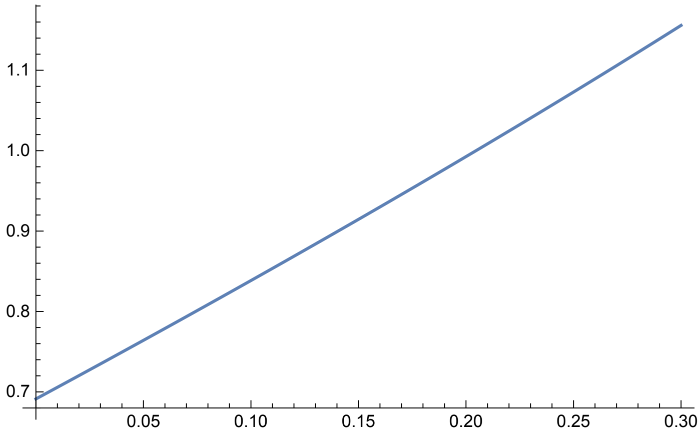

```
(* here comes the angle estimate stuff; first the part for small angles *)

(* p 70 block *)

bt[a_, l_, ap_] := ArcSin[Sin[3 ArcSin[1 / 3] + 2 ap] / a /
   Sqrt[l^2 + (1 / a) ^2 + 2 l / a * Cos[3 ArcSin[1 / 3] + 2 ap]]]

(* this below test was too loose and fails *)

Plot[{bt[0.85, 0.595 - ap / 15, ap], ArcSin[1 / 3] + ap}, {ap, 0, 0.1}]
```



```
(* p. 71-73 block *)

delta[a_, k_, eta_] :=
 If[Cos[eta] < -k a, Pi - ArcSin[Sin[eta] / Sqrt[1 + a^2 k^2 + 2 a k Cos[eta]]],
  ArcSin[Sin[eta] / Sqrt[1 + a^2 k^2 + 2 a k Cos[eta]]]]

(* if k=|\Dl| is above this, then the angle is small enough *)

kmax[a_, ap_] :=
 Sin[eta - (ArcSin[1 / 3] + ap)] / a / Sin[eta] /. {eta → 3 ArcSin[1 / 3] + 3 ap}
(* if k is above this number, then automatically the angle at 0 is ≤
 ArcSin[1/3]+ap ; so if this number <= 1, then done *)

Plot[{kmax[0.85, ap], 0.595 - ap / 15}, {ap, 0, 0.3}]
```

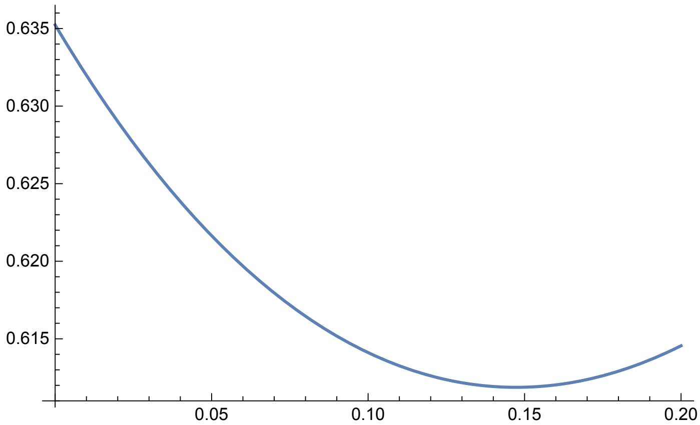



```
(* but you see is above 1, so we have to work *)

fctx[a_, k_, ap_] :=
 1 / a * Sin[delta[a, k, 3 * ArcSin[1 / 3] + 2 ap] - ArcSin[1 / 3] - ap] /
   Sin[delta[a, k, 3 * ArcSin[1 / 3] + 2 ap]]

Plot[delta[0.85, 0.595 - ap / 15, 3 * ArcSin[1 / 3] + 2 ap], {ap, 0, 0.3}]
```



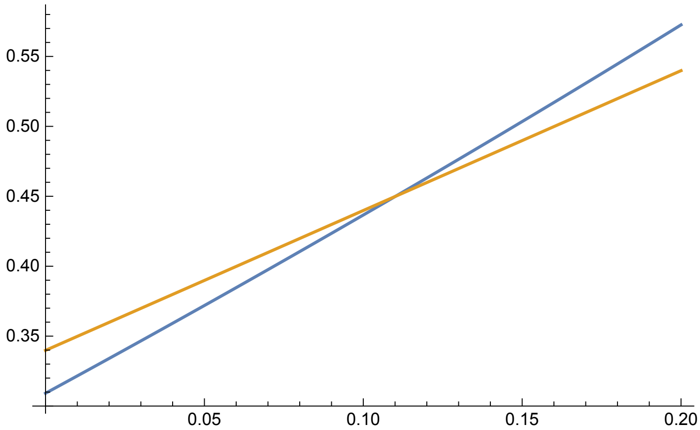

```
(* the lower bounds on the norms of 1/ap_i to prove for
    the small angle proof S2 without lemma 2.3 ; Case 1 *)
```

```
Plot[fctx[0.85, 0.595 - ap / 15, ap], {ap, 0, 0.2}]
```




```
(* Case 2 ok *)
```

```
Plot[{delta[ 2 Sqrt[2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]],
   0.595 - ap / 15, 2 * ArcSin[1 / 3] + 2.5 ap], ArcSin[1 / 3] + ap}, {ap, 0, 0.2}]
```



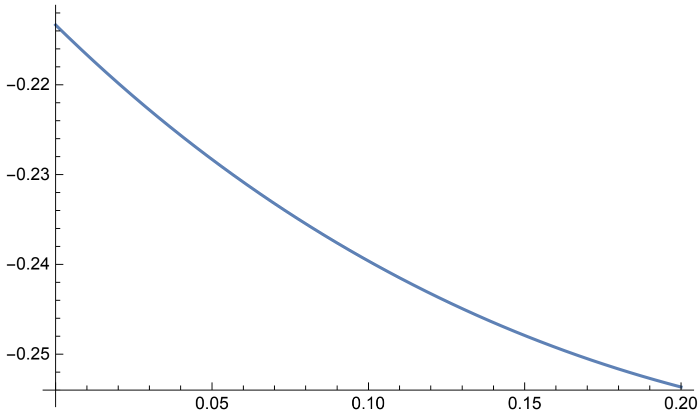

```
FindRoot[delta[ 2 Sqrt[2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]],
   0.595 - ap / 15, 2 * ArcSin[1 / 3] + 2.5 ap] - (ArcSin[1 / 3] + ap), {ap, 0.11}]
```

```
{ap → 0.110115}
```

```
(*Case 3 *)
```

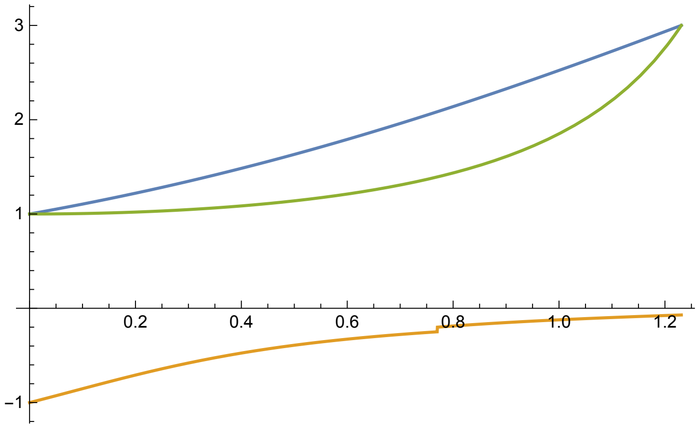
```
fctt[ap_, k_] := fctx[1 / 0.7040644, k, ap] - 1 / 2;
```

```
Plot[fctt[ap, 0.595 - ap / 15], {ap, 0, 0.2}]
```

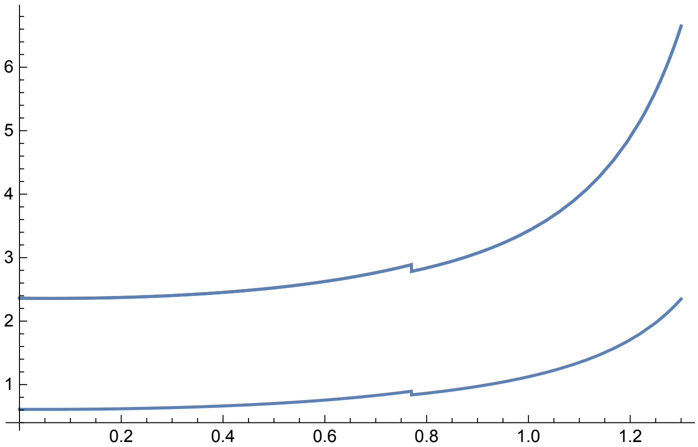



```
(* here the small angle case is done *)


(* this is the general alpha case *)

sind[ap_, l_] := (* sin of delta *)
 Sin[ 2 ArcSin[1 / 3] + ArcTan[Sin[ap] Cos[ap] / (1 + Cos[ap] ^2)] + ap] /
   Sqrt[4 (4 + Tan[ap] ^2) l ^2 + 1 / Cos[ap] ^2 -
     4 Sqrt[4 + Tan[ap] ^2] l Cos[2 ArcSin[1 / 3] +
         ArcTan[Sin[ap] Cos[ap] / (1 + Cos[ap] ^2)] + ap] / Cos[ap]] / Cos[ap]


MaxQ[a_, b_, c_, d_] := If[-b / (2 a) < d, a d^2 + b d + c, -b^2 / (4 a) + c] /; a < 0

lap[ap_] := If[ap > 0.77, (0.61 - ap / 30) / Cos[ap] ^ (11 / 10 - ap / 60),
   (0.61 - ap / 30) / Cos[ap] ^ (13 / 10 - ap / 60)];
(* the function l(alpha) for the general case w/ lemma 2.3 all alpha *)

Gm[ap_, l_, k_] := 2 Sqrt[k^2 + 2 Cos[ap] k + 1] - k / l;

Phi[ap_, l_] := Gm[ap, l, Max[Min[2 - Cos[ap] + Sqrt[Cos[ap] ^2 - 4 Cos[ap] + 3],
    -Cos[ap] + Sin[ap] / Sqrt[4 l^2 - 1]], 1 / Cos[ap]]]

(* for the general proof Phi \ge l *)
```

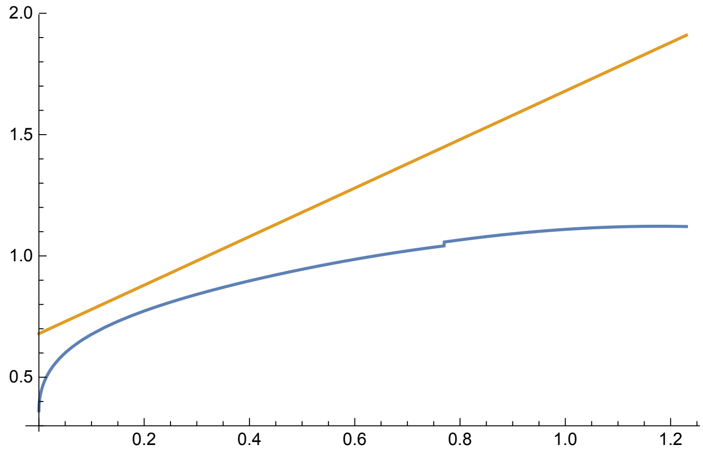
```
Plot[{2 - Cos[ap] + Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3],
  -Cos[ap] + Sin[ap] / Sqrt[4 lap[ap]^2 - 1], 1 / Cos[ap]}, {ap, 0, ArcCos[1 / 3]}]
```



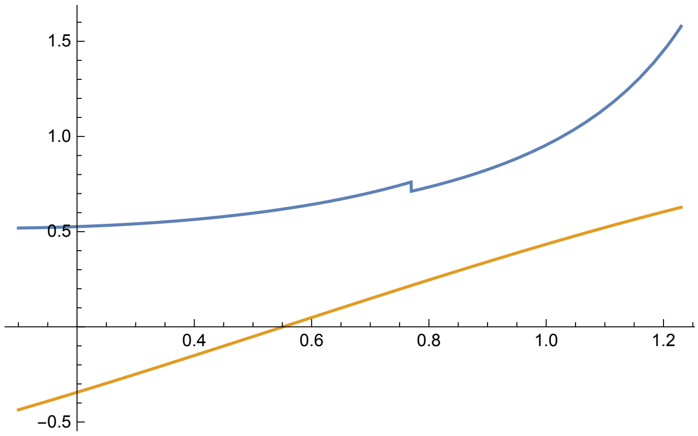

```
Plot[{2 Sqrt[4 + Tan[ap]^2] - 1 / l / Cos[ap], Phi[ap, l], l} /. {l → lap[ap]},
 {ap, 0, 1.3}]
```



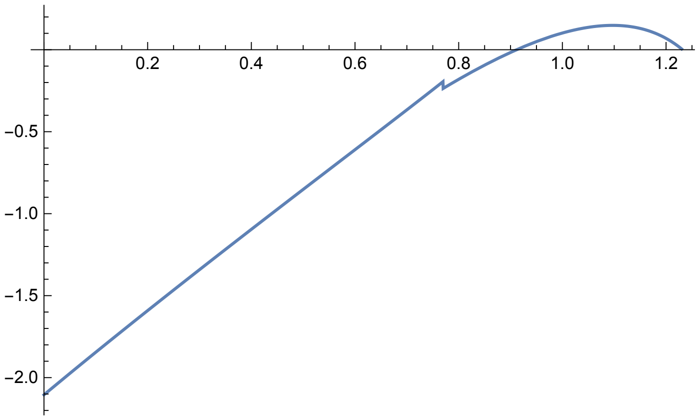

```
Primeap[ap_] := ArcCos[2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]]

beta[l_, ap_] := ArcSin[sind[Primeap[ap], l]] + ArcTan[Tan[Primeap[ap]] / 2]
(* delta in paper *)

Plot[{beta[lap[ap], ap], 2 ArcSin[1 / 3] + ap}, {ap, 0, 1.23}]
```



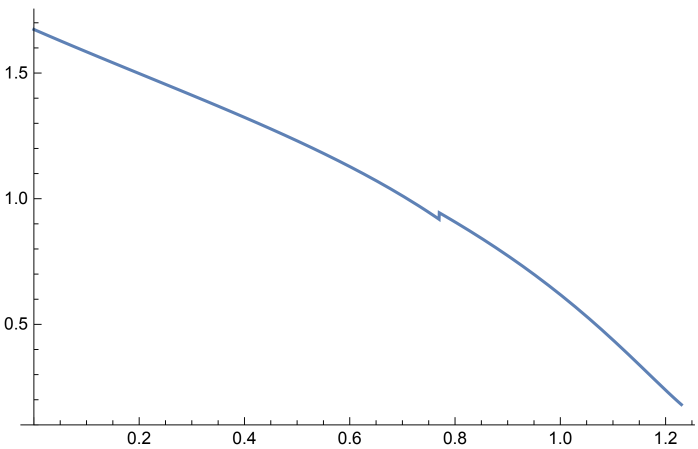
```
(* Case 1.1.1 *)

Plot[{0.85 * lap[ap], -Cos[3 * ArcSin[1 / 3] + ap]}, {ap, 0.1, 1.23}]
```



```
Plot[-Cos[3 * ArcSin[1 / 3] + ap] / (a^2 - l^2 Cos[ap]^2) - l //.
  {a → 0.85, l → lap[ap]}, {ap, 0.0, 1.23}]
```

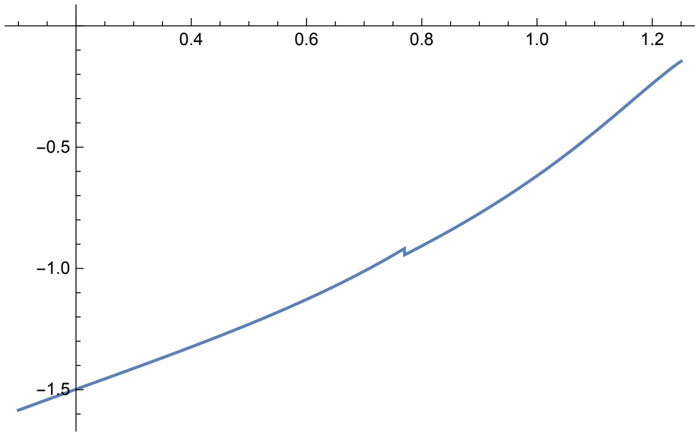


```
(* the above is not alw ≤0, so it's dangerous to test as immed follows *)

Plot[1 + a^2 l^2 + 2 a l Cos[3 ArcSin[1 / 3] + ap] - l^4 Cos[ap]^2 //.
  {a → 0.85, l → lap[ap]}, {ap, 0.0, 1.23}]
```

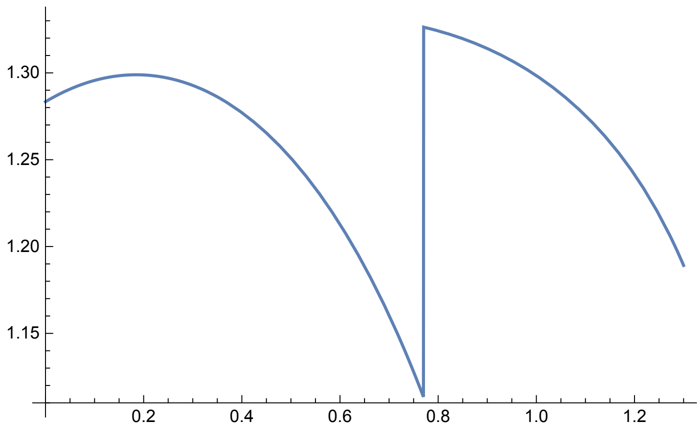


```
(* instead of the prev two plots it's cleaner to test as below *)
```

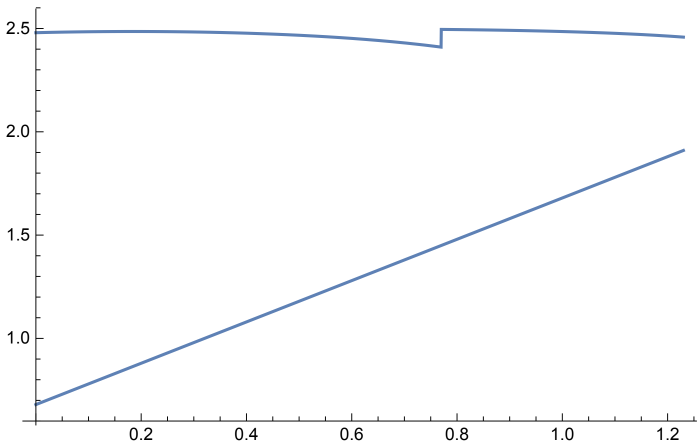
```
Plot[MaxQ[-a^2 + l^2 Cos[ap]^2, -2 a Cos[3 ArcSin[1/3] + ap], -1, 1] //.
  {l → lap[ap], a → 0.85}, {ap, 0.1, 1.25}]
```




```
Plot[Sqrt[8] - 3 l^2 Cos[ap]^2 / a^2 //. {a → 0.85, l → lap[ap]}, {ap, 0.0, 1.3}]
```

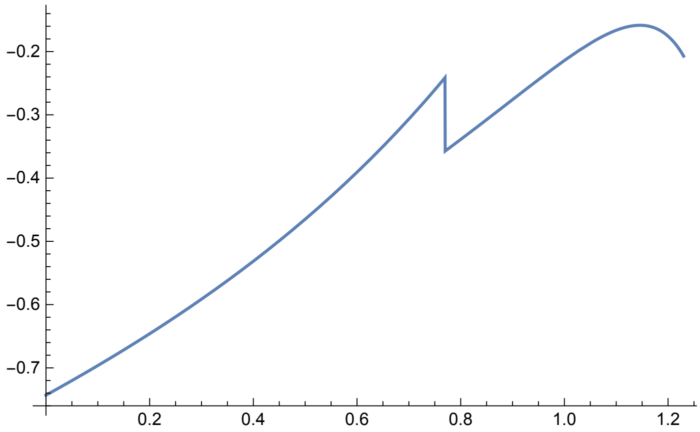



```
Plot[{If[Sqrt[8] - 3 l^2 Cos[ap]^2 / a^2 > 0,
    Pi + ArcTan[-1 / (Sqrt[8] - 3 l^2 Cos[ap]^2 / a^2)],
    ArcTan[-1 / (Sqrt[8] - 3 l^2 Cos[ap]^2 / a^2)]],
   2 ArcSin[1/3] + ap} //. {a → 0.85, l → lap[ap]}, {ap, 0, 1.23}]
```

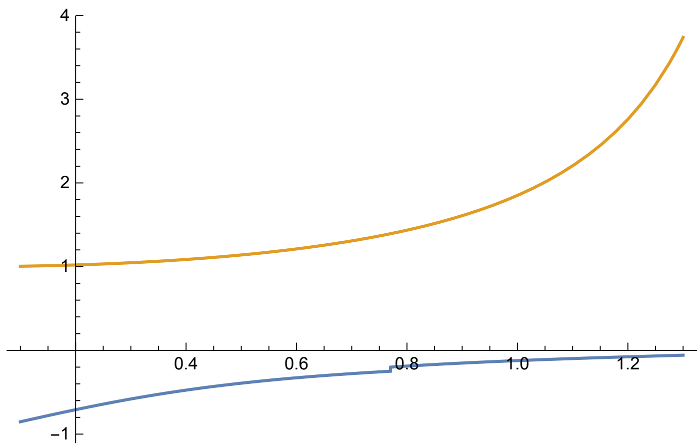



```
(* this min is not needed if we check that 2\as +\ap is smaller *)
```

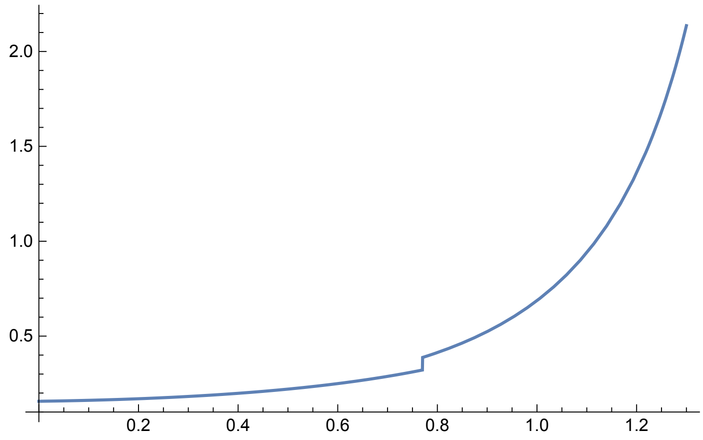
```
Mf[a_, l_, ap_] :=
 Min[If[Sqrt[8] - 3 l^2 / a^2 > 0, Pi + ArcTan[-1 / (Sqrt[8] - 3 l^2 / a^2)],
   ArcTan[-1 / (Sqrt[8] - 3 l^2 / a^2)]], 2 ArcSin[1 / 3] + ap]

(* the old 1st case; is uses that 2\as +\ap is smaller  *)
Test111[ap_, l_, a_] := MaxQ[l^2 Cos[ap]^2 - a^2,
   2 a (-Cos[2 ArcSin[1 / 3] + ap] * (Sqrt[8] / 3 - l^2 Cos[ap]^2 / a^2) +
      Sin[2 ArcSin[1 / 3] + ap] / 3 ), (l Cos[ap] / a)^2 - 1, l];

Plot[Test111[ap, lap[ap], 0.85], {ap, 0, 1.23}]
```



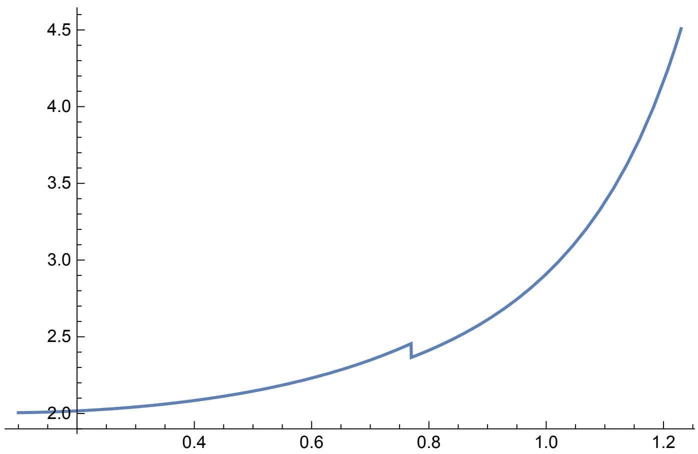
```
(* MaxQ[l^2Cos[ap]^2-a^2,2l^2Cos[ap]^2/a-2a Cos[3*ArcSin[1/3]+ap] ,
 (l Cos[ap]/a)^2-1,l],the old version *)

(* Case 1.1.3 *)

Plot[{-Cos[ap] + Sin[ap] / Sqrt[4 lap[ap]^2 - 1], 1 / Cos[ap]}, {ap, 0.1, 1.3}]
```



```
Plot[1.1 Phi[ap, l] - 1 - l Phi[ap, l] * Cos[ap] /. {l → lap[ap]}, {ap, 0, 1.3}]
```

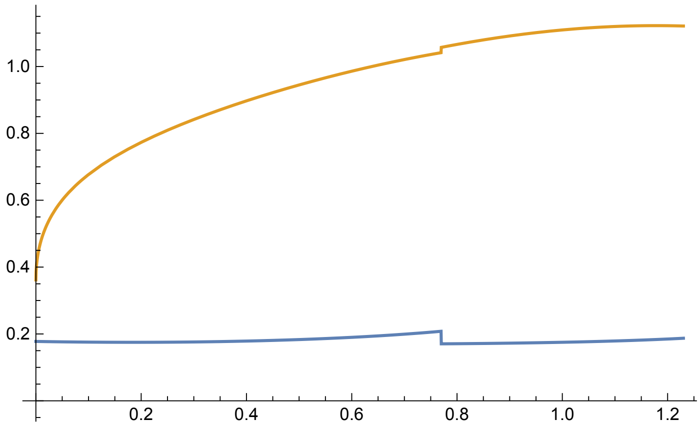



```
(* Case 1.1.3.2.1 *)
Eta3[ap_, l_, a_, b_] := ArcTan[l^2 Cos[ap]^2 / (3 a b - Sqrt[8] l^2 Cos[ap]^2)];

(* need to repeat Phi from small angle part *)

Gm[ap_, l_, k_] := 2 Sqrt[k^2 + 2 Cos[ap] k + 1] - k / l;

Phi[ap_, l_] := Gm[ap, l, Max[Min[2 - Cos[ap] + Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3],
    -Cos[ap] + Sin[ap] / Sqrt[4 l^2 - 1]], 1 / Cos[ap]]]

Plot[0.85 Phi[ap, lap[ap]], {ap, 0.1, 1.23}] (* should be ≥1 *)
```

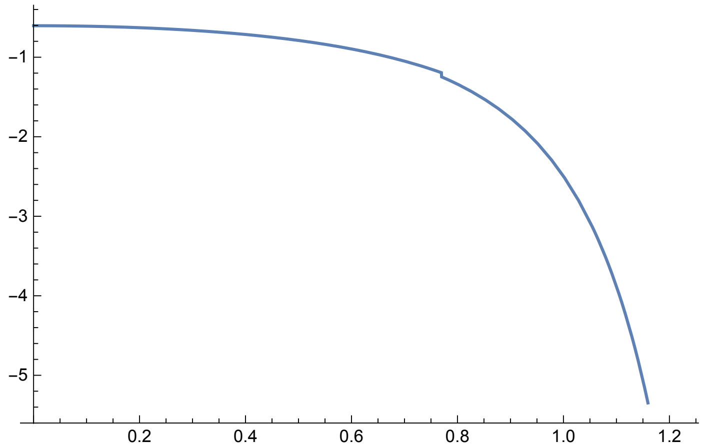



```
Test87[ap_, l_, a_, b_] := MaxQ[l^2 Cos[ap]^2 - a^2,
  2 ((a - l^2 Cos[ap]^2 Sqrt[8] / 3 / b) Cos[Min[beta[l, ap], Eta3[ap, l, a, b]]] +
     l^2 Cos[ap]^2 / (3 b) Sin[Min[beta[l, ap], Eta3[ap, l, a, b]]]),
  (l Cos[ap] / b)^2 - 1, Phi[ap, l]];
```

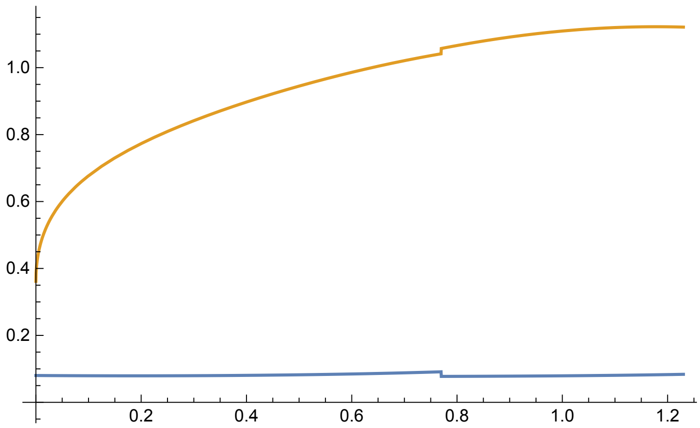
```
Plot[{Eta3[ap, lap[ap], 0.947, 1.1], beta[lap[ap], ap]}, {ap, 0, 1.23}]
```




```
Test11321[ap_, l_] := Test87[ap, l, 0.947, 1.1];

Test11321[0.1, 0.9]

0.933195

Plot[Test11321[ap, lap[ap]], {ap, 0, 1.23}]
```

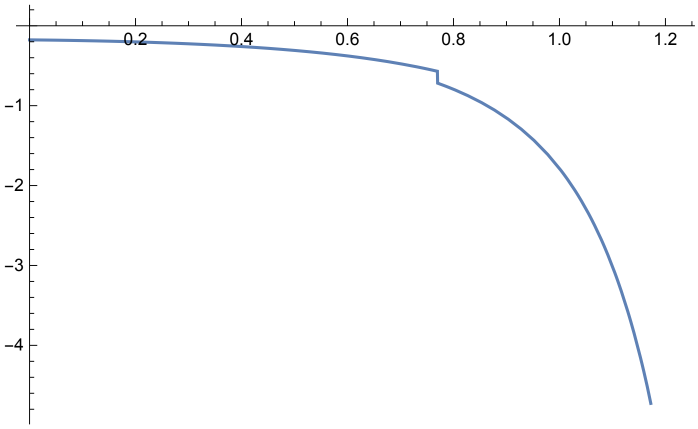



```
(* Case 1.1.3.2.2 *)

Plot[{Eta3[ap, lap[ap], 0.947, 2], beta[lap[ap], ap]}, {ap, 0, 1.23}]
```

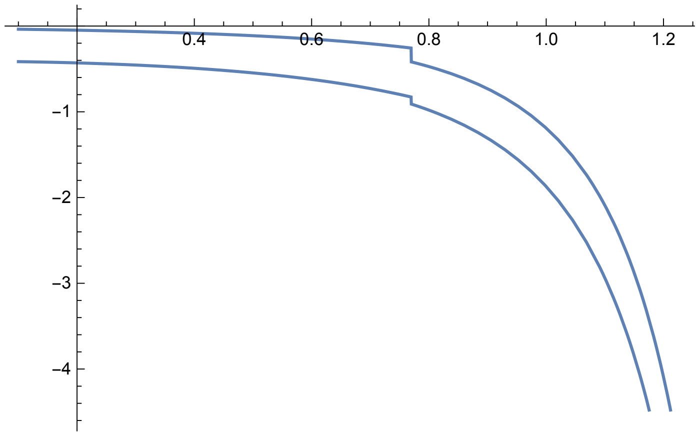



```
Test11322[ap_, l_] := Test87[ap, l, 0.947, 2];
```

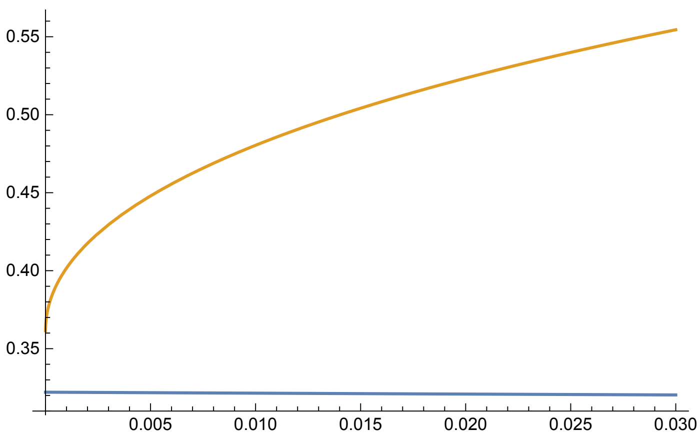
```
Plot[Test11322[ap, lap[ap]], {ap, 0, 1.23}]
```



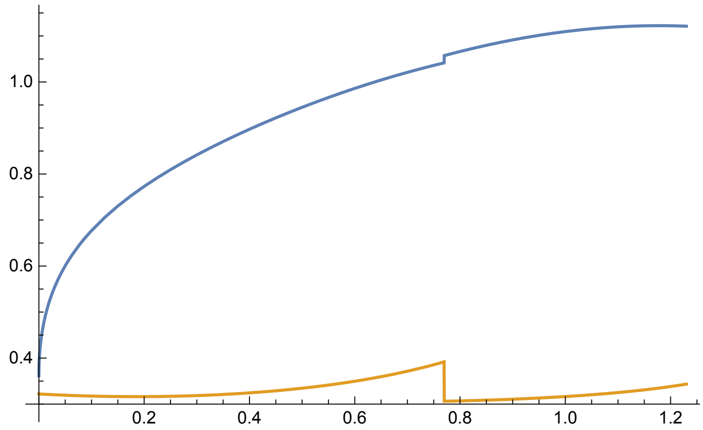

```
(* Case 1.1.3.3 old second case *)

(* m=3,4,5 *)

Test1133[ap_, l_] :=
  {Test87[ap, l, 0.885285, 0.896456], Test87[ap, l, 0.885285, 1.5]};

Plot[Test1133[ap, lap[ap]], {ap, 0.1, 1.23}]
```




```
(* Case 1.1.3.4 *) (* m≥ 6 *)

gm[ap_, l_, a_] := ArcTan[l^2 Cos[ap]^2 / (3 a^2 - Sqrt[8] l^2 Cos[ap]^2)]

Plot[{gm[ap, lap[ap], 0.85], beta[lap[ap], ap]}, {ap, 0, 0.03}]
```

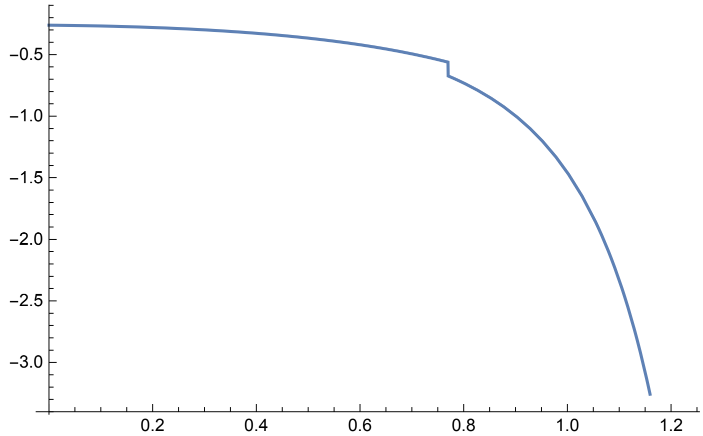


```
Plot[{beta[lap[ap], ap], gm[ap, lap[ap], 0.85]}, {ap, 0, 1.23}]
```




```
Test1134[ap_, l_, a_] := MaxQ[l^2 Cos[ap]^2 - a^2,
  2 / a * ((a^2 - Sqrt[8] l^2 Cos[ap]^2 / 3) * Cos[Min[beta[l, ap], gm[ap, l, a]]] +
     l^2 Cos[ap]^2 / 3 * Sin[Min[beta[l, ap], gm[ap, l, a]]]),
  (l Cos[ap] / a)^2 - 1, Phi[ap, l]]
```

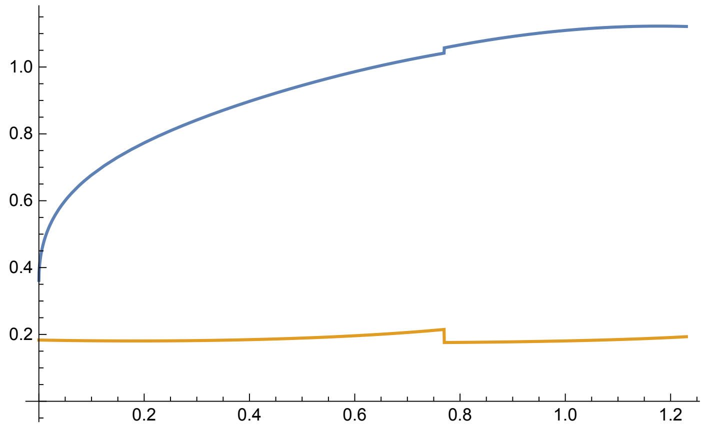
```
Plot[Test1134[ap, lap[ap],  0.85], {ap, 0, 1.23}]
```



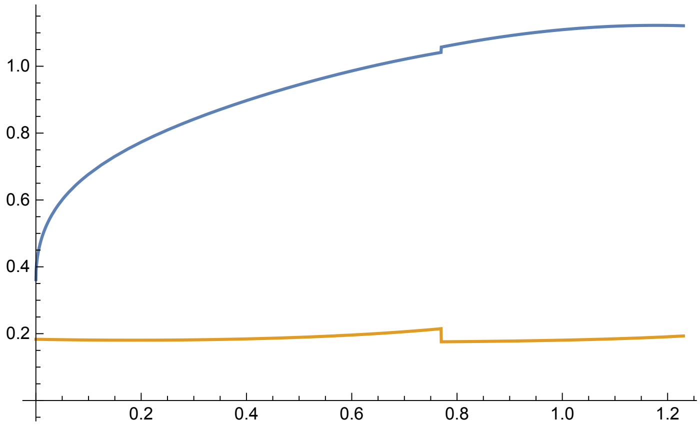

```
(* this graphic is repeated as 1st comp of Test1134[_,_] below *)
```

```
Eta2[ap_, l_, a_] := ArcTan[5 l^2 Cos[ap]^2 / (18 a - 5 Sqrt[8] l^2 Cos[ap]^2)];
```

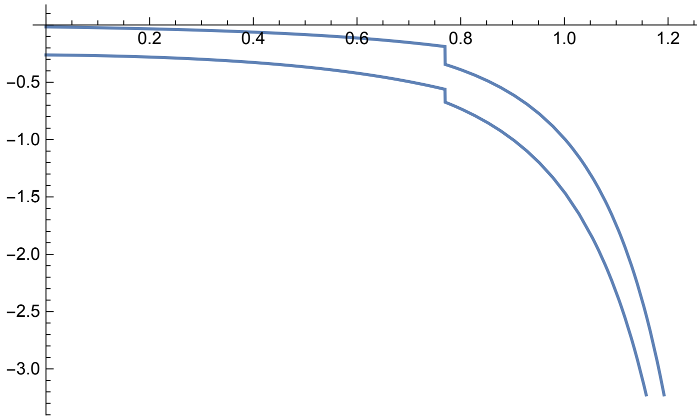
```
Plot[{beta[lap[ap], ap], Eta2[ap, lap[ap], 0.85]}, {ap, 0, 1.23}]
```



```
(* old test with 1.5 does not work

  Test11332[ap_,l_,a_]:= MaxQ[l^2Cos[ap]^2-a^2,
  2*((a-2Sqrt[8]l^2Cos[ap]^2/9)*Cos[Min[beta[l,ap],Eta2[ap,l,a]]]+
     2l^2Cos[ap]^2/9*Sin[Min[beta[l,ap],Eta2[ap,l,a]]]),
  (l Cos[ap]/a)^2-1,Phi[ap,l]] *)

Plot[{beta[lap[ap], ap], Eta3[ap, lap[ap], 0.85, 1.2]}, {ap, 0, 1.23}]
```

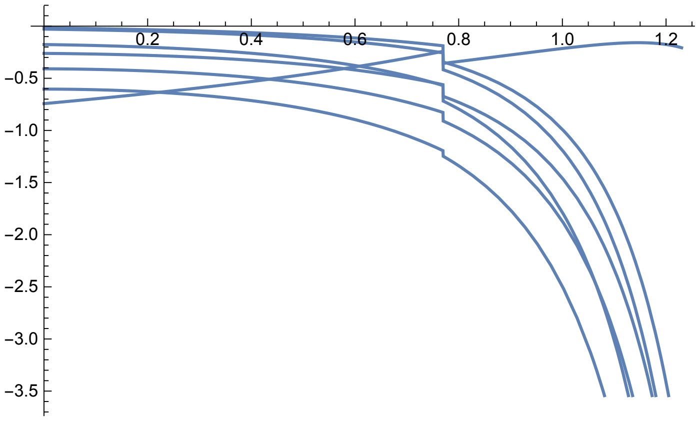



```
Test1134[ap_, l_] := {Test87[ap, l, 0.85, 0.85], Test87[ap, l, 0.85, 1.2]};

Plot[Test1134[ap, lap[ap]], {ap, 0, 1.23}]
```

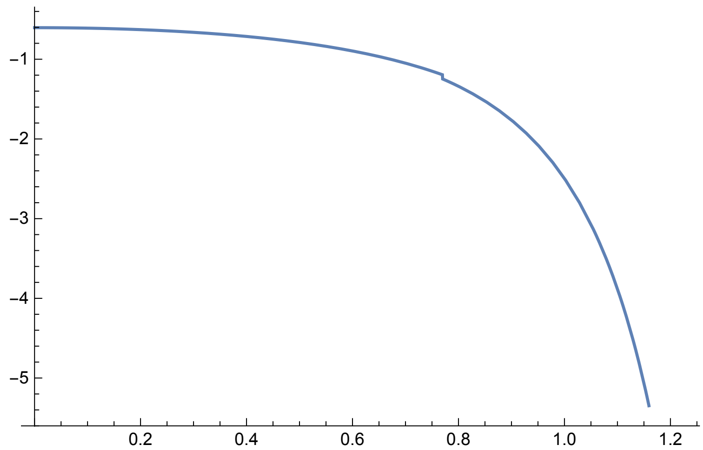



```
(* this part does work now ! *)

(* all parts together *)

Fct[a_, l_, ap_] := {Test111[ap, l, a], Test11321[ap, l],
  Test11322[ap, l], Test1133[ap, l], Test1134[ap, l]}
```

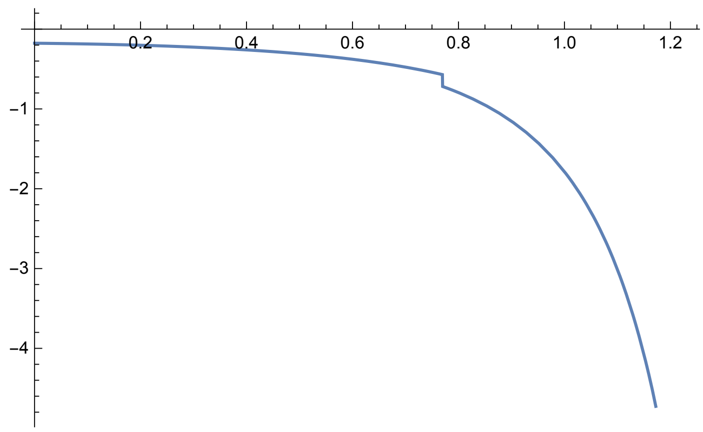
```
Plot[Fct[0.85, lap[ap], ap][[#]] & /@ {1, 2, 3, 4, 5}, {ap, 0, 1.23}]
```



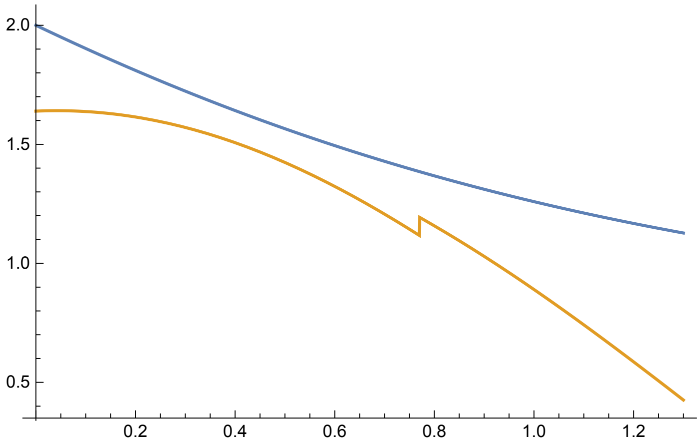

```
Plot[Fct[0.85, lap[ap], ap][[2]], {ap, 0.00, 1.23}]
```



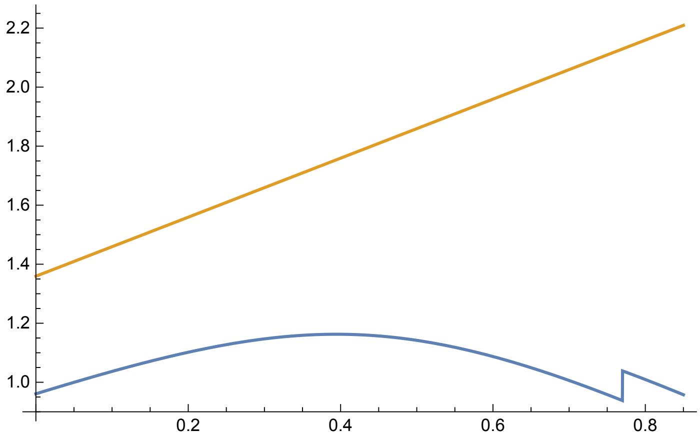

```
Plot[Fct[0.85, lap[ap], ap][[3]], {ap, 0.00, 1.23}]
```



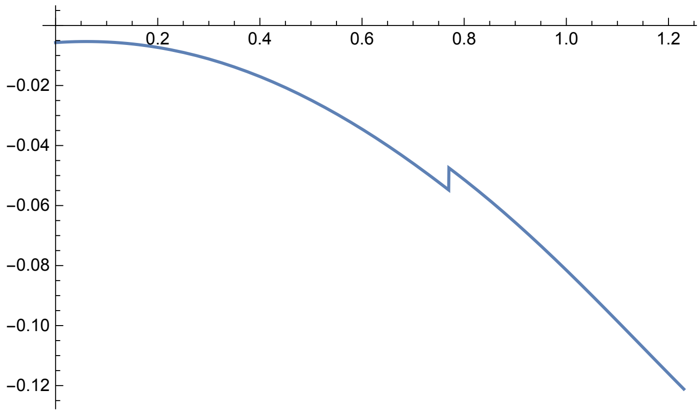
```
(* Case 1.2.1 *)
Plot[
 {2 Sqrt[2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]], 1 / lap[ap]}, {ap, 0, 1.3}]
```




```
Plot[{ArcSin[1 / (2 * lap[ap] * Sqrt[2 - Cos[ap] - Sqrt[Cos[ap]^2 - 4 Cos[ap] + 3]])],
  ap + 0 * ArcTan[Tan[ap] / 2] + 4 ArcSin[1 / 3]}, {ap, 0.0, 0.85}]
```

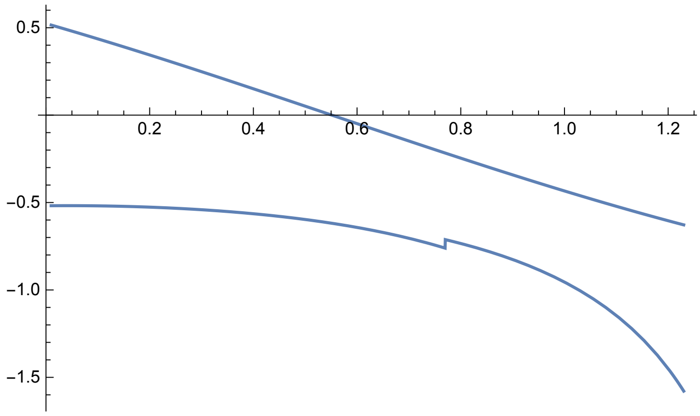



```
H[a_, l_, al_] := Sin[(ArcSin[1 / 3]) / 2] / a / Sqrt[Phi[al, l]^2 + 1 / a^2 -
      2 / a * Phi[al, l] * Cos[(ArcSin[1 / 3]) / 2]] - Sin[(ArcSin[1 / 3]) / 2]

Plot[H[0.85, lap[ap], ap], {ap, 0, 1.23}]
```

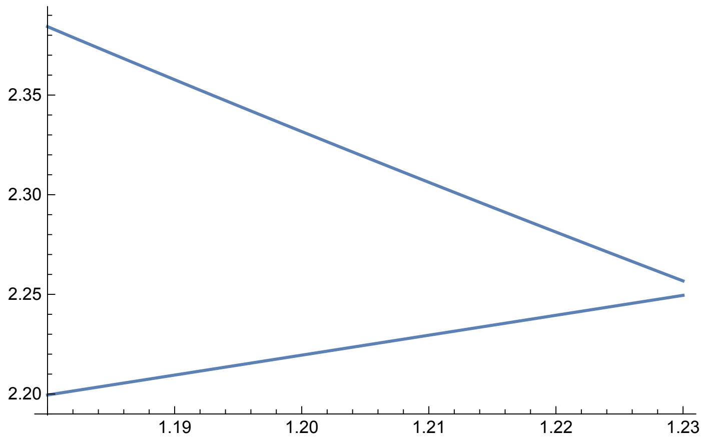



```
(* end new stuff *)

(* here comes the angle estimate stuff - Lemma 4.2 *)
```

```
(* the lower bounds on the norms of 1/ap_i to prove for
    the general proof with lemma 2.3 *)

Mm[a_, k_, ap_] := If[a k > 1,
  Min[3 * ArcSin[1 / 3] + 1.0 ap, Pi / 2 + ArcSin[1 / (a k)]], 3 * ArcSin[1 / 3] + 1.0 ap]

Plot[{Cos[3 * ArcSin[1 / 3] + ap], - (a lap[ap])} /. {a → 0.85}, {ap, 0.01, 1.23}]
```




```
Plot[{3 * ArcSin[1 / 3] + 1.0 ap, Pi / 2 + ArcSin[1 / (a lap[ap])]} /. {a → 0.85},
 {ap, 1.18, 1.23}]
```

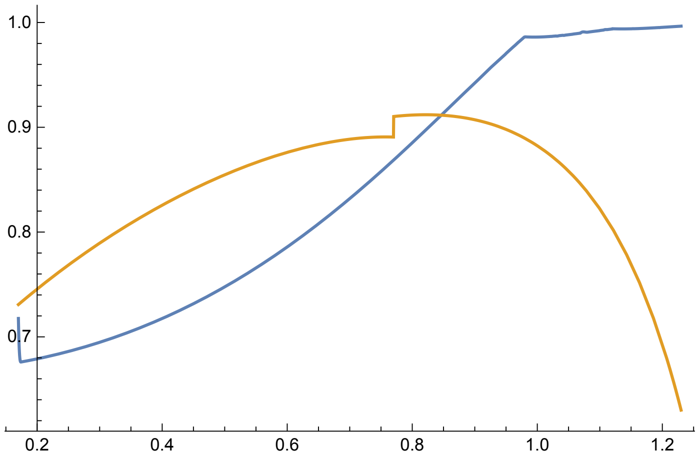



```
(* repeat def of delta from above *)

delta[a_, k_, eta_] :=
 If[Cos[eta] < -k a, Pi - ArcSin[Sin[eta] / Sqrt[1 + a^2 k^2 + 2 a k Cos[eta]]],
  ArcSin[Sin[eta] / Sqrt[1 + a^2 k^2 + 2 a k Cos[eta]]]]

(* Case 1.1.1 *)

fctx[a_, k_, ap_] := 1 / a * Sin[delta[a, k, Mm[a, k, ap]] - ArcSin[1 / 3] - 0.0 ap] /
   Sin[delta[a, k, Mm[a, k, ap]]]
```

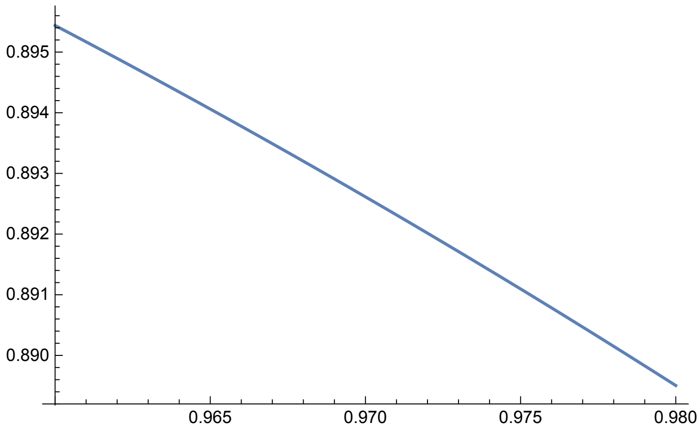
```
Plot[{1/almax3[ap], fctx[0.85, lap[ap], ap]}, {ap, 0.17, 1.23}]
```

InterpolatingFunction::dmval :

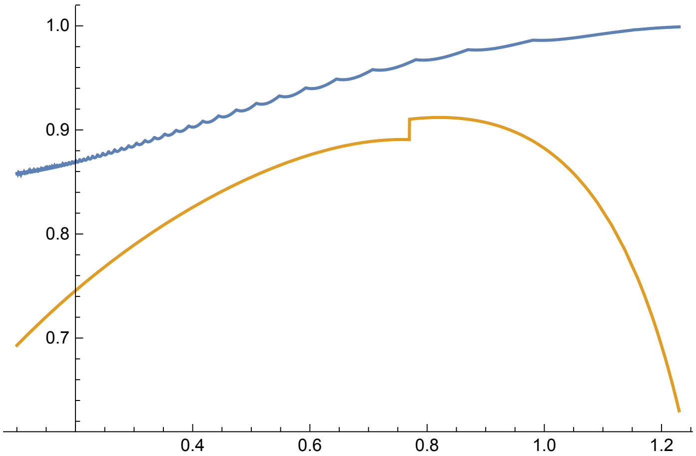
Input value {0.170022} lies outside the range of data in the interpolating function. Extrapolation will be used. >>



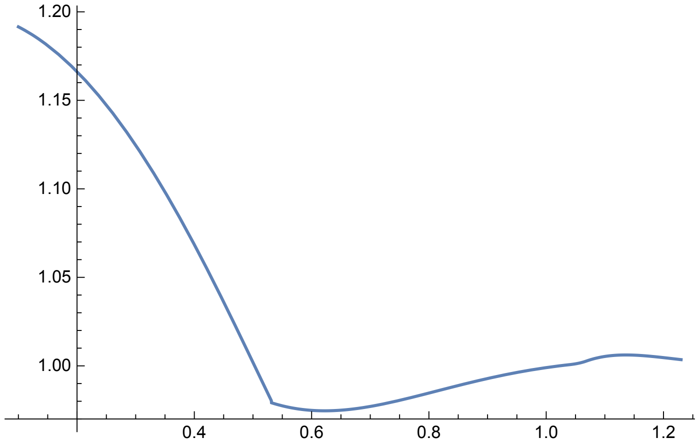

```
Plot[fctx[0.85, lap[ap], ap], {ap, 0.96, 0.98}]
```



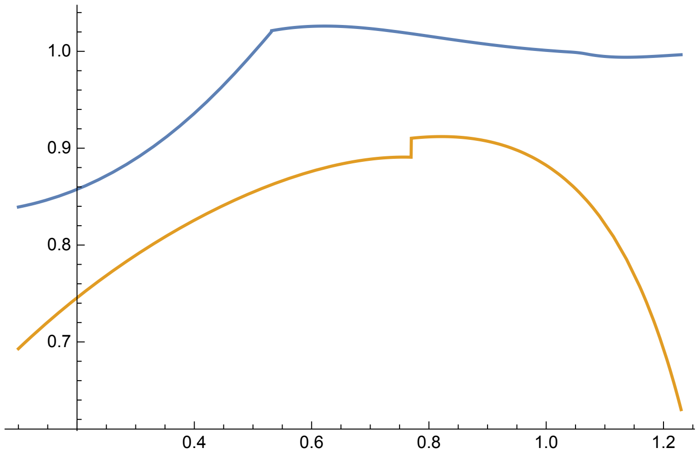

```
(* this works now for ap≥0.9, but for smaller ap need to break up in cases *)

(* enough to work for ap≤ 0.9 *)

(* m+1\ge 7 *)

Plot[{1/almax7[ap], fctx[0.85, lap[ap], ap]}, {ap, 0.1, 1.23}]
```



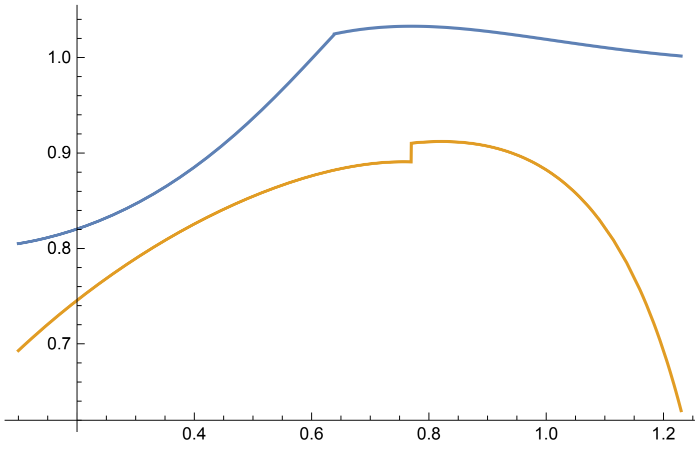
```
Plot[AlMax[ap, 6], {ap, 0.1, 1.23}]
```




```
Plot[{1/AlMax[ap, 6], fctx[0.85, lap[ap], ap]}, {ap, 0.1, 1.23}]
```

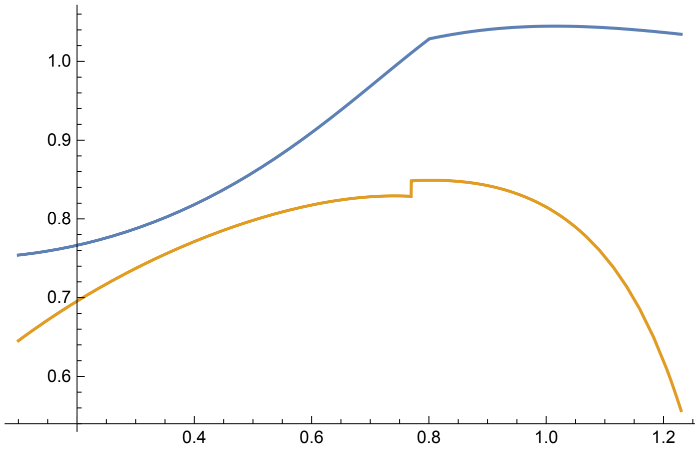



```
Plot[{1/AlMax[ap, 5], fctx[0.85, lap[ap], ap]}, {ap, 0.1, 1.23}]
```

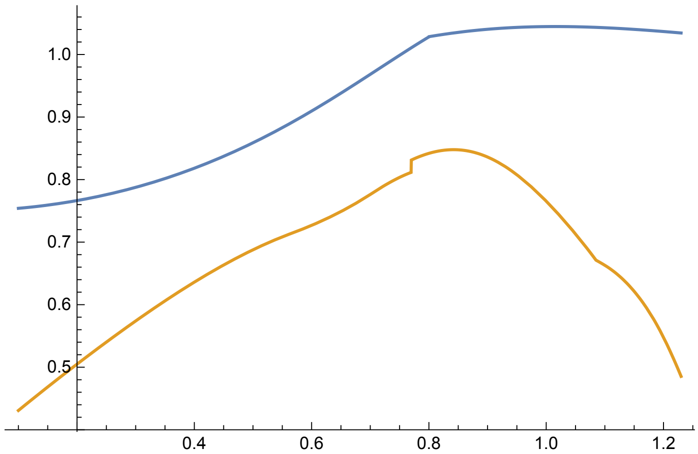



```
(* works so we dont need to do this crunching right below *)

Plot[{1/AlMax[ap, 5], fctx[Min[AlMin[ap, 4], AlMin[ap, 6]], lap[ap], ap]},
 {ap, 0.17, 1.23}]

(* Min[AlMin[ap,3],AlMin[ap,5] over all ap is 0.896456 *)
```

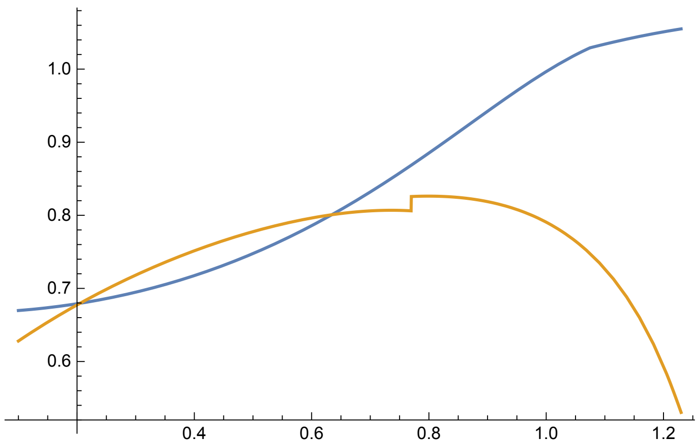
```
Plot[{1/AlMax[ap, 4], fctx[0.896456, lap[ap], ap]}, {ap, 0.1, 1.23}]
```



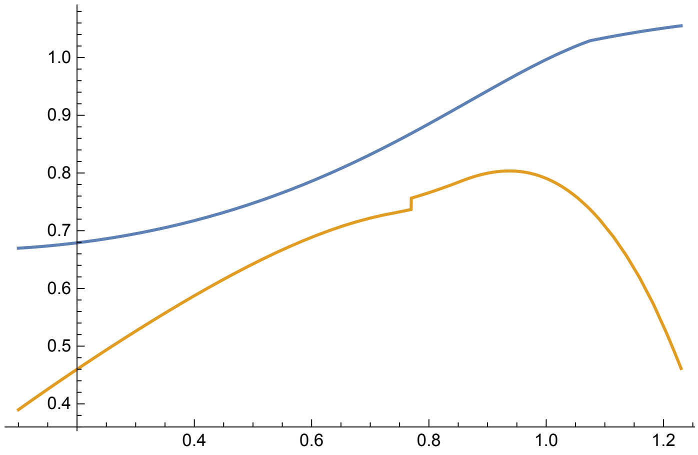

```
Plot[{1/AlMax[ap, 4], fctx[Min[AlMin[ap, 3], AlMin[ap, 5]], lap[ap], ap]},
 {ap, 0.1, 1.23}]
```



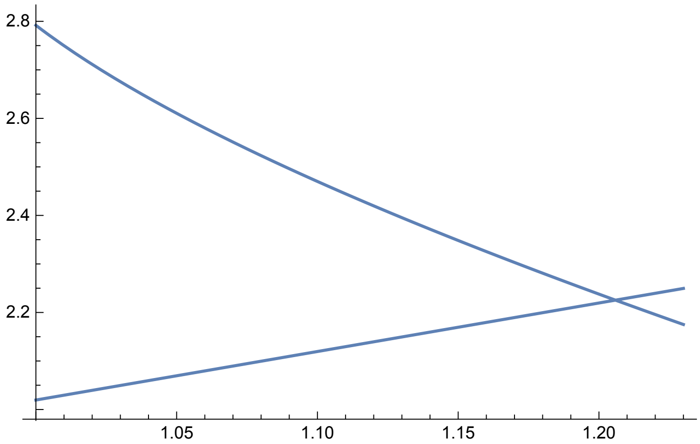

```
Plot[{1/AlMax[ap, 3], fctx[0.91465, lap[ap], ap]}, {ap, 0.1, 1.23}]
```



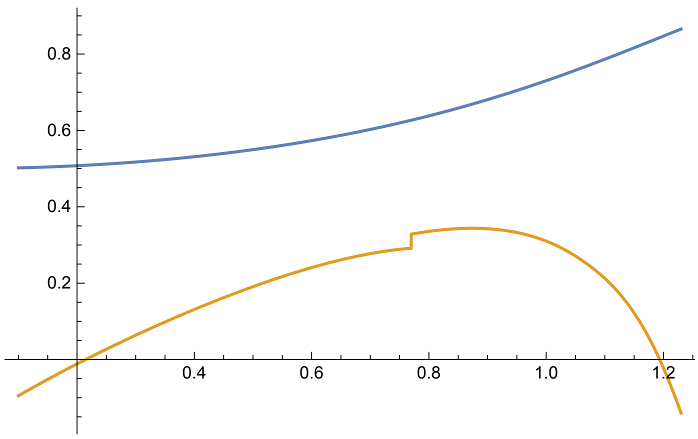
```
Plot[{1/AlMax[ap, 3], fctx[Min[AlMin[ap, 2], AlMin[ap, 4]], lap[ap], ap]},
 {ap, 0.1, 1.23}]
```



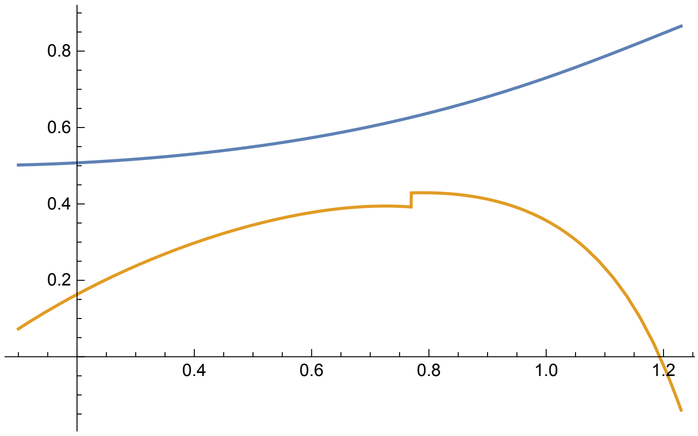

```
(* Case 1.2.1 *)

Plot[{3*ArcSin[1/3] + 1.0 ap, Pi/2 + ArcSin[1/(a lap[ap])]} /. {a → 0.94744},
 {ap, 1., 1.23}]
```



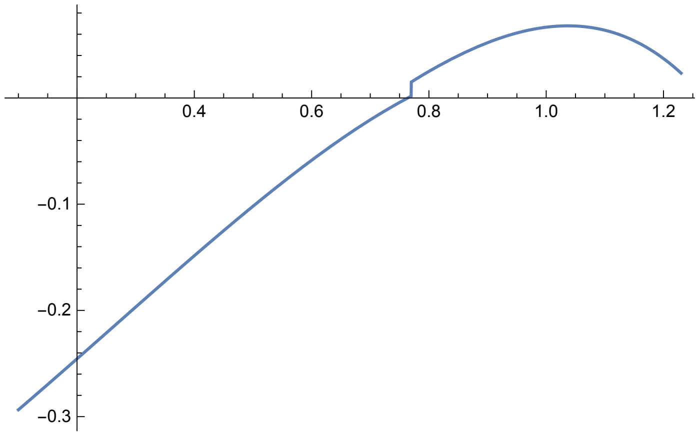

```
fcty[a_, k_, ap_] := 1/a*Sin[delta[a, k, Mm[a, k, ap]] - 2 ArcSin[1/3] - 0.0 ap]/
   Sin[delta[a, k, Mm[a, k, ap]]]

Plot[{1/AlMax[ap, 2], fcty[AlMin[ap, 3], lap[ap], ap]}, {ap, 0.1, 1.23}]
```



```
Plot[{1/Sqrt[1 + 3 * Cos[ap]^2], fcty[0.94744, lap[ap], ap]}, {ap, 0.1, 1.23}]
```

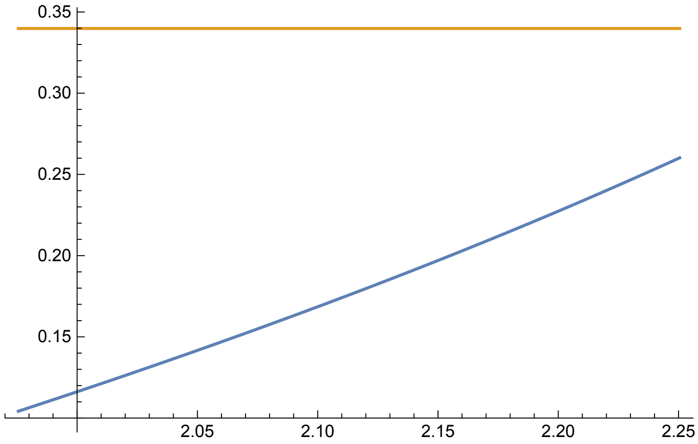



```
(* this case was moved out of lemma 4.1 and seems not needed now *)
Plot[delta[AlMin[ap, 2], lap[ap], ap] - ArcSin[1/3], {ap, 0.1, 1.23}]
```

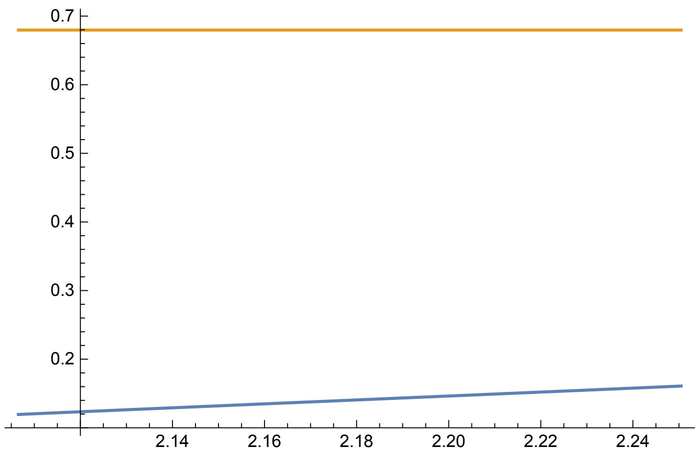



```
(* resolve AlMax[ap,2]= Sqrt[1+3*Cos[ap]^2];
  simplify w/ Lemma 2.5 AlMin[ap,2]=
 2Sqrt[2-Cos[ap]-Sqrt[Cos[ap]^2-4Cos[ap]_3]]≥3/Sqrt[2] ,
AlMin[ap,3]≥0.94744, Min[AlMin[ap,4],AlMin[ap,6]]≥0.85 .... *)

(* Case 1.1.2 *)

k[eta_, M_] := -Sqrt[3 + Sqrt[8]] / (Sqrt[6] * M * Cos[eta - ArcSin[1/3]/2]);

eps[a_, eta_, M_] :=
 ArcSin[Sin[eta] / Sqrt[1 + a^2 k[eta, M]^2 + 2 a k[eta, M] Cos[eta]]] -
  ArcSin[Sin[eta] / Sqrt[1 + 1.2^2 k[eta, M]^2 + 2 * 1.2 * k[eta, M] Cos[eta]]]

almax3[Pi/2 - 2.5 ArcSin[1/3]]
1.18666

(* so can take 1.2 *)
```

```
Plot @@ ({{eps[a, eta, 1.2], ArcSin[1/3]},
    {eta, Pi/2 + ArcSin[1/3]/2 + ArcSin[a Sqrt[3 + Sqrt[8]]/(3 Sqrt[6] * 1.2)],
     Pi/2 + ArcSin[1/3] * 2}} /. {a → 0.85})
```

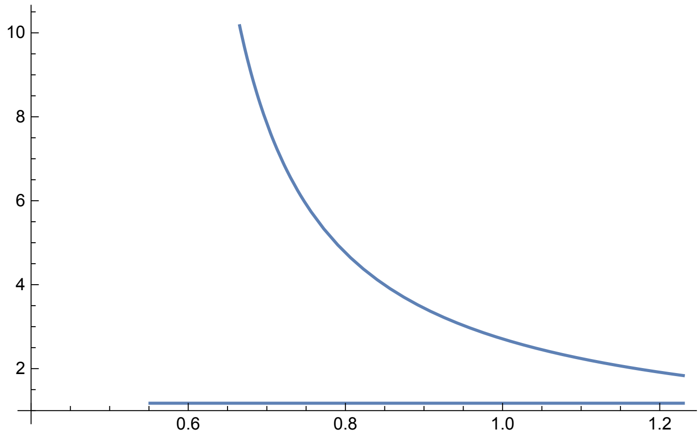



```
(* Case 1.2.2 *)

Plot @@ ({{eps[a, eta, Sqrt[1 + 3 Cos[eta - 3 ArcSin[1/3]]^2]], 2 ArcSin[1/3]},
    {eta, Pi/2 + ArcSin[1/3]/2 + ArcSin[2 Sqrt[8] a
        Sqrt[3 + Sqrt[8]]/(9 Sqrt[6] * Sqrt[1 + 3 Sin[2.5 ArcSin[1/3]]^2])],
     Pi/2 + ArcSin[1/3] * 2}} /. {a → 0.94744})
```

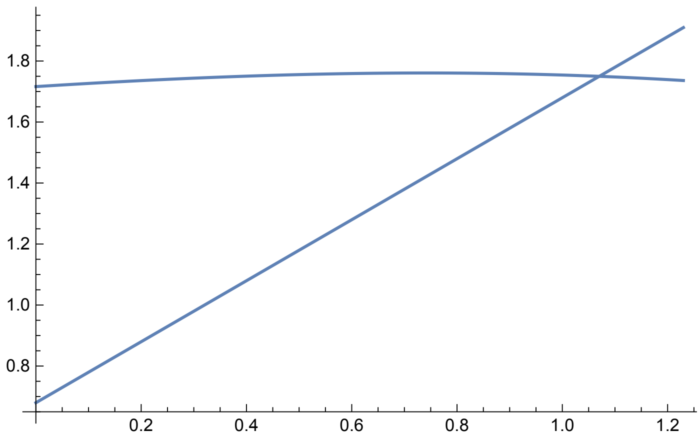



```
(* the foll test is indep on choice of lap[] *)


(* OLD STUFF *)

UpperShaft[a_, ap_] := If[3 * ArcSin[1/3] + ap > Pi/2,
  {1/(a Sin[3 * ArcSin[1/3] + ap - Pi/2]) *
    Sin[ArcSin[1/3] + Eta[a, 3/a, 3 * ArcSin[1/3] + ap]], 1/a}]
```

```
Plot[UpperShaft[0.85, ap], {ap, 0.4, 1.23}]
```

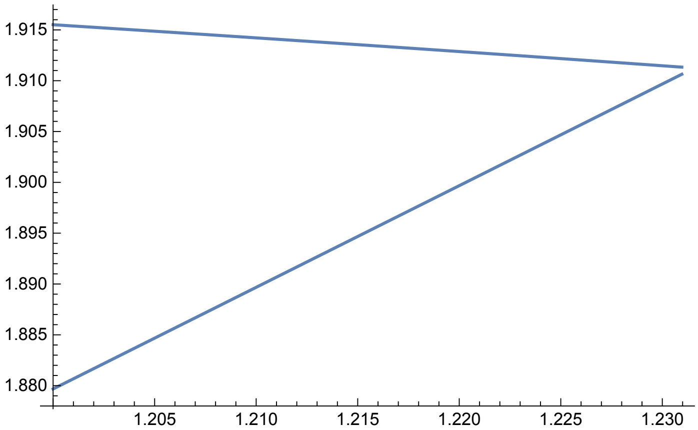



```
(* END OLD STUFF *)

(* STUFF added from .m file *)

(* Case 2.1 *)

Plot[{2 ArcSin[1 / 3] + ap,
   Pi / 2 + ArcSin[Sin[3 ArcSin[1 / 3] + ap] / Sqrt[1 + 9 m^2 / (4 a^2) +
        3 m / a Cos[3 ArcSin[1 / 3] + ap]]]} /. {a → 0.85, m → 3}, {ap, 0, 1.23}]
```



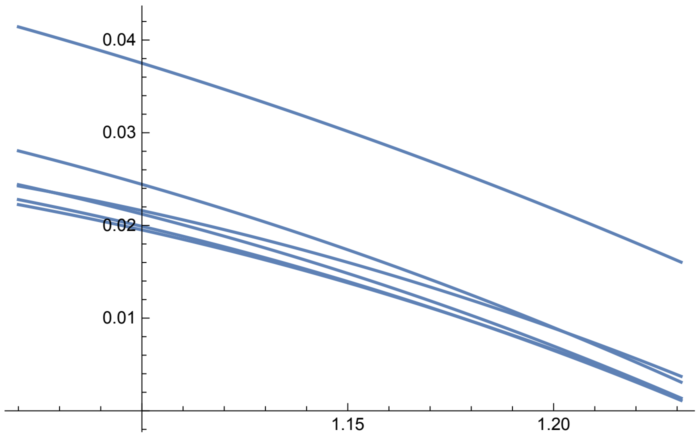

```
FindRoot[2 ArcSin[1 / 3] + ap ==
   Pi / 2 + ArcSin[Sin[3 ArcSin[1 / 3] + ap] / Sqrt[1 + 9 m^2 / (4 a^2) +
        3 m / a Cos[3 ArcSin[1 / 3] + ap]]] /. {a → 0.85, m → 3}, {ap, 1.07}]

{ap → 1.07016}
```

```
Plot[{2 ArcSin[1/3] + ap, Pi/2 + ArcSin[Sin[3 ArcSin[1/3] + ap]/
      Sqrt[1 + 9 m^2/(4 a^2) + 3 m/a Cos[3 ArcSin[1/3] + ap]]]} /.
  {a → 0.85, m → 1.6}, {ap, 1.2, ArcCos[1/3]}]
```




```
Ls = {3, 2.1, 1.81, 1.69, 1.635, 1.609, 1.6};
```

```
Plot[
 (M * Sin[ArcSin[1/3] + ArcSin[Sin[3 ArcSin[1/3] + ap]/Sqrt[1 + 9 m^2/(4 a^2) + 3
            m/a Cos[3 ArcSin[1/3] + ap]]]] - 1 /. {a → 0.85,
      m → Ls[[#]], M → Ls[[# + 1]]}) & /@ Range[6], {ap, 1.07, ArcCos[1/3]}]
```




```
(* here essentially the proof should be done !! *)
```

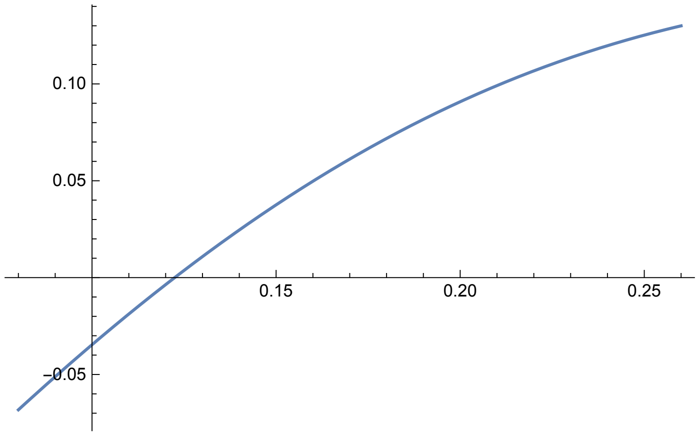
```
(* some old and likely redundant attempts *)
```

```
limit := 0.315
```

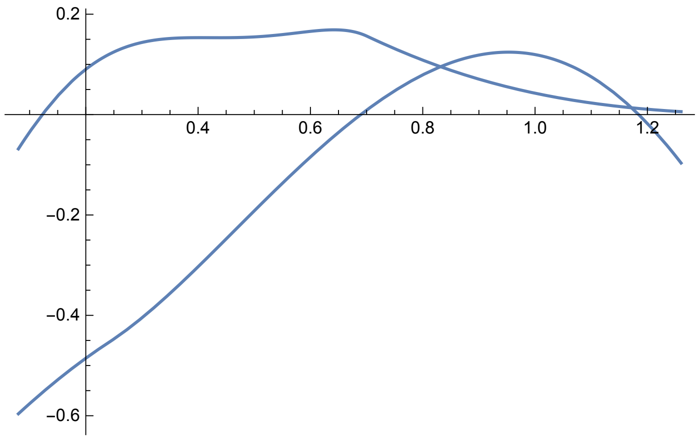
```
sind[ap_, l_] :=
 Sin[2 ap + ArcSin[limit] + ArcTan[Sin[ap] Cos[ap]/(1 + Cos[ap]^2)]]/
  Sqrt[4 (4 + Tan[ap]^2) l^2 + 1/Cos[ap]^2 - 4 Sqrt[4 + Tan[ap]^2] l
     Cos[2 ap + ArcSin[limit] + ArcTan[Sin[ap] Cos[ap]/(1 + Cos[ap]^2)]]/Cos[ap]]
```

```
Fct[a_, l_, ap_] := {MaxQ[l^2 Cos[ap]^2 - a^2,
    2 l^2 Cos[ap]^2 / a - 2 a Cos[3 * ArcSin[limit] + 2 ap], (l Cos[ap] / a)^2 - 1, l],
  MaxQ[l^2 Cos[ap]^2 - a^2,
    2 a - 2 l^2 Cos[ap]^2 (Cos[ArcSin[limit]] * Cos[beta[l, ap]] -
          Sin[ArcSin[limit]] * Sin[beta[l, ap]]) / (a),
    (l Cos[ap] / a)^2 - 1, 2 Sqrt[4 + Tan[ap]^2] - 1 / l / Cos[ap]]}
```

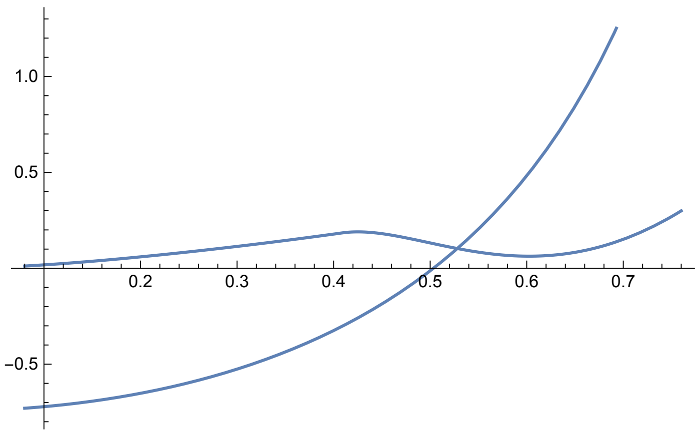
```
Plot[Fct[0.8527, 0.61 - ap / 6, ap][[2]], {ap, 0.08, 0.26}]
```



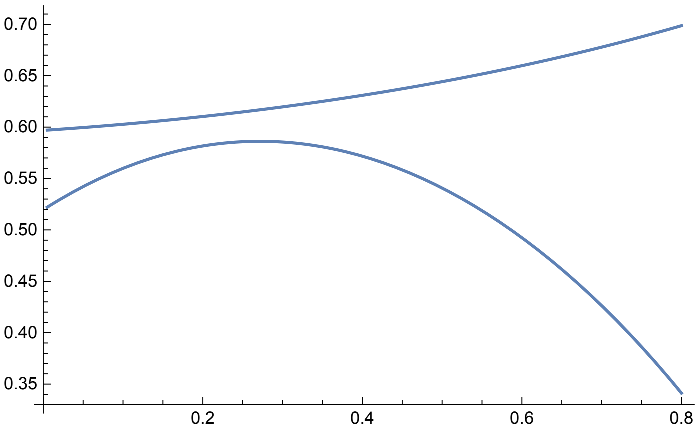

```
Plot[Fct[0.8527, (0.61 - ap / 6 - ap^2 / 8 - ap^3 / 70), ap], {ap, 0.08, 1.26}]
```



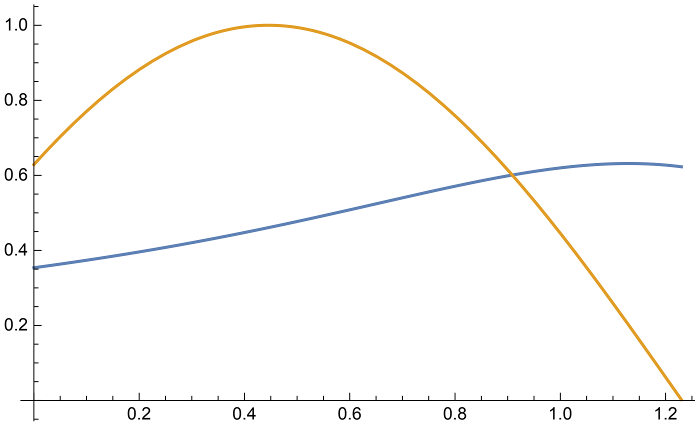

```
Plot[Fct[0.8527, l, 0.57], {l, 0.08, 0.76}]
```



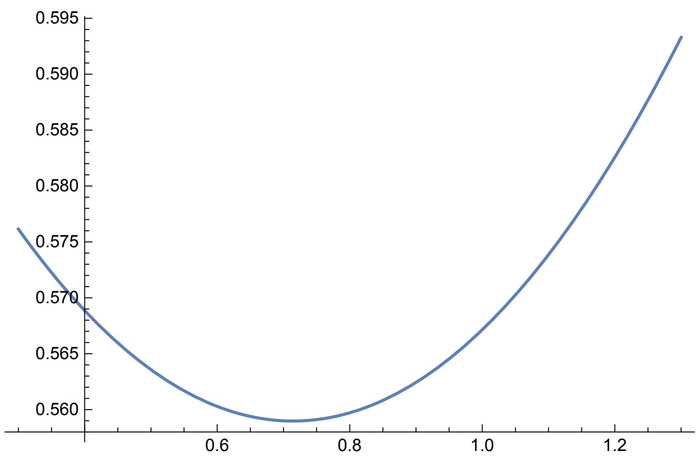
```
(* so if we prove |arg ap|≤arcsin(0.315) for ap le 0.275..=pi/2-3\as/2 ,
then ok  and |ap_n|ge 0.8527 *)

Plot[{((0.61 - ap / 6) * Sin[2 ap + 3 ArcSin[1 / 3]]) / Cos[ap],
   1 / (2 a Cos[(ArcSin[1 / 3] + ap) / 2])} /. {a → .85}, {ap, 0.005, 0.8}]
```



```
sind[ap_, l_] := Sin[ 2 ArcSin[1 / 3] + ArcTan[Tan[ap] / 2]] /
   Sqrt[4 (4 + Tan[ap] ^2) l^2 + 1 / Cos[ap] ^2 - 4 Sqrt[4 + Tan[ap] ^2]
      l Cos[2 ArcSin[1 / 3] + ArcTan[Tan[ap] / 2]] / Cos[ap]] / Cos[ap]

Plot[{sind[ap, 0.61 - ap / 13], Sin[2 ArcSin[1 / 3] + 2 ap]}, {ap, 0, 1.23}]
```

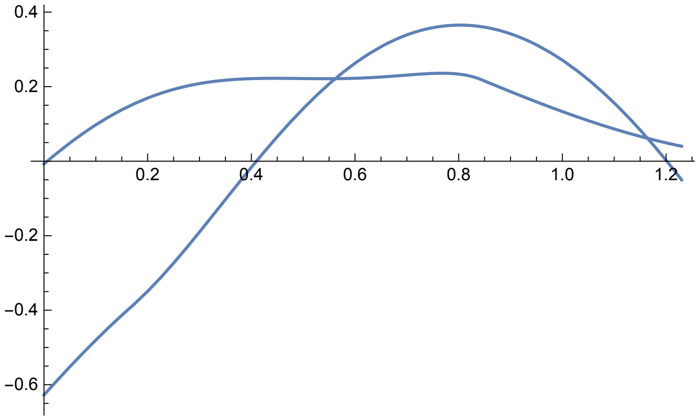


```
(* now see what you get with |arg ap_n/ap_n-1|<\as *)

Plot[0.61 - ap / 7 + ap^2 / 10, {ap, 0.3, 1.3}]
```

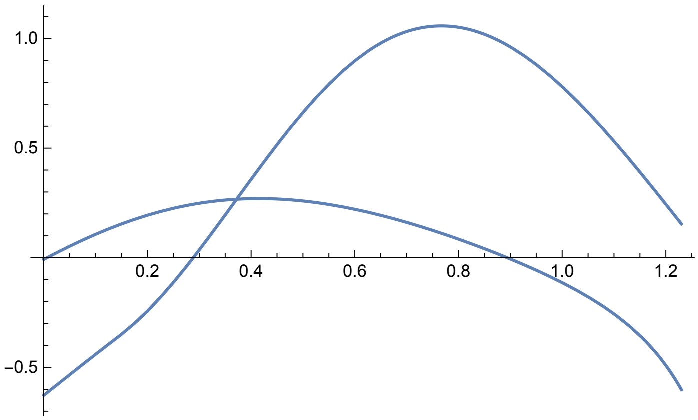


```
Fct[a_, l_, ap_] := {MaxQ[l^2 Cos[ap]^2 - a^2,
    2 l^2 Cos[ap]^2 / a - 2 a Cos[3 * ArcSin[1 / 3] + ap] , (l Cos[ap] / a)^2 - 1, l],
  MaxQ[l^2 Cos[ap]^2 - a^2, 2 a - 2 / a * l^2 * Cos[ap]^2 *
      (Sqrt[8 / 9] * Cos[beta[l, ap]] - Sqrt[1 / 9] * Sin[beta[l, ap]]),
    (l Cos[ap] / a)^2 - 1, 2 Sqrt[4 + Tan[ap]^2] - 1 / l / Cos[ap]]}
```

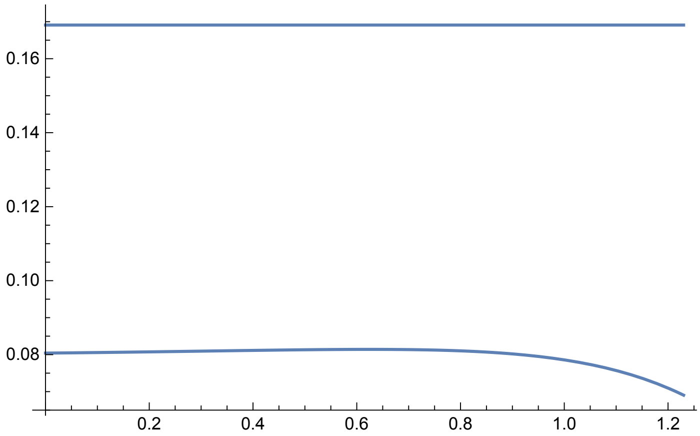
```
Plot[Fct[0.851, 0.61 - ap / 16 - ap^2 / 8 - ap^3 / 70, ap], {ap, 0.00, 1.23}]
```



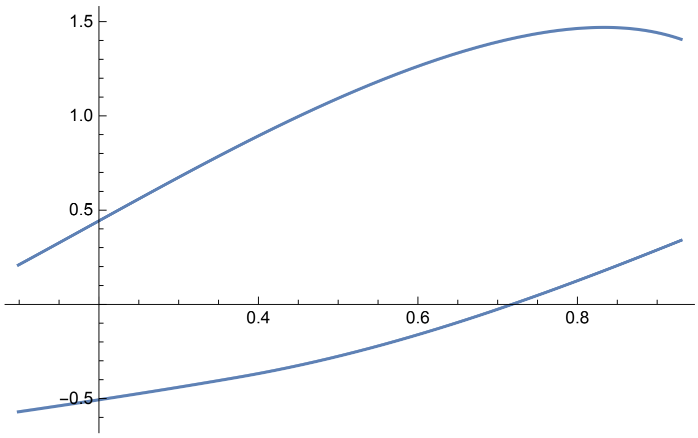

```
Plot[Fct[0.851, 0.61 + ap / 30, ap], {ap, 0.00, 1.23}]
```



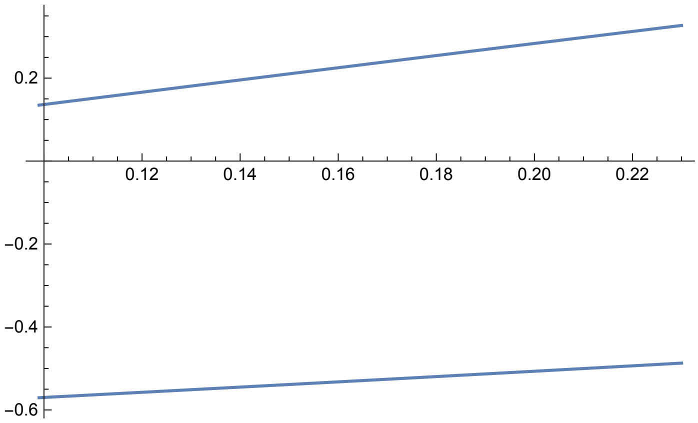
```
Plot[
 {Sin[ArcSin[1/3]/2]/a/Sqrt[(2 Sqrt[4 + Tan[ap]^2] - 1/l/Cos[ap])^2 + 1/a^2 -
      2/a * (Sqrt[4 + Tan[ap]^2] - 1/l/Cos[ap]) Cos[ArcSin[1/3]/2]],
   Sin[ArcSin[1/3]/2]} /. {l → 0.61 - ap/30, a → 0.85}, {ap, 0, 1.23}]
```



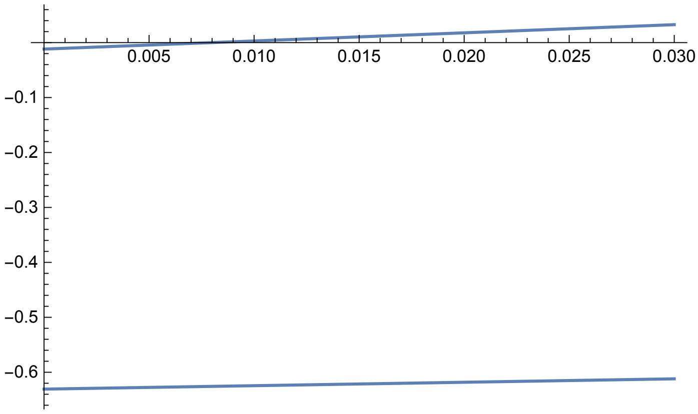

```
(* the modified approach with z^{-n/2}\Dl *)

MaxQ[a_, b_, c_, d_] := If[-b/(2 a) < d, a d^2 + b d + c, -b^2/(4 a) + c] /; a < 0

sind[ap_, l_] :=
 Sin[ 2 ArcSin[1/3] + ArcTan[Tan[ap]/2]]/Sqrt[4 (4 + Tan[ap]^2) l^2 +
    1 - 4 Sqrt[4 + Tan[ap]^2] l Cos[2 ArcSin[1/3] + ArcTan[Tan[ap]/2]]]

beta[l_, ap_] := ArcSin[sind[ap, l]] + ArcTan[Sin[ap] Cos[ap]/(1 + Cos[ap]^2)]

Fct[a_, l_, ap_] := {MaxQ[l^2 - a^2,
   2 l^2/a - 2 a * Cos[3 * ArcSin[1/3] + ap] , (l / a)^2 - 1, l / Cos[ap]],
  MaxQ[l^2 - a^2, 2 a - 2 l^2/a *
     (Sqrt[8/9] * Cos[beta[l, ap] + ap] - Sqrt[1/9] * Sin[beta[l, ap] + ap]),
   (l / a)^2 - 1, 2 Sqrt[4 + Tan[ap]^2] - 1/l]}

Plot[Fct[0.852, 0.61 - ap/12, ap], {ap, 0.099, 0.93}]
```



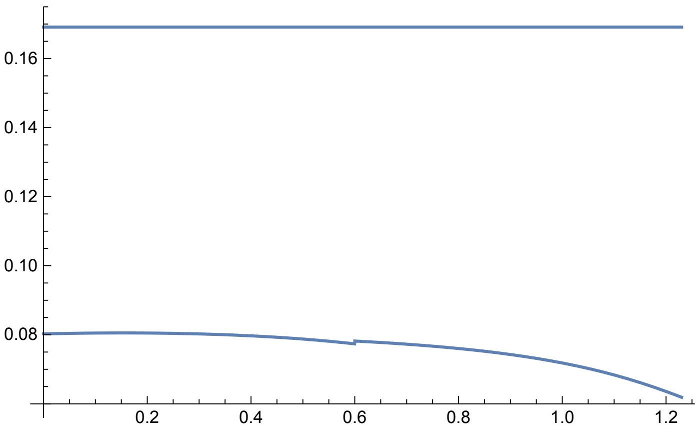
```
Plot[Fct[0.852, 0.61 - ap / 12, ap], {ap, 0.099, 0.23}]
```



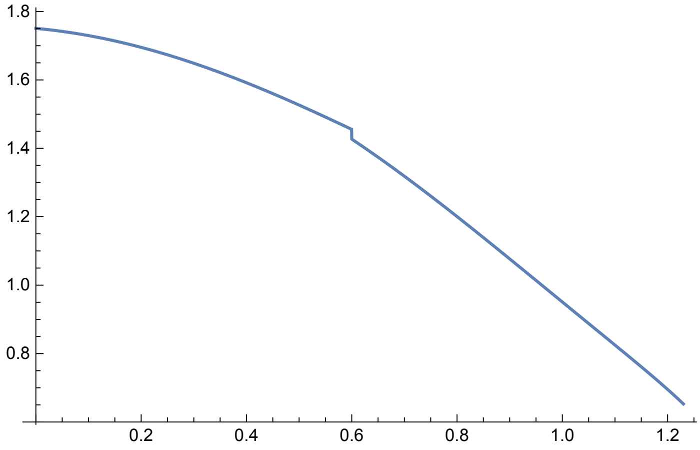
```
Plot[Fct[0.852, 0.61 - ap / 13 + ap^2 / 10, ap], {ap, 0.0, 0.03}]
```



```
Plot[{Sin[ArcSin[1 / 3] / 2] / a / Sqrt[(2 Sqrt[4 + Tan[ap] ^2] - 1 / l) ^2 + 1 / a^2 - 2 / a *
      (Sqrt[4 + Tan[ap] ^2] - 1 / l) Cos[ArcSin[1 / 3] / 2]], Sin[ArcSin[1 / 3] / 2]} /.
  {l → If[ap < 0.6, 0.61 - ap / 12, 0.61 - ap^2 / 8 - ap^3 / 10], a → 0.852}, {ap,
  0, 1.23}]
```



```
Plot[2 Sqrt[1 + 3 Cos[ap] ^2] - Cos[ap] / l - 1 /.
  {l → If[ap < 0.6, 0.61 - ap / 12, 0.61 - ap^2 / 8 - ap^3 / 10]}, {ap, 0, 1.23}]
```



```
limit := 1 / 3

sind[ap_, l_] := Sin[2 limit + ArcTan[Tan[ap] / 2]] / Sqrt[4 (4 + Tan[ap] ^2) l^2 +
    1 - 4 Sqrt[4 + Tan[ap] ^2] l Cos[2 limit + ArcTan[Tan[ap] / 2]]]

beta[l_, ap_] := ArcSin[sind[ap, l]]

Fct[a_, l_, ap_] := {MaxQ[l^2 - a^2,
   2 l^2 / a - 2 a * Cos[3 * ArcSin[limit] + ap] , (l / a) ^2 - 1, l / Cos[ap]],
  MaxQ[l^2 - a^2, 2 a - 2 l^2 / a *
     (Cos[ArcSin[limit]] * Cos[beta[l, ap] + ap] - limit * Sin[beta[l, ap] + ap]),
   (l / a) ^2 - 1, 2 Sqrt[4 + Tan[ap] ^2] - 1 / l]}

Plot[Fct[0.852, 0.61 - ap / 12, ap], {ap, 0.009, 0.83}]
```