NON-TRIVIALITY OF THE JONES POLYNOMIAL AND THE CROSSING NUMBERS OF AMPHICHEIRAL KNOTS

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Abstract. Using an involved study of the Jones polynomial, we determine, as our main result, the crossing numbers of (prime) amphicheiral knots. As further applications, we show that several classes of links, including semiadequate links and Whitehead doubles of semiadequate knots, have non-trivial Jones polynomial. We also prove that there are infinitely many positive knots with no positive minimal crossing diagrams. Some relations to the twist number of a link, Mahler measure and the hyperbolic volume are given, for example explicit upper bounds on the volume for Montesinos and 3-braid links in terms of their Jones polynomial.

Keywords: amphicheiral knot, Jones polynomial, twist number, hyperbolic volume, Mahler measure, semiadequate link, positive knot, crossing number

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1. Introduction

The ultimate goal of this paper is to prove

Theorem 1.1 For each natural number $n \geq 15$, there exists a prime amphicheiral knot of crossing number n.

This result thus determines the crossing numbers of prime amphicheiral (or achiral) knots, i.e. those isotopic to their mirror images. It can be considered as concluding a topic with an illustrious history. The problem (given also as question 1.66 in [Ki]) goes back to the origins of knot theory some 120 years ago, when Tait started compiling knot tables and sought the amphicheiral knots therein (see for example [HTW]). He actually might have believed that only even n occur, based on evidence from alternating knots (which are highly over-represented in small crossing numbers). The proof that alternating amphicheiral knots have even crossing number was a major step that came with the discovery of the Jones polynomial V [J] about 20 years ago. It was obtained as a consequence of the crossing number result for alternating knots of Kauffman, Murasugi and Thistlethwaite [Ka, Mu, Th2]. Their work also easily allows one to realize all even $n \geq 4$ (solving part (B) or the op. cit. question in Kirby's book). The odd n (part (A) of the question) are far more difficult to deal with. A knot for $n = 15$ was found by Hoste and Thistlethwaite in a purely computational manner, during the compilation of knot tables. However, exhaustive enumeration quickly becomes impossible with increasing crossing numbers, and no generally applicable methods are known to handle such examples. The approach leading to our result will emerge from a much more advanced treatment of the Jones polynomial, which has further noteworthy implications. A brief summary of our work, with a few more details on its historical background, is given in the announcement [St19].

The Jones polynomial, which has led to a significant change-of-face of low-dimensional topology in recent years, remains despite its popularity far from understood. Yet we do not know much about the direct appearance of this invariant. For example, we can still not answer Jones' question if it distinguishes all knots from the trivial one (see [Bi, Ro, DH]). In [Ka, Mu, Th2] it was proved that the span (difference between minimal and maximal degree) gives a lower bound for the crossing number, which is exact on alternating diagrams, thus confirming a century-old conjecture of Tait. For such links the coefficients alternate in sign. In [LT] (semi)adequate links were introduced, motivated by further extensions of non-triviality and crossing number results. Adequate links are a generalization of alternating links, but in practice relatively few non-alternating links are adequate. Semiadequate links are a much larger class. It contains, beside alternating links, also other important classes like positive and Montesinos links. In [Th], Thistlethwaite obtained a description of certain "critical line" coefficients of the Kauffman polynomial of semiadequate links in terms of graph-theoretic invariants. For the Jones polynomial it was observed in [LT] that (almost by definition) for semiadequate links the leading coefficient becomes ± 1. Apart from this insight, a while there was no further information we had about coefficients of the polynomial.

This paper aims to provide some understanding of coefficients 2 and 3 of the Jones polynomial. Our starting point will be formulas for these coefficients in semiadequate diagrams, shown in §3 (propositions 3.1 and 3.3). The formula for coefficient 2 was motivated by, and is a common generalization of the one found by Dasbach-Lin [DL] in alternating diagrams and myself in [St] for positive diagrams. With these formulas we thus offer a unifying concept of the two approaches. The formula for the third coefficient was obtained independently and simultaneously by Dasbach-Lin in [DL2]. However, rather than just focusing on the formulas themselves, our main contribution here will be to apply them to several, incl. long-standing problems in knot theory. These applications occupy the later sections, and can be briefly grouped as follows.

Non-triviality of the Jones polynomial. Using the formula for coefficient 2 and some knowledge of properties of positive links (some of which are prepared in §3), we can conclude this non-triviality in many cases, including semiadequate links (theorem 4.1), Whitehead doubles of semiadequate knots (proposition 4.3), 3-braid and Montesinos links (corollary 4.1), and some strongly n-trivial knots (proposition 4.4). These results are treated in §4 (except for 3-braids, which occupy §6). They extend all previously known results (of [Ka, Mu, Th2] for alternating, [LT] for adequate links, and [Th2] for the Kauffman polynomial).

Estimates of hyperbolic volume. The Jones polynomial coefficients are related, as Dasbach-Lin [DL] observe, to a quantity called twist number. This number occurs in recent work of Lackenby-Agol-Thurston [La] as a way of estimating hyperbolic volume. We can thus amplify Dasbach-Lin's volume inequalities for alternating links, by similar (though slightly more involved) upper estimates of the volume for adequate, 3-braid and Montesinos links in §7 (propositions 7.2 and 7.3). Such relations aim at providing a less exact, but more practical alternative to the so-called Volume conjecture [MM] (which states for a general link an exact, but very involved, formula for the volume in terms of the *colored* Jones polynomial). They also relate to the open problem whether one can augment hyperbolic volume but preserve the Jones polynomial.

3-braid links. It turns out that 3-braid links are semiadequate (corollary 6.3), and we obtain several applications to them. We readily have non-triviality of the polynomial (which, *inter alia*, also gives a new proof of the faithfulness of the Burau representation). We solve the problem of minimal conjugacy length in the 3-braid group (theorem 6.1), which was asked for by Birman-Menasco [BM], and extend their classification of amphicheiral 3-braid links to those which are unorientedly amphicheiral (theorem 6.2). We can also describe the 3-braid links of unsharp Morton-Williams-Franks braid index inequality (corollary 6.5).

Positive and amphicheiral knots. The applications that require the most substantial effort are the proof that infinitely many positive knots have no minimal crossing diagrams in §5 (propositions 5.1 and 5.2), and its extension to show theorem 1.1. In latter's proof, there is a certain dichotomy between crossing numbers $n = 15+4k$ and $n = 17+4k$, resulting from the somewhat different nature of the examples we focus on. Accordingly, the proof of theorem 1.1 is divided into the last two sections.

The formulas of §3 are the central and unifying theme in all these results. The only part of the paper they do not directly appear in is §8, which studies the relation between the leading and trailing coefficients of the Jones polynomial, twist numbers and Mahler measure. From this relation, Dasbach-Lin obtained, and then further empirically speculated about, certain relations between coefficients of the Jones polynomial and hyperbolic volume. We give a qualitative improvement of the Dasbach-Lin result [DL], showing that *every* coefficient of the Jones polynomial gives rise to a lower bound for the volume of alternating knots (corollary 8.5). This supplies with some more understanding their experimental questions.

Besides polynomial non-triviality, and apart from the above particular series of (achiral) examples, one should hope to say something on crossing numbers of semiadequate links in general. The only previous result, in [Th], is that these links are non-trivial (i.e. their crossing number is not 0). We develop a method to obtain estimates on crossing numbers of semiadequate links in a separate paper [St15]. Among others, we prove that a semiadequate link has only finitely many reduced semiadequate diagrams, and there are only finitely many semiadequate links with the same Kauffman polynomial.

2. Basic preliminaries

The Jones polynomial is useful to define here via Kauffman's state model [Ka]. Recall, that the Kauffman bracket $\langle D\rangle$ of a link diagram D is a Laurent polynomial in a variable A, obtained by summing over all states S the terms

$$A^{\#A(S)-\#B(S)}\left(-A^2-A^{-2}\right)^{|S|-1}, \tag{1}$$

where a *state* is a choice of *splicings* (or *splittings*) of type A or B for any single crossing (see figure 1), $\#A(S)$ and $\#B(S)$ denote the number of type A (resp. type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in S.

We call the *A-state* the state in which all crossings are A-spliced, and *B-state* is defined analogously. We call a trace a in the A-state *dual* to a trace b in the B-state, if a and b correspond to the same crossing as in figure 1.

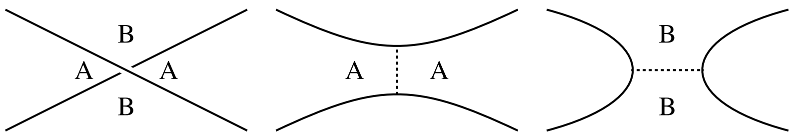


Figure 1: The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing. It is useful to put a "trace" of each splitted crossing as an arc connecting the loops at the splitted spot.

The Jones polynomial of a link L can be specified from the Kauffman bracket of some diagram D of L by

$$V_L(t) = \left(-t^{-3/4}\right)^{-w(D)}\langle D\rangle\Big|_{A=t^{-1/4}}, \tag{2}$$

with $w(D)$ being the writhe of D.

Let S be the A-state of a diagram D and S' a state of D with exactly one B-splicing. If $|S|>|S'|$ for all such S', we say that D is *A-adequate*. Similarly one defines a B-adequate diagram D. See [LT, Th]. Then we set a diagram to be

$$\begin{aligned} \text{adequate} &= \text{A-semiadequate } \textit{and} \text{ B-semiadequate}, \\ \text{semiadequate} &= \text{A-semiadequate } \textit{or} \text{ B-semiadequate}, \\ \text{inadequate} &= \textit{neither} \text{ A-semiadequate } \textit{nor} \text{ B-semiadequate}. \end{aligned}$$

(Note that inadequate is a stronger condition than not to be adequate.)

A link is called (A or B-)adequate, if it has an (A or B-)adequate diagram. A link is *semiadequate* if it is A- or B-adequate. A link is *inadequate*, if it is neither A- nor B-adequate.

As noted, semiadequate links are a much wider extension of the class of alternating links than adequate links. For example, only 3 non-alternating knots in Rolfsen's tables [Ro2, appendix] are adequate, while all 55 are semiadequate.

Although the other polynomials will make only a secondary appearance in the paper, we include a description at least to clarify conventions.

The *skein (HOMFLY) polynomial P* is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1}\,P(\times) + l\,P(\times) = -m\,P(\,)\,(\,)\,. \tag{3}$$

This convention uses the variables of [LM2], but differs from theirs by the interchange of l and l^{-1}. We call the three diagram fragments in (3) from left to right a *positive* crossing, a *negative* crossing and a *smoothed out* crossing (in the skein sense).

A diagram is called *positive*, if all its crossings are positive. A(n oriented) link is positive, if it admits a positive diagram (see for example [St6]).

The *Kauffman polynomial* [Ka2] F is usually defined via a regular isotopy invariant $\Lambda(a,z)$ of unoriented links.

We use here a slightly different convention for the variables in F, differing from [Ka2, Th] by the interchange of a and a^{-1}. Thus in particular we have the relation $F(D)(a,z) = a^{w(D)}\Lambda(D)(a,z)$, where $w(D)$ is the writhe of a link diagram D, and $\Lambda(D)$ is the writhe-unnormalized version of the polynomial. Λ is given in our convention by the properties

$$\begin{gathered}\Lambda(\times) + \Lambda(\times) = z\,\big(\,\Lambda(\asymp) + \Lambda(\,)\,(\,)\,\big)\,,\\ \Lambda(\,) = a^{-1}\,\Lambda(\,|\,); \quad \Lambda(\,) = a\,\Lambda(\,|\,)\,,\\ \Lambda(\bigcirc) = 1\,.\end{gathered}$$

Note that for P and F there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity.

The Jones polynomial V can be obtained from P and F (in our conventions) by the substitutions (with $i=\sqrt{-1}$; see [LM2] or [Ka2, §III])

$$V(t) \quad = \quad P(-it, i(t^{-1/2}-t^{1/2})) \quad = \quad F(-t^{3/4}, t^{1/4}+t^{-1/4})\,. \tag{4}$$

By $[P]_M$ we denote the coefficient of the monomial M in the polynomial P. If P has a single variable, then we use the exponent rather than the whole monomial for M. (For example, $[V]_3 = [V]_{t^3}$ for $V \in \mathbb{Z}[t^{\pm 1}]$.)

Definition 2.1 Two crossings of a link diagram D are twist equivalent if up to flypes they form a clasp. (As in [St5], this means that they are either $\sim$-equivalent or $\tilde{\underset{*}{\sim}}$-equivalent.)

Let $t(D)$ be the *twist number* of a diagram D, which is the number of its twist equivalence classes. We call such equivalence classes also simply *twists*. For a knot K we define its twist number by

$$t(K) := \min\{\,t(D) : D \text{ is a diagram of } K\,\}\,.$$

(In order to distinguish the twist number from the variable of the Jones polynomial, we will always write an argument in parentheses behind it.)

Let $c(D)$ be the crossing number of a diagram D, and $c(K)$ the crossing number of a knot or link K,

$$c(K) := \min\{\,c(D) : D \text{ is a diagram of } K\,\}\,.$$

Thistlethwaite proved in [Th] that (with our convention for Λ) for a link diagram D of $c(D)$ crossings we have $[\Lambda(D)]_{z^l a^m} \neq 0$ only if $l+|m| \leq c(D)$, and that D is A resp. B-adequate iff such a coefficient does not vanish for some l and m with $l-m=c(D)$ resp. $l+m=c(D)$. These properties imply most of his results, incl. the main one, the crossing number minimality of adequate diagrams.

The coefficients of $\Lambda(D)$ for which $l \pm m = c(D)$ form the "critical line" polynomials $\phi_{\mp}(D)$. Thistlethwaite expresses these polynomials in terms of some graph invariants, so they clearly encode combinatorial information of the diagram. Unfortunately, we do not know how to interpret most of this information, i.e., to say what tangible features of the diagram it measures.

Still the minimal and maximal degree of $\phi_{\pm}(D)$ do have a "visual" meaning. In §5 of [St15] we translated this meaning to our present context. One degree can be expressed also in terms of the writhe and Jones polynomial, thus giving a new obstruction to semiadequacy (settling in particular the undecided 12 crossing knot in [Th]). However, for semiadequate links this obstruction (consequently vanishes and) gives no new information. In contrast, the other degree (see theorem 9.2 below) gives a new invariant for semiadequacy. This will be used crucially in the last sections, together with our work of "decoding" in such a visual way the Jones polynomial coefficients.

We also require occasionally the *Alexander polynomial* Δ. We define it here by being 1 on the unknot and a relation involving the diagrams occurring in (3) of triples of links L_+, L_- and L_0, differing just at one crossing,

$$\Delta(L_+) - \Delta(L_-) = \left(t^{1/2} - t^{-1/2}\right)\Delta(L_0)\,. \tag{5}$$

This relation is clearly a special case of (3). Consequently, there is the substitution formula (see [LM2]; i is again the complex unit),

$$\Delta(t) = P(i, i(t^{1/2} - t^{-1/2}))\,,$$

expressing Δ as a special case of P.

Definition 2.2 When

$$V_K = a_0 t^k + V_1 t^{k+1} + \ldots + a_d t^{k+d} \tag{6}$$

with $a_0 \neq 0 \neq a_d$ is the Jones polynomial of a knot or link K, we will write for d the *span* $\operatorname{span} V_K$ of V, for k the *minimal degree* $\min\deg V_K$ and for $k+d$ the *maximal degree* $\max\deg V_K$. We will use throughout the paper the notation $V_i = V_i(K) = a_i$ and $\bar{V}_i = \bar{V}_i(K) = a_{k-i}$ for the the $i+1$-st or $(i+1)$-last coefficient of V (since these terms will occur often, and to abbreviate the clumsier alternative $[V]_{\min\deg V+i}$ resp. $[V]_{\max\deg V-i}$).

Knots of ≤ 10 crossings will be denoted according to Rolfsen's tables [Ro2, appendix], and for ≥ 11 crossings according to Hoste and Thistlethwaite's program KnotScape [HT], appending for given crossing number the non-alternating knots after the alternating ones. The obverse (mirror image) of K is denoted by $!K$. The mirroring convention we use for knots in the tables, including for ≤ 10 crossings, is this of [HT].

Dasbach-Lin (among other authors) noticed that $t(D)$ is an invariant of alternating *diagrams* D of an alternating knot K. This follows, *inter alia*, from [MT]. So we may define

Definition 2.3 The *alternating twist number* $t_a(K)$ is the twist number of an alternating diagram D of K.

Remark 2.1 In general $t_a(K) > t(K)$, for example 7_4 has $t_a(7_4) = 3$ and $t(7_4) = 2$.

In [DL], Dasbach-Lin consider $T_i(K) := |V_i| + |\bar{V}_i|$ and prove

Lemma 2.1 ([DL]) For an alternating knot K, we have $t_a(K) = T_1(K)$.

They apply this to obtain bounds on the volume of alternating knots in terms of $|V_1|$ and $|\bar{V}_1|$. This led us to consider these values closer. In what follows we will explain the outcome. In particular, one can extend some of their results in a weaker form to adequate links. On the other hand, T_1 seems to be useful also in other situations, as we will see.

3. Jones polynomial of (semi)adequate links

3.1. The second coefficient

We consider the bracket [Ka] (rather than Tutte) polynomial. The A-state of D, the state with all splicings A, is denoted by $A(D)$. (Occasionally, we omit the argument D in this notation, if no ambiguity arises.) For us a state is always understood as a planar picture of loops (solid lines) and traces connecting these loops (dashed lines). Then it is clear that and how to reconstruct D from $A(D)$.

Definition 3.1 One fundamental object exploited in this paper is the *A-graph* $G(A) = G(A(D))$ of D. It is defined as the planar graph with vertices given by loops in the A-state of D, and edges given by crossings of D. (The trace of each crossing connects two loops.) The analogous terminology is set up also for the B-state.

Clearly the A-state determines the A-graph, but not conversely. Their distinction is relevant in some situations. However, $G(A(D))$ (including its planar embedding) determines $A(D)$ if D is alternating; then sometimes $G(A(D))$ is called the Tait graph of D. Note also that, for alternating D, the duality of crossing traces between $A(D)$ and $B(D)$ corresponds to the duality (in the usual graph-theoretic sense) of edges in the planar graphs $G(A(D))$ and $G(B(D))$.

Let $v(G)$ and $e(G)$ be the number of vertices and edges of a graph G. Let G' be G with multiple edges removed (so that a simple edge remains). We call G' the *reduction* of G.

We will write sometimes

$$s_+(D) = v(G(A(D))) = v(G(A(D))'), \qquad s_-(D) = v(G(B(D))) = v(G(B(D))').$$

The definition of A-adequate can be restated saying that $G(A(D))$ has no edges connecting the same vertex. For B-splicings the graph $G(B(D))$ and the property B-adequate are similarly defined (and what is stated below proved).

In the following, we shall explain the second and third coefficient of the Jones polynomial in semiadequate diagrams. Bae and Morton [BMo] and Manchon [Ma] have done work in a different direction, and studied the leading coefficients of the bracket (which are ± 1 in A-adequate diagrams) in more general situations.

Then we have

Proposition 3.1 If D is A-adequate connected diagram, then in the representation (6) of V_D we have $V_0 = \pm 1$, $V_1 V_0 \le 0$, and

$$|V_1| = e(G(A(D))') - v(G(A(D))') + 1 = b_1(G(A(D))') \tag{7}$$

is the first Betti number of the reduced A-graph.

Proof. This will follow from the proof of proposition 3.3. Alternatively, see [St] where the formula is proved for D positive (then $A(D)$ is the Seifert graph). The adaptation to A-adequate diagrams is easy. □

The formula (7) was also explained in [DL2], and later generalized in [DFK+], relating the second coefficient to certain combinatorial maps obtained from the diagram.

Lemma 3.1 If G is a planar simple graph (no multiple edges), then $b_1(G) \le \left\lfloor \frac{2}{3} e(G) \right\rfloor - 1$. This inequality is sharp for proper G when $e(G) > 2$.

(Here and below $\lfloor x \rfloor$ is the largest integer not greater than x.)

Proof. The plane complement of G has $b_1(G) + 1$ cells (including the one at infinity). Each cell has at least 3 edges, since G has no multiple edges, and each edge bounds at most two cells. Now to make the inequality sharp, start with a triangle, repeatedly connect (with 3 edges) a new inner vertex in one of the faces of the graph. If $e \equiv 2 \bmod 3$, finally add a vertex on an edge h between $v_{1,2}$ and connect it to v, where $v \ne v_{1,2}$ is a vertex of one of the faces that h bounds. □

In the following positivity arguments will be essential. It is well-known that if a diagram D is positive, then it is A-adequate: the A-state of D is just the Seifert picture of D (and the A-state loops are the Seifert circles). Since A-adequacy is an unoriented condition, D would remain A-adequate even if we alter orientation of some components. We say that an unoriented diagram D *admits a positive orientation*, or is *positively orientable*, if it arises from such a diagram by forgetting orientation. The following lemma, which will be repeatedly used below, specifies which A-adequate diagrams are positively orientable.

Lemma 3.2 Let D be A-adequate. Then D is positively orientable iff $G(A(D))$ is a bipartite graph.

Proof. If D is positively orientable, its graph $G(A(D))$ is the Seifert graph of a (positive) diagram, and hence bipartite. Conversely, if $G(A(D))$ is bipartite, it is possible to orient the loops in $A(D)$ so that each trace looks locally like . Then it is clear that it is possible to extend this loop orientation to an orientation of D, and with that orientation D becomes positive. $\square$

Corollary 3.1 Let L be an A-adequate link, with an A-adequate connected diagram D, then

$$1-\left\lfloor \frac{2}{3}c(D)\right\rfloor \leq V_0V_1 \leq 0\,.$$

If $V_1=0$, then D admits a positive orientation, and (with this orientation) L is fibered.

Proof. We have $V_1=-V_0b_1(G(A(D))')$. Now $G(A(D))'$ is planar, so to its b_1 we apply lemma 3.1.

Now if $V_1=0$, then $G(A(D))'$ is a tree, so in particular bipartite. So the loops of the A-state of D can be oriented alternatingly, and then they become Seifert circles, and with the inherited orientation the crossings become positive. That L is then fibered follows from [St]. $\square$

This gives a new semiadequacy test. Beside that Thistlethwaite's condition on the positive critical line coefficients [Th] involves the Kauffman polynomial, which is considerably slower to calculate, the Jones polynomial features sometimes prove essential, as show the following examples.

Example 3.1 The knots $!9_{46}$, $!9_{47}$ and 9_{48} (mirrored as in KnotScape) have positive critical line coefficients of the Kauffman polynomial, but $|V_0|=2$.

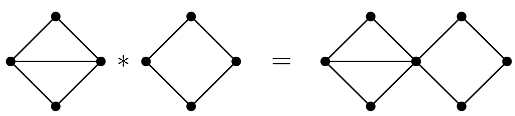
Example 3.2 The knot 14_{22068} has positive critical line coefficients, but $V_0V_1>0$.

The minimal number $e(n)=e_n$ of edges needed for a planar simple graph to have given $b_1=n$ is by lemma 3.1

$$e_n=\begin{cases}\frac{3}{2}(n+1) & n \text{ odd}\\ \frac{3}{2}n+2 & n \text{ even}\end{cases}.$$

For the next example, and for later discussion, it is useful to define some properties of graphs (see also [MS]).

Definition 3.2 The *join* (or *block sum*, as called in [Mu3]) '$*$' of two graphs is defined by



This operation depends on the choice of a vertex in each one of the graphs.

We call v a *cut vertex* of a graph G, if G gets disconnected when deleting all edges incident to v and *additionally* v itself. (When we delete an edge, we understand that a vertex it is incident to is *not* to be deleted too.)

Every connected non-trivial (i.e. with at least one edge) graph G can be written as a join $G_1*\ldots*G_n$ for some non-trivial connected graphs G_i, such that no G_i has a cut vertex. We call G_i the *join factors* of the graph G. The number $a(G)=n$ of join factors of G is called *atom number* of G.

Example 3.3 The knot 14_{46350} has positive critical line coefficients, but $V_0V_1 = -8$. Similarly, 16_{484942}, 16_{487600} and 16_{564314} have $V_0V_1 = -10$. They are non-alternating. Since the A-state of a non-alternating A-adequate diagram has a separating loop, we see that the A-graph $G(A(D))$ of an A-adequate diagram D must be the join of two graphs, i.e. have a cut vertex. Since

$$\min\big(\min\{\, e_a+e_b\,:\,a+b=8,\ a,b>0\,\},\ e_8+1\,\big)=15\,,$$

we see that an A-adequate diagram of 14_{46350} must have at least 15 crossings. Similarly an A-adequate diagram of the 16 crossing knots has at least 18 crossings. This is still a non-trivial information in comparison to the Kauffman polynomial.

In fact even without the assumption that the 16 crossing knots in the previous example are non-alternating, we can deduce this.

Corollary 3.2 If L is non-split and $e(|V_1|) > \operatorname{span} V(L)$, then L is non-alternating.

Proof. If D is an alternating diagram of L, then $c(D) = \operatorname{span} V(L)$, but we have $c(D) = e(A(D)) \geq e(G(A(D))') \geq e(b_1(G(A(D))'))$, and since D is A-adequate, $e(b_1(G(A(D))')) = e(|V_1|)$. □

For the 16 crossing knots in example 3.3 we have $e_{10} = 17$, while $\operatorname{span} V = 14$. So we see that these knots have no A-adequate minimal crossing diagram, and so can not be alternating, not even adequate. Contrarily their Jones polynomials are alternating, and monic on both sides. Thus we have new information even compared to previous conditions on the Jones polynomial.

3.2. Conditions for positivity

Note that in corollary 3.1 the case $V_1 = 0$ poses strong additional restrictions to A-semiadequacy. The change of component orientation alters V only by a positive unit [LM], so that several positivity criteria still apply. Among others, with $n(L)$ the number of components of L, we have $V_0 = (-1)^{n(L)-1}$, and for any $t \in (0,1]$ we have $(-1)^{n(L)-1}V_L(t) \geq 0$ (see [St7]). For knots positively orientable is the same as positive, and we have a series of further properties. For example $\min\deg V_L = g(L)$, the genus of L, which is in particular always positive.

Example 3.4 The knot $!11_{405}$ has $V_0 = 1$ and $V_1 = 0$, but $\min\deg V = -2 < 0$. The knot $!12_{1531}$ has $|V_0| = 1$ and $V_1 = 0$, but $V_0 = -1$, and so is not A-semiadequate. (It has also $\min\deg V < 0$, but taking its iterated connected sums with positive trefoils we can find an example with $\min\deg V > 0$. Alternatively consider $!16_{1133621}$, where $\min\deg V = 2$.) The knots have positive critical line polynomials.

Similarly one can examine the inequalities for the Vassiliev invariants

$$v_2(K) = -\frac{1}{6}V''(1) \geq \frac{c(K)}{4}\,, \quad \text{and} \quad v_3(K) = -\frac{1}{12}V''(1) - \frac{1}{36}V'''(1) \geq \frac{3}{8}c(K) - \frac{3}{4}\,, \tag{8}$$

proved in [St6] (up to the different normalization of v_3 we use here). Again all these, and the previous, conditions eventually dominate in the one or other direction.

Example 3.5 The knot $!15_{120559}$ has $V_0 = 1$ and $V_1 = 0$, and $v_2 = 5$ and $v_3 = 16$, which satisfy the conditions of [St6], but $\min\deg V = 0$. The knot 14_{26659} has $\min\deg V = 3$, $V_0 = 1$ and $V_1 = 0$, and $v_2 = 4 > 14/4$, but $v_3 = 4 < {}^3\!/_8 \cdot 14 - {}^3\!/_4$. The knot 15_{136877} has $\min\deg V = 2$, $V_0 = 1$ and $V_1 = 0$, and $v_3 = 5 > {}^3\!/_8 \cdot 15 - {}^3\!/_4$, but $v_2 = 3 < 15/4$.

Here is another application of the positive case.

Definition 3.3 We call a link *weakly adequate*, if it is both A-adequate and B-adequate.

Example 3.6 The knot of [St14], 11_{550}, has a B-adequate 11 crossing diagram, which is not A-adequate, and an A-adequate (because positive) 12 crossing diagram. So it is weakly adequate but not adequate. Another such example is Perko's knot 10_{161}.

Proposition 3.2 A non-trivial weakly adequate *knot* has $T_1 > 0$.

Proof. $T_1(K) = 0$ implies that both K and its mirror image are positive, which is impossible. □

Example 3.7 Examples of adequate *links* with $T_1 = 0$ include the split union of a figure-8-knot with an unknot, the Hopf link, and iterated connected sums and disconnected twisted blackboard framed cables of it. More such (incl. hyperbolic) examples will be given below.

For knots a simple source of $T_1 = 0$ examples is

Corollary 3.3 Let K be a weakly adequate knot that has the Jones polynomial of a (m,n)-torus knot. Then $m = 2$. In particular, the only weakly adequate torus knots are the $(2,n)$-torus knots.

The second claim implies the claim for adequate torus knots, that can be alternatively proved by combining work of [Th] with [Mu2], and in particular the property for alternating torus knots, for which now numerous further proofs are known (see [St7] for some discussion).

Proof. Since a torus knot is positive and fibered we see directly $V_1 = 0$ for the Jones polynomial of any torus knot. Then (after possible mirroring) K is positive. Now we would like to prove that $\bar{V}_1 = 0$ when $m,n > 2$. We use Jones' formula for the polynomial of a torus knot (see [J, prop. 11.19]).

$$V(T_{m,n}) = -\frac{t^{m+n} - t^{m+1} - t^{n+1} + 1}{t^2 - 1} t^{(m-1)(n-1)/2}. \tag{9}$$

Then, by conjugating and adjusting the degree, we see that $\bar{V}_1 = 0$ is equivalent to the vanishing of the t-derivative of $\frac{t^{m+n} - t^{m-1} - t^{n-1} + 1}{1 - t^2}$ at $t = 0$. This is directly to verify when $m,n > 2$. Since one can recover (m,n) from $V(T_{m,n})$, the Jones polynomial distinguishes all torus knots, and the second claim follows. □

Example 3.8 It is not clear what non-torus knots can have the Jones polynomial of a torus knot. For $m = 2$ infinitely many such knots are known from Birman [B] (see §6.3), but the case $m \geq 3$ is open. Such examples do not occur among prime knots up to 16 crossings. The simplest non-torus knot examples of $T_1 = 0$ are $16_{1095701}$ and $16_{1318182}$ (which are hyperbolic).

The next corollary determines asymptotically the minimal value of $V_0 V_1$ for positive link diagrams.

Corollary 3.4 Let L be a positive $n(L)$-component link, with a positive connected diagram D. Then

$$-\frac{1}{2}c(D) \leq (-1)^{n(L)-1} V_1 \leq 0,$$

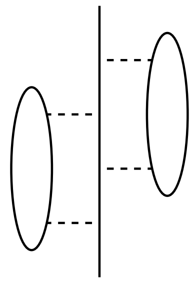
and the left inequality is asymptotically sharp (for large $c(D)$).

Proof. Now $G(A(D))'$ is bipartite, so the smallest cell is a 4-gon. The rest of the argument is as for lemma 3.1. To see sharpness consider $A(D)$ to be the graph made by subdividing a square into $n \times n$ smaller squares and let $n \to \infty$. □

3.3. The third coefficient

The third coefficient in semiadequate diagrams can still be identified in a relatively self-contained form. It depends, however, on more than just $G(A(D))'$ or $G(A(D))$. Originally, the motivation for the below formula was to estimate the volume of Montesinos links (see §7). However, the final sections show much more important applications of it.

Definition 3.4 We call two edges $e_{1,2}$ in $G(A(D))'$ *intertwined*, if the following 3 conditions hold:



1. $e_{1,2}$ have a common vertex v.
2. The loop l of v in the A-state of D separates the loops $l_{1,2}$ of the other vertices $v_{1,2}$ of $e_{1,2}$.
3. $e_{1,2}$ correspond to at least a double edge in $A(D)$, and there are traces of four crossings along l that are connected in cyclic order to l_1, l_2, l_1, l_2.

It should be made clear that intertwinedness of edges in $G(A(D))'$ still depends on the *state* $A(D)$, rather than just the graph $G(A(D))'$ or $G(A(D))$: in the graphs the intertwining information becomes lost.

Definition 3.5 A *connection* in $A(D)$ is the set of traces between the same two loops, i.e. an edge in $G(A(D))'$. A connection e in $A(D)$ is said *multiple*, if it consists of at least two crossing traces. More generally we can define the *multiplicity* of a connection as the number of its traces.

Then we can speak also of intertwined (multiple) connections. We will later occasionally relax terminology even more and speak just of intertwined loops M and N, when their connections to a third loop L are intertwined. This is legitimate, because one can determine L from (M,N) uniquely.

Call below a loop in $A(D)$ *separating* if it is connected by crossing traces from either side. So if connections (M,L) and (N,L) are intertwined, then L would be separating, and connected by M and N from opposite sides. Clearly there can be only one such L for given M and N. (Moreover, almost throughout where we will apply this terminology below, there will be in fact only one separating loop in $A(D)$.)

Definition 3.6 Define the *intertwining graph* $IG(A(D))$ to consist of vertices given by multiple connections in $A(D)$ (or multiple edges in $G(A(D))$), and edges connecting pairs of intertwined connections.

This is the second graph associated to D, which plays a fundamental role in the whole paper. Note that this is a simple graph (no multiple edges), but not necessarily planar or connected. It may also be empty. For example, for an alternating diagram this graph has no edges, so is a (possibly empty) set of isolated vertices. With the preceding remark, $IG(A(D))$ is determined by the state $A(D)$, but not (in general) by the graph $G(A(D))$.

Proposition 3.3 If D is A-adequate, then

$$V_0V_2 = \binom{|V_1|+1}{2} - \triangle A(D)' + \chi(IG(A(D))) = \binom{|V_1|+1}{2} + e_{++}(A(D)) - \triangle A(D)' - \delta A(D)',$$

where $\triangle A(D)'$ is the number of triangles (cycles of length 3) in $G(A(D))'$, $\delta A(D)' = e(IG(A(D)))$ the number of intertwined edge pairs in $G(A(D))'$, and $e_{++}(A(D)) = v(IG(A(D)))$ the number of multiple connections in $A(D)$.

This formula was obtained independently and simultaneously by Dasbach-Lin [DL2]. Their proof is longer, but it unravels the underlying combinatorics completely, while here we will help ourselves with some (skein theoretic) work from [St].

Remark 3.1 Note that in alternating diagrams $\delta A(D)' = 0$, since there are no separating loops, while in positive diagrams $\triangle A(D)' = 0$ because $G(A(D))'$ is bipartite. Note also that a pair of intertwined edges does not occur in a triangle.

Proof of proposition 3.3. The states that contribute to V_2 have B-splicings only in at most two edges in $G(A(D))'$, or in 3 edges that form a triangle.

The zero edge case is the A-state that defines V_0, and this gives a contribution of $V_0\binom{s_+-1}{2}$ to V_2 and $V_0(s_+-1)$ to V_1, where $s_+(D)$ is the number of loops in the A-state.

The one edge case consists of the states that determine V_1. If we have a connection e in $A(D)$ of multiplicity k, then the contributions of these states to V_1 are

$$V_0\left(-k+\binom{k}{2}-\binom{k}{3}+\dots\right) = -V_0\,. \tag{10}$$

With the term from the zero edge case this proves proposition 3.1. Now the contributions to V_0V_2 are

$$-k(s_+-2)+\binom{k}{2}(s_+-1)-\binom{k}{3}s_+ +\binom{k}{4}(s_++1)-\dots = (s_+-2)\cdot(-1)+\left[\binom{k}{2}\cdot 1-\binom{k}{3}\cdot 2+\binom{k}{4}\cdot 3-\dots\right].$$

The bracket evaluates to 1 for multiple connections ($k \geq 2$) and to 0 for single ones ($k = 1$).

Now we must consider the states that have B-splitting in two edges in $G(A(D))'$. For most pairs of edges one has the product of two evaluations as in (10) (but with opposite sign, since we have one loop less), which leads to $\binom{e_+}{2}$, with e_+ the number of edges of $G(A(D))'$. So far we arrived, in collecting terms in V_0V_2, at

$$\binom{s_+-1}{2}-\left[e_+(s_+-2)-e_{++}\right]+\binom{e_+}{2} = \binom{e_+-s_++2}{2}+e_{++} = \binom{|V_1|+1}{2}+e_{++}\,. \tag{11}$$

Herein all pairs of edges in $G(A(D))'$ were counted as if they would contribute 1. Now, there are two types of pairs of edges where this may not be so, the intertwined edge pairs, and those contained in triangles.

So now we need to take care of these "degenerate pairs of edges" in $G(A(D))'$, and of states with a triple of B-splicings in triangles of $G(A(D))'$. Below we show how to find the corrections to (11) coming from these.

Consider triangles, and edge pairs contained in triangles. (Any edge pair is contained in at most one triangle.) Fix an ordering of edges in $G(A(D))'$, and then of traces in $A(D)$, so that if $e_i < e_j$ in $A(D)$ then $e'_i < e'_j$ in $G(A(D))'$. Let $g_{1,2,3} \in G(A(D))'$ be the edges of a fixed triangle with $g_1 < g_2 < g_3$. We consider all states that have a B-splitting in preimages of at least two of $g_{1,2,3}$. Set H_i to be the set of preimage traces of g_i in $A(D)$.

Let $i, j \in \{1,2,3\}$ with $i \neq j$. We sort the states by the minimal (in $A(D)$ trace order) preimages $h_i \in H_i$, $h_j \in H_j$ of g_i, g_j that are B-split. The case that in all of $H_{1,2,3}$ a B-splicing is done is incorporated into $i = 1, j = 2$. If one of h_i, h_j is not maximal in H_i resp. H_j, then the contributions of the states that B-split any (possibly empty) set of crossings in $H_{i,j}$ bigger than $h_{i,j}$, and possibly in H_3 if $i = 1, j = 2$, is the product of two or three alternating binomial coefficient sums, at least one of which evaluates to 0. So we need to count only contributions of states where $h_{i,j}$ are both the maximal elements in $H_{i,j}$. However, if $\{i, j\} = \{1,2\}$, we still have the non-trivial (and therefore evaluating to 0) alternating binomial coefficient sum coming from splicing edges in H_3. So the non-zero contributions come from $\{i, j\} \neq \{1,2\}$, and they are two, rather than 3, as counted in $\binom{e_+}{2}$. Thus the number of triangles must be subtracted from (11).

Finally we need to take care of intertwined edge pairs. Here the combinatorics is even messier, but we can help ourselves with our previous work. Clearly, the contribution will not depend on the remaining loops, and edges connecting them. So we may evaluate it on a diagram what consists only of the three relevant loops. Such are exactly the diagrams of prime closed positive 3-braids. From [St] we know that $V_0V_2 = 1$ for such links. Since we can evaluate all other terms easily, we obtain that intertwined edge pairs give no contribution, and so must be subtracted from the $\binom{e_+}{2}$ in (11). With this the desired formula is proved. □

Here is an application to positive links.

Proposition 3.4 If K is a positive link, then with $\gamma(K) = 2\min\deg V_K - |V_1| = 1 - \chi(K) - |V_1|$, we have

$$\min\left(0, \gamma(K) - \left(\frac{\gamma(K)}{2}\right)^2\right) \leq V_0V_2 - \binom{|V_1|+1}{2} \leq \gamma(K)\,.$$

Moreover, these inequalities are sharp (i.e. equalities) for links of arbitrarily small χ.

Proof. If D is a positive diagram of K, then $1-\chi = b_1(G(A(D)))$ and $|V_1| = b_1(G(A(D))')$, so

$$\gamma(K) \geq e_{++}(A(D))\,. \tag{12}$$

We know also that $\triangle A(D)' = 0$. So it remains to see

$$\delta A(D)' \leq \left(\frac{e_{++}}{2}\right)^2. \tag{13}$$

Now one can color the crossing traces (and then also the edges in $G(A(D))'$) black and white, so that traces of different (resp. same) colors connect to any loop from different (resp. the same) side. Then only edge pairs of opposite color can be intertwined in $G(A(D))'$. For given number of multiple traces, the maximal number of pairings occurs when all black traces are intertwined with all white traces. In this situation the maximal number of intertwinings is when exactly one half of the multiple traces have either color.

To make the inequalities sharp, note that (12) is sharp whenever $A(D)$ has no ≥ 3-ple traces. Many such diagrams have $\delta = 0$, for example the $(2,2,\dots,2)$-pretzel diagrams, which thus realize the right inequality sharply.

To make the left inequality sharp we need to make (13) sharp. Consider a positive diagram with one separating Seifert circle a and n Seifert circles $c_1,\dots,c_n$, $d_1,\dots,d_n$ on each side of a. Let them be attached to a by crossings in cyclic order $c_1,\dots,c_n,d_1,\dots,d_n,c_n,\dots,c_1,d_n,\dots,d_1$. □

Remark 3.2 Note also that $e_{++}(D) \leq {}^1\!/_2 c(D)$. This gives some estimate on the crossing number of a positive diagram. It seems, however, unlikely strong enough to prove that some positive knots have no minimal (crossing number) positive diagram. To find infinitely many such knots is a problem was proposed by Nakamura [Na]. We will later give a solution (see proposition 5.1), although using more elaborate arguments.

Corollary 3.5 If K is a fibered positive link, then

$$2\min\deg V - \min\deg V^2 \leq V_0V_2 \leq 2\min\deg V\,, \tag{14}$$

and when K is prime, then $V_0V_2 \leq 1$.

Proof. Now $|V_1| = 0$. The inequalities (14) follow except the cases of $(1-\chi)/2 = \min\deg V < 2$ in the left inequality. If $1-\chi \leq 3$, then the left inequality is still valid when (12) is sharp. Since for a fibered positive link, $G(A(D))'$ is a tree, all connections in $A(D)$ can be assumed multiple (otherwise D can be reduced). So if (12) is unsharp, then at least a triple connection exists. Then for $1-\chi \leq 3$ we have at most 3 Seifert circles. Such diagrams are of closed positive 3-braids. Then we know $V_0V_2 = 1$, which satisfies the inequalities.

If now K is prime, then D is prime (see [O]). Now let $\star$ be an equivalence relation on edges in $G(A(D))'$ defined by the transitive expansion of the relation 'intertwined'. That is, $k\star l$ if they can be related by a sequence of consecutively intertwined edges. Now it remains to argue that, since $G(A(D))'$ is a tree and D is prime, $\star$ must have a single equivalence class, and therefore $\delta A(D)' \geq e_{++} - 1$.

If $\star$ has ≥ 2 equivalence classes, elements $x_{1,2}$ in such classes connect some loop l in the A-state (from which side is not specified). For a loop x connecting l by a connection of n crossing traces, inscribe an n-gon, consisting of n chords, into l with vertices the endpoints of the traces. Then if $x_{1,2}$ are in different $\star$-equivalence classes, the chord diagram[1] in l will be disconnected, with the chords coming from $x_{1,2}$ in different components. So one can put a interval γ that intersects l twice and contains parts of the chord diagram on both sides. Because $G(A(D))'$ is a tree, none of the loops inside or outside l can be connected except through l. So γ can be extended to a closed curve that gives a non-trivial factor decomposition of D, a contradiction. □

[1] Here we have chords with coinciding basepoints, so this is not the sort of chord diagram from Vassiliev invariant theory.

Remark 3.3 The examples realizing the left inequality sharply in proposition 3.4 remain valid also for corollary 3.5. The right inequalities in the corollary are sharp for positive braids. On the other hand, positive braids allow an estimate also of V_3 [St], which seems hard in general. See also remark 7.3.

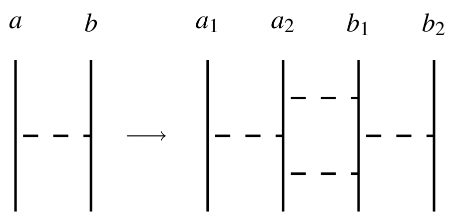
3.4. Cabling

In the following we briefly explain the effect of the formulas for $V_{1,2}$ under cabling.

Lemma 3.3 Let D be an A-adequate link diagram (not necessarily reduced) and D_2 the (blackboard framed) 2-parallel of D. Then $V_0V_1(D_2) = V_0V_1(D)$ and

$$V_0V_2(D_2) = \binom{|V_1(D)|}{2} + 1 - \triangle(A(D)).$$

Proof. To come from D to D_2, one applies the following modifications



Let $k(A(D))$ be the number of loops in $A(D)$ connected by only one trace to some other loop. We call such loops isolated.

- All edges in $A(D_2)$ are multiple except $k(A(D))$ because all a-b connections are doubled in $a_2 - b_1$, and all loops in $A(D)$ except $k(A(D))$ are connected by at least two traces to some other loop(s). So
$$e_{++}(A(D_2)) = e(G(A(D_2))') = v(A(D)) + e(G(A(D))') - k(A(D)). \tag{15}$$
- Triangles in $A(D_2)$ correspond bijectively to triangles in D,
$$\triangle(A(D_2)) = \triangle(A(D)). \tag{16}$$
- Now consider intertwined edges in $G(A(D_2))'$. First, intertwined connections (or edge pairs in $G(A(D_2))'$) correspond to
$$\big((a_2,b_1)\text{-connection, }(a_1,a_2)\text{-connection}\big)\,\big((a_2,b_1)\text{-connection, }(b_1,b_2)\text{-connection}\big),$$
such that the (a_1,a_2)-connection resp. (b_1,b_2)-connection are multiple, i.e. a resp. b is not an isolated loop in $A(D)$. So for each edge $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}$ in $G(A(D))'$ there exist exactly two intertwined edge pairs in $G(A(D_2))'$, but from these $2e(G(A(D))')$ pairs $k(A(D))$ must be excluded. (Here we use also that $a \neq b$.) So
$$\delta(G(A(D_2))') = 2e(G(A(D))') - k(A(D)). \tag{17}$$

Then using (15), (16), and (17),

$$\begin{aligned} V_0V_2(D_2) &= \binom{|V_1(D_2)|+1}{2} - \triangle(A(D_2)) + e_{++}(A(D_2)) - \delta(A(D_2)) \\ &= \binom{|V_1(D)|+1}{2} - \triangle(A(D)) + v(A(D)) - e(G(A(D))') \\ &= \binom{|V_1(D)|+1}{2} - \triangle(A(D)) + 1 - |V_1(D)| \\ &= \binom{|V_1(D)|}{2} - \triangle(A(D)) + 1. \qquad \square \end{aligned}$$

Corollary 3.6 The quantities $\triangle(A(D))$ and $\chi(IG(A)) = e_{++}(A(D)) - \delta(A(D))$ are invariants of A-adequate diagrams D of a link L. We call them together with $\chi(G(A))$ the *A-state (semiadequacy) invariants* of L.

Proof. Let $D_{1,2}$ be A-adequate diagrams of a link. By adding kinks (which does not change the quantities) we may assume that $D_{1,2}$ have the same writhe. Then their 2-parallels are isotopic, and we see that $\triangle(A(D))$ is invariant. The invariance of $\chi(IG(A(D)))$ follows from looking at $V_0V_2(D_i)$. $\square$

Remark 3.4 It is easily observed that the formulas remain correct if we replace 2-cables by n-cables for $n > 2$. (This means also that we obtain no new invariant information from higher cables.)

The advantage of lemma 3.3 and corollary 3.6 is to extract invariant information out of V and its cables, which is much more direct to compute than the whole (NP-hard; see [JVW]) polynomial. This can, for example, be applied to verify for a given knot a large number of diagrams. Note also that the meaning of the various quantities encountered is rather visual from a semiadequate diagram – in contrast, for example, to the coefficients of the Kauffman polynomial studied in [Th2] in this context. The work of sections 5, 9 and 10 exploits heavily this big advantage.

Let us note the following, which will be useful below,

Corollary 3.7 Semiadequacy invariants are mutation invariant.

Proof. It is well-known [MT] that the Jones polynomial and all its cables are mutation invariant. $\square$

4. Non-triviality of the Jones polynomial

4.1. Semiadequate links

An important application of corollary 3.1 is

Theorem 4.1 There is no non-trivial semiadequate link L with trivial Jones polynomial (i.e., polynomial of the same component number unlink), even up to units.

This is an extension of the results of [Ka, Mu, Th2] for alternating links, [LT] for adequate links, [St6] for positive *knots*, and [Th2] for the Kauffman polynomial of semiadequate links. While the existence of non-trivial links with trivial polynomial is now settled for $n(L) \geq 2$ [EKT], the (most interesting) case $n(L) = 1$ remains open. In fact, except for the above cited (meanwhile classical) results, and despite considerable (including electronic) efforts [Bi, Ro, DH, St9], even nicely defined general classes of knots on which one can exclude trivial polynomial are scarce. Our theorem seems thus to subsume all previous self-contained results in this direction.

Proof. We use induction on $n(L)$. Let first $n(L) = 1$. If the Jones polynomial of a *knot* is determined up to units, it is uniquely determined, because $V(1) = 1$ and $V'(1) = 0$. Thus $V = 1$, and we have $V_1 = 0$. Now by corollary 3.1, L must be a positive knot, but there is no non-trivial positive knot with trivial Jones polynomial [St6]. Now let $n(L) > 1$. By [Th, corollary 3.2], if L is split, so is any semiadequate diagram D of L, and in this case we can argue by induction on the number of components of the split parts of D. Else D is connected (non-split), but since up to units $V = (-t^{1/2} - t^{-1/2})^{n(L)-1}$, we have $V_0V_1 = n(L) - 1 > 0$ and a contradiction to corollary 3.1. $\square$

4.2. Montesinos links

To avoid confusion, let us fix some terminology relating to Montesinos links. The details on Conway's notation can be found in his original paper [Co], or for example in [Ad].

Let the *continued* (or *iterated*) *fraction* $[[s_1, \ldots, s_r]]$ for integers s_i be defined inductively by $[[s]] = s$ and

$$[[s_1, \ldots, s_{r-1}, s_r]] = s_r + \frac{1}{[[s_1, \ldots, s_{r-1}]]}\,.$$

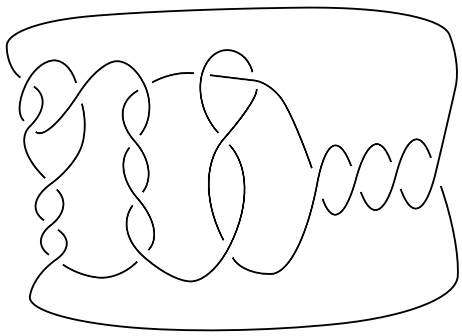


Figure 2: The Montesinos knot $M(3/11,-1/4,2/5,4)$ with Conway notation $(213,-4,22,40)$.

The rational tangle $T(p/q)$ is the one with Conway notation $c_1\ c_2\ \dots c_n$, when the c_i are chosen so that

$$[[c_1,c_2,c_3,\dots,c_n]] = \frac{p}{q}\,. \tag{18}$$

One can assume without loss of generality that $(p,q)=1$, and $0<|p|<q$. A *rational (or 2-bridge) link* $S(q,p)$ is the closure of $T(p/q)$.

Montesinos links (see e.g. [BZ]) are generalizations of pretzel and rational links and special types of arborescent links. They are denoted in the form $M(\frac{p_1}{q_1},\dots,\frac{p_n}{q_n},e)$, where e,q_i,p_i are integers, $(q_i,p_i)=1$ and $0<|p_i|<q_i$. Sometimes e is called the *integer part*, and the $\frac{p_i}{q_i}$ are called *fractional parts*. They both together form the *entries*. If $e=0$, it is omitted in the notation. Our convention follows [Oe] and *may differ* from other authors' by the sign of e and/or multiplicative inversion of the fractional parts. For example $M(\frac{p_1}{q_1},\dots,\frac{p_n}{q_n},e)$ is denoted as $\mathfrak{m}(e;\frac{q_1}{p_1},\dots,\ \frac{q_n}{p_n})$ in [BZ, definition 12.28] and as $M(-e;(q_1,p_1),\dots,\ (q_n,p_n))$ and the tables of [Kw].

If all $|p_i|=1$, then the Montesinos link $M(\pm\frac{1}{q_1},\dots,\pm\frac{1}{q_n},e)$ is called a *pretzel link* $P(\pm q_1,\dots,\pm q_n,\varepsilon,\dots,\varepsilon)$, where $\varepsilon=\mathrm{sgn}(e)$, and there are $|e|$ copies of it. We also say it is a $(\pm q_1,\dots)$-pretzel link.

To visualize the Montesinos link from a notation, let q_i/p_i be continued fractions of rational tangles $c_{1,i}\dots c_{n_i,i}$ with $[[c_{1,i},c_{2,i},c_{3,i},\dots,c_{l_i,i}]]=\frac{q_i}{p_i}$. Then $M(\frac{p_1}{q_1},\dots,\frac{p_n}{q_n},e)$ is the link that corresponds to the Conway notation

$$(c_{1,1}\dots c_{l_1,1}),(c_{1,2}\dots c_{l_2,2}),\dots,(c_{1,n}\dots c_{l_n,n}),e\,0\,. \tag{19}$$

An example is shown in figure 2.

An easy exercise shows that if $p_i>0$ resp. $p_i<0$, then

$$M(\dots,p_i/q_i,\dots,e) = M(\dots,(p_i\mp q_i)/q_i,\dots,e\pm 1)\,, \tag{20}$$

i.e. both forms represent the same link. In our convention the identity (20) can be read naturally to preserve the sum of all entries, and an integer entry can be formally regarded as a fractional part. Theorem 12.29 in [BZ] asserts that the entry sum, together with the vector of the fractional parts, modulo $\mathbb{Z}$ and up to cyclic permutations and reversal, determine the isotopy class of a Montesinos link L. So the number n of fractional parts is an invariant of L; we call it the *length* of L. If the length $n<3$, an easy observation shows that the Montesinos link is in fact a rational link.

We can now conclude some work on Montesinos links that could only be partially done in [LT] using adequacy. Adequacy is clearly too restrictive, for instance the knot 9_{46} is a pretzel knot that is not adequate, but semiadequate; see example 3.1. (As the paper [HTY] shows, the problem to determine the crossing number of a general Montesinos link can unlikely be solved using the Jones polynomial only; in [LT] the Kauffman polynomial was used.)

Corollary 4.1 No Montesinos link has trivial Jones polynomial up to units.

Proof. Let
$$M(p_1/q_1,\dots,p_n/q_n,e) \tag{21}$$
be a representation chosen up to mirroring so that $n\ge 2$, $0<p_i/q_i<1$. If $e\ge 0$, the canonical diagram to this representation is alternating, if $e<-1$ it is semiadequate by splicing so that the $-e$ crossings create $1-e$ loops in between them (and using that no p_i/q_i represents a single crossing tangle). Let this be, say, the A-state. Now consider $e=-1$. Then write $M(p_1/q_1,\dots,p_{n-1}/q_{n-1},p_n'/q_n)$ such that $-1<p_n'/q_n<0$. If now $n=2$, we have a rational link, so let $n>2$. Then since the tangle representing p_n'/q_n does not consist of a single crossing, the B-state shows semiadequacy. □

We can say something more on the Jones polynomial, also in cases not covered in [LT].

For the rational link we have a new handy condition, which is a simple straightforward observation.

Proposition 4.1 If L is a rational link, then $||V_1|-|\bar V_1||\le 1$. □

Let $D=M(p_1/q_1,\dots,p_n/q_n,e)$ be a Montesinos link diagram of a representation chosen so that $n\ge 3$, $0<p_i/q_i<1$. (For $n\le 2$ we obtain 2-bridge links.) Then if $e\ge 0$ or $e\le -n$, the link is alternating. If $e<-1$ and $e>1-n$ then the link is adequate and considered in [LT]. We write $S(q,p)$ in Schubert form for the rational link with fraction p/q.

Proposition 4.2 (1) If $e<-1$ in (21) then
$$|\bar V_1|=|\bar V_1(S(q_1,p_1)\#\dots\#S(q_n,p_n))|+\begin{cases}1 & \text{if } e<-2\\ 0 & \text{if } e=-2\end{cases}$$
(2) If $e=-1$ and $\delta(L):=\#\{\,i:p_i/q_i\ge 1/2\,\}-1$ is non-zero, then $|\bar V_0|=\delta$, and $\operatorname{span}V(D)=c(D)-3$.

Proof. The diagram D is B-adequate, and the B-state has one separating loop that contains the e twists. They contribute to $b_1(A(D))$ depending of whether $e=-2$ or $e<-2$. This explains part (1).

In part (2), the diagram D is almost alternating. So it is neither A- nor B-adequate. If we resolve the dealternator by the bracket relation into diagrams D_A and D_B, we observe that D_A and D_B are both alternating (and in particular adequate). The lowest terms in A in $A\langle D_A\rangle$ and $A^{-1}\langle D_B\rangle$ will cancel (since D is not B-adequate), but δ is exactly the difference of the second lowest terms. It is easy to see that the highest terms in A in $A\langle D_A\rangle$ and $A^{-1}\langle D_B\rangle$ again cancel (since D is not A-adequate), but the second highest terms differ by ±1 and do not cancel. So if $\delta\ne 0$, the span of $V(D)$ is by two less than the difference between the highest and lowest $t=A^{-1/4}$ terms, which is $(c(D)+s_+(D)+s_-(D)-2)/2$. Since by direct check $s_+(D)+s_-(D)=c(D)$, the claim follows. □

Remark 4.1 Note in particular that part (2) allows to construct Montesinos links whose leading or trailing Jones polynomial coefficient is arbitrary up to sign. (See also [Ma, HTY] for similar families of such examples.)

Example 4.1 It was at first suggestive that one could extend corollary 4.1 also to arborescent links, but a proof met difficulties that led to examples like the one in figure 3. This arborescent knot, with Conway notation $(-3,-3,-3)2,(3,3,-3)1$, has Jones polynomial with leading and trailing coefficient ±3, and so is neither A- nor B-adequate (though the Kauffman polynomial is positive on both critical lines). Another example, the arborescent knot $(3,3,3)-2,(3,3,3)-2,(-3,-3,-3)2$ has $V_0V_1=2$ and non-positive B-critical line of the Kauffman polynomial.

4.3. Whitehead doubles

Untwisted Whitehead doubles have trivial Alexander polynomial, and are one suggestive class of knots to look for trivial Jones polynomial. (Practical calculations have shown that the coefficients of the Jones polynomial of Whitehead doubles are absolutely very small compared to their crossing number.)

Proposition 4.3 Let K be a semiadequate non-trivial knot. Then the untwisted Whitehead doubles $Wh_\pm(K)$ of K (with either clasp) have non-trivial Jones polynomial.

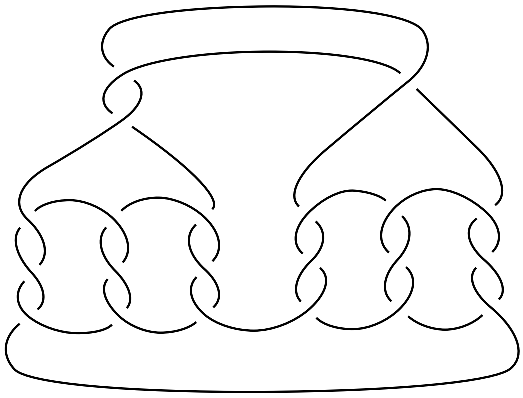


Figure 3: An arborescent inadequate knot (-3,-3,-3)2,(3,3,-3)1.

Remark 4.2 Because V determines v_2 (from (8)), and also $v_2 = {}^1\!/_2\Delta''(1)$, among Whitehead doubles only untwisted ones may have trivial Jones polynomial.

Proof. Let D be an A-adequate diagram of K. (B-adequate diagrams are dealt with similarly.) Without loss of generality assume that $w(D) > 0$ (possibly add positive kinks). If one of $Wh_\pm(K)$ has $V = 1$, then by a simple skein/bracket relation argument, the blackboard framed, disconnected 2-cable diagram $\tilde{D}$ of D must have the Jones polynomial $V_w = V(\tilde{D})$ of the $(2,2w)$-torus link up to units. Since D is A-adequate, so is $\tilde{D}$, and since $w > 0$, the polynomial V_w has $V_1 = 0$. Thus $\tilde{D}$ must admit a positive orientation. In particular, all its components' self-crossings must be positive, that is, D is positive. But we know from [St6] that the untwisted Whitehead double of a positive knot has non-trivial Jones polynomial. $\square$

This generalizes a result for adequate knots in [LT] and positive knots in [St6] and considerably simplifies the quest for trivial polynomial knots among Whitehead doubles. In particular, one can extend the verification of [St9] to establish that no non-trivial knot of ≤ 16 crossings has untwisted Whitehead doubles with trivial Jones polynomial.

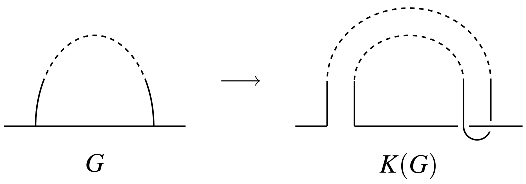
Remark 4.3 We do not claim that $Wh_\pm(K)$ are themselves semiadequate. For example, consider the alternating 11 crossing knots 11_{62} and 11_{247}. The Jones polynomial of $Wh_-(11_{62})$ has $V_0 = 2$, while $Wh_-(11_{247})$ has Jones polynomial with $\bar{V}_1 = 0$ (but is clearly not negative). Thus neither A- nor B-semiadequacy need to be preserved under taking *untwisted* Whitehead doubles. (Note also that the Kauffman polynomial semiadequacy test for such examples would have been *considerably* less pleasant.)

4.4. Strongly n-trivial knots

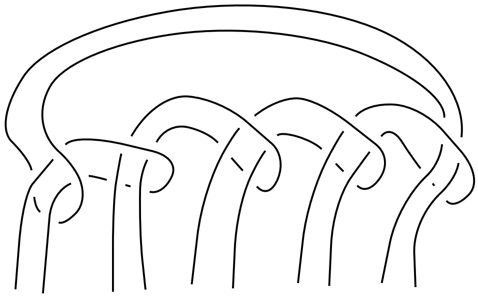
The next application concerns strongly n-trivial knots. They were considered first around 1990 by Ohyama, and studied more closely recently [To, HL, AK]. While one can easily verify by calculating the Jones polynomial that a given example is non-trivial, the proof of non-triviality for a family of knots with arbitrarily large n remained open for a while. (For $n > 2$ the Alexander polynomial is trivial [AK].) A proof that partially features the Jones polynomial value $V(e^{\pi i/3})$ was given in [St10], but nothing about the Jones polynomial of the examples directly could be said. How to evaluate the Jones polynomial was also asked by Kalfagianni [Kf]. Now we can deal with a different class of examples, proving the polynomial non-trivial.

Proposition 4.4 There exist for any n strongly n-trivial knots with non-trivial Jones polynomial.

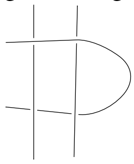
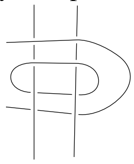
Proof. Apply the construction of [AK]



on the Suzuki graph $G = G_n$ obtained by gluing into a circle the ends of the following family of tangle diagrams (shown for $n = 5$):



(22)

(The signs of the clasps at the arrow hooks are irrelevant.) Assume now some of the resulting knots $K_n = K(G_n)$ for $n \geq 2$ has trivial Jones polynomial. Then smoothing out the crossings of the strong n-trivializer (the hook clasps), we obtain a $(1+n)$-component link diagram $D = D_n$ with trivial polynomial. This link diagram is regularly isotopic to a diagram $D' = D'_n$ that is obtained from G in (22) by replacing  by  and  by  .

But by a direct check, D' is A-adequate. □

Remark 4.4 By using proposition 3.1, one can also show that the polynomials of K_n are pairwise distinct for different n. While non-triviality of K_n seems provable also using [HL, AK], distinctness seems not so easy to see. For example, I do not know if the genera of K_n are all distinct (although by [HL] they have an increasing lower estimate).

The next applications require a longer discussion and are put into separate sections. In relation to non-triviality, let us conclude (for now) by saying that from our present result theorem 4.1, and with some help of Bae and Morton [BMo], we proved in [St16]:

Theorem 4.2 ([St16]) Non-trivial k-almost positive knots have non-trivial Jones polynomial for $k \leq 3$ and non-trivial skein polynomial for $k \leq 4$.

5. Minimal positive diagrams

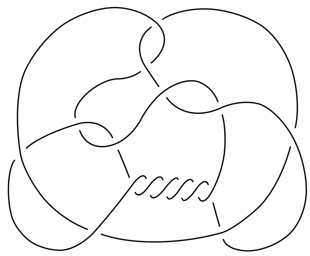
A further application relates to remark 3.2 and Nakamura's problem [Na]. We will exhibit two types of infinite families of positive knots with no minimal (crossing number) positive diagram. The first example of such a knot was given in [St14]. There we noticed also that such examples give instances of a negative solution to the problem of Cromwell [Cr] whether homogeneous knots have homogeneous minimal diagrams (as do alternating knots).

The approach we use will be set forth in the last sections for the exhibition of the families of odd crossing number achiral knots. So this section can be considered a preparation for latter's proof, though much more substantial argumentation remains.

5.1. Genus three knots

Proposition 5.1 There are infinitely many positive knots of genus 3 that have no minimal positive diagram.

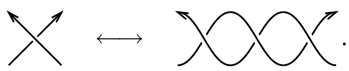
Proof. Consider the following family of positive knot diagrams of genus 3 (where the 6 half-twists in the group are indefinitely augmented).



(23)

Let D_n be the diagram in this family that has $2n$ crossings, and K_n the knot it represents. As the original example D_6 in [St14], these diagrams reduce to diagrams $\tilde{D}_n$ on $2n-1$ crossings. The diagrams $\tilde{D}_n$ are almost positive, and can be checked to be B-adequate. Now it follows from [Th] that then either (a) $c(K_n) = 2n-2$, and K_n has an adequate positive diagram D'_n of $2n-2$ crossings, or (b) $c(K_n) = 2n-1$. In case (b) we will show that no $2n-1$-crossing diagram D'_n of K_n is positive.

Assume first (a). We use some computation based on the generator theory explained in [St5, St13]; for length reasons we repeat only a minimum on details concerning the related notions. Two crossings are $\sim$*-equivalent* if they form a reverse clasp up to flypes. A $\bar{t}'_2$*-move* (or *twist*) adds two new elements/crossings to some $\sim$-equivalence class of a diagram:



(24)

(Such a move may also replace a positive crossing by 3 such.) A *generator* is a diagram all whose $\sim$-equivalence classes have at most two crossings; its *series* is the set of diagrams obtained from the generator by crossing changes, and successive flypes and $\bar{t}'_2$-moves. The importance of these concepts lies in the fact that knot diagrams of given (canonical) genus decompose into finitely many series [St5]. For genus 3, there are 4017 prime alternating generator knots [St13].

A minimal crossing (adequate positive) diagram $D = D'_n$ lies in the series of some genus 3 generator D'. Since K_n is prime by [O], so is D, and D' has even crossing number. Prime genus 3 generators are determined as explained in [St13].

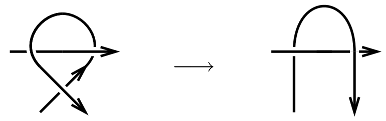
One calculates that $V_1(K_n) = -1$. Now we claim that D has at most one $\sim$-equivalence class of more than 2 crossings. Assume it has 2 such classes. Then $G(A(D))'$ would contain two paths $\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet$ of length ≥ 3. If the paths form parts of a single cycle, and this is the only cycle in $G(A(D))'$, then the crossings would be $\sim$-equivalent. So there exist two distinct cycles, and then $V_2 \leq -2$, so the diagram cannot depict any of the K_n. (Keep in mind that $V_0 = +1$ in positive knot diagrams.)

So D is obtained from D' by twisting (possibly repeatedly) at a single crossing. Now adequacy is invariant under $\bar{t}'_2$ twists, after one twist is performed.

So we select prime even crossing genus 3 generators D' with $V_1 = 0$ or $V_1 = -1$, apply one $\bar{t}'_2$ twist at any crossing (*not* simultaneously, that is, generating a new diagram separately for each crossing), and check whether $V_1 = -1$ and the (twisted) diagram is adequate. Such diagrams happen to occur only for one generator, 10_{154}. Now $\bar{t}'_2$ twists (even before a $\bar{t}'_2$ twist is applied previously) are also easily seen to preserve $\bar{V}_1$. But $\bar{V}_1(K_n) = -3$, while on all twisted diagrams $\bar{V}_1$ equals $\bar{V}_1(10_{154}) = -2$. So case (a) is ruled out.

Now turn to case (b). Assume D is a (minimal crossing number) positive diagram of $2n-1$ crossings. We consider odd crossing number generators. Adequacy tests are no longer valid, but still we can use V_1. So we check which positive generators have $V_1 = -1$ or 0, apply a $\bar{t}'_2$ twist at any crossing, and check $V_1 = -1$. We can then also check whether $V_2 = 1$, as it is for K_n. Again V_2 does not change under further $\bar{t}'_2$ twists, after one twist is performed. (Keep in mind that our diagrams D are positive, and so $\triangle A(D)' = 0$.)

Only 4 positive generators D' produce diagrams D that satisfy these conditions. The positive generators all have a fragment that admits a reducing move called second reduction move in [St6].



(25)

Thus, even after arbitrary twists, such diagrams are not of minimal crossing number. This contradiction completes the proof. □

In [St], we observed that for a positive link the condition $V_1 = 0$ is equivalent to being fibered. The observation made in the proof shows

Corollary 5.1 If for a positive link $V_0V_1 = -1$, then the minimal genus Seifert surface is unique and is obtained from a (positively and at least twice full-)twisted annulus by iterated Hopf plumbing.

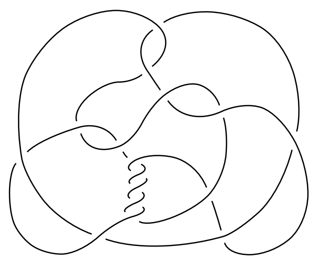
Proof. The canonical surface from a positive diagram is a minimal genus surface. Then $V_0V_1 = -1$ and the above argument shows that such a surface is obtained from a twisted annulus by Murasugi sum of $(2,.)$-torus link fiber surfaces. The work of Kobayashi [Ko] shows then that the surface is a unique minimal genus surface. Then the argument in the proof of proposition 5.1 of [GHY] shows that one can reduce Murasugi sum with the $(2,.)$-torus link surfaces to Hopf plumbing. □

5.2. Fibered knots

Using refinements and extensions of the preceding arguments we can settle the problem also for Nakamura's series of fibered positive knots. This proof is longer, but uses less computation, and initiates the technique needed later for our main result.

Proposition 5.2 There are infinitely many fibered positive knots that have no minimal positive diagram.

Proof. Now we modify the initial example $K_6 = 11_{550}$ to diagrams D_n as in [Na] like



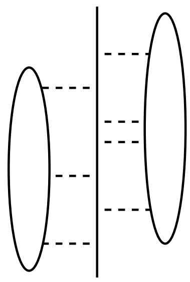
(26)

Again let K_n be the knot represented by D_n. The diagrams D_n reduce to almost positive B-adequate diagrams of $2n-1$ crossings. Now K_n are fibered, of increasing genus. We have the previous 2 cases.

In case (a) we must have a positive adequate diagram with 5 Seifert circles (=loops in the A-state). Since $G(A(D))'$ is a tree, it has 4 edges. All edges are multiple (a single edges would give a nugatory crossing). Now $V_2 = -1$, which implies that we have 5 intertwined edge pairs. We recall the intertwining graph $IG(A(D))$ defined by vertices being the (multiple) edges in $G(A(D))'$, and edges given by intertwined edge pairs. If we color the regions in the A state of D even-odd then each edge in $G(A)'$ receives a color. Since only oppositely colored edges can be intertwined, $IG(A(D))$ is a bipartite graph. But no bipartite graph on 4 vertices has 5 edges. This rules out case (a).

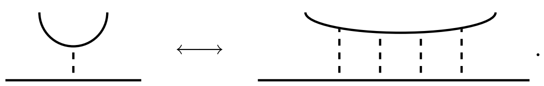
Now consider case (b) of $c(K_n) = 2n-1$. Assume $K = K_n$ for some n has a positive diagram D of $2n-1$ crossings. D has 6 Seifert circles. We know also that D is not (B-)adequate. By the previous argument we see that $IG(A(D))$ has 5 vertices and 6 edges, so $IG(A(D)) = K_{2,3}$ (the complete bipartite graph on $2,3$ vertices) is the only option. This means that D has a single separating Seifert circle, with two resp. three other Seifert circles, attached (by multiple crossings) from either side, so that each pair of Seifert circles from opposite sides is intertwined.

Let the *intertwining index* of an edge pair be half the number of interchanged connections from eithers side of l. For example the intertwining index of



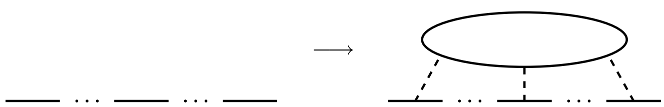
is 3, and edges are intertwined iff their intertwining index is ≥ 2. (We will from now on, to save space, draw in diagrams only a part of l that contains its basepoints. So the straight line, that represents l, is understood to be closed up.)

We assume in the following that an *edge* in $A(D)$ stands for a possible (but non-empty) collection of parallel traces:


(27)

(The term "edge" thus assumes a clear separation between the state $A(D)$ and the graph $G(A(D))$.) Traces between loops a and b are *parallel* if between their basepoints on both a and b no traces connecting a or b to other loops occur. With this convention we identify, and do not display several, parallel traces in diagrams. A connection in $A(D)$, which is the set of all traces or edges that connect the same two loops, in general decomposes into several edges.

Now one can obtain $A(D)$ by starting with some single loop, and then attaching new loops with all their traces.


(28)

Since $G(A(D))'$ is a tree, we can assume that we attach the traces of the new loop to fragments of the same previous loop.

Let

$$\gamma(D) := c(D) + 2 - v(A(D)) - v(B(D)) . \tag{29}$$

This number is always even, and non-negative. Its relevance lies in the property (which easily follows from the Kauffman bracket; see [LT]) that

$$\operatorname{span} V(D) \leq c(D) - \gamma(D) , \tag{30}$$

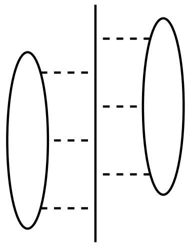
for any diagram D. It is possible to some extent to track $\gamma(D)$, and thus we obtain constraints on our D from the span of the Jones]polynomial. This will be useful below, and with much more importance in §9,10. Also, equality holds in (30) if and only if the extreme degree contributions of the A- and B-states in the bracket are not cancelled. This occurrs when D is A resp. B-adequate, but there are more general situations, which were studied by Bae and Morton [BMo]. We will use a part of their description shortly.

It is easy to see that (28) never reduces γ. Our diagram D must, by calculation of $V(K_n)$, have the property that $\operatorname{span} V(D) = c(D) - 2$, and therefore $\gamma(D) \le 4$.

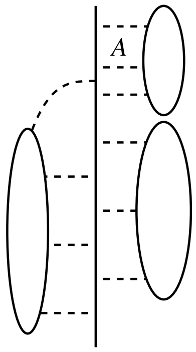
This means that if after some move (28), γ becomes ≥ 6, then we can rule out any completion of this loop insertion out. In particular, it is easy to see that no two loops have intertwining index ≥ 4.

Now we distinguish two cases.

Case 1. First assume there is a pair of loops of intertwining index 3.

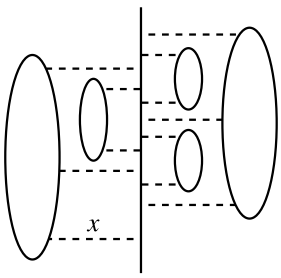


Then $\gamma = 4$ already after inserting these two loops. So we are not allowed to increase it when putting in the other two loops. If we put a loop so as to create a new region A, not connected by a trace from the other side of l,



then A must be joined by such a trace after putting one of the subsequent loops (otherwise the diagram will not be prime, or the edges in the boundary would remain parallel, in opposition to our convention). But one observes easily that this loop insertion would augment γ.

Thus all 3 remaining loops can be attached only between connections from the other side of l. They must also be intertwined among each other. One then observes that, up to symmetries, the only option is



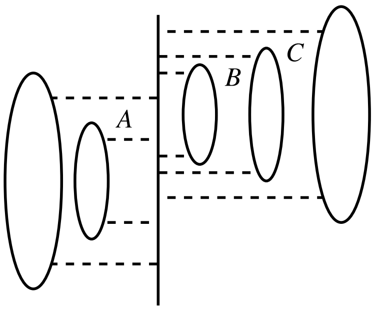
.

A direct check shows that this diagram is adequate if and only if x is a multiple edge. If x is single, then in $B(D)$ we have only one trace that connects the same loop. By [BMo], we conclude then that the extreme B-term in the bracket vanishes. Then, since $\gamma = 4$,

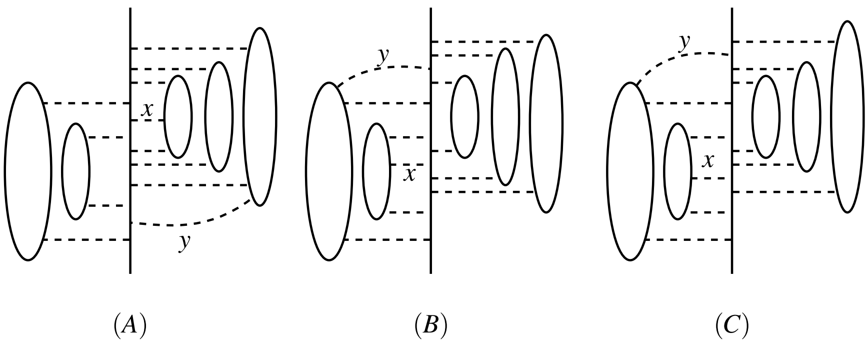
$$\operatorname{span} V(D) < \frac{c(D) + v(A(D)) + v(B(D)) - 2}{2} = c(D) - 2\,,$$

with a contradiction. So the argument in this case is complete.

Case 2. Now assume all loops have intertwining index 2. Then our A-state is obtained from



by adding edges. If we do not create equivalent edges, then we can attach edges that connect a loop to one of regions A, B or C. (To connect to two, or all three of A, B, C would lead again to $\gamma(D) > 4$ in (29).) In all three cases, which we denote after the region concerned, we can add at most two edges x and/or y.

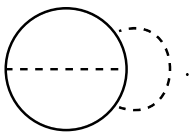


(31)

It is easy to see that adding only x or only y is equivalent, and that the cases B and C are equivalent up to mutations.

Case 2.1. Neither x nor y is added. By direct verification, D is adequate.

Case 2.2. Both x and y are added. As in case 1, we observe the B-state of D. There are now two traces that connect the same loops, which we call *inadequate*, but one has no *linked pair*, i.e. no pair of inadequate traces at the same loop in the mutual position

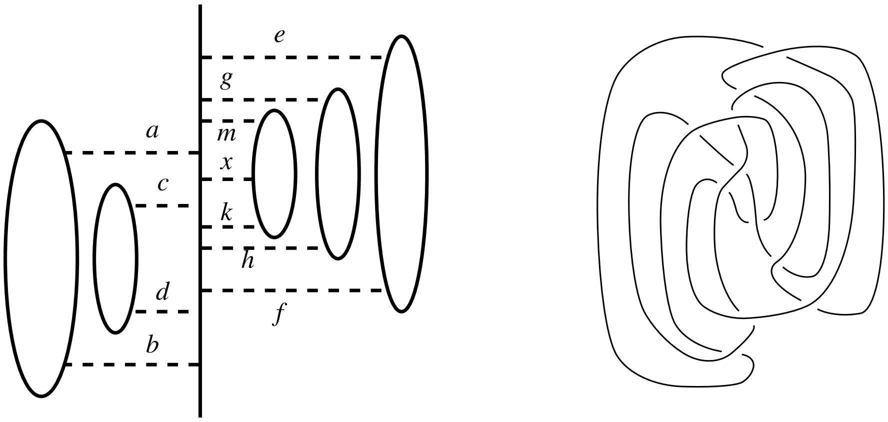


(32)

Then again by replacing edges by (possibly multiple) traces, either we have an adequate diagram, or if not, by [BMo] the extreme B-state term in the bracket vanishes. Since again $\gamma = 4$, we obtain $\operatorname{span} V(D) < c(D) - 2$.

Case 2.3. The cases when, say, x (but not y) is added are more problematic.

Case 2.3.1. Consider first a trace connecting A. Name the other crossings as indicated on the left of (33). The resulting link diagram D' (if all edges are replaced by a single trace, i.e. have multiplicity 1) is shown on the right.



(33)

Then (for single multiplicity edges) in the B state the trace of x (that connects the same loop), is linked with a,b,c,d. Latter four are not linked among each other. Since one observes that for all diagrams D we have $\gamma = 4$ in (29), we must deal with the situation where x is a single crossing (and one of a,b,c,d is too).

Now we require some computation. Let $x = (x_1,\dots,x_{10}) \in \mathbb{N}_+^{10}$ with $x_i \geq 1$ and assign to x a diagram D_x obtained from (33) by replacing a by x_1, b by x_2, ..., m by x_{10} parallel edges, as in (27). Define $\tilde{x} \in \{1,2\}^{10}$ by $(\tilde{x})_i = \min(x_i,2)$.

We generate the 1024 diagrams $D_{\tilde{x}}$ obtained by taking $\tilde{x} \in \{1,2\}^{10}$, and calculate their Alexander polynomial. (In fact, 288 diagrams are enough, as we can assume up to flypes that $(2,1) \notin \{(a,b),(c,d),(e,f),(g,h)\}$, and $1 \in \{a,b,c,d\}$.) Recall our normalization of Δ by the skein relation (5), or equivalently by the conditions $\Delta(1) = 1$ and $\Delta(t) = \Delta(1/t)$. Let $\bar{\Delta}_i = [\Delta]_{\max\deg\Delta - i}$. Then for all $D_{\tilde{x}}$ we find

$$\bar{\Delta}_0 = \bar{\Delta}_1 = 1\,. \tag{34}$$

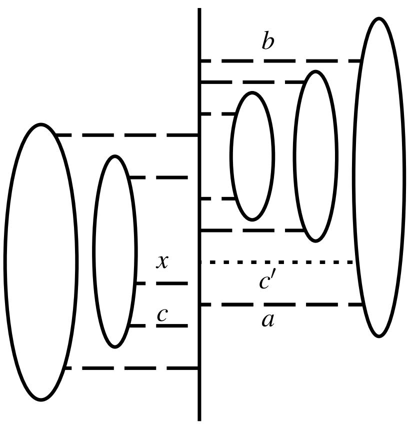
(The properties $\bar{\Delta}_0 = 1$ and $2\max\deg\Delta = 1-\chi(D_{\tilde{x}})$ are clear by theory *a priori*.) A direct skein argument using (5) shows then that for all $x \in \mathbb{N}_+^{10}$ we have $\bar{\Delta}_2(D_x) = \bar{\Delta}_2(D_{\tilde{x}})$. Now it is easily found that for our knots $\bar{\Delta}_2(K_n) \in \{-6,-7\}$. Contrarily another calculation reveals that

$$\bar{\Delta}_2(D_x) < -7\,, \tag{35}$$

except for x with at least nine $i \in \{1,\dots,10\}$ having $x_i = 1$. (Because of (34) it is enough to check (35) for the finitely many $x = \tilde{x}$. The only such x for which (35) fails, are those with at most one entry $x_i = 2$, occurring for some of $i = 5,\dots,10$.)

So we need to consider only diagrams D_x where x has at most one entry $x_i > 1$. But since $D' = D_{(1,\dots,1)}$ in (33) is a 3-component link diagram, such D_x will have at least two components.

Case 2.3.2. Finally, consider a trace connecting B. We name the edges in the diagram (31) (C) as indicated below. (For a moment think of c' as not being there.)



It suffices to consider only inadequate diagrams D where $B(D)$ has a linked pair (32) of traces. Now for a simple edge x in $B(D)$, i.e. one consisting of only one crossing trace, this trace is linked with (the trace of) the edge a or/and b, if these are simple too, and no further (potential) linked pairs occur. Thus we need to deal with the cases when x is single, and one of a and b is. Up to a flype, we may assume that a is single. Then using Reidemeister III moves, one can move the crossings in c into c', and thus returns to case 2.3.1. So our proof is now complete. □

6. 3-Braids

6.1. Semiadequacy of 3-braids

Let β be a braid word. We write

$$\beta = \prod_{j=1}^{k} \sigma_{p_j}^{q_j}$$

with σ_i the Artin generators, $p_j \neq p_{j+1}$ and $q_j \neq 0$. We will deal mostly with 3-braids, since here the most interesting applications are possible. For 3-braids p_j are interchangingly 1 and 2. Thus β is (at least up to conjugacy) determined by the vector (q_j) (up to cyclic permutations), which is called the *Schreier vector*. It has even length (unless the length is 1). *We will assume for the rest of this section that the Schreier vector has length* > 2. (The braids $\sigma_1^k\sigma_2^l$ are easy to deal with directly.) Any braid word β gives a link diagram $\hat{\beta}$ under closure, as a braid gives a link. Let $\delta = \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$ be the square root of the center generator of B_3.

The following is shown by a careful observation:

Lemma 6.1 A diagram of closed 3-braid word β (with Schreier vector of length > 2) is A-adequate if and only if it satisfies one of the following two conditions:

1. it is positive, or
2. (1) it does not contain $\delta^{-1} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$ as subword and
   (2) positive entries in the Schreier vector are isolated (i.e., entries cyclically before and after positive entries are negative). □

Corollary 6.1 A diagram of closed 3-braid word β is adequate if and only if

1. it is positive and does not contain δ as subword, or
2. it is negative and does not contain δ^{-1} as subword, or
3. it is alternating. □

Remark 6.1 For positive/negative braids to contain $\delta^{\pm 1}$ is equivalent to the Schreier vector containing no entries ± 1.

Corollary 6.2 If a positive word β contains no δ as subword, then it is not conjugate to a positive word that does contain δ as subword.

Proof. The diagram $\hat{\beta}$ is adequate, and by [Th] any other same crossing number diagram must be adequate too. But if β' contains no δ, then $\hat{\beta}'$ is not B-adequate. □

Theorem 6.1 Let β be a 3-braid word. Then the following are equivalent:

(a) $\hat{\beta}$ is semiadequate,

(b) β has minimal length in its conjugacy class,

(c) Some $\gamma \in \{\beta, \bar{\beta}\}$ satisfies one of the conditions enumerated in lemma 6.1.

Before the proof we make some remarks on consequences and related problems. The implication (c) $\Rightarrow$ (b) says in particular that if a word β has minimal length up to cyclic permutations, it has minimal length up to conjugacy. We say then it has *minimal conjugacy length*.

There is a problem due to Stallings (see [Ki, problem 1.8]) whether minimal length braid words are closed under end extension (replacing a final $\sigma_i^{\pm 1}$ by $\sigma_i^{\pm k}$ for $k > 1$). A natural modification of this problem is

Question 6.1 Are minimal conjugacy length words closed under (general) extension (replacing any $\sigma_i^{\pm 1}$ by $\sigma_i^{\pm k}$)?

We can confirm this for 3-braids. To describe minimal length or minimal conjugacy length words in general braid groups seems extremely difficult, though.

A further version of question 6.1 is to replace minimal conjugacy length by minimal length among Markov equivalent braids (a) of the same or (b) of arbitrary strand number. Using theorem 6.1 and [BM] we find that version (a) holds for 3-braids. A computer quest yielded a negative answer to part (b) for 5-braids.

Example 6.1 The knots 14_{42676} and 14_{42683} have 14 crossing 5-braid representations, which after twofold extension give rise to 16 crossing braids that correspond to 8 different 15 crossing knots. Since these knots have 4-braid 15 crossing representations, and the braid index of 14_{42676} and 14_{42683} is 5, we see that neither minimal length, nor strand number minimality are generally preserved under extension. The examples also yield a negative answer to a similar problem: is the property of a braid word to realize the crossing number of its closure knot or link (i.e. the closed braid diagram to be of minimal crossing number) preserved under extension? (Here somewhat simpler examples were found previously.)

The proof of theorem 6.1 is mainly contained in the following two lemmas.

Lemma 6.2 If β has minimal length up to cyclic permutations and contains δ (resp. δ^{-1}), then β is positive (resp. negative).

Proof. It suffices to prove in the positive case. The proof is given constructively by direct braid word transformation. We assume β is not positive and find a word reduction. To accomplish this, apart from braid relations $\sigma_1^{\pm_1 1}\sigma_2^{\pm_2 1}\sigma_1^{\pm_2 1} \leftrightarrow \sigma_2^{\pm_2 1}\sigma_1^{\pm_2 1}\sigma_2^{\pm 1}$ (with the $\pm$ signs chosen consistently with their indices and possible interchange of $\sigma_{1,2}$) we use the "wave" moves $\sigma_1^{\mp 1}\sigma_2^k\sigma_1^{\pm 1} \leftrightarrow \sigma_2^{\pm 1}\sigma_1^k\sigma_2^{\mp 1}$.

We assume that a negative letter is on the right of δ in β. The case when it is on the left is obtained by reversing the order of the letters in the word. The sequence of words below gives a sample word reduction (an entry $\pm i$ for $i > 0$ means $\sigma_i^{\pm 1}$). More formally one uses that one can 'pull' letters through δ, interchanging indices $1 \leftrightarrow 2$, and can bring δ close to the negative letter, where it cancels. Here is a simple example:

```
1 2 1 1 1 2 2 2 1 1 1 2 2 2 -1
1 2 1 1 1 2 2 2 1 1 -2 1 1 1 2
1 2 1 1 1 2 2 -1 2 2 1 1 1 1 2
1 2 1 1 -2 1 1 2 2 2 1 1 1 1 2
1 -1 2 2 1 1 1 2 2 2 1 1 1 1 2
     2 2 1 1 1 2 2 2 1 1 1 1 2
```

□

Lemma 6.3 If β has Schreier vector that has length-2-subsequences of both positive and negative entries, then β is word reducible.

Proof. Take a positive and negative length-2-subsequence p,q of the Schreier vector that have minimal distance to each other. Assume w.l.o.g. p comes before q and is of the form $\sigma_1^k\sigma_2^l$. Then the entries in the Schreier vector between (and including) the second entry for p and the first entry for q must alternate in sign. Then q is of the form $\sigma_1^{-k'}\sigma_2^{-l'}$.

One brings then the subword '1 2' from the pair p of consecutive positive entries close to q using $\sigma_1\sigma_2^k\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^k\sigma_2$ and $\sigma_1\sigma_2\sigma_1^{-k} = \sigma_2^{-k}\sigma_1\sigma_2$, and then cancels. Again we give a sample word reduction.

```
1 1 1 2 2 2 -1 -1 -1     2 2 2 -1 -1 -1      2 2 2 -1 -1 -1 -2 -2 -2
1 1 -2 1 1 1 2 -1 -1     2 2 2 -1 -1 -1      2 2 2 -1 -1 -1 -2 -2 -2
1 1 -2 1 1    -2 -2 1 2 2 2 2 -1 -1 -1      2 2 2 -1 -1 -1 -2 -2 -2
1 1 -2 1 1    -2 -2 -2 1 1 1 1 2 -1 -1      2 2 2 -1 -1 -1 -2 -2 -2
1 1 -2 1 1    -2 -2 -2 1 1 1     -2 -2  1 2 2 2 2 -1 -1 -1 -2 -2 -2
1 1 -2 1 1    -2 -2 -2 1 1 1     -2 -2 -2 1 1 1 1  2 -1 -1 -2 -2 -2
1 1 -2 1 1    -2 -2 -2 1 1 1     -2 -2 -2 1 1 1 -2 -2 1  2 -2 -2 -2
1 1 -2 1 1    -2 -2 -2 1 1 1     -2 -2 -2 1 1 1 -2 -2 1       -2 -2
```
□

Proof of theorem 6.1. (a) $\Rightarrow$ (b). If β is not of minimal length in its conjugacy class, $\hat{\beta}$ is regularly isotopic to a diagram of smaller crossing number. This contradicts the result of Thistlethwaite [Th, corollary 3.1].

(b) $\Rightarrow$ (c). This is accomplished by lemmas 6.2 and 6.3.

(c) $\Rightarrow$ (a). Use lemma 6.1. □

6.2. Some applications

Now we put together some consequences of the preceding treatment of semiadequacy of 3-braids. The most obvious one is

Corollary 6.3 3-braid links are semiadequate, and so have non-trivial Jones polynomials up to units. □

The material in this section was originally motivated by the question whether there is a non-trivial 3-braid knot with trivial Jones polynomial. I was initially aware of Birman's claim of a negative answer (see [B, St11]). However, Birman did not give any reference or further comment, and I knew of no proof. Only after the present proof was obtained, I was pointed to the paper [Ta]. There explicit (but very complicated) formulas for polynomials of closed 3-braids are given, and the Jones polynomial non-triviality result (for knots) is claimed after an 18-page long calculation.

Apart from dealing with links too, our proof should be considered more conceptual. We thereby also extend the result on the non-triviality of the skein polynomial of [St11], although our approach here is quite different from [B] or [St11].

The feature of excluding trivial polynomials up to units should be contrasted to the existence of a (2-component) 4-braid link with trivial polynomial up to units. Such a link, the closure of $(\sigma_2\sigma_1\sigma_3\sigma_2)^2\sigma_1^3\sigma_3^{-3}$, is given in [EKT]. It also relates to the problem of the (possible) faithfulness of the Burau representation. It is known that the Burau matrix determines the Jones polynomial for 3- or 4-braids. So a non-faithful Burau representation on 4-braids will (in practice; see [Bi]) imply a 4-braid knot or link L with trivial polynomial, but the above example cautions about a possible reverse conclusion. To put our work into this context, it seems more likely (though not easy) to develop a combinatorial proof that no L exists and deduce that Burau is faithful (as we can now for 3-braids, providing a numerous proof) rather than the other way around.

Remark 6.2 There seems no easy way to decide what of A- or B-adequate 3-braid representations exist for a given link, starting from another semiadequate diagram of it. For example 5_2 and 10_{161} are A-adequate, but have only B-adequate 3-braid representations. Latter knot thus shows that a weakly adequate 3-braid link may not be adequate, while former knot shows that an adequate 3-braid knot may not have an adequate 3-braid (word) representation.

Contrarily, if a link has 3-braid representations, one of which is A-adequate and one of which is B-adequate, then it also has an adequate representation. This follows, because A- and B-adequacy always appear in minimal length 3-braid words. Using this remark, we can extend a result of Birman-Menasco [BM].

Definition 6.1 We call an oriented link *positively/negatively (orientedly) amphicheiral, or achiral*, if it is isotopic to its mirror image with the orientation of no/all components reversed, and components (possibly and arbitrarily) permuted. We call the link *unorientedly achiral*, if it is isotopic to its mirror image with the orientation of some (possibly no, but also not necessarily all) components reversed (and components possibly permuted).

It is clear that oriented achirality (of some sign) implies unoriented one, and for *knots* both notions coincide. The Hopf link is an example of a link which is unorientedly achiral, but not orientedly so.

Theorem 6.2 Let L be an unorientedly achiral 3-braid link. Then $L=\hat{\beta}$, where $\beta\in B_3$ is either (a) an alternating braid with Schreier vector admitting a dihedral (anti)symmetry, or (b) $\beta=\sigma_1^k\sigma_2^l$ and $|k|,|l|\le 2$.

Proof. We assume β is a minimal length word up to conjugacy. The cases that β has a Schreier vector of length two or less are easy to deal with, and lead to case (b). So we assume the Schreier vector of β is of length at least 4.

By the above remark, semiadequacy appears in minimal length braid words. We know that any minimal length word is A-adequate or B-adequate, but since L is achiral and semiadequacy is independent on component orientation, β must be both, i.e. it must be adequate.

In case β is alternating, we can apply [MT]. An easy observation shows that the diagram is determined by the Schreier vector up to dihedral moves (cyclic permutations and reversal of order), and the only possible flypes in a diagram of a closed 3-braid occur at Schreier vector of length 4, with an entry ± 1 (see [BM]). In that case, however, the flypes just reverse the orientation of (all components of) the link. So we obtain no new symmetries. We arrive at case (a).

It remains to exclude the case that β is not alternating. In that event, we know that β is positive or negative, and has no Δ as subword. Positive (or negative) links are not orientedly achiral, so the possibility that L is a knot or that the achirality is oriented is easily ruled out.

It is clear that all components of an unorientedly achiral link are achiral knots, or mirror images in pairs. Latter option does not lead to anything new for closed 3-braids, so we ignore it. Then if $L=\hat{\beta}$ is a 2-component link, two of the strands x,y of β form (under closure) a positive achiral knot. This knot must be trivial, and so x,y have only one common crossing. Then an easy observation shows that β contains Δ as subword, a contradiction.

So assume that $L=\hat{\beta}$ is a 3-component link. Then a linking number argument shows that two of the components of L must have no common crossing. This means that in the Schreier vector all entries are even and (up to mirroring) positive. Then reversal of component orientation makes the (say) positive diagram $D=\hat{\beta}$ into a negative diagram D'. Now D' must be orientedly isotopic to the mirror image of D. We know that in positive and negative diagrams the canonical Seifert surface has minimal genus. So to find a contradiction, we just need to count the Seifert circles in D in D' and see that they are not equally many. □

In Birman-Menasco [BM] the oriented achirality result was obtained (which leads to case (a) above), so our result is a slight extension of theirs. However, the proof in [BM] requires their extremely involved proof of the classification of closed 3-braids. It is therefore very useful to obtain simpler proofs of at least consequences of their work. So more important than the drop of orientations here is the complete bypassing of the method in [BM], even though the insight in [BM] motivated the present proof, and with [MT] we make use of another quite substantial result.

With a similar argument, we easily see

Corollary 6.4 A non-split 3-braid *knot* with $T_1=0$ is a closed positive or negative 3-braid.

Proof. Take a minimal word length 3-braid representation β. The closure of β is a semiadequate diagram, since $V_1=\bar{V}_1=0$, it must be positive. □

The case of *links* is quite different, and we will describe them now, since in the following it will be somewhat important to understand how one can make 3-braids more complicated without altering T_1.

Proposition 6.1 Assume β is an A-adequate 3-braid word and $V_1=0$. Then, up to interchange $\sigma_1\leftrightarrow\sigma_2$,

(a) β is positive, or

(b) $\beta = \sigma_1^{-2}\sigma_2^k$ for $k > 0$, or

(c) β can be written as

$$\beta(k_1,l_1)\dots\beta(k_n,l_n)\,, \tag{36}$$

where

$$\beta(k,l) := \sigma_1^{-1}\sigma_2^k\sigma_1^{-1}\sigma_2^{-1}\sigma_1^l\sigma_2^{-1}\,. \tag{37}$$

Proof. The first two cases are direct to see, so assume β is not positive and is of Schreier vector length > 2. So the Schreier vector has isolated positive entries. The form (36) is then observed by directly drawing the picture of the A-state. □

Since the form in (37) will be needed later, let us call $\beta(k,l)$ *exceptional syllables* and the words in (36) *exceptional words*. Note that by braid relations, an exceptional word can be made to have only one of $k_1,l_1,\dots,k_n,l_n$, say l_n, being non-zero, and can be rewritten then as $\delta^{-2n}\sigma_{1,2}^{l_n+2n}$. Here $\delta^2 = (\sigma_1\sigma_2)^3$ generates the center of B_3.

A first small but useful application of proposition 6.1 is

Theorem 6.3 Let L be a 3-braid link. Then L is hyperbolic if and only if it is not, up to mirror image, among the links with the following 3-braid representations.

1. $\sigma_1^k\sigma_2^l$ with $k,l \in \mathbb{Z}$ (unknot, composite and split links, $(2,.)$-torus links)
2. $\sigma_1\sigma_2^2\sigma_1\sigma_2^k$ with $k \in \mathbb{Z}$ (the $(2,-2,k)$-pretzel links; Seifert fibered)
3. $(\sigma_1\sigma_2\sigma_1)^k$ with k odd (Seifert fibered)
4. $(\sigma_1\sigma_2)^k$ with $|k| \geq 3$ (torus links; Seifert fibered), and
5. $(\sigma_1\sigma_2^2\sigma_1)^k\sigma_2^l$ for $k,l \in \mathbb{Z}$, $|k| > 1$ (toroidal).

Proposition 6.1 is used in combination with the description of [BMu] for the proof of the Seifert fibered cases. For the toroidal ones, Tetsuya Ito pointed us to the paper of Birman and Menasco [BM3]. Our initial goal was to supply a simpler independent proof based on the argument of Williams [Wi]. Sadly, this not only failed, but led also to the discovery of a gap in the proof of Williams' cabling result for the braid index.

Proof. Let us exclude the composite and split links. These were found by Murasugi [Mu4] to be only those of form 1. (The split ones follow also from the study of the Alexander polynomial in [St11], and the composite ones also from [BM2].) The toroidal 3-braid links are those of form 5, as shown by Birman and Menasco in Corollary 1 of [BM3]. (In the braid form there the values of the parameters p,q should be corrected from $|p| \geq 2$, $|q| \geq 1$ to $|q| \geq 2$, $p \in \mathbb{Z}$; see also the erratum and [Ng], although this does not concern Corollary 1.)

So we deal here with the Seifert fibered links. These links were described in [BMu], and are there divided into 3 types: those with $n = 0,1$ or 2 (link components being) exceptional fibers of a Seifert fibration of S^3. As unoriented links, these are torus links (p,q), with optionally adding some of the cores of the two complementary solid tori.

Denote these links by $H(p,q,n)$. Up to mirroring, we have $p,q \geq 0$. For $q = 0$ or $p = 0$ the links are just unlinks and connected sums of Hopf links, so we consider $p,q > 0$. For $n = 1$, we distinguish which core is added by writing $H(p,q,1)$ or $H(q,p,1)$, and we treat the cases separately in forms 2 and 3 below. Otherwise $H(p,q,n) = H(q,p,n)$, so we may assume $p \geq q > 0$.

Consider first the links with the component orientation as a closed positive braid.

- For $n = 0$ exceptional fibers, we have the torus links. Since the braid index of the torus links is well-known (see [FW]), we have form 4.
- For 1 exceptional fiber, by looking at the braid index of sublinks, we see that we have forms 2 and 3.
- For 2 exceptional fibers, we obtain again form 2 for even k.

However, we have the result only up to orientation. We must show (for links) that if we reverse orientation of some component(s) in Burde-Murasugi's links, the resulting link is not a 3-braid link, except in considered cases. Since we cannot have more than 3 components, it is enough to reverse exactly one component.

If $q+n>3$, then by using the fact that the braid index of a link is at least the sum of the braid indices of its components, we see that, even with orientations reversed, no 3-braid link occurs. The only orientation-reversed link for $q+n\le 2$ of braid index 3 is the reverse $(2,4)$-torus link, which occurs in from 3.

So consider $q+n=3$, and we have an orientation-reversed link for some of the above forms 2 to 4.

For form 2, this is easy: if the component involved only in $\sigma_2\sigma_1^2\sigma_2$ is reversed, then we get the mirror image of the form for $-k$. For k even and $|k|>2$ we can reverse one other component, but then get a $(2,k)$-reverse torus link, with $b\ge 3$, as proper sublink, so $b(L)>3$, a contradiction. (For $k=\pm 2$ we get after reversing again the link for $-k$ up to mirror image.)

For form 3, the same sublink argument deals with the cases except that k is odd, $|k|>1$, and in L the unknotted component is reversed.

Assume this orientation-reversed link $L'=L'_k$ is a closed 3-braid. We can assume w.l.o.g. that this braid β is semiadequate. Now, L' is one of Traczyk's links [Tr]; the Jones polynomial is of the form $t^r(-t^{1/2}-t^{5/2})$ (for $r\in\mathbb{Z}$), and so $V_1=\bar V_1=0$. Thus, by proposition 6.1, β must be either positive or negative, or of the from 5. Latter case is easily ruled out by the parity of linking numbers, remembering that k is odd.

It remains to exclude that L' is a closed positive/negative braid. This is done by looking at the degrees of V. With the positive orientation, we have $\min\deg V(L)=\dfrac{1-\chi(L)}{2}=\dfrac{3k-2}{2}$, and so by the reversing property of V (see e.g. [LM]), $\min\deg V(L')=\dfrac{-3k-2}{2}$, and thus $\max\deg V(L')=\dfrac{2-3k}{2}$. Assume w.l.o.g. (up to mirroring) that $k>1$. Then L'_k can be only a negative braid. In this case $\max\deg V(L')=-\dfrac{1-\chi(L')}{2}$, so $\chi(L')=3-3k$. But this is easily refuted by looking at the canonical surface and counting in the Seifert circles in the $3k$-crossing diagram.

For form 4, the only case of links is $3\mid k$, and then we have form 3 with even k. □

We will use in a separate paper [St12] semiadequacy to answer positively the suggestive question: If a link is the closure of a positive braid, and has braid index 3, is it always the closure of a positive 3-braid? This question is motivated in particular by counterexamples of 4-braids.

Although we restricted ourselves to 3-braids, of course some statements for general strand number are possible.

Proposition 6.2 Let β be a braid such that when writing

$$\beta=\prod_{j=1}^{k}\sigma_{p_j}^{q_j}$$

with $p_j\ne p_{j+1}$ and $q_j\ne 0$, we have $q_jq_k>0$ for $p_j=p_k$ (that is, β is homogeneous) and $q_j\ne -1$. Then $\hat\beta$ has non-trivial Jones polynomial. □

Remark 6.3 The knot 9_{47} has a homogeneous braid representation, but is not weakly adequate, so that a general homogeneous braid representation does not suffice to prove non-triviality of the Jones polynomial, at least using the present approach.

6.3. 3-braids with polynomials of $(2,.)$-torus links

In a separate paper [St12], we explained how to find the closed 3-braids for a given Jones polynomial. This method will in general not help to deal with infinitely many polynomials at a time. A tool in that vein is probably too much to hope for, given that the distribution of Jones polynomials on closed 3-braids has surely some degree of difficulty (as became evident to Birman [B]). Dealing with a particular interesting infinite family, here we will apply the formulas for $V_{1,2}$, together with some representation and skein theoretic arguments, to determine exactly which closed 3-braids have the Jones polynomials $V_{(2,q)}$ of the $(2,q)$-torus links $T(2,q)$.

Lemma 6.4 Let $q > 1$. Then there is no positive 3-braid different from the $(2,q)$-torus link with the same Jones polynomial as $T(2,q)$.

Proof. Since $2\min\deg V = 1-\chi$, we know that a positive 3-braid β with $V(\hat\beta) = V_{(2,q)}$ has $q+1$ crossings. Now we know (see [LT]) that

$$\operatorname{span} V(D) \le \frac{1}{2}\Big(c(D) + v(A(D)) + v(B(D)) - 2\Big). \tag{38}$$

Applying this to $D = \hat\beta$, we find that $\operatorname{span} V = q = c(D) - 1$ holds only if the Schreier vector of β has length 2 or 4. In former case we are easily done, and in latter case we have additionally that the inequality (38) is sharp, so that the extremal B-states (in the language of [BMo]) have a non-cancelling contribution. Using (for example) the formalism in [BMo], one easily sees that then either one σ_1 and one σ_2 syllable are trivial (and then the braid reduces to a $(2,.)$-torus braid), or all 4 syllables are non-trivial. The we have an adequate diagram D, and $|\bar V_1| = 1$ implies that the syllables have exponent 2 except exactly one. So we have $\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^k$ for $k \ge 3$, and we can exclude these braids easily by induction on k using the skein relation for V. □

Let $[\beta]$ be the exponent sum of β and $\delta = \sigma_1\sigma_2\sigma_1$. We write $V(\beta)$ for $V(\hat\beta)$. The *Birman dual* [B] of $\beta \in B_3$ is $\beta^* = \beta^{-1}\delta^{4p}$, where we assume $[\beta] = 6p$ is divisible by 6. Birman introduced this construction in order to obtain different 3-braid links with the same Jones polynomial, proving $V(\beta^*) = V(\beta)$.

Theorem 6.4 Let $q \ge 0$. Then

(a) If $q \ge 7$ and $q \equiv \pm 1\,(6)$, then except the $(2,q)$-torus knot, there is exactly one other closed 3-braid with the same Jones polynomial. It is obtained by taking the Birman dual of $\sigma_1^{q\mp 1}\sigma_2^{\pm 1}$.

(b) For all other q, the $(2,q)$-torus link is the only closed 3-braid with this Jones polynomial.

Proof. Modulo mirroring assume β is an A-adequate 3-braid with the polynomial of a $(2,q)$-torus link, where now $q \in \mathbb{Z}$. We can exclude $|q| \le 1$, since these are trivial polynomials that were dealt with before.

First assume $q \ge -2$. Then $V_1 = 0$. By direct observation (see proposition 6.1), β is of the following forms (after conjugation and braid relations):

- $\sigma_1^{-2}\sigma_2^k$, and $\sigma_1^{-1}\sigma_2^k$,
- positive, or
- $(\sigma_1\sigma_2^2\sigma_1)^{-k}\sigma_2^l$, with $k,l \ge 0$.

The first form is easy to handle, and the second one is handled by lemma 6.4, in both cases with the expected outcome. So consider the third form. We can evaluate the Jones polynomial using the trace of the (reduced) Burau representation ψ. We have

$$\psi(\sigma_1) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix}, \qquad \psi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}.$$

So for $k \in \mathbb{Z}$

$$\psi(\sigma_1^k) = \begin{bmatrix} (-t)^k & \frac{1-(-t)^k}{1+t} \\ 0 & 1 \end{bmatrix}, \qquad \psi(\sigma_2^k) = \begin{bmatrix} 1 & 0 \\ t\frac{1-(-t)^k}{1+t} & (-t)^k \end{bmatrix}.$$

Then $\operatorname{tr}\psi(\sigma_1^k) = 1 + (-t)^k$ and

$$\operatorname{tr}\psi(\sigma_1^k\sigma_2^l) = (-t)^k + (-t)^l + t\,\frac{(1-(-t)^k)(1-(-t)^l)}{(1+t)^2}. \tag{39}$$

Also $\psi(\delta^2) = t^3 \cdot Id$, where $\delta^2 = (\sigma_1\sigma_2)^3$ is the center generator. Then (see [B] or [J])

$$V_{\hat\beta}(t) = \left(-\sqrt{t}\right)^{e-2}\left[t\cdot\operatorname{tr}\psi + (1+t^2)\right], \tag{40}$$

with $e=[\beta]$ the exponent sum. So with $\beta=(\sigma_1\sigma_2^2\sigma_1)^{-k}\sigma_2^l$, we find

$$V(\hat{\beta}) = V(\delta^{-2k}\sigma_2^{l+2k}) = \left(-\sqrt{t}\right)^{l-4k-2}\left[t\cdot t^{-3k}(1+(-t)^{l+2k})+(1+t^2)\right]. \tag{41}$$

We can assume $k,l>0$. Otherwise we have for $k=0$ a positive braid, and can apply the lemma 6.4, or for $l=0$ a negative braid. In latter case, since we assumed $q\geq -2$, the only possibility is the negative Hopf link polynomial, and since for a negative braid $\max\deg V=-1/2$ determines $\chi(\hat{\beta})$, we are easily done.

Now $k,l>0$. Then from (41) we have

$$\min\deg V = \frac{l-4k-2}{2}+1-3k = \frac{l}{2}-5k \geq -\frac{5}{2}, \tag{42}$$

the last inequality because of $q\geq -2$. Also, since $l+2k>2$, we find

$$\max\deg V = \frac{l-4k-2}{2}+\begin{cases} 1+l-k & 1+l-k>2 \\ 2 & 1+l-k<2 \\ 2 & 1+l-k=2 \text{ and } l \text{ even} \\ 0 & 1+l-k=2 \text{ and } l \text{ odd}\end{cases}$$

In particular, $\operatorname{span}V\geq l+2k-2$. But from (41) we see also that V has at most 4 monomials, and by observing the shape of $V_{(2,q)}$, we conclude $\operatorname{span}V=q\leq 4$, so that

$$l+2k\leq 6. \tag{43}$$

No integers $k,l>0$ satisfy simultaneously the inequalities in (42) and (43). This finishes the case $V_1=0$.

Now let $V_1=1$, that is, $q\leq -3$. By a similar observation we can determine the forms of β (up to conjugacy and braid relations):

- $\sigma_1^k\sigma_2^{-l}$, $l>2$, $k\geq -2$,
- $\sigma_1^k\sigma_2^{-l}\sigma_1^m\sigma_2^{-n}$, $k,l,m,n>0$, or
- $\sigma_1^{-k}(\sigma_1\sigma_2^2\sigma_1)^{-l}\sigma_2^m$, with $k>0$.

The first case is again easy to settle. The second case gives an alternating (braid) diagram. Using the twist number as in [DL], or the theorem in [Th2], one easily verifies that among alternating links the $(2,q)$-torus links are the only ones with their Jones polynomial. So the third form is again the only interesting one.

We can assume again that $l,m>0$. Now we use proposition 3.3. Consider the A-adequate diagram $D=\hat{\beta}$ and its A-state. We have $V_1=1$ and by direct observation,

$$\begin{array}{l} e_{++}=2l, \\ \delta=2l-1, \end{array} \quad \text{and} \quad \triangle A(D)'=\begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases}.$$

So

$$V_0V_2=\begin{cases} 2 & \text{if } k>1 \\ 1 & \text{if } k=1 \end{cases}.$$

Since for $V_{(2,q)}$ we have $V_0V_2\in\{0,1\}$, we find $k=V_0V_2=1$ (and also $V_2\neq 0$, so that $q\neq -3$). Using (39) and (40), we have again

$$\begin{aligned} V_{\hat{\beta}} = V(\delta^{-2l}\sigma_2^{m+2l}\sigma_1^{-1}) &= \left(-\sqrt{t}\right)^{-4l+m-3}\left[t\cdot t^{-3l}\left[(-t)^{m+2l}-\frac{1}{t}+\frac{1-(-t)^{m+2l}}{1+t}\right]+1+t^2\right] \\ &= \left(-\sqrt{t}\right)^{-4l+m-3}\Big[\underbrace{t^{1-3l}\sum_{i=-1}^{m+2l}(-t)^i}_{(*)}+1+t^2\Big]. \end{aligned} \tag{44}$$

If m is even (and $\hat{\beta}$ is a 2-component link), then $V_0 = 1$, which is not the case for any $V_{(2,q)}$ when $q \le -3$ (having $V_0 = -1$). So assume m is odd (and $\hat{\beta}$ is a knot). The term (*) in (44) gives a coefficient list $-1\,1\,\dots -1\,1\,-1$, with maximal polynomial degree, say, y. In order to obtain a coefficient list $-1\,1\,\dots -1\,1\,-1\,1\,0\,1$ by adding $t^x(1+t^2)$, there are two options, $x = y+1$ and $x = y-2$. Then $1-3l+m+2l \in \{-1,2\}$, so $m = l-2$ or $m = l+1$.

In these cases we have the polynomials $V_{(2,q)}$ for some $q \le -2$ up to units. Then "up to units" can be removed, either by direct check of the degree, or more easily by the previous argument using $V(1) = 1$ and $V'(1) = 0$, since we have knot polynomials. So we have the forms

- for $l = m+2$ odd, $\sigma_1^{-1}(\sigma_1\sigma_2^2\sigma_1)^{-l}\sigma_2^{l-2}$, and
- for $l = m-1$ even, $\sigma_1^{-1}(\sigma_1\sigma_2^2\sigma_1)^{-l}\sigma_2^{l+1}$.

It is direct to check that the first family is the (mirrored) Birman dual of $\sigma_1^{6p-1}\sigma_2$ with $l = 2p-1$ and the second family is the dual of $\sigma_1^{6p+1}\sigma_2^{-1}$ with $l = 2p$.

Birman duality is trivial (i.e., the duals are equal) for $p = 0$, or in the first family if $p = 1$. In the other cases one can use (as in [B]) the signature σ to distinguish the duals. For the first family, first change a crossing in $\alpha = \sigma_1^{6p-1}\sigma_2$ to obtain an alternating braid $\alpha_0 = \sigma_1^{6p-1}\sigma_2^{-1}$, and calculate $\sigma(\alpha_0^{-1}\delta^{4p})$ using Murasugi's formulas. Then observe that it differs for $p \ge 2$ at least by 4 from $\sigma(T(2,6p-1)) = 6p-2$, and that σ changes at most by 2 under the crossing change, that turns $\alpha_0^{-1}\delta^{4p}$ into $\alpha^{-1}\delta^{4p} = \alpha^*$. $\square$

The following consequence, due to El-Rifai, classifies 3-braids whose Morton-Williams-Franks inequality for the braid index is inexact. Here $MWF(\beta) = 1 + (\max\deg_l P - \min\deg_l P)/2$ with $P = P(\hat{\beta})$ is a notation of [Na2] for this bound on (the closure of) a braid β.

Corollary 6.5 (El-Rifai) A braid $\beta \in B_3$ has $MWF(\beta) \le 2$, if and only if β is (up to mirroring and conjugacy) as specified in theorem 6.4.

Proof. It is an easy observation (using for example [LM2, proposition 21]; see remark 1.4 in [Na2]) that $MWF(\beta) \le 2$ implies that $P(\hat{\beta}) = P_{(2,q)}$ for some q, so in particular $V(\hat{\beta}) = V_{(2,q)}$. In the reverse direction, note (as in [B]) that Birman duality preserves not only V but also P. $\square$

Remark 6.4 El-Rifai's result is obtained in his (unpublished) thesis [El], which I learned about from [Na2]. It was the interest in this problem (and the inavailability of El-Rifai's solution) that motivated theorem 6.4.

Theorem 6.4 and corollary 6.5 also relate to some evidence we have that closed 3-braids with the same Jones polynomial have in fact the same skein polynomial P. From [St12] we know that for given V at most three different P occur, and P is unique if $\max\deg V \cdot \min\deg V < 0$. This condition is not fulfilled, though, by torus link polynomials, so it is justified to consider corollary 6.5 (still only) as a consequence of theorem 6.4.

7. Upper bounds on the volume

In this short section we explain how our formulas for $V_{1,2}$ relate to the twist number of §2, and volume. Another relationship is shown for all V_i for alternating links in §8. For the treatment of a general adequate link, some related discussion is given also in §4 of [FKP].

7.1. Adequate links

Definition 7.1 Let $t_k(D)$ be the number of twist equivalence classes of k elements in a diagram D.

Proposition 7.1 For an adequate diagram D of a knot or link K we have

$$t(D) \geq |V_1| + |\bar{V}_1| - 2c(K) + 2\,\mathrm{span}\,V(K) \geq -t_1(D) - t_2(D), \tag{45}$$

and

$$t(D) \geq 2c(K) - 2\,\mathrm{span}\,V(K). \tag{46}$$

Proof. We have as before $|V_1| = e(G(A(D))') - v(G(A(D))') + 1$, and similarly for $\bar{V}_1$ and $B(D)$.

Now we claim that

$$c(D) - t_2(D) - t_1(D) \leq e(G(A(D))') + e(G(B(D))') \leq c(D) + t(D). \tag{47}$$

To see this note that we may assume (as in [DL] or [La]) that twist equivalent crossings form a twist tangle (i.e., that the diagram is *twist reduced* in the terminology of [La]). Then in at least one of the A and B-state of D, the traces of all crossings in a twist equivalence class will connect the same two loops.

This explains the second inequality in (47). For the first inequality, observe that in the other state, the splicings form a small (inner) loops between p and q, and p and q connect the same loops only if there are no more twist equivalent crossings to p and q. (Only then the outer loop can be the same.) Also no other crossings connect the same pairs of loops. The identification of crossings single in their twist equivalence class can not be controlled easily in $A(D)$ or $B(D)$. So

$$t_2(D) + \sum_{k>2} k \cdot t_k(D) \leq e(G(A(D))') + e(G(B(D))'),$$

which is the left inequality in (47).

If D is now (A and B-)adequate, then by [Th] we have $c(D) = c(K)$ and by [LT] that

$$v(A(D)) + v(B(D)) = v(G(A(D))') + v(G(B(D))') = 2\,\mathrm{span}\,V(D) - c(D) + 2. \tag{48}$$

So

$$|V_1| + |\bar{V}_1| - 2c(K) + 2\,\mathrm{span}\,V(K) = e(G(A(D))') + e(G(B(D))') - c(D),$$

and (45) follows. To see (46), we use (48) and must show that at least $c(D) - t(D) + 2$ loops exist in $A(D)$ and $B(D)$ together. Now each equivalence class of k crossings gives rise to $k-1$ (inner) loops in one of the states, and each state has at least one more loop to connect the outer splicings. □

Corollary 7.1 If K is adequate, for any adequate diagram D of K we have (with $v_0 = \mathrm{vol}(4_1)/2$ the ideal tetrahedral volume)

$$\mathrm{vol}(K) \leq 10v_0\left(2\,\mathrm{span}\,V(K) + t_1(D) - c(K) - 1\right).$$

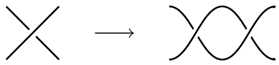
Proof. Use $c(D) - t(D) + t_1(D) \geq t(D)$. The claim follows using the inequality $\mathrm{vol}(K) \leq 10v_0(t(D) - 1)$ of Lackenby-Agol-Thurston [La]. □

Remark 7.1 Alternating diagrams are adequate with the additional property that the A- and B-state loops are connected by crossing traces only from outside (modulo change of the infinite region). Then it is easy to see that crossings connecting the same vertices in $A(D)$ or $B(D)$ are twist equivalent, and so the right inequality of (47) is exact. Then we have a different proof of the Dasbach-Lin lemma 2.1.

Remark 7.2 I do not know whether $V(K)$ determines $c(K)$ for adequate K. One should expect that it does not, but among adequate knots of ≤ 16 crossings there is no pair with different crossing number but same Jones polynomial. Thistlethwaite [Th] proved that the Kauffman polynomial determines the crossing number for adequate links. Also, the obvious fact that the left hand-side of (48) is at least 2 implies that $\mathrm{span}\,V(K) \geq c(K)/2$, so that $V(K)$ at least bounds above $c(K)$.

Example 7.1 It is in general not true that $t(D)$ is an invariant of adequate diagrams D of K. The simplest counterexample is 11_{440}, which has adequate diagrams of 6 and 7 twist equivalence classes. (This corrects my initial blunder to believe the right inequality of (47) is exact also for general A-adequate diagrams, and then that so would be the inequality in the corollary.) Also $T_1(K) = |V_1| + |\bar{V}_1| \le t(D)$ is not always true, as show the examples 15_{253246} and 15_{253273}.

The terms $t_{1,2}(D)$ seem a bit artificial, but they are necessary at least for estimating the twist numbers, since in general adequate diagrams a big discrepancy between $T_1(D)$ and $t(D)$ may occur. To illustrate this, we give a few constructions of such examples. Call the replacement of a crossing in a link diagram by a clasp of the same checkerboard sign a *clasping*.



(This may preserve or alter the number of components dependingly on the crossing and orientation of the clasp.)

Example 7.2 The diagrams D_k given by the closures of the n-braids $(\sigma_1\sigma_2\dots\sigma_{n-1}\sigma_{n-1}\dots\sigma_1)^k$ for $n > 3$ fixed and $k \to \infty$ have $t_{1,2} \to \infty$, but the contribution of the crossings with no twist equivalent one to $e(G(A(D_k))') + e(G(B(D_k))')$ is bounded. Also, the crossings in equivalent pairs are identified in both $G(A(D_k))'$ and $G(B(D_k))'$. Thus the left inequality in (47) is exact up to a bounded difference. If one likes knot diagrams, apply claspings at the last $n-1$ braid crossings, making $\sigma_{n-1}\dots\sigma_1$ into $\sigma_{n-1}^2\dots\sigma_1^2$. So the $t_{1,2}$ terms on the left of (47) are inavoidable. Whether the $t_1(D)$ term is indeed essential in corollary 7.1 is not evident, though, since our examples have bounded volume.

Example 7.3 Let K_1 be an adequate knot, and K_{n+1} be constructed from K_n as follows. Take a disconnected twisted with blackboard framing 2-cable of an adequate diagram D_n of K_n, and make a clasping at one of the mixed crossings of the resulting 2-component link diagram, so that the result is a knot and not a cable. We have the diagram D_{n+1} of K_{n+1}, which is easily seen to be adequate (similarly to [LT]). One can check using lemma 3.3 that cabling preserves (adequacy and) $T_1 = |V_1| + |\bar{V}_1|$, while clasping augments it by one. Thus $T_1(K_{n+1}) = T_1(K_n) + 1$, while $c(K_{n+1}) = 4c(K_n) + 1$, and the twist numbers of D_n grow similarly. Thus $T_1(D_n) = O(\log_4 t(D_n))$ as $n \to \infty$.

Still one can ask (with little hope, though)

Question 7.1 Are there only finitely many adequate or hyperbolic knots of given T_1?

Example 7.4 The existence of $T_1 = 0$ knots immediately implies (by connected sums) that for knots which are non-hyperbolic and not adequate the answer is negative. If the number of link components may grow unboundedly, the following sequence of links gives a negative answer: take a trivial 3-braid and place circles around strands $1,2$ and $2,3$ interchangingly (as in fig. 13 of [CR]). It is now easy to check that these links are adequate, with $T_1 = 0$, and their volume grows unboundedly.

We can thus also answer negatively the question of Dasbach-Lin whether T_1 bounds (above) the volume at least for links. For knots, it would be interesting to know the behaviour of $\operatorname{vol}(K_n)$ in example 7.3. It might show that for an adequate link an upper bound on the volume, if it at all exists, no better than exponential in terms of T_1 is possible (in contrast to Dasbach-Lin's linear bound for alternating links).

7.2. 3-braid and Montesinos links

Concerning 3-braids, the finiteness theorem in [St12] implies the existence of some upper bound on the volume in terms of the Jones polynomial. We can make now a more concrete estimate:

Proposition 7.2 Let L be a 3-braid link. Then there is a constant C independent on L such that $\mathrm{vol}(L) \le C\,(T_1'(L)+1)$, where $T_1'(L) = |V_1'| + |\bar{V}_1'|$, and

$$V_1' = \left\{ \begin{array}{cl} V_1 & \text{if } V_1 \neq 0 \\ \min\deg V & \text{if } V_1 = 0 \end{array} \right. , \quad \bar{V}_1' = \left\{ \begin{array}{cl} \bar{V}_1 & \text{if } \bar{V}_1 \neq 0 \\ \max\deg V & \text{if } \bar{V}_1 = 0 \end{array} \right. .$$

Proof. Assume L has an A-adequate 3-braid representation β. The links with Schreier vector length 2 are non-hyperbolic. If A-adequacy of β comes from a positive 3-braid representation, the number of crossings of β is bounded linearly by the degree of V. So assume A-adequacy comes from isolated positive Schreier vector entries. From a straightforward generalization of the argument proving proposition 6.1 it follows that V_1 bounds linearly the length of the Schreier vector entries in β, that do not belong to exceptional subwords. Now, we observed after proposition 6.1 that exceptional subwords can be made to a single syllable by factoring out a power of δ^2. Such powers can be collected at the end of the braid, because δ^2 is central. So $\beta = \beta'\delta^{2n}$, and the Schreier vector length of β' is bounded linearly by V_1. The rest follows by Thurston's hyperbolic surgery theorem.

A similar argument applies for B-adequate 3-braid representations. $\square$

The case of Montesinos links is more difficult, and the additional work on V_2 is needed. We resume the notation of §4.2.

Proposition 7.3 If L_i are Montesinos links and have the same Jones polynomial, then $\mathrm{vol}(L_i)$ are bounded. More exactly, for every Montesinos link L, we have $\mathrm{vol}(L) \le C' \cdot (T_1(L) + T_2(L))$, with a constant C' independent on L.

Proof. Choose the general form of representation

$$L_i = M(p_{i,1}/q_{i,1} \ldots, p_{i,n_i-1}/q_{i,n_i-1}, p_{i,n_i}/q_{i,n_i}, e_i),$$

with $p_{i,j}, q_{i,j} \ge 1$. W.l.o.g. assume L_i are B-adequate, so that $e_i \neq -1$.

First observe that, from part (1) of proposition 4.2, if L_i are adequate, i.e. $e_i \neq 1 - n_i$, then the twist number is bounded from a multiple of T_1. So we can assume that $e_i = 1 - n_i$.

By looking at $\bar{V}_1(L_i)$ and using part (1) of proposition 4.2 and proposition 4.1, we see that for fixed $\bar{V}_1$ the twist number may go to infinity only if the lengths n_i go to infinity and almost all $q_{i,k} = p_{i,k} + 1$ (i.e. for given i the number of k not satisfying this condition remains bounded as a sequence over i when $i \to \infty$). Otherwise $\bar{V}_1$ bounds the twist number.

Now consider $\bar{V}_2(L_i)$. We need to study the behaviour of the terms independent from $\bar{V}_1$, which are $\triangle B(D)'$, $e_{++}(B(D))$ and $\delta B(D)'$. The first quantity remains bounded. Every $q_{i,k} = p_{i,k} + 1$ contributes one to $e_{++}(B(D))$. This contribution can be equilibrated by $\delta B(D)'$ only if $e_i = -2$. (Otherwise there is no intertwined pair of connections in $B(D)$.) Then, however, for length $n_i > 3$, the L_i are adequate. Thus a non-zero multiple of all twists will be detected by $|\bar{V}_2| + |\bar{V}_1|$, and we are done. $\square$

Remark 7.3 Unfortunately, I still don't know how to conclude (if true) that there are only finitely many Montesinos links with the same Jones polynomial, since controlling the crossing number from $\mathrm{span}\,V$ is non-trivial. There are situations, for example the $(-3,k,l)$-pretzel links with $k,l \ge 4$, where in the almost alternating representation with $e = -1$, the third terms in $A\,\langle D_A\rangle$ and $A^{-1}\,\langle D_B\rangle$ of the proof of part (2) of proposition 4.2 cancel also. In the paper [HTY] it was shown that the difference between $\mathrm{span}\,V(L)$ and $c(L)$ for certain pretzel links can actually be made arbitrarily large, so that even a study of V_3 etc. may not be helpful.

8. Mahler measure and twist numbers

8.1. Estimating the norm of the Jones polynomial

We use the notation $t(K)$ and $t_a(K)$ from §2.

Theorem 8.1 For all numbers $n \in \mathbb{N}$ there are numbers c_n, d_n such that if $t(K) \le n$, then $(t+1)^n V_K(t)$ has at most c_n non-zero coefficients, each of which has absolute value at most d_n.

In other words, the 1-norm $||X||_1 = \sum_i |X_i|$ (with X_i the coefficients of X) of $X = (t+1)^n V_K(t)$ is bounded when $t(K) \le n$.

Remark 8.1 One can write down explicit values for c_n and d_n, which would be exponential in n.

In simultaneous joint work with Dan Silver and Susan Williams [SSW] we give similar results to theorem 8.1 for the Alexander and skein polynomial. Latter result implies a different, but more technical, proof of theorem 8.1.

Dan Silver also informed me of recent related work by I. Kofman and A. Champanerkar [CK]. While they work in a more general situation concerning (multi-strand) twisting and (colored) Jones polynomial, our proofs are more conceptual and less computational, and owe to the fact that we have done some of the relevant work previously, in [St3, St4, St2]. They should also emphasize the relation to the questions in [DL].

Corollary 8.1 $||(t+1)^n V_K||_{L^2(S^1)}$ is bounded for $t(K) \le n$.

Here S^1 is understood as the set of unit norm complex numbers.

Proof. By [St2, §6], we have $||X||_{L^2}^2 = \sum |X_i|^2$, where X_i are the coefficients of X. $\square$

The Mahler measure $M(P)$ of a polynomial P is defined as the product of norms of all (complex) roots outside the unit circle S^1 and the norm of the leading coefficient. For the relevance of this concept and further discussion see for example [GH, SW].

Corollary 8.2 The Mahler measure $M(V_K)$ is bounded for $t(K) \le n$.

Proof. Apply [St2, lemma 6.1]. $\square$

Corollary 8.3 The Mahler measure $M(V_K)$ bounds below increasingly $\mathrm{vol}(K)$ when K is alternating. (That is, if $M(V_K)$ grows to infinity, then so does $\mathrm{vol}(K)$.)

Proof. Use Lackenby's inequality [La], $\mathrm{vol}(K) \ge v_0(t_a(K) - 2)/2$ for K alternating. $\square$

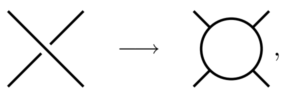
Corollary 8.4 Let V_K be the Jones polynomial represented as in (6). Then for all l and n there are numbers $c_{n,l}$ such that if $t(K) \le n$, then $|V_l| \le c_{n,l}$.

Proof. The possible values for $a_0 t^k + a_1 t^{k+1} + \ldots + a_l t^{k+l}$ are determined by the first $l+1$ (with respect to the minimal degree) coefficients of $(t+1)^n V_D$, and there are finitely many such choices. $\square$

Corollary 8.5 Any coefficient $|V_l|$ in the representation (6) (that is, fixed with respect to the minimal degree of V) bounds below increasingly $\mathrm{vol}(K)$ when K is alternating. $\square$

Similarly, so do $T_l = |V_l| + |\bar{V}_l|$, which explains rigorously some of the experimental observations of Dasbach-Lin [DL]. By remark 8.1, explicit estimates obtained this way are not optimal, so we clearly lose some efficiency for the sake of generality. Possibly one can prove better estimates with more effort from the graph theoretical setting of [DL], but at least for non-alternating knots, where the graph approach fails, we have new information from our main theorem. A few more applications follow after its proof.

Proof of theorem 8.1. Let T be a diagram *template*. It is obtained from a link diagram by replacing some crossings by slots



as was done for tangles in [ST]. We represent any twist sequence of link diagrams by a template T, into whose slots twist tangles are inserted. We say that we *associate* a diagram to the template in this way. We meet the convention that twist tangles inserted into template slots have an even number of half-twists (possibly leave a crossing outside the slot).

We assign a sign to each slot of T, which is positive if the twists in the slot are parallel and negative if the twists are reverse. (This means that not necessarily any sign choice can be realized by proper link component orientation.)

Let $s(T)$ be the number of slots of T, $c(T)$ the number of crossings (which do not include slots) and $s_p(T)$ the number of positive (parallel twist) slots of T.

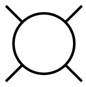
Lemma 8.1 For each template T there are numbers c_T and d_T such that if D is a diagram associated to T, then $(t+1)^{s(T)}V_D(t)$ has at most c_T non-zero coefficients, each of which has absolute value at most d_T.

Since for diagrams D with $t(D) \le n$ we need to consider templates T with $s_p(T) \le s(T) \le n$ and $c(T) \le n$ (the crossings left outside the slots to adjust half-twist parity), which are finitely many, the theorem follows.

We prove the lemma inductively on $s_p(T)$.

First the lemma is easy to verify when $s(T) = 1$ and $c(T) = 0$ (the $(2,n)$-torus links). For $s_p(T) = 0$ it follows from formula (8) of [St3] (as explained in [St4], including the correction of the misprint in the original version).

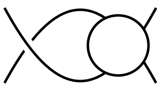
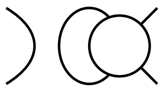
Now let $s_p(T) > 0$ and consider a parallel twist slot

 , (49)

of T, in which we assume the twists to be vertical. Replace $V(D)$ by the Kauffman bracket $[D]$ of D (see [Ka]). We have

$$V_D(t) = \left(-t^{-3/4}\right)^{-w(D)} [D]\Big|_{A=t^{-1/4}},$$

so the claims for $(t+1)^n V(D)$ and $(A^4+1)^n [D]$ are equivalent. The bracket of (49) is a linear combination of those of

 and  .

The first diagram has smaller s_p (since one of the components in the twist slot has now reverse orientation), while the other one is a connected sum of a $(2,n)$-torus link and a diagram of smaller s and s_p, and the claim follows easily. □

8.2. Link families with growing twist numbers

We have now a simpler proof of

Proposition 8.1 Let $V, X \in \mathbb{Z}[t,t^{-1}]$ with $X \ne 0$ and $V \ne 1$, and K_n be *knots* (not links) with $V(K_n) = XV^n$. Then $t(K_n) \to \infty$.

Proof. If V is zero or a unit in $\mathbb{Z}[t,t^{-1}]$, then for all but finitely many n there is no knot with Jones polynomial XV^n. (If the Jones polynomial of a *knot* is determined up to units, it is uniquely determined, because $V(1) = 1$ and $V'(1) = 0$.) Thus $||V||_{L^2} > 1$, and so (see [St2, §6]) $\max_{|t|=1} |V(t)| > 1$. Then $|V| > 1+\varepsilon$ and $|\hat{X}| > \varepsilon$ on an arc of S^1 for any non-zero polynomial $\hat{X}$, so that $||\hat{X}V^n||_{L^2} \to \infty$ when $n \to \infty$. Now taking $\hat{X} = (t+1)^p X$ we would have a contradiction to $t(K_{n_i}) \le p$ for any subsequence of K_n. □

In particular, taking V to be the polynomial of a $T_1 = 0$ knot, we have

Corollary 8.6 There exist $T_1 = 0$ knots of arbitrarily large twist number. □

Taking V to be the polynomial of the figure-8-knot, we have

Corollary 8.7 There exist (even alternating) knots with $M(V_K) = 1$ and arbitrarily large twist number (not only alternating twist number). □

More generally all the work of [St3] done on the relation between Jones polynomial and canonical (weak) genus, can be extended to twist numbers. Among others, we can now state:

Corollary 8.8 Let K be knot that has a (not necessarily untwisted) Whitehead double W_K, whose Jones polynomial $V(W_K)$ has an absolute coefficient $|V_i| \geq 4$ for some i. Then any sequence of Whitehead doubles of $\#^n K$ has unbounded twist number when $n \to \infty$. □

Note that all the preceding examples are non-hyperbolic, so that the volume cannot be used to estimate the twist number.

From [St3] we obtain also

Corollary 8.9 Let $|t| = 1$ and $t(K) \leq n$. Then for $t \neq -1$ we have $|V_K(t)| \leq C_{t,n}$ for some constant $C_{t,n}$ depending only on t and n. Also $|V_K(-1)| \leq C_n(\operatorname{span} V_K)^n$ for some constant C_n depending only on n. □

Latter property can be used for example to easily deduce

Proposition 8.2 For $n \geq 2$ there exist n-component links with trivial Jones polynomial and arbitrarily large twist number.

Proof. By [EKT], there are such links that have as a component a 2-bridge knot whose alternating Conway notation contains an iterative subsequence. Clearly the twist number of a link is not smaller than the one of its components. But it is easy to see that the determinant $|V(-1)|$ of such 2-bridge knots grows exponentially (with the number of occurrences of the subsequence), while their crossing number (or span of the Jones polynomial) grows linearly. □

Question 8.1 Are there infinitely many *knots* with the same Jones polynomial that have arbitrarily large twist number?

Remark 8.2 Kanenobu [K] found infinitely many knots with the same Jones polynomial, but these knots have bounded twist number.

The following questions arise (and were posed also by others) as a natural follow-up to theorem 8.1 and its applications.

Question 8.2 Are there knots/links of unbounded volume, which

(1) (Dan Silver) have Jones polynomial of bounded Mahler measure? Or, as possible improvements,

(2) are there alternating such knots or

(3) (Efstratia Kalfagianni [Kf]) are there knots with Jones polynomial of bounded 1-norm $\sum |V_i|$, or

(4) even with the same polynomial?

Remark 8.3 Since the 1-norm bounds the Mahler measure (see [St2]), a positive answer to part 3 implies such for part 1. Part 2 is related to the question how to find *prime* alternating knots whose Jones polynomial is a power of $V(4_1)$ (or more generally cyclotomic; for this problem see [CK2]). Since for alternating knots the 1-norm of the Jones polynomial is equal to the determinant, a series of knots answering positively part 3 (or even 4) are *a forteriori* non-alternating. Since bounded twist number implies bounded volume, a positive answer to part 4 for knots will imply such to question 8.1. In particular, again Kanenobu's series can not be used. Similarly fails now also the construction in [EKT], which yields non-hyperbolic links. It is worth remarking that for the Alexander polynomial most of these questions can be effectively decided [Kf, SSW, St8].

9. Odd crossing number achiral knots

9.1. Theorem and initial remarks

We conclude with the most substantial application of the work in this paper. It relates, as explained, to one of the oldest problems in knot theory, concerning the crossing number of achiral knots, dating back to more than a century ago, when it was observed by Tait in his pioneering work on tabulating the simplest knots. For alternating knots, this problem was solved in [Ka, Mu, Th2] as a consequence of the proof of another of Tait's conjectures, the minimality of reduced alternating diagrams. This was one of the celebrated achievements of the Jones polynomial. Contrarily, we can state now

Theorem 9.1 For all numbers $n = 11+4k$, $k > 0$, there exists a prime achiral knot K_k of crossing number n. Specifically, K_k is the closure of the 5-braid $-12^2 3^{2k} 4-32-1(-2)^{2k}(-3)^2 4-23$.

(In the braid word, $i > 0$ stands for the Artin generator σ_i, and $-i$ for σ_i^{-1}.) One easily sees that the given knots are negative amphicheiral. The case $k = 1$ is $K_1 = 15_{224980}$, Thistlethwaite's example of an amphicheiral 15 crossing knot (see [HTW]). It apparently came up in routine knot tabulation, and the insight that this knot has crossing number 15 is a result of generating all diagrams of fewer crossings, and being able to distinguish K_1 from these diagrams. Even though candidates to generalize such an example are straightforward, the major problem one faces when the crossing number increases is that exhaustive diagram verification quickly becomes impossible.

So one is led to seek general tools to rule out all potential fewer crossing diagrams. Our formulas for the coefficients of V seem the first tool that is powerful enough to accomplish the main part of the work, although a lot remains to be done, and again computer aid at some point seems very helpful. (A careful reading will reveal that a minor part of our arguments is in fact redundant. Even aware of this, we preferred not to omit them, since the complexity of proof makes the danger serious to err on the opposite side and introduce a gap.)

Our result settles the first half of theorem 1.1 and bases on Thistlethwaite's example. We will later deal with the other half of odd crossing numbers by the same method, though that case poses added difficulty. (The proof of theorem 9.1 will explain why; see remark 9.1 after it.)

For the proof of theorem 9.1 we need most of the preparations, in particular the invariants in §3.4. It is useful to assume below that $k > 1$. That is, we waive on unnecessarily reproving Thistlethwaite's example, for which our argument would work, but would need slight modification.

A few simple observations are useful to collect in advance.

Lemma 9.1 K_k are non-alternating.

Proof. We have $V_0V_1 = \bar{V}_0\bar{V}_1 = -1$, so if alternating, these knots must have twist number two. Then they are either rational of genus 1, or positive. Former is excluded looking at the Alexander polynomial, and latter, for example, using the signature. □

Lemma 9.2 K_k are prime.

Proof. K_k have braid index $b \le 5$. We use the Birman-Menasco result [BM2] on the (1-sub)additivity of b under connected sum. If composite, K_k must be either connected sum of

(i) four braid index 2 factors,
(ii) an amphicheiral $b = 3$ knot and two braid index 2 factors,
(iii) two amphicheiral $b = 3$ knots, or
(iv) a $b = 3$ knot with its mirror image.

By Birman-Menasco [BM], or the work in §6 here, amphicheiral $b = 3$ knots are alternating, and since braid index 2 knots are too, but K_k are not, we can have only option (iv). In that case look at the skein polynomial P. Its second highest m-coefficient $X = [P]_{m^{\max\deg_m P-2}}$ must be of the form $f(l)+f(l^{-1})$ for some $f \in \mathbb{Z}[l]$, in particular $[X]_{l^0}$ must be even. But by direct calculation it is an odd integer. □

9.2. Start of proof of theorem 9.1: basic setup

The braid representation in the theorem gives a diagram of K_k of $12+4k$ crossings, that reduces (by a generalized version of the move for Thistlethwaite's example) to a semiadequate diagram $\tilde{D}_k$ of writhe ± 1 and $11+4k$ crossings. By [Th], we have $c(K_k) \geq 10+4k$, and if $c(K_k) = 10+4k$, then K_k is adequate. We assume, fixing k, that $D_k = D$ is an adequate diagram of K_k. The proof will consist in successively ruling out all possibilities for D.

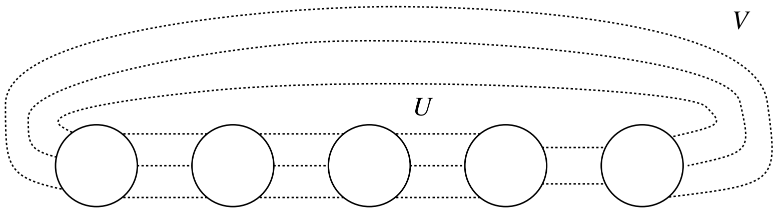
First we remark that D is prime, because K_k is prime and an adequate (connected sum factor) diagram represents a non-trivial knot.

We use the invariants of §3.4. Either by looking at the semiadequate diagrams $\tilde{D}_k$, or (more tediously) by calculation of the Jones polynomial and its 2-cable, we find that D_k must satisfy

$$\triangle = 0, \quad \chi(G) = 0, \quad \chi(IG) = 0 \ \text{ for } \textit{both } A\text{- and } B\text{-state}, \quad \text{and} \quad \gamma = 4\,. \tag{50}$$

These properties will be used throughout the whole proof. The quantity γ refers to (29).

We will use a similar approach as in §5.1 and again construct the A-state of D by successively attaching loops as in (28). A difference to §5.1 is that now $\chi(G) = 0$, so G contains (exactly) one cycle, and we must start building $A(D)$ with this cycle rather than a single loop. We call the cycle of loops in $A(D)$ the *outer cycle*[1]. Since D is not positive, the length n of the outer cycle is odd, and since $\triangle = 0$, we have $n > 3$.



(51)

Let us for convenience assume D drawn so that the loops in the outer cycle are always connected from the outside, i.e. the unbounded region in their complement. We define *interor* and *exterior* of these loops according to this convention.

Every loop l which is not in the outer cycle is attached to a single previous loop m. This means that all traces of crossings, that connect l at one end connect to m on the other. The parallel equivalence classes of such traces are called below *legs*. So legs are edges in $A(D)$ connecting an attached loop. The number of legs of l is called the *valence* of l. As in (27), and unless stated clearly otherwise, we group parallel traces into a single one (with multiplicity indicated, or explained from the context) when drawing loop diagrams. So a dashed line in a diagram starting from a loop attached by (28) (usually) stands for a leg.

Note that γ does not depend on the size (multiplicity) of parallel equivalence classes of traces in $A(D)$ or $B(D)$. Again the addition of legs, and consequently the move (28), never reduces γ, so if we reach a stage that $\gamma > 4$, or we can delete legs and see that $\gamma > 4$, we can discard any (diagram obtained by) continuation of loop attachments.

It is clear that the legs of each attached loop form a multiple connection of traces. So the existence of attachments forces the intertwining graph $IG(A)$ to be non-empty (i.e. have at least one vertex). Then the condition $\chi(IG) = 0$ means that we must have a cycle in $IG(A)$, and similarly in $IG(B)$. The existence of this cycle will be helpful at several places below.

Convention 9.1 We also meet the convention that we consider diagrams *up to mutations*. All our tests involve (quantities and properties determined by) invariants that do not change under mutations (see in particular corollary 3.7), so this restriction is legitimate.

Note that we have the same conditions for $B(D)$. We will try to use these conditions to restrict the possible ways in which we can build $A(D)$ successively by attaching loops.

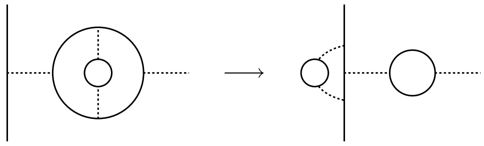
[1] Beware in particular of confusing the words 'loop' and 'cycle'; the meanings they are used in here are very different! A loop is a piece of a diagram obtained after splicing all crossings, and a cycle in a set of loops connected by splicing traces in an appropriate manner.

Now we define the *depth* of loops in $A(D)$ inductively over the order of attachments as follows. The depth of the loops in the outer cycle, and of loops attached to them from the exterior, is 0. Assume next a loop M'' is attached to a loop M. If M has depth k and is connected to a loop M' of depth $k' \le k$, then any loop M'' connected to M from the opposite side to M' has depth $k+1$. If M'' is attached to M from the same side as M', then we set the depth of M'' to be k. For example in part (a) of figure 4 (page 51), we have an outer cycle loop L (of depth 0), other depth-0 loops E,F,G (not on the outer cycle), and depth-1 loops A,B,C,D.

If $k > 0$, we call the side of a loop that contains loops of not larger depth the *exterior* of M. The other region is the *interor* of M. If the interor contains no further loops, we call M *empty*.

We call *a attached* to b if there is a leg of a connected to b. If a is attached inside b then a has higher depth, and if a is attached outside, it has the same depth, but a higher distance to a loop of smaller depth.

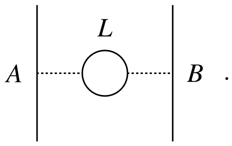
By mutations we assume that the total depth of loops is the smallest possible. In particular, the following flypes/mutations are applied to reduce the total depth:


(52)

We call a diagram *flat* if it has minimal total depth (sum of depth of its loops) in its mutation equivalence class. *So we assume from now on that D is flat.*

The next assumption we can make is that, among diagrams of the same total depth, we choose the one with the fewest edges, i.e. *we assume D is edge-reduced*.

Lemma 9.3 The two legs of a valence 2-loop L are not attached to the same region, i.e. $A \neq B$ in



(53)

Proof. Straightforward, because D is prime. □

In the following we say in (53) that L *connects* or *identifies* regions A and B. The meaning comes from looking at $B(D)$. There A and B correspond to loops, and the attachment of L has the effect of joining these loops.

9.3. Controlling the number of attachments

We need to extend lemma 9.3 to higher valence attachments. For this we decisively use $\gamma = 4$ in (50). We need also the following piece of information.

Definition 9.1 Assume D is A-adequate. A loop in $A(D)$ is separating if it is connected by crossing traces from both sides (see §3.3). Define a *block component* of $A(D)$ to be a region of the complement of the set of loops in $A(D)$ together with the loops in its boundary and all crossing traces within that region.

A connected (sum) component of a block component is defined as for link diagrams: a block component decomposes as connected sum along a curve intersecting it in two points on the loops, and in no point on crossing traces. An A-state *atom* of D is defined to be a connected component of a block component of D. We write $a_A(D)$ for the (A-state) *atom number* of D, the number of its A-state atoms. Obviously, the same definitions can be set up for the B-state. We will sometimes write $a(D)$ if it is clear from the context if the A-state or B-state is meant.

A paraphrasing of $a_A(D)$ in terms of the A-state graph $G = G(A(D))$ is that $a_A(D) = a(G)$, where $a(G)$ is the atom number from definition 3.2.

A first illustration is the case when D is positive. Then the (separating) A-state loops are the (separating) Seifert circles, the notion of block(component)s coincides with the one of Cromwell [Cr], and (A-state) atoms are connected components of blocks.

We visualize the notion also with the special case we will use in our proof: the outer cycle in (51) is an (A-state) atom of D, and each attachment (28) adds one new atom. So in our case the atom number counts the number of attachments (plus one). A decisive merit of this quantity is that it is controllable from a link invariant. The following is proved in [St15] using Thistlethwaite's work on the Kauffman polynomial[1].

Theorem 9.2 (see proposition 5.3 in [St15]) For A-adequate diagrams D of a link L, the number $c(D) - a_A(D)$ is an invariant of L.

Then calculation shows

Corollary 9.1 To construct D_k, we must perform exactly 3 attachments (28) on the outer cycle loop. □

This information will be used for the crossing numbers $15+4k$ only secondarily. A banal reason was that the author became aware of it only after most of this proof was completed, and corollary 9.1 is used only for minor simplifications and fixes of the argument. However, the corollary became the main motivation for attempting the case $17+4k$. Indeed, this corollary becomes a major tool in that part, for without it the proof would have been irrecordably complicated.

In particular, we have

Corollary 9.2 There are at least two attachments (of depth 1) inside an outer cycle loop. In particular, there is at most one attachment outside an outer cycle loop.

Proof. Were there no two attachments inside an outer cycle loop, we would have no cycle in $IG(A(D))$. □

The next simplifications need a definition.

Definition 9.2 Call a crossing of a diagram D to be A-inadequate if its trace in $A(D)$ connects the same loop. (So D is A-adequate if it has no A-inadequate crossing.) Similarly one defines B-inadequate crossings.

Let D' be obtained from D by making all edges in $A(D)$ simple (i.e. of multiplicity 1). We call an edge in $A(D)$ to be *inadequate* if its trace in D' comes from a B-inadequate crossing (of D'). A loop of valence 2, attached by (28) in $A(D)$, is inadequate if its both legs are inadequate.

An example, and simultaneously the important special case we will use below, is as follows. Attach inside a loop L another loop M, and let R be a region outside L touched by a leg basepoint of M. If no other loop inside L has a leg basepoint touching R, and no loop outside L has non-empty interior, then M is inadequate.

Lemma 9.4 If e is an inadequate edge of $A(D)$, then e has multiplicity 2.

Proof. Clearly $e \geq 2$ by A-adequacy of D. If e has multiplicity ≥ 3, we have a cycle of $B(D)$ that consists of simple connections between the loops only (dual to the traces of the edge e). This must be then the only cycle. However, then with corollary 9.1, we see that by 3 attachments we cannot create then a cycle in $IG(B(D))$. □

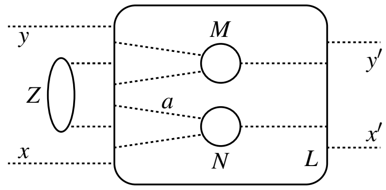
Lemma 9.5 If D has an inadequate loop P, then D cannot be one of our hypothetic diagrams D_k.

Proof. Let x, y be the multiplicities of the legs of P in D. Clearly $x, y \geq 2$, and by connectivity, one is at least 3. But this contradicts lemma 9.4. □

[1] Note, though, that $a_A(D)$ is called $m(D)$ in [St15], and $a(D)$ is used there with a different meaning.

Lemma 9.6 There is no region touched in two different segments by legs of attachments (28) on the opposite side.

Proof. The situation in which two such legs could occur is that they connect to the same region on both sides of an outside attachment, like



(54)

If both segments of L left and right of Z are touched by a leg of the same loop M only, then we could reduce the number of legs by a mutation, or an attachment inside M that prevents us from that would make $\gamma > 4$. Thus we can assume they are touched by at least two, and then exactly two by corollary 9.1, loops M,N inside L. This also implies, by corollary 9.2, that the analogous case when Z is inside L, and M,N would be outside, does not occur. So we have (54), with the option that some of a,x',y' may be zero.

Since we have an outside attachment Z, the other two attachments are inside by corollary 9.2. So Z is the sole outside attachment, and there are no attachments of depth > 1. Note that Z cannot have valence more than 2. Otherwise we could not keep $\gamma \leq 4$ and simultaneously make the diagram prime with the inside attachments. But from (54) we see that Z would be inadequate, contradicting lemma 9.5. □

In particular with $\gamma = 4$, we have

Lemma 9.7 There are no attached loops of valence different from 2 or 3.

Proof. Valence 1 means a composite diagram. Also, we can assume that traces of attached loops that connect to the same region are parallel (because the diagram is prime). So valence ≥ 4 for a loop attached inside some other loop will account in $\gamma \geq 6$. If a loop M of valence at least 4 is attached outside to an outer cycle loop L, there are at least 4 intervals of L to which traces of M are attached from inside (3 between the legs of M, and one more to make the diagram prime). Then we have again $\gamma \geq 6$. □

9.4. Properties of edge multiplicities

As in §5.1, we can assume up to flypes that *D has at most one $\sim$-equivalence class of more than 2 crossings*. Since for the time being we trace only the number of components, adequacy, and the invariants in (50), we change the multiplicity of this class to 4 or 5 crossings, preserving parity. The only possible change in the invariants in (50) can occur if the multiplicity 3 is increased to 5, and a triangle in $B(D)$ is destroyed. But we know that there are no triangles anyway. So, for the sake of working with the invariants in (50), we can assume that

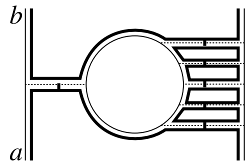
parallel equivalence classes have 1, 2, 4 *or* 5 *elements, and at most one class has* 4 *or* 5 *elements.* (55)

Only at the very final stage of the proof, in §9.8, we will change 4 and 5 back to $2n$ and $2n+1$.

Convention 9.2 Below we will draw in diagrams the B-state in thicker (solid for loops and dashed for traces) lines than the A-state. Often drawing A- and B-state in the same diagram makes it hard to parse, though, and drawing separate diagrams takes extra space we like to save. So we will several times waive on drawing the B-state even when we need to argue with it. The indicated examples should advise how to obtain it in the other cases. In other situations, to enhance legibility, we will draw only the loops of the B-state, but not the traces, which are easy to reconstruct.

Lemma 9.8 Assume D has an empty loop of valence 2. Then the leg valences are $(1,1)$, $(1,4)$ or $(2,5)$.

Proof. Multiplicities $(2,2)$ and $(2,4)$ are ruled out because of connectivity. (The tangle has a closed component, so the diagram is not of a knot.) $(1,5)$ is ruled out because in



if $a = b$, the diagram is not adequate, else $B(D)$ has an even length cycle. □

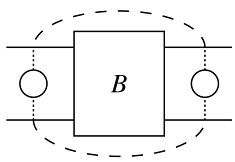
Convention 9.3 We denote an edge (parallel equivalence class of traces) and its multiplicity by the same letter. For example '$x = 2$' means that the edge x has multiplicity 2.

9.5. Restricting parallel loops

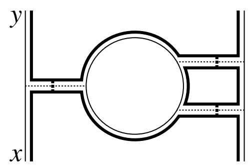
We call loops *parallel* if they have both valence 2 and their legs connect the same regions.

Lemma 9.9 Assume D has a pair of parallel loops. Assume both loops are empty. Then their valencies are $(1,1)$ and $(1,4)$ resp.

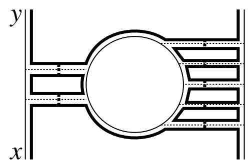
Proof. By mutations we can assume that the box B in



is empty. (The dashed lines on top and bottom should indicate that one can connect the basepoints of the legs of the two loops without crossing any other traces or loops.) Let (a,b), (c,d) be the multiplicities of the two leg pairs. (We assume $a,b,c,d \in \{1,2,4,5\}$.) For any of both pairs, $(1,2)$ is ruled out:



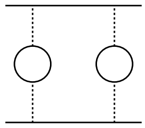
if $x = y$, the diagram is not adequate, else $\triangle > 0$. Then if $(a,b),(c,d) \neq (1,1)$, then any of those creates a cycle in $B(D)$, so $\chi(B(D)) < 0$. So we can assume that w.l.o.g. $(a,b) = (1,1)$, and we need to sort out only the possibilities for (c,d). The cases $(c,d) = (2,2),(2,4),(1,5)$, are ruled out as before. The case $(2,5)$ is ruled out because in that case $(a,b) = (1,1)$ accounts for the loops x,y in



being different, and so again $B(D)$ has two different cycles. □

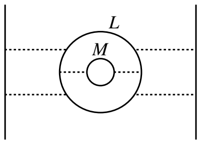
Lemma 9.10 Assume D has a pair of parallel loops. Then both loops are empty.

Proof. We can assume again that we have


. (56)

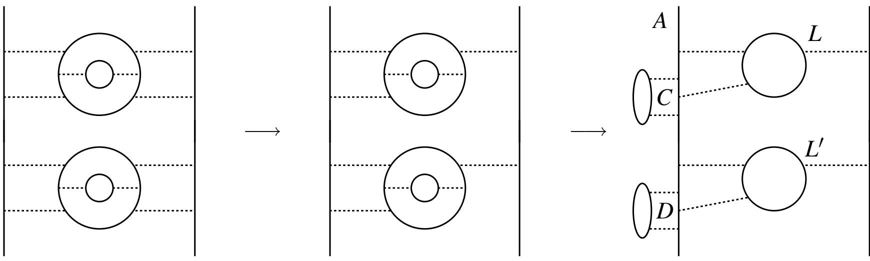
We can assume now using corollary 9.1 that at least one of the loops is non-empty, and that (56) is a pair of such loops of maximal depth.

Now if a loop is non-empty, we can flype its interor out by (52) (and have a contradiction to the flatness condition), unless we have a super-configuration of



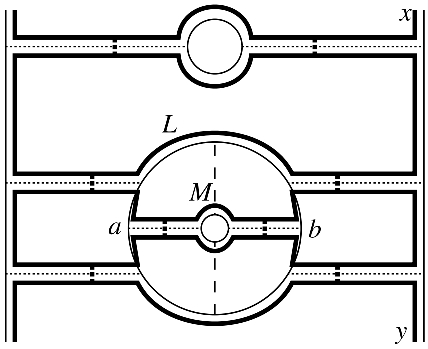
. (57)

(X is a super-configuration of Y, if Y arises from X be deleting some number of, possibly no, legs.) If such a situation occurs in both loops in (56), then we delete one of the 4 legs in each copy of (57), flype the interor out, and see $\gamma > 4$, since the legs of the (low empty loops) L, L' identify 4 different regions A, B, C, D.



(58)

Thus we can assume that exactly one of the loops in (56) is a super-configuration of (57). Deleting a leg, and flyping the interor out, as in (58), we see that $\gamma = 4$.



(59)

Adding a leg at some of the vertical dashed positions, joins two loops in $B(D)$, so γ gets > 4.

By corollary 9.1 we may assume that the interor of M is empty and M has no parallel loop, and no loop is attached outside to L. Let a, b be the multiplicities of the legs. If one of a or b is single, then D is not adequate. If $a = b = 2$, then D is not a knot diagram. So $\max(a,b) \geq 3$. But then (59) contains one cycle in $B(D)$, and there must be another one, that links x and y via the rest of the diagram ($x \neq y$ because $\triangle = 0$), because D is prime. So $B(D)$ has at least two cycles, a contradiction. □

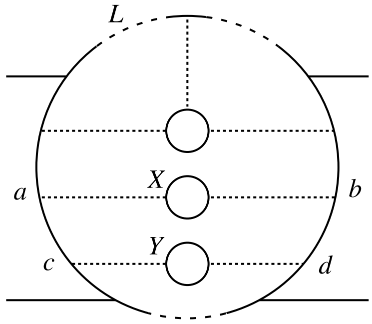
Lemma 9.11 There is no triple of parallel (empty) loops, and no two different pairs.

Proof. Use $\chi(G(B(D))) = 0$ and corollary 9.1. □

Lemma 9.12 There exist no parallel (empty) loops in D.

Proof. In order to create a cycle in $IG(A)$ with three attachments (corollary 9.1), a parallel pair X,Y of loops in D must be attached (with depth 1) inside an outer cycle loop L, and intertwine at least two multiple edges. (We assume X,Y are empty by lemma 9.10.) In particular, X,Y do not identify any of the regions U or V in (51). Now the third attachment is supposed to establish $\gamma=4$ and not be an inadequate loop (by lemma 9.5).

Since X,Y do not identify U or V, this means that a loop of valence 2 attached within an outer cycle loop L' different from L would be inadequate. The condition $\gamma=4$ also rules out the option of attaching the third loop outside to L, or attaching a loop of valence 3 inside some $L'\neq L$. Then the only option is



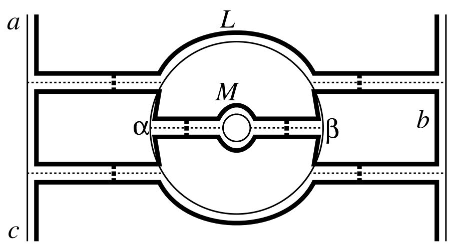
(60)

(The dashed pairs of the loop indicate that something may be attached to it there from the outside.) Now, by connectivity reasons, not all of a,b,c,d are 1, and then we have two cycles in $G(B(D))$. □

9.6. Ruling out nesting of loops

Lemma 9.13 All valence 2-loops of depth 1, or such of depth 0 outside the outer cycle, are empty.

Proof. Assume L is a valence 2-loop with non-empty interior. As before, we have a similar picture to (57):



. (61)

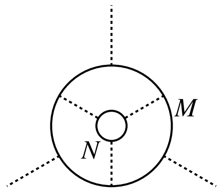
(Here α,β denote the leg multiplicities of M.) If $a\neq c$, then we argued to have two cycles in $B(D)$ (see the proof of lemma 9.9). So $a=c$. Then again $\gamma=4$.

Assume first M has no parallel loop inside L. Now, regardless of the way the $B(D)$ state loops are joined by possible $A(D)$-state loops attached to M, for an adequate diagram we must have $\alpha,\beta\geq 2$. For connectivity reasons then again $\max(\alpha,\beta)\geq 3$, so we have one cycle in $B(D)$. Now (again for both ways the loops are connected inside M), for connectivity reasons, one of the outer 4 legs of L must be multiple (else the tangle in (61) contains a closed component). But then, since $b\neq a=c$, we have a second cycle in $B(D)$.

If M has a parallel loop inside L, we have by lemma 9.9 an edge of multiplicity 4 inside L, which accounts for one cycle in $B(D)$, and the argument with the outer legs of L applies as before to yield a second cycle in $B(D)$. □

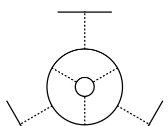
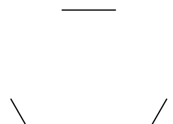
Lemma 9.14 Assume M is a valence 3-loop of depth ≤ 1 (but not an outer cycle loop if of depth 0). Then M is empty.

Proof. Assume M has a non-empty interior. By flatness we can assume we have a loop N inside M like



. (62)

Observe that

$$\gamma\left(\right.$$  $$\left.\right) = \gamma\left(\right.$$  $$\left.\right) + 4, \qquad (63)$$

and remember that $\gamma(D) \leq 4$.

If M has depth 0, then removing it with its interor results in a diagram that still has a depth 1 loop (by primeness), and (63) shows $\gamma \geq 6$. So assume that M is a depth-one loop.

First we argue that we have no loop attached to M from the exterior. Namely, such a loop must be intertwined with some loop in M's interior (because D is prime), but then $\gamma \geq 6$.

We claim next that all loops attached inside M are empty. If a valence 2-loop N' attached inside M is non-empty. then (by deleting some legs if necessary), we flype its interior out, and have a loop attached outside to M, and $\gamma \geq 6$. If we have a non-empty valence 3-loop attached inside M, we find by flatness a fragment (62) inside M. Using (63) and that M remains of depth 1, we have $\gamma \geq 6$.

This means that $IG(A)$ has no cycle resulting from edges inside M.

Also, from (63) we see that removing M and its interior we have a diagram with $\gamma = 0$, i.e. no loops of non-zero depth. Since D is non-alternating and has loops of positive depth, in particular, M must be the only loop attached inside an outer cycle loop. But then one easily sees that $IG(A)$ is a forest, and so $\chi(IG(A)) > 0$. □

We should note the following important implications of the last two lemmas.

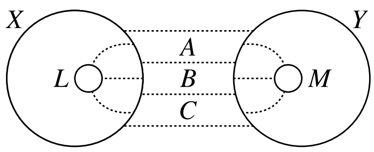
Corollary 9.3 There are no loops attached (neither from the interior, nor from the exterior) to depth 1 loops, or to depth 0 loops which are not outer cycle loops.

Proof. For the interior this is proved in the previous two lemmas. For the exterior this follows then because D is prime. □

Corollary 9.4 There are no loops of depth > 1 in D. □

Lemma 9.15 Only one loop of the outer cycle is non-empty.

Proof. Assume two loops X and Y of the outer cycle are non-empty. Let L and M be loops attached to X and Y inside.

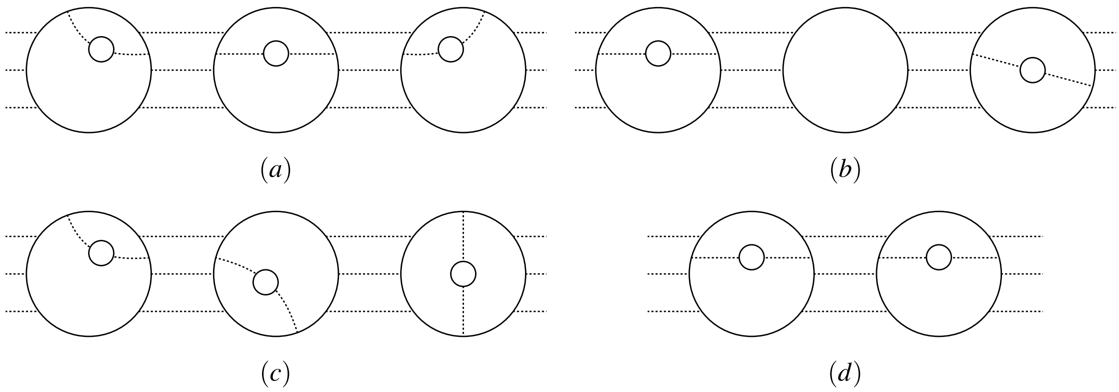


(64)

If, say, L has valence ≥ 3, then it is 3, and we have $\gamma = 4$ already after attaching L. Let A, B, C be the regions connected by the legs of L. Then M must connect the same triple (if valence 3) or 2 of the three (if valence 2) of A, B, C. If M has valence 2, one can mutate it into X. If M has valence 3, then L and M are both empty inside. Else one could as before, after possibly deleting some legs, flype out their interor and see that they do not connect the same triple of regions, so $\gamma > 4$. Also, since the length of the outer cycle is > 2, the coincidence of the 3 regions connected forces L and M to be intertwined with at most one multiple edge in the outer cycle. Then $IG(A)$ is a forest, and $\chi(IG(A)) > 0$, a contradiction. The same argument applies if inside X or Y several valence-2 loops are attached that identify three different regions A,B,C.

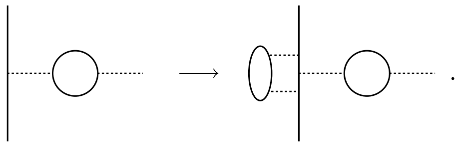
So we can assume that loops attached inside X and Y identify only two pairs of regions. In particular there is only one such loop for each X and Y, because two such are parallel, which we excluded with lemma 9.12. So L and M are the single loops attached inside X and Y, and have both valence 2. Let (A,B) and (C,D) be the region pairs their legs connect. (Pairs are unordered.) If $(A,B) = (C,D)$ we can move by mutation L and M into X, and see that L and M are parallel, again in contradiction to lemma 9.12.

So assume $|\{A,B,C,D\}| \geq 3$. Then $\gamma = 4$ after installing L and M. If L or M are non-empty, flatness forces again fragments like (57). Then delete a leg and flype to see that $\gamma > 4$. So L and M are empty. Then we have some subconfiguration of (i.e. obtained by possibly deleting some depth-1-loops from) one of the following:



(65)

where we may add a depth-0 loop for some of the depth-1 loops like



(66)

However, we see that in all cases of (65), we have $\chi(IG(A)) > 0$, and (66) does not change $\chi(IG(A))$. □

Lemma 9.16 The non-empty outer cycle loop has at least two loops attached inside.

Proof. The graph $IG(A)$, which most be non-empty, must have a cycle. There is no way of creating a cycle with edges of depth > 1 (because there are no such), and if only one loop is attached inside the non-empty outer cycle loop, then $IG(A)$ cannot have a cycle at all. □

9.7. Diagram patterns

The preparations so far yield a very restricted type of combinatorial shapes of $A(D)$, which we call below *patterns*. Formally, a pattern is a set of A-state diagrams, which are obtainable from a given one by replacing (traces of) simple edges e by a number of parallel traces (i.e. a multiple edge), including 0 (i.e. deletion of e).

Depending on the choice of regions (a triple, a pair, or two pairs) identified by the loops attached inside the outer cycle loop L we have three types of patterns we refer to below as (a), (b) and (c), see figure 4. Each edge in figure 4 stands for a group of parallel traces. In the following we identify an edge with its multiplicity. (We allow in general this multiplicity to be 0, i.e. to delete the edge. Then we delete also loops becoming isolated, i.e. when all their legs are 0.) We assign to an edge a letter as indicated in figure 4.

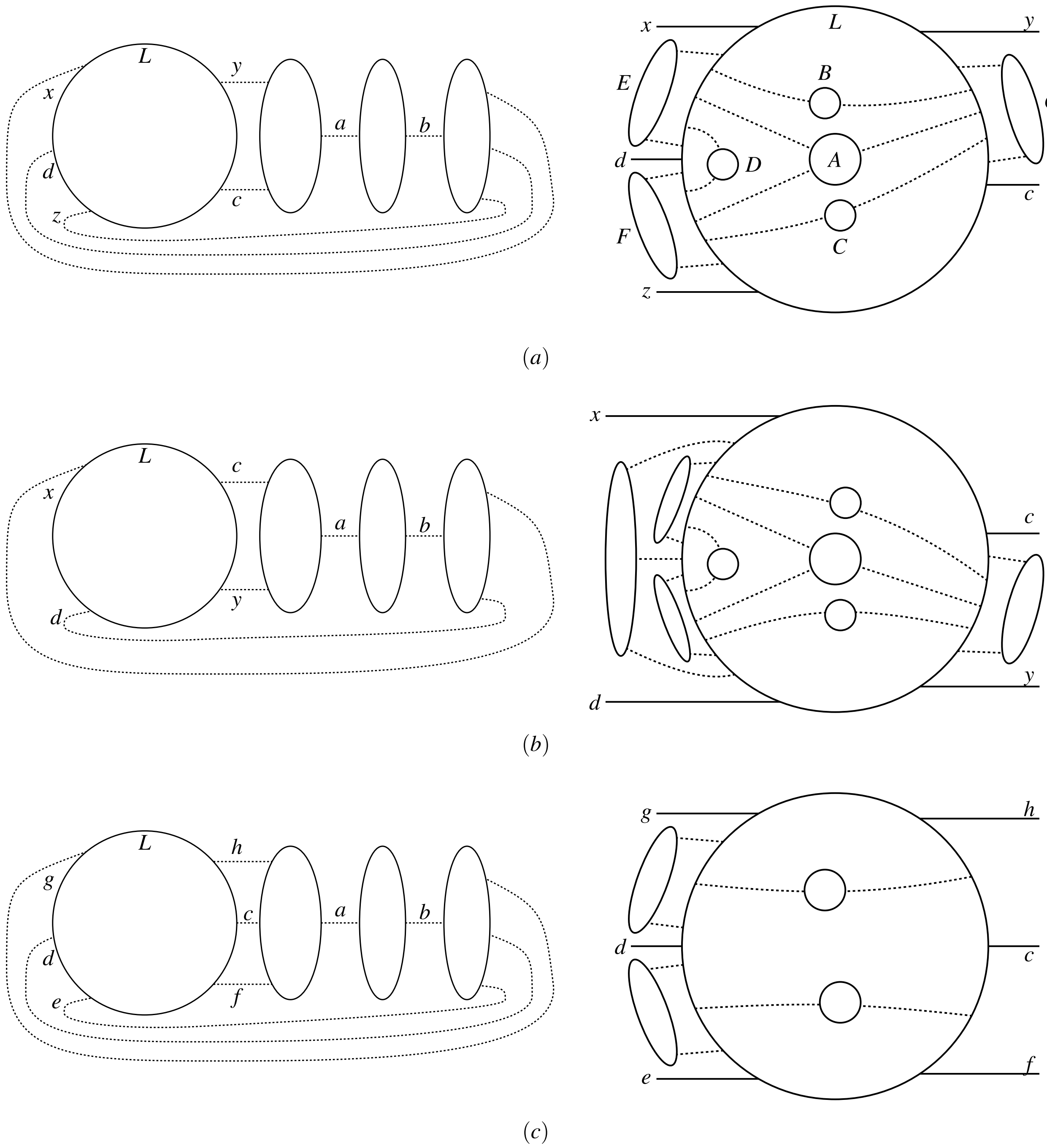
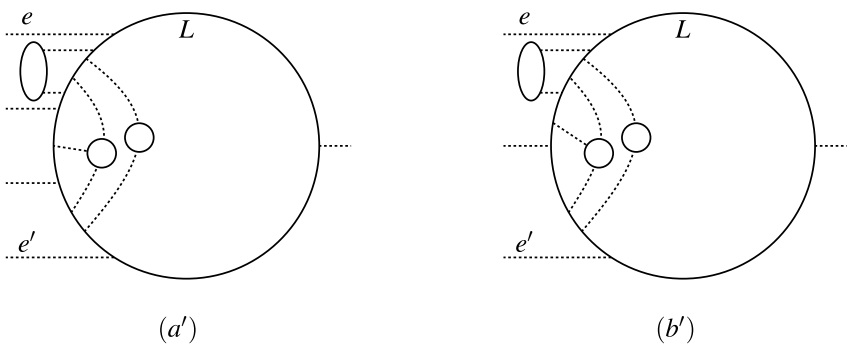


Figure 4: Diagram patterns. The left pictures show the outer cycle loops and the edges connecting them. The right parts show the non-empty outer cycle loop L and the ways other loops can be attached to it, up to omissions and parallel loops.

Now, by lemma 9.12, a pair of parallel loops is excluded in all patterns. Also in patterns (c) the only choice of two pairs of regions so that $IG(A)$ contains a cycle is as indicated. We allow in patterns edges that connect L to a neighboring loop in the outer cycle to be of multiplicity 0 (i.e. to be deleted) when generating diagrams D'. Still many of these cases can be discarded because of repetitions, or because D' (and so D) becomes a composite diagram. In particular in figure 4 we chose the naming of edges in all 3 cases so that initial alphabet letters, a to h, stand for multiplicities that can be assumed to be at least 1 (by symmetry, because D is prime, or because, in case (c), a cycle must occur in $IG(A)$). The names a and b always stand for edges that connect two empty outer cycle loops. The omitable (i.e. possibly of multiplicity 0) edges are named by end-alphabet letters x,y,z. In case (c), if c or d is missing, we have a special case of (b).

It is possible that all regions identified by loops attached inside L are on the same side, that is, between traces/edges that connect L to the same neighboring loop L' in the outer cycle. One easily sees that in the pattern (c) both loops of depth 1 (which identify two disjoint pairs of regions) must intertwine two connections of the outer cycle. So we need to look at patterns (a) and (b), and we call the modified patterns of type (a′) and (b′). Here are examples for either case (keep in mind corollary 9.1):



(67)

Now, by a mutation, we can move down the uppermost (possibly multiple) edge e between L and L' (the loop left of L, which is not drawn), making e parallel to e'. This way we convert patterns as in (67) into such included in figure 4 (with some outer cycle connection of multiplicity 0).

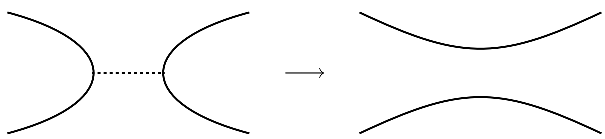
There is one more peculiar instance of patterns of types (b) and (b′), in which one may attach inside L two loops of valence three, as shown in (54). (This diagram is for patterns (b); in the analogous picture for patterns (b′), one pair of edges (x,x') or (y,y') may have the same two loops connected, or even unify to the same edge.) However, such patterns were ruled out by lemma 9.6 (because the loop Z is inadequate).

Next observe that patterns (c) are excluded by lemma 9.5. We need to test only types (a) and (b). While one can rule out by hand many of these cases, still too many different possibilities remain to make a "manual" continuation worthwhile. It appears more helpful to check these patterns with a computer. We found no (technically) easy way to calculate the A- and, particularly, the B-state invariants from the patterns directly, so we relied on a calculation of their Jones polynomials. To perform this calculation, a few more preparations are necessary. The first is a technical help.

Lemma 9.17 There exists at most one multiple edge between empty loops of the outer cycle. Consecutively, there exist at least two simple edges.

Proof. That the second claim follows from the first claim is a consequence of the fact that the outer cycle has length at least 5, and only one non-empty loop. To see the first claim, assume there were at least two multiple edges. They form then isolated vertices in $IG(A)$. However, by three attachments it is impossible to create three (homologically independent) cycles in $IG(A(D))$, and so $\chi(IG(A(D)))$ would remain positive. □

We can now *desplice* a single edge between two empty outer cycle loops and join them.



(68)

Then we continue until we have a diagram D' with an outer cycle of length 4. Thus (and this is to be kept in mind) the number of desplicings that turn D into D' is odd. We can recover D by choosing an outer cycle loop and *splicing* it again (the reverse operation to (68)).

The reason for the change to D' is computational. For the calculation of the Jones polynomials, we tried to apply the program of Millett-Ewing in [HT] that uses a skein algorithm. It requires a choice of orientation on all components, but this choice is not natural for an odd length outer cycle. (The way components are connected varies drastically for the different combinations of edge multiplicities.) In opposition, for even length we have a canonical positive orientation on D'.

In our situation the (de)splicings between D' and D do not change adequacy and the invariants of $A(D)$. Let a,b be the (multiplicities of) the outer loop edges that do not connect the loop with non-empty interior. By mutations we can assume that $a=1$.

The effect of (68) on $B(D)$ is that (up to mutations that preserve all B-state invariants) an edge is reduced to multiplicity 1 (if $b>1$) or 2 (if $b=1$). This does not change $\chi(G(B(D)))$ and $\triangle(B(D))$, and the only effect it can have on $IG(B(D))$ is that an isolated vertex is removed if $b>1$. So we can conclude:

- The A-state invariants (in particular also both $V_{1,2}$) coincide on D,D'.
- We have $\chi(B(D'))=\chi(B(D))=0$, and $\triangle(B(D'))=\triangle(B(D))=0$.
- $\chi(IG(B(D')))$ is -1 or 0 (so $\bar V_0\bar V_2=0,1$), and if $b=1$, then $\chi(IG(B(D')))=0$ (and $\bar V_0\bar V_2=1$).

The generation process of diagrams in the above patterns follows the general rules:

- a loop except the outer cycle loops may or may not be there,
- exactly 3 (non-outer cycle) loops are attached in total (corollary 9.1),
- no pair of parallel loops occurs (lemma 9.12),
- there are at least two depth-one loops (lemma 9.16),
- leg multiplicities for valence two loops are given in lemma 9.8, and for valence three loops are not all even (for connectivity reasons),
- leg multiplicities follow (55),
- every depth-0 loop which is not an outer cycle loop is intertwined with some depth-one loop (else the diagram is not prime), and
- $c(D')$ is odd (because $c(D)$ is even and the number of desplicings is odd).

There is a large but finite number of cases for D'. We wrote a computer program to generate them. In case (b) we simultaneously calculated the invariant $\chi(IG(A))$ from the data (since now this is possible directly without calculation of the whole polynomial!), to minimize the number of cases in advance. (The values of γ, $\chi(G(A))$ and $\triangle(A)$ are correct by construction.) This way the list reduces to about 120,000 diagrams D', the most complicated ones having 27 crossings.

We calculate the Jones polynomial of these diagrams (it takes a few minutes) and check that we have a diagram D' with

1. $\operatorname{span} V=c(D')-2$ (still useful to test, since despite $\gamma=4$, we do not know if D' is B-adequate),
2. $V_0V_1=\bar V_0\bar V_1=-1$, and $V_0V_2=1$.
3. If in the outer loop of D' we have $(a=)b=1$, then we can test also $\bar V_0\bar V_2=1$, otherwise this product is 0 or 1.
4. D' has one or two components. (This was for technical reasons easier to check using the value $V(1)$ rather than at the time of generating the diagrams.)

Thus it is legitimate to *discard* during the generation process outer loop edge multiplicities that satisfy some of the following properties:

1. Type (a): $a > 1$, which can be avoided by mutations, or $x = y = 0$, because then regions connected by depth-1 loops identify, and we have a special case of type (b).
2. Type (b): $a > 1$, or $(a =)b = 1$ and $c + y > d + x$ (then can flip by 180 degree or do mutations)
3. Types (a) and (b): for some outer cycle loop all edges connecting it to neighboring outer cycle loops are of even multiplicity (then D' and D are disconnected)

Most combinations are ruled out by the Jones polynomial conditions: only 40 patterns of diagrams of type (a) and 2 patterns of diagrams of type (b) remain. The next part of the proof handles these cases.

9.8. Concluding part: Generic and sporadic cases

Let us call in mind how one recovers all possible diagrams D from D'. First one can, reversing (68), by (iterated) splicing create single edges in the outer cycle of $A(D)$. Then remember our convention from the beginning of the proof that we worked with parallel equivalence classes of 4 or 5 edges in $A(D)$. This was done so as to have only a finite number of diagrams to test with the Jones polynomial, and because the test is equivalent for all the others. Now we must reconvert these multiplicities to $2n$ or $2n-1$ (for $n \geq 2$). So if D' has an edge of 4 or 5 traces, we have two sites to twist at to generate diagrams D, while otherwise we have only one (the outer cycle).

Let us call a diagram D' *sporadic* if among the diagrams D that result from D' there is only a finite number with $w(D) = 0$ and $c(D) - 2g(D) \leq n$ for any fixed number n. (Here $g(D)$ refers to the canonical genus of the diagram.) Otherwise call D' *generic*.

The (hypothetic) diagrams $D = D_k$ of K_k satisfy these conditions for $n = 6$ (by direct calculation of the Alexander polynomial). So generic diagrams D' give infinitely many candidates for diagrams D, while sporadic D' give only finitely many. The assumption D' to be generic leads to further significant computational simplifications.

Lemma 9.18 If D' is generic then it has one (and then only one) edge of 4 or 5 traces.

Proof. Twisting at only one site (the outer loop in $A(D)$ when recovering D from D') can give at most one diagram D of writhe 0. □

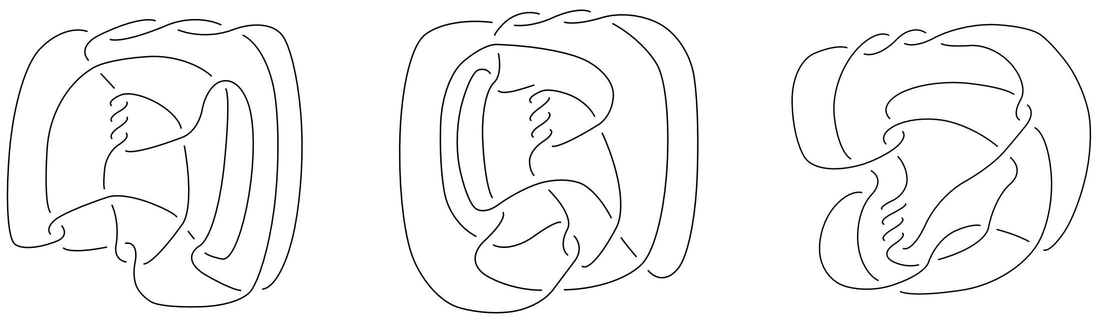
Lemma 9.19 If D' is generic, we can assume that D (or D') has no multiple edges between empty outer cycle loops.

Proof. We can assume henceforth, because $w(D) = 0$, that D' has an edge of 4 or 5 traces, and the recovery of D requires twisting at two different sites, with the number of twists having a bounded difference. By lemma 9.17 we know already that all but finitely many edges between empty outer cycle loops in D are simple. Since the twists added by these edges must be parallel (else $g(D)$ will grow at most like $c(D)/4$), all multiplicities are odd. Since at most one edge has multiplicity ≥ 3, at most one edge is multiple. But if it is multiple, then it is the only edge in $A(D)$ of multiplicity ≥ 3. So in recreating D we would have to twist at that site. Now one easily observes that if the twists that correspond to edges in the outer cycle are parallel, those that correspond to the multiple edge are reverse. So again $g(D)$ will grow at most like $c(D)/4$, and not $c(D)/2$. □

Then one easily observes:

Corollary 9.5 If D' is generic, the change (68) from D to D' (or vice versa) preserves all A- and B-state invariants. That is, it is legitimate to test the conditions (50) on D'. □

With the same argument as in lemma 9.19 we have



(a) (b) (e)

Figure 5: The diagrams $\hat{D}_2$ in the remaining 3 flype-inequivalent generic sequences.

Lemma 9.20 If D' is generic, then for each of both neighbors L' of the non-empty loop L in the outer cycle, there is an edge of odd multiplicity between L and L'. □

Recall also that D' must have at most two components.

The conditions on generic D' we just put together rule out the 2 remaining possibilities of patterns (b), and leave (from the 40) only 24 diagrams D' of patterns (a). All these diagrams have two components. The vectors describing them are below:

A	B	C	D	E	F	G	outer cycle		A	B	C	D	E	F	G	outer cycle		
112	11	00	00	00	00	11	1152011	$*$	142	11	00	00	11	00	00	1111101	(f)	
112	11	00	00	00	00	14	1112100	$*$	142	11	00	00	11	00	00	1121010	(g)	
112	11	00	00	11	00	00	1111401	$*$	211	00	00	11	00	11	00	1111025	$	
112	11	00	00	14	00	00	1111101	$*$	211	00	00	11	00	14	00	1111200	$	
112	11	00	00	14	00	00	1121010	$*$	211	00	00	11	11	00	00	1114101	$	
112	14	00	00	00	00	11	1112100	(a)	211	00	00	11	11	00	00	1121510	$	(69)
112	14	00	00	11	00	00	1111101	(b)	211	00	00	11	14	00	00	1111101	$	
112	14	00	00	11	00	00	1121010	(c)	211	00	00	14	00	11	00	1111200	(h)	
121	00	11	00	00	00	11	1112150	#	211	00	00	14	11	00	00	1111101	(k)	
121	00	11	00	00	14	00	1111020	#	214	00	00	11	00	11	00	1111200	(l)	
121	00	14	00	00	11	00	1111020	(d)	241	00	00	11	11	00	00	1111101	(m)	
124	00	11	00	00	11	00	1111020	(e)	412	11	00	00	00	00	11	1112100	(n)	

These vectors give the multiplicities of edges in D'. The legs of the loop A are listed in counterclockwise order with the first leg connecting to the right side in figure 4 (between the legs of G). The edges of the outer cycle are listed in the order a,b,c,d,x,y,z. Up to mutations (actually flypes) this information specifies the diagrams. Twelve of the 24 diagrams D' do not give a knot diagram D by undoing the desplicings (68) (i.e. crossings that connect the empty outer cycle loops of $A(D')$ do not involve both components). These patterns have leg multiplicities $A=(1,1,2)$ and $B=(1,1)$ (the vectors of these cases are marked with an asterisk), $A=(1,2,1)$ and $C=(1,1)$ (marked with a double cross), or $A=(2,1,1)$ and $D=(1,1)$ (marked with '$'). Consider the other twelve diagrams, to which we assign labels (a) to (n) as above. (In calculation 10.1 of the next section we will have an easier, and available for verification, way to confirm the outcome of these 12 diagrams.)

For all twelve D' we obtain (indeed) diagrams D of achiral adequate knots that satisfy all criteria we could impose so far coming from the Jones polynomial (and as well the Kauffman polynomial in corollary 9.1). Let us call these diagrams $\hat{D}_k$, reintroducing the twist parameter k from the beginning of the proof.

The 12 initial diagrams $\hat{D}_2$ for the series (a) to (n) have 18 crossings, and they have exactly two (parallel) twists of 3 or 4 crossings. Each next diagram $\hat{D}_{k+1}$ arises from the previous one $\hat{D}_k$ by a adding two crossings each at its two twists. Now it turns out that up to mirroring and flypes from these 12 series only 3 are inequivalent. Since flypes persist under twists, it is enough to deal with these 3. They are shown in figure 5. In fact, we have only two sequences of knots, since the series (b) and (e) are not flype equivalent but still represent the same knot. (The division of the 12 series between the two knots is evident from (70).)

Now we must use some further information to distinguish these series of $\hat{D}_k$ from the hypothetic diagrams D_k of our odd crossing number knot candidates. We use the degree-2 Vassiliev invariant v_2 (from (8)).

It is well-known (see [St18]) that in a sequence of diagrams $\hat{D}_k$, where $\hat{D}_{k+1}$ is obtained from $\hat{D}_k$ by a pair of parallel twists at the same sites, $v_2(\hat{D}_k)$ is a(n at most) quadratic polynomial in k. So to determine $v_2(\hat{D}_k)$ for all $k \geq 0$, it is enough to evaluate v_2 on the first three diagrams. Actually one could also switch crossings in the twists, as we did, to use simpler diagrams instead. (This gives a meaning to $\hat{D}_k$ and D_k also when $k < 1$.) The same behavior is exhibited by v_2 on our candidate knots. The calculation shows

k	-1	0	1	2
$v_2(\hat{D}_k)$ for series (a),(c),(d),(f),(k) and (m)	-5	-1	5	13
$v_2(\hat{D}_k)$ for series (b),(e),(g),(h),(l) and (n)	-4	0	6	14
$v_2(D_k)$	-3	1	7	15

(70)

So the polynomials for v_2 for all eight sequences of $\hat{D}_k$ do not coincide with those for our hypothetic diagrams D_k (of the same crossing number). As explained, the coincidences of v_2 on the two groups of 6 series are because there are only two series of knots.

Now turn to the sporadic cases. We obtain by calculation 16 diagrams D' of pattern (a) in figure 4 and 2 of pattern (b). (This is the rest of the 42 cases from the end of §9.7 after removing the 24 generic ones we just treated.)

There are 4 sporadic diagrams D', where L is connected to an outer cycle loop only by even valence edges (i.e. the conclusion of lemma 9.20 is violated). Since all these diagrams have two components, and $a = b = 1$, it is easy to see that (even an odd number of) splicings reverse to (68) yield no knot diagrams.

Consider the other 14 sporadic diagrams D'. We can handle them similarly to the described procedure for the generic ones, but there is a (technically) more convenient way. As it turns out, we have that for all patterns $a = b = 1$, and $c(D') \in \{11, 15\}$. This means that one desplicing should give an adequate knot diagram D, with $c(D) \in \{12, 16\}$, $\gamma(D) = 4$ and all A- and B-state invariants as for K_k. We can thus check in the tables of [HT] for adequate knots K with $\operatorname{span} V(K) = c(K) - 2$, $V_0V_1 = \bar{V}_0\bar{V}_1 = -1$ and $V_0V_2 = \bar{V}_0\bar{V}_2 = 1$. There are five 14 crossing knots, which are not relevant, and three 16 crossing knots, $16_{1166743}$, $16_{1223228}$, $16_{1303786}$. However, beside the first and last three coefficients, we have one more piece of information (which we ignored so far for the sake of testing V directly), say, the triangle number. By simply drawing a 16 crossing diagram of each of these three knots, one observes that one of the A or B state has a triangle.

This argument shows that the diagrams D coming from our 18 sporadic diagrams D' either have multiple components, or $B(D)$ has a triangle. We can thus rule out all sporadic diagrams D' too. The exclusion of these final remaining cases completes the proof of theorem 9.1. □

Remark 9.1 The reason for the insufficiency of this proof in crossing numbers $17 + 4k$ is that I found so far no generalization of Thistlethwaite's example with $\chi(G(A)) = 0$, while for $\chi < 0$ the combinatorics of the argument seems to become considerably more complicated. The last section completes the work by dealing with this difficulty using all tools we have set up so far.

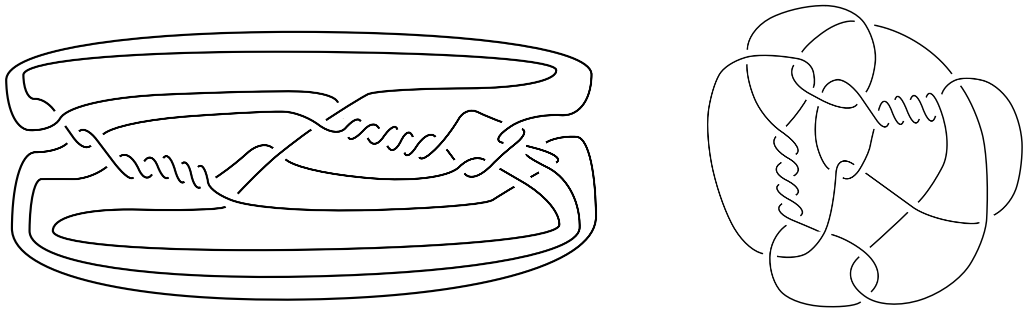
10. Proof of main result

10.1. Examples

To complete the proof of theorem 1.1, we need to settle crossing numbers $17 + 4k$. The examples we find are as follows. Let δ_i be the tangle (with n strings) obtained by (ignoring orientations and) replacing the crossing  in the braid generator tangle $\sigma_i \in B_n$ by , and write $\bar{\delta}_i$ for the mirror image of δ_i, containing a . (This is no longer an inverse in a group theoretic sense.)

Theorem 10.1 The knots K_k, given by the (braid-like, but now unoriented) closure of the tangles $T_k = -12^2 3^{2k+2} 4 - 32 - 1(-2)^{2k+2} 3^{-2} 4\bar{\delta}_2\delta_3$ are prime, amphicheiral, and have crossing number $17+4k$.

Below on the left is shown the diagram of the theorem for $k=2$:



(71)

Amphicheirality of K_k is again evident, and it is not too hard to see that the $18+4k$ crossing diagrams as closure of the specified tangle reduce by one crossing, as in the right diagram of (71). The minimality of this diagram is again the object of difficulty.

The two replacements of braid crossings by the δ and $\bar{\delta}$ preserve the connectivity, but alter some geometric properties. For example, the knots from the previous section all have the same braid index (5), in opposition to our K_k, which in turn have the same genus. The genus (equal to 6) can be inferred from [St17], using that the knots are semi-homogeneous in the sense introduced there, as their diagrams from the left of (71) are Murasugi sums of (special) alternating diagrams.

The $17+4k$ crossing diagrams of K_k (of writhe ± 1) are again semiadequate. So again a $16+4k$ crossing diagram D_k, which is to be ruled out, must be adequate. The Jones invariants (for both A- and B-state of D_k) become as follows:

$$\chi(G) = -1\,, \quad \chi(IG) = -1\,, \quad \triangle = 1\,, \tag{72}$$

except again for $k=0$, where $\triangle = 2$. These changes to the previous case will lead to almost entirely different proof, even though its spirit is similar. First note that the triangle number becomes important, in that it controls the length of one of the two cycles in the state graph G. In contrast, the other cycle must be of length ≥ 4 (for $k>0$). There is no parity condition on this length anymore, since the triangle spoils the bipartacy of G. We call again the longer cycle the *outer cycle*, while the shorter will still be the *triangle*.

Furthermore, again we have

$$\gamma := c(D_k) + 2 - s(A(D_k)) - s(B(D_k)) \,=\, 4\,. \tag{73}$$

Let us point out also the following, which was not (needed to be) used before. Since we must have writhe $w(D_k)=0$, and the Jones polynomial is reciprocal, the A-and B-state of D_k must have equal number of loops, and from (73) we have

$$s(A(D_k)) = s(B(D_k)) = c(D_k)/2 - 1 = 7 + 2k\,. \tag{74}$$

This property is easy to test from the A-state directly, and we use it in both our computations (see §10.4.3) and arguments (see lemma 10.20). Contrarily, the (vanishing of) the writhe can be tested only from the knot diagram. We thus postpone its application to the very end of the proof in §10.9, where all semiadequacy invariant tests are exhausted, and we recover the knot diagram.

Even if just a triangle, the second cycle in $G(A)$ nonetheless complicates things drastically. The length of the proof results mainly from the extraordinary care that is needed not to leave out some cases how the attachments are performed (similar to §9, but now with the new option (81), as we discuss below). Once the cases are identified, the invariants we have exercise a control strong enough to easily rule out most of them; only a few must be entrusted to the computer.

To restrict the attachments, we will rely now more heavily on the atom number invariant of theorem 9.2. By verification, we see that corollary 9.1 holds further for our present family (with a certain complication, which leas to the distinction of cases A,B,C in §10.3.2). Another circumstance that comes extremely to our help is that IG has now also two cycles.

The atom number again easily implies that K_k are non-alternating. (Another argument can go via using $V_1 = \bar{V}_1 = \pm 1$ and analyzing alternating diagrams with at most 4 twists.) In contrast, due to the different geometric picture, the primeness proof requires a different, and somewhat longer, discussion. We start with this. (The primeness proof now does not use non-alternation.)

10.2. Primeness

Lemma 10.1 The knots K_k are prime. Hence so are their hypothetic diagrams D_k.

We lean on the work of Kirby and Lickorish [KL]. The notion of a rational and prime tangle is explained there. However, we need a modification to tangles with > 2 strings.

Let us make the basic assumption that a tangle T is considered here to have an arbitrary number $n \geq 2$ of strings, intervals properly embedded in a ball B. We call then T an *n-(string) tangle*. We will exclude closed components, and keep strands unoriented, unless explicitly indicated. Tangles are considered equivalent up to homeomorphisms of the ball B that keep its boundary sphere S fixed.

Definition 10.1 Let D be a disk properly embedded in B, and not meeting the punctures $S \cap T$.

a) D is called a *separating disk* of T if $D \cap T = \varnothing$, but for any of the two balls B_i of $B \setminus D$, we have $B_i \cap T \neq \varnothing$. We call T *separable* if T has a separating disk.

b) The disk D is a *decomposing disk* of T if $D \cap T$ is one point, and for any of the two B_i, the intersection $B_i \cap T$ is not a single unknotted arc.

c) A sphere $S' \subset S$ is a *decomposing sphere*, if $|S' \cap T| = 2$ and the interor ball of S' in B contains something different from an unknotted arc of T. If T has a decomposing sphere, we say that T has a *connected sum factor*. We call T *decomposable* if T has a decomposing disk or sphere.

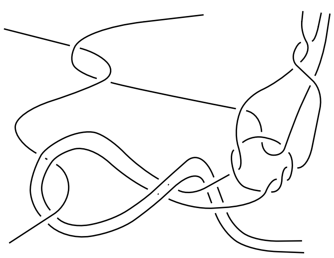
An n-string tangle T is *prime* if it is inseparable and indecomposable.

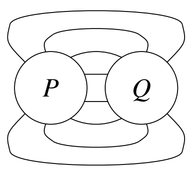
Note that if D is a decomposing disk for a 2-string tangle T, imperatively one $B_i \cap T$ contains a single string, and thus T has a connected sum factor.

Kirby and Lickorish define a 2-tangle two be prime if it is inseparable and has no connected sum factor. They observe that if a 2-tangle has no connected sum factor and is separable, then it is rational. This motivates the following definiton, which helps identifying separable n-string tangles.

Definition 10.2 An n-string tangle T is a *generalized rational tangle* if there is a division $C \cup D$ of the strings of T into two subsets (with $C \neq \varnothing \neq D$, and $C \cap D = \varnothing$), such that each 2-string subtangle of T made up of a string from C and one from D is a rational tangle, with possible connected sum of some string with non-trivial knots K'.

Below on the left is an example of 4 strings (with $|C| = 1$, $|D| = 3$), that visualizes also the meaning of K' and the occurrence of different rational subtangles.



$P * Q =$ 

(75)

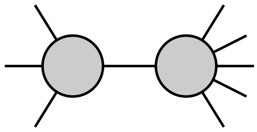
We have, the *composition* $P * Q$ of two n-tangles P and Q (shown above on the right for $n = 4$; for 2 strings Kirby and Lickorish call it 'sum'). Note that this operation depends on some cyclic matching of tangle ends in particular diagrams of P and Q.

Remark 10.1 The local connected sum factors K' at a string s in definition 10.2 can be often excluded, for example if s is unknotted. Also, if $|C| = 1$, then excluding a connected sum factor prohibits K' for the string $s \in C$. In particular, for 2-string tangles T, where $|C| = |D| = 1$, the K' can be ignored when defining a prime tangle, and our definition coincides with the one of Kirby and Lickorish. (They allow for closed tangle components, but for tangle compositions which are supposed to be knots, such components do not occur, so our initial constraint makes sense.)

We observed that separable tangles are generlized rational. Then the following contraposition is rather easy to see.

Corollary 10.1 Let T be an n-string tangle. Assume for each two strings s, s' of T there is a sequence of strings $s = s_1, s_2, \ldots, s_k = s'$ such that for all $1 \le i \le k-1$ the 2-string subtangle of T made of s_i and s_{i+1} is a prime tangle (in the sense of [KL]). Then T is inseparable (in the sense of definition 10.1). □

The identification of indecomposable n-string tangles is a little more difficult. If T is decomposable, its decomposing disk can be brought to standard form

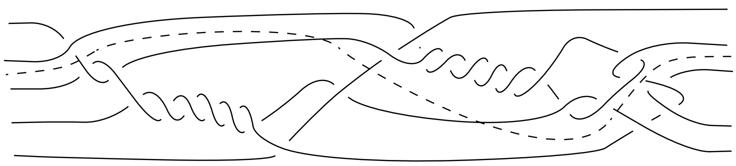
$T =$ 

by some homeomorphism of the ball B, which induces a homeomorphism of the boundary sphere S preserving (but not necessarily fixing) the punctures $S \cap T$. This homeomorphism can be undone when composing T with its mirror image $!T$ as on the right of (75), in such a way that a string is connected to its mirror image. Thus we obtain a link $L = T * !T$ of as many components as T has strings. If T is decomposable, L must be a composite link.

Theorem 10.2 A knot K, which is the composition $P * Q$ of two prime tangles P, Q as on the right of (75), is prime.

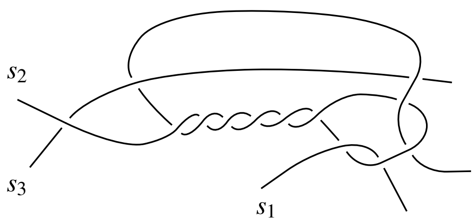
Proof. This is an easy modification of Kirby-Lickorish's proof. One needs to look again at how a sphere that determines a factor decomposition of K would intersect the sphere of its tangle sum decomposition. The primeness of tangles was defined so as to exclude all possibilities for this intersection. □

Proof of lemma 10.1. When the tangles T_k, whose closure is shown on the left of (71), are separated as by the dashed line,



(76)

we obtain a sum decomposition of the closure K_k into a 3-string tangle T'_k and its mirror image, $K_k = T'_k * !T'_k$.

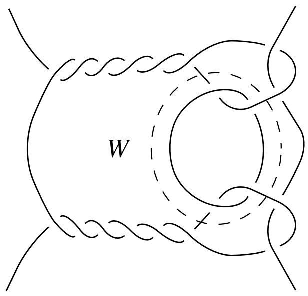
$T'_2 =$ 

(77)

Therefore, it suffices to prove that T_k' are prime. To see that T_k' is inseparable, we use corollary 10.1. With the ordering of the strings of T_k' as in (77), it is easy to verify, using [KL], that (s_1,s_2) and (s_2,s_3) are prime 2-string tangles.

To see that T_k' is indecomposable, build $T_k' * !T_k'$ so that each string of T_k' is closed off with its mirror image. The resulting 3-component link can be shown to be prime using a (different) 2-tangle decomposition, and [KL] thus.

The strings s_2 and s_3 of T_k' will form a (prime) $(2,-2)$-tangle. Its complementary tangle T_k'' (2 strings with a closed component) is shown to be prime as follows.

$T_2'' =$  (78)

The tangle T_k'' contains another $(2,-2)$-tangle W (from s_1 and s_2). This is prime, and thus a separating disk or decomposing sphere of T_k'' can be made disjoint from the tangle ball X of W. But all (three) strings of T in $B\setminus X$ are unknotted, and there is no closed component there. Thus there is no decomposing sphere of T_k''. Then, if T_k'' is separable, it is rational, and otherwise it is prime. If T_k'' rational, the closure link L_k'' of T_k'' given by connecting the top two and bottom two ends in (78) by unknotted arcs is a 2-component rational (2-bridge) link.

To exclude this final option, one can observe that L_k'' is a Montesinos link, and use the classification of such links (see, e.g., [BZ]). Another argument goes thus. A rotation of the ball of T_k'' around its horizontal axis turns T_k'' into its mirror image. One can orient the strings of T_k'' so that this rotation changes the orientation of all strings simultaneously. This implies that L_k'' has (with some component orientation) vanishing Alexander polynomial. The only such rational link is the (2-component) unlink. It remains thus to see that L_k'' is not the unlink, which can be done using some invariant.

This completes the primeness proof for K_k. That for D_k follows again at once from the non-triviality of (semi)adequate links. □

Remark 10.2 There is another proof of the primeness of K_k (including those in theorem 9.1), which goes via identifying them as 3-bridge knots. If K_k is composite, by Schubert's theorem, it has two 2-bridge factors. The value $V_1(K_k) = \pm 2$ will show these 2-bridge knots to have in their alternating diagrams 1 or 2 twists, and then only a tiny list of possibilities remains to be ruled out. This argument will in fact show that K_k are hyperbolic.

10.3. Basic cases

For the crossing number proof, we need again several preparations.

10.3.1. The 17 crossing knot

Let us first deal with the case $k=0$. We must rule out that K_0 is an adequate non-alternating prime 16 crossing knot. A check of the Jones polynomial (here $V_2 = \bar{V}_2 = 0$ because we have two triangles) of these knots, obtainable from the tables of [HT], shows only one knot with matching polynomial, 16_{934760}. But the skein polynomial distinguishes it from our (therefore) 17 crossing example K_0. We thus assume for the rest of the discussion that we have diagrams D_k of ≥ 20 crossings, for $k>0$, and so $\triangle = 1$.

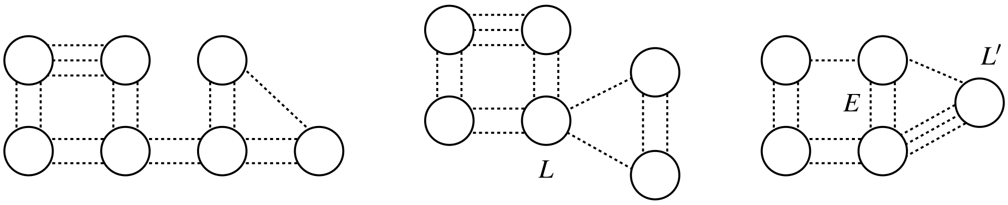
Again our proof will consist in trying to construct $A(D_k)$ successively by loop attachments, and seeing that we can never do it right. The two cycles change, though, the configuration we start with. Here we should introduce a case distinction, which will accompany us though most of the proof. It bases on the mutual position of the two cycles in $A(D_k)$.

Let us again draw the outer cycle in the plane so that all its edges lie outside the loops, i.e. in the region of their complement that contains infinity. This position then defines a notion of *interior* and *exterior* (resp. *in/outside*) for all loops, incl. those to be subsequently attached. For example, if a loop M is attached inside (i.e. in the interior) of L, then L lies in the exterior of M. A loop is *empty* if its interior is empty, i.e. contains no traces or other loops. (Note that for a semiadequate diagram, no loops implies no traces.) A loop is separating if both its interior and exterior are non-empty (see §3.3).

10.3.2. Cases A,B,C

Let us first consider the case that the (edges of the) triangle lie(s) on the same side of the outer cycle loops as the outer cycle.

Case A. The triangle and outer cycle have at most one common loop. This case will be easily out. If there is no common loop, then we have, prior to attachments, something like the figure (A1) in (79).



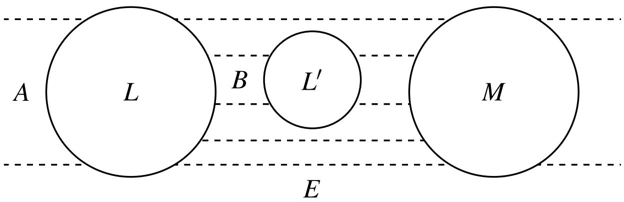
$(A1)$ $(A2)$ (B) (79)

To these cycles we must attach loops as in (28). However, we see that (A1) of (79) has already 3 connected sum factors, and so atoms. Thus we have only one attachment left by corollary 9.1, but this cannot make the diagram D prime.

If the outer cycle and triangle have one common loop, then we have the figure (A2) in (79), and two attachments left. In order to create two cycles in $IG(A(D))$, we must attach both loops inside the common loop L, so that their legs become intertwined with at least three edges outside L. This is handled in §10.6.

We are next left to treat the cases B and C.

Case B. The triangle and outer cycle have two common loops. This is case (B) in (79). Here we have again 3 attachments (28) to perform. There is a degenerate instance of this case, which we call case B', in which the common connection E of the outer cycle and triangle goes on both sides of the triangle loop L'.

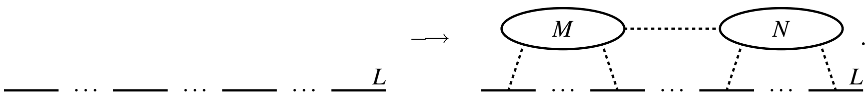


(80)

This peculiarity will cost us some extra effort later. At least we can assume that, if M is empty, L has attachments inside that connect the regions A and B. (Otherwise we can return to case B by a mutation.)

Still we will see below that Cases B and B' are relatively simple. The other case requires the most work.

Case C. The triangle and outer cycle lie on opposite sides of a loop. In that case, we start with the outer cycle (51) (without insistence of odd length), and have three attachments to perform, but among them there is one of the sort



. (81)

The loops M,N will be often referred to as the *triangle loops*. Again the same arguments concerning attachments and case distinction apply for the B-state of D_k. Often it will help to switch from $A(D_k)$ to $B(D_k)$, and to see that there some feature becomes violated. We will exhibit several features in the next lemmas. These properties must apply to both the A- and B-state.

Let us in advance fix the following *complexity*. We regard case A as simpler than case B, which in turn is simpler than case C. We choose D_k mirrored so that $A(D_k)$ is the simplest w.r.t. this hierarchy. Thus, for example, if in the treatment of ($A(D_k)$ being of) case C we establish that $B(D_k)$ is of case A or B, then we can consider this D_k as dealt with. Note that to establish case A or B, it is enough to identify two cycles whose edges do not lie on opposite sides of some loop.

Next let us reintroduce the notion of *depth* of loops. This is relatively easily adapted from the previous section.

In our case A or B the additional loop(s) L' on the triangle but not on the outer cycle has/have depth 0. Loops attached outside to L' (i.e. on the side that contains the edges of the triangle and outer cycle) have depth 0, loops attached inside L' have depth 1, loops attached inside those have depth 2 etc.

In case C the assignment of depth to the loops needs to be clarified only for the triangle attachment (81). This is done in the obvious way, by saying that these 2 loops would have the depth of a loop that would be attached instead of them by (28).

Let us again call an *edge* a parallel equivalence class of crossing traces in $A(D)$. The multiplicity of an edge and the property to be simple are then obviously defined. Again we often regard a letter naming an edge as a variable that indicates its multiplicity.

With this set up, we assume again $D = D_k$ is *flat*, i.e. of minimal loop depth sum in its mutation equivalence class. Also we assume it *edge-reduced*, i.e. within the diagrams of the same minimal total loop depth, it should have the smallest number of edges (equivalence classes of parallel traces). In the following, we will several times apply a mutation to bring a diagram into a more favorable shape. While we do not say it each time explicitly, it is to be understood that such a mutation can be, and is, chosen so as to preserve the minimal loop depth and edge-reducedness.

10.4. Preparatory considerations

10.4.1. Control on position of attachments

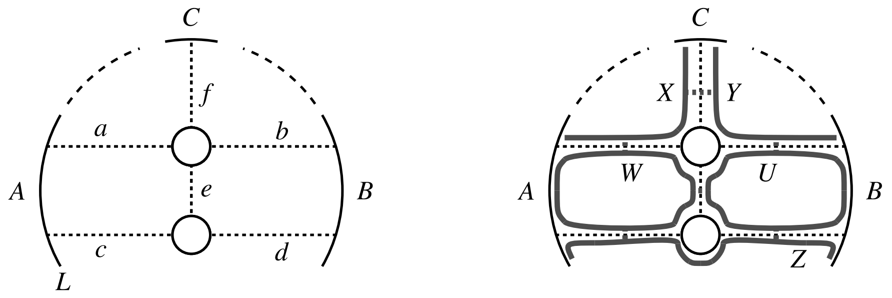
In order to deal with cases B and C we will need several preparatory lemmas. We postpone case B' to §10.7.

Lemma 10.2 The attachment (81) in case C is made inside an outer cycle loop (i.e. L is an outer cycle loop, and M,N are attached inside L, and of depth 1).

Proof. Let L be the loop the attachment (81) is made at. If L is not an outer cycle loop, then L is itself an attached loop. So after L and (81) we still have no cycle in $IG(A)$, but only (at most) one attachment (28) left. It is easy to see that, in which way we even choose to make this attachment, we cannot create two cycles in $IG(A)$. So L is an outer cycle loop. Then by definition of case C, (81) is made inside L. (Otherwise we have case A.) □

An edge adjacent to an attached loop is called a *leg*. Edges between the two attached loops M,N in (81) may not be regarded as legs. The valence of an attached loop using (28) is as before the number of its legs. This number coincides with the number of regions touched by the legs, under exclusion of peculiarities (54). Because of primeness and $\gamma = 4$, the number of regions touched is always 2 or 3. We do not use a notion of valence for the loops M,N of (81), but we will distinguish cases by the number of regions (again 2 or 3) touched by legs of M or N.

Under exclusion of peculiarities (54) (which we will soon accomplish with lemma 10.11), the triangle attachment (81) is of the shape shown on the left:



. (82)

Here we assume the regions A,B,C outside L along each segment drawn in solid line to be the same (i.e. solid line segments are not touched by traces from outside L). For future reference, let us call the edges of the triangle a to f. Hereby it is possible that some of $a,\dots,f$ is (of multiplicity) zero. In fact, this necessarily occurs, as shows the following lemma.

Lemma 10.3 We have in case C no triangle attachment of the type (82) with empty triangle loops and none of a,b,c,d being 0. In other words, one of the loops of the triangle has a leg that touches at most one of A or B.

Proof. First consider the case $f=0$ or $f=2$. By connectivity again at least of a,b,c,d is ≥ 2, and then by direct observation $B(D)$, whose loops are shown on the right of (82), has two cycles not lying on opposite sides of a loop. Thus $B(D)$ is of type B.

Next assume $f\geq 3$. If $X=Y$, we have a cycle in $B(D)$ made up of X and the loops between the traces of f. The triple X,V,W gives a second cycle, so $B(D)$ is of type A.

If $X\neq Y$, then we still have the cycles U,W,Z and the one containing U,W,X,Y. They are distinct even if Z is one of X or Y. So $B(D)$ is of type B. Finally, if $f=1$, then by adequacy of D necessarily $X\neq Y$, and then the same two cycles exist. □

Lemma 10.4 There is at most one non-empty outer cycle loop in case A.

Proof. To create a cycle in $IG(A)$, we need both attachments to be within the same loop. □

Lemma 10.5 We can assume, up to mutation, that there is at most one non-empty outer cycle loop in case C.

Proof. Compare the proof of lemma 9.15. We assume we have loops attached within two outer cycle loops and derive a contradiction.

If we had an attachment inside some of the triangle loops, then this would be the third attachment, and we cannot create 2 cycles in $IG(A)$. So the triangle loops are empty.

Then if the triangle attachment identifies only two regions A and B, we have (82) with $f=0$, but a,b,c,d non-zero to have two cycles in $IG(A)$. We ruled this out by lemma 10.3. (Note that, with only A and B touched, we cannot have two legs touching the same region on either side of an attached loop, as in (54), because then D would not be prime.)

Thus the triangle attachment identifies 3 regions, say A,B,C. (Here we must include the situation similar to (54), but with M,N connected by a trace.) Now, an attachment in any other outer cycle loop, must identify only 2 or all 3 of these regions. In former case, one can achieve by a mutation (which does not spoil flatness or edge-reducedness) that only one outer cycle loop is non-empty. In latter case, we have a modification of (64), in which X and Y may contain several loops, but their legs still touch only A,B,C. Then, however, again $IG(A)$ has no cycle. □

Lemma 10.6 There is at most one non-empty outer cycle or triangle loop in case B.

Proof. If we had 3 attachments (28) divided into several outer cycle or triangle loops, then we must create two cycles in $A(D)$ with two of them M,N, attached in some (outer cycle or triangle) loop L. So each of the loops M,N inside L must be intertwined with 3 edges outside L. Now, since we have 3 attachments, and one is inside another loop different from L, we cannot have outside attachments to L. This means that M,N must be intertwined with 3 edges outside L from the outer cycle or triangle. While there are two loops L with three outside connections, one cannot (unlike in case B', which we will treat later) install M,N to intertwine with all of them, without their legs intersecting, or $\gamma > 4$. □

Let (as before) $\tilde{D}$ be obtained from $D = D_k$ by making all edges in $A(D)$ simple (i.e. a single trace). We call again an edge (parallel equivalence class of traces) in $A(D)$ to be *inadequate* if, when reduced to a single crossing, this crossing is B-inadequate in $\tilde{D}$, i.e. has a trace in $B(\tilde{D})$ that connects a loop to itself.

Lemma 10.7 If x is an inadequate edge, then $x = 2$.

Proof. In order D_k to be B-adequate, we must have $x \geq 2$. If $x > 2$, then we have a cycle (triangle, or outer cycle) in $B(D)$ of simple edges, i.e. each pair of neighbored loops on this cycle is connected only by one trace. Moreover, this cycle has only one loop with non-empty interior, and all its other loops have only two adjacent traces. If $B(D)$ is of type A, then it is easy to see that with two attachments (28) one cannot create two cycles in $IG(B(D))$. This means that $B(D)$ must be of type C. However, since one of its cycles has only simple edges, one sees again that with the two other attachments (28) we cannot create two cycles in $IG(B)$. □

Again an attached loop (though not some in (81)) of valence 2 is inadequate, if both its legs are inadequate.

Corollary 10.2 There are no inadequate loops.

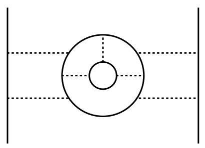
Proof. If both their legs have multiplicity 2, the diagram is disconnected. □

Lemma 10.8 There is no valence three loop, attached by (28), which has non-empty interior.

Proof. Case A is easy, so consider cases B, B' and C. Assume L were such a loop, and M, resp. M,N, the loop(s) attached inside L using (28), resp. (81) (in case C). The attachments of L, M and possibly N do not create any cycle in $IG(A)$. By assuming flatness of D, we see also that attaching L, M and possibly N augments γ by at least 4 (see (63)). This means that the third attached loop P is either outside an outer cycle loop (or the extra triangle loop in case B), or inside L. In either case it is easy to see that this attachment cannot create any cycle in $IG(A)$. □

Lemma 10.9 There is no valence two loop, attached by (28), which has non-empty interior.

Proof. Again case A is easily settled, and excluded. Assume L were such a loop, and M, resp. M,N, the loop(s) attached inside L using (28), resp. (81) (in case C). If (81) is used inside L, we have only one instance of (28) left, but cannot create two cycles in $IG(A)$ with it. The same situation occurs if we use (28) twice inside L. So we can assume that the loop M is the only loop inside L. If now M has valence 2, then by flatness of D, we must have the picture (57), and M is inadequate, in contradiction to corollary 10.2. Thus M must have valence 3, and we have

 ,

or a super-configuration thereof, but then $\gamma \geq 6$. □

Corollary 10.3 There are no loops of depth ≥ 3, and the only possible ones of depth 2 are (in case C and) attached inside triangle loops (81). □

We will later rule out, during the detailed treatment of case C, depth-2 loops completely (lemma 10.28).

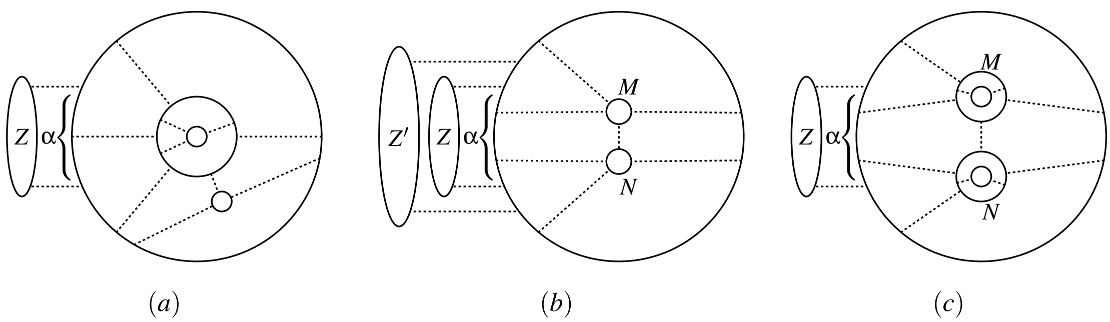
The next step is to get disposed of things like (54).

Lemma 10.10 In case A or B, there is no region touched in two different segments by legs of (possibly different) attachments (28) on the opposite side.

Proof. In case A, we have no outside attachment to L (both attachments must be inside to create cycles in $IG(A)$). In case B, the proof is the same as for lemma 9.6. $\square$

Lemma 10.11 In case C, there is no region touched in two different segments by legs of attachments (28) or (81) on the opposite side.

Proof. If Z in (54) were attached inside an outer cycle loop L, then we would have at most one loop outside attached to L (the second attachment (28)), and thus would reduce the number of legs by a mutation. Thus Z is attached outside L.



(83)

Again Z has valence 2, else we would have $\gamma > 4$, or a composite diagram D. If Z is not inadequate, then either (i) there is a depth-2 loop inside one of the triangle loops M,N, as in (83)(a), or (ii) there is a second attachment outside L, as in (83)(b). In either case it is impossible to have the triangle (81) and the second attachment (28) inside L of depth 1. In particular, the segments of L on either side of α must be touched from the inside by legs of some of the two triangle loops, and only of those loops.

If (i) holds, then we need both M,N to be intertwined with Z, in order to have two cycles in $IG(A)$. In this case, however, we need depth-2 loops within both M and N to avoid that Z is inadequate, as in (83)(c). However, then we need 4 attachments.

Thus we have a second attachment Z' outside L. We claim that Z and Z' are parallel. For the argument compare also the proof of lemma 10.19 below.

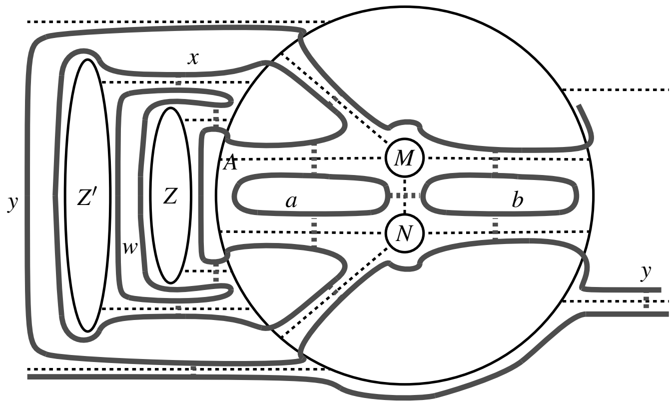
Let us say that Z and Z' *enclose* if they look like in (86) or the left diagram in (88), but not like in (87) and the right diagram in (88).

If Z and Z' would not enclose, then there are three regions outside L (the two separated by the legs of Z, and the one enclosed between L and Z') that must be touched by legs inside L, and because $\gamma = 4$, there are no further ones. Now because D is prime, we must have Z and Z' being separated by traces from L to some other outer cycle loop (like in (87), and unlike the right diagram in (88)). But then again Z would be inadequate.

Thus Z and Z' enclose. Then, since there is no loop of depth > 1, again it is easy to see that if Z and Z' were not parallel, one would be inadequate. (See case 1 in the proof of lemma 10.19.)

Now we have established that Z and Z' are parallel. Again if some of M or N has no leg on the segment α of L, then it intertwines with at most two connections outside L, and then we have at most one cycle in $IG(A)$. Therefore, we must have the picture (83)(b).

We may assume w.l.o.g., using the two cycles in $IG(A)$ and up to mutation, that M,N intertwine the connection between L and an outer cycle loop on the left as drawn below. Similarly, we can assume that between the two outer cycle edges to the right at least the lower one is there. Then we have (recall convention 9.2):



(84)

Note that the loop y (of $B(D)$) closes on the left as shown, because of the neighboring loop to L on the outer cycle (which we omitted).

Now if we look at the B-state of (84), we see that there is a cycle of length at least 4 (containing a,b,x,y), which then must be the outer cycle. (This cycle may get longer if multiplicities of edges in the A-state are augmented.) Now we observe also that this cycle has two loops, x and y, with non-empty interior. This contradicts lemmas 10.4, 10.5 and 10.6, though.

Note that in lemma 10.5 we took the freedom to apply a mutation. However, the situation in its proof where this was necessary does not occur here. The valence-2 attachment w inside x touches a region A not touched by traces from within y. □

So in particular all valencies of attachments (28) are 2 or 3, and (81) has *always* the shape (82), with some of a to f being 0, except for possible depth-2 loops (see corollary 10.3) attached inside the triangle loops.

Lemma 10.12 We can assume, up to mutation, there is at most one edge of multiplicity ≥ 3.

Proof. An edge x of multiplicity ≥ 3 gives rise to a cycle of length ≥ 4. To see this, let a and b be the loops in $B(\tilde{D}_k)$ connected by the trace of the simple crossing x. As before, if $a \neq b$, then by primeness of D_k, a and b must be connected outside x (i.e. via traces not dual to those in x) in $B(D_k)$. If this cycle is of length 3, then $x = 3$, and $a = b$, i.e. x is inadequate. This contradicts lemma 10.7, though.

Now we have only one cycle of length at least 4. So if $x,y \geq 3$, then they must give rise each to a single cycle in $B(D_k)$, and both cycles must coincide. Then again it is easy to see that by mutations (that preserve the total loop depth, in order to not to collide with our assumption the diagram to be flat), one of x or y can be made to ≤ 2. □

Lemma 10.13 The outer cycle has length at least 5.

Proof. That the outer cycle has length at least 5 follows directly from (74), with our assumption $k > 0$, and the observation that there are exactly 4 loops in $A(D)$ outside the outer cycle. □

Lemma 10.14 There exist no two pairs of parallel loops, and no triple of parallel loops.

Proof. The two pairs are out immediately, since we have only three attachments. The triple is out similarly, since, by the lack of further attachments, we would have $\gamma = 2$. □

10.4.2. Going over to initial diagrams

Let us now, again, temporarily eliminate the dependence of D_k on k by considering only the simplest diagram that comes in question. (In §10.4.3 below we explain how we recover the whole sequence from that diagram.) Again we will see

Lemma 10.15 Assume there is at most one non-empty outer cycle loop and at most one multiple edge on the outer cycle between two empty outer cycle loops. Then connectivity (number of components) of D_k and all invariants we test so far, namely those in (72), (73), and corollary 9.1, are not changed if

1. we change, keeping the parity, an edge of multiplicity > 2 to 4 or 5, or
2. change the length of the outer cycle to 5 or 6 by joining pairs of simple edges between outer cycle loops:

(85)

Proof. To justify this we need to exclude one pathological situation in either points.

The only troublesome situation in case 1 is when we may destroy a triangle in $B(D)$ by changing a leg of multiplicity 3 to 5. This leg must be then inadequate. (Otherwise all cycles it belongs to have length at least 4.) In the proof of lemma 10.12, we argued this possibility out, though.

For case 2 we will need to see that by adjusting the length of the outer cycle, we do not make a multiple edge in $B(D)$ simple, which may affect $IG(B(D))$ (by deleting an isolated vertex). But since at least two simple edges remain in the outer cycle of $A(D)$ by assumption, the graph $IG(B(D))$ is not affected either. □

Observe that the premise of the lemma is needed only for part 2. So again we can assume

all edges have multiplicities 1,2,4,5, and at most one has 4 or 5.

Let us call the edge of multiplicity 4 or 5 the *twist edge*.

With this assumption we have

Lemma 10.16 The leg multiplicities of valence-2-loops are $(1,1)$, $(1,2)$, $(1,4)$, $(1,5)$, and $(2,5)$.

Proof. Similar to that of lemma 9.8. However, now $(1,2)$ and $(1,5)$ come in, because we have a triangle resp. may have an outer cycle of even length in $B(D)$. □

It is easy to prove that the adjustment (also in point 2) of lemma 10.15 can be done in case B. Case C is more awkward and, after lemma 10.5, the completion will be postponed to lemma 10.27. Case B' is dealt with in lemma 10.23, and corollary 10.4 (a direct consequence of lemma 10.24).

Lemma 10.17 In case B there is at most one outer cycle loop with non-empty interior, and there are no multiple edges between empty outer cycle loops.

Proof. That there is at most one non-empty outer cycle loop was shown in lemma 10.6. It is easy to see that with 3 attachments we can create at most 2 cycles in $IG(A(D))$: by the argument in the proof of lemma 10.6, there cannot be two loops inside L intertwined (each) with 4 connections outside L in case of an outside attachment, and intertwined with 3 connections outside L in case of 3 attachments inside L. (Note again that (54) is excluded, because of lemma 10.10.) Therefore, there cannot be a multiple edge between the other at least 4 outer cycle loops, for it would crate an isolated vertex in $IG(A)$. □

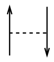
Lemma 10.18 In case A there is at most one outer cycle loop with non-empty interior, and there is at most one multiple edge between empty outer cycle loops.

Proof. We argued that case (A1) in (79) is excluded. In (A2) we need both attachments to be inside L, and with L having at most 4 multiple outside connections, we have at most 3 cycles in $IG(A)$. So we can have at most one multiple edge between empty outer cycle loops (giving an isolated vertex in $IG(A)$). □

10.4.3. Computations

It will be again inavoidable at some points of the proof to use a computer and check a certain number of explicit patterns. (Or at least, their manual case-by-case discussion would lead to no reasonable argument and exposition.) While it is not helpful to enter into full (implementational) details, we clarify some features of this procedure in advance, and concentrate on the mathematical part of the work later.

Since we have two cycles now, in turned in practice (due to the patterns to check) too technically cumbersome to try to make the diagrams positive (by making both cycles to be of even length). In other words, the convenience we would have from using pre-existing software to calculate V would not longer compensate for the effort in adapting the input to and output from this computation. So it was better to write an entirely new program which tests all Jones and Kauffman semiadequacy invariants directly from the A-state.

The state was encoded by recording the basepoints of the traces in cyclic order along each loop. Hereby we chose the orientation of loops to be *coherent*, i.e. so that each trace looks locally like . In contrast to the positive orientation in the proof of lemma 3.2, the coherent orientation always exists, but it does not necessarily extend to a link diagram orientation. This explains the inconvenience we overcome when abandoning the calculation from (oriented) link diagrams.

The coherent orientation also allows one to switch easily between the A- and B-state (and test latter's invariants too). We also programmed a component number test.

We have a total of 7 places where we use a computation. These are designated explicitly, as below. The programs used are available at [St20].

Calculation 10.1 After we implemented the aforementioned checks, we ran first as a test the patterns (a) and (b) of figure 4 in the previous section. We obtained the 12 vectors (a)-(n) of (69) within about a minute. The previous calculation of V took about half an hour. This speed-up should be related also to the feature of the Jones and Kauffman semiadequacy invariants to be of polynomial complexity, in contrast to the full polynomial invariants they are derived from, which are NP hard.

To make things even more convenient, we implemented then also the recovery of the sequence of D_k from the pattern. We used as input patterns in which the outer cycle has length 5. First, if the total number of crossings is odd, then we extended the outer cycle to length 6 (by putting an extra loop connected by two simple edges to its neighbors).

If all legs have multiplicity at most 2, then we have a sporadic case, in which we can only augment the length of the outer cycle by multiples of 2 (by adding two loops connected by only simple edges, in the opposite direction to (85)). Now we used that A- and B-state must have the same number of loops (74). Thus there is at most one diagram we can obtain ("at most" because the B-state may have already fewer loops than the A-state before we even extend the outer cycle in $A(D)$). We designed our program to report this diagram, but in practice it turned out that this case never occurred.

If we have an edge of multiplicity 4 or 5, the twist edge, then first we adjust it (within its parity) so that A- and B-state have the same number of loops. (In practice it turned out that in most cases we had to change here multiplicity 5 to 3.)

Then the sequence of diagrams $\hat{D}_k$ is obtained by adding 2 to the multiplicity of that edge and simultaneously extending the outer cycle by 2. Since this augments the crossing number of the diagram by 4, we can discard diagrams which (after equating the number of A- and B-state loops) are of crossing number not divisible by 4. Otherwise, we designed our program to output the DT code of the first four diagrams of each such series (which were then subjected to Vassiliev invariant tests).

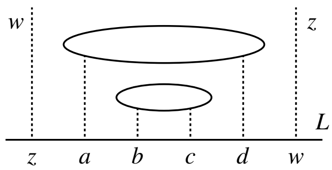
We collect at the end of the proof all sequences coming out of our computations. We refer at this place already to figure 10 on p. 92, where the diagram of each sequence is shown for $k = 1$. See §10.9 for more explanation. We will in fact encounter some sequences repeatedly. For the calculations it is too cumbersome and not worthwhile to implement all conditions that lead to the particular case where the calculation is performed, and that make this case to be mutually exclusive from the others. (In fact, extraneous checks lend more safety to the calculation.) Even during the calculation in the same case there are various symmetries which are too hard to see and remove in advance.

10.5. Restrictions on outside attachments

In this subsection, before we start a detailed separate discussion of cases A, B and C, we derive restrictions on how loops can be attached (using (28)) to loops on the outer cycle (or in case B the extra triangle loop). Case A is trivial from this point of view, so we exclude it from our treatment.

Lemma 10.19 If two loops of valence 2 are attached using (28) outside to outer cycle loop L, then *either* they are parallel *or* they are not within the same two outside loop edges.

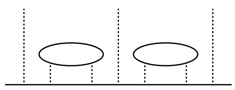
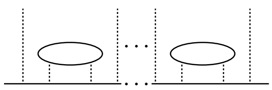
Let us illustrate the two options by pictures. (We discussed, though, the situation already partially in the proof of lemma 10.11.) Parallel loops attached outside to L are



. (86)

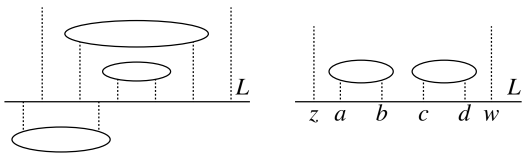
Here the outside of L is above in the diagram, and the edges w, z are understood to connect L to some other outer cycle loop. Also (unlike below) we understand that no further loops are attached on the part of L drawn as horizontal line, *except* between b and c.

Loops not within the same two outside loop edges look like:

 or more generally  . (87)

(Again the outside is above L, but something may now be attached from below to L.)

So what this lemma says is, for example, that the following scenarios are out:



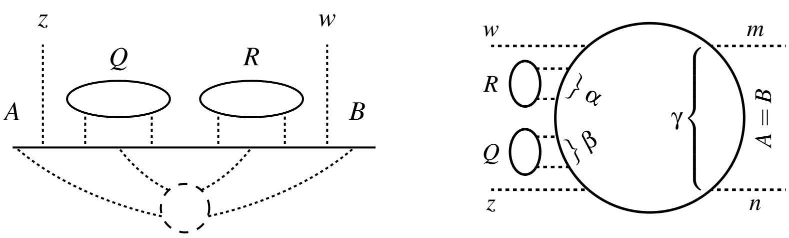
(88)

Proof. If we have two loops attached outside, in order $IG(A)$ to have a cycle, we must have case C, and the triangle (81) is the only attachment inside L.

Assume the two loops M, N are attached within the same arc of L separated by basepoints of edges to other outer cycle loops.

Case 1. M and N are enclosed. This is the situation (86), but allowing for attachments from below on the horizontal line. Under exclusion of inadequate loops, b cannot be alone on an interval between trace basepoints from inside L. So it must be grouped with a or c on the same interval. In latter case D is composite, so b is grouped with a. Similarly c must be grouped with d, and then M and N are parallel.

Case 2. M and N are not enclosed. This is the situation in the right picture of (88). The argument in the previous case modifies to show that b is grouped with c, d with w, and a with z. So we have the picture (a) in (89).



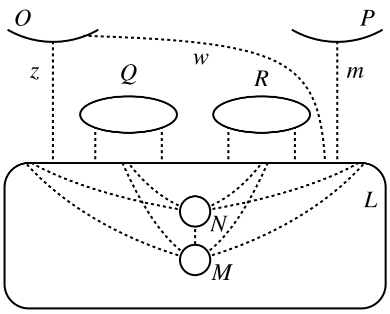
(89)

(a) (b)

The dashed circle in the lower part means that we do not specify how traces are connected to the loops M,N inside the circle. A trace that touches regions A or B must be there by primeness of D. However, since $\gamma = 4$, we see that actually then $A = B$. So outside L we have the picture (b) in (89), with α, β, γ being the segments of L on which legs of M and N are attached from inside L.

We assume that z,w resp. m,n connect L to its two neighboring loops on the outer cycle O resp. P. We must allow that some of z,w,m,n be 0. This covers all possibilities how the edges outside L connect to O,P, with one exception. This exception is the option that A lies between edges that connect L to the same outer cycle loop O as z and w. This situation is, however, recurred to the other cases by a mutation, as explained after (67).

Case 2.1. Assume first that some of z,w,m,n is 0. Up to mutations, we may assume w.l.o.g. that $n = 0$. Then the connections of L to the two inside loops M and N are intertwined with at most one outer cycle connection from L. To have two cycles in $IG(A)$, this connection must be then indeed intertwined with both M and N, i.e. $z \neq 0 \neq w$. So we have the following picture, up to symmetry:

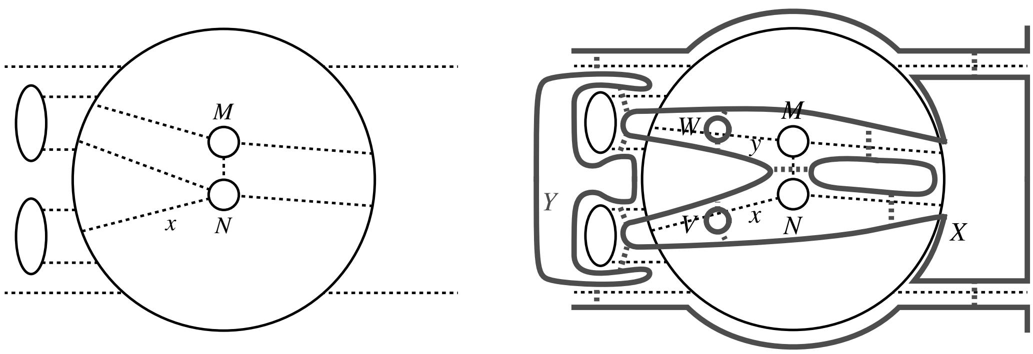


The understanding of legs whose traces cross is that for each such intersecting pair at least one leg must be of multiplicity 0. All edges outside L are of non-zero multiplicity.

Now the connection between L and P is not intertwined with any of the (connections to the) loops inside L. In order to have then two cycles in $IG(A)$, we need that M and N are intertwined with all of Q, R and O. However, then legs from M and N must be installed inside L on all intervals α, β, γ of picture (b) in (89). That is easily seen to be impossible without leg intersections.

Case 2.2. So we have the case that in (89) (b) all of z,w,m,n are non-zero.

Since $\chi(IG(A)) = -1$, we see that M and N are intertwined with at least 3 connections outside L. Thus they must both have legs on γ of picture (b) in (89). A simple check shows then that we have (up to symmetry) only two options left. These cases are shown below, assuming all edges drawn have non-zero multiplicity. In case (b) we have drawn simultaneously the B-state (following convention 9.2).



(90)

(a) (b)

Case (a) can be handled as in lemma 10.3. The edge x is inadequate, so $x = 2$. Then by connectivity, one of the other legs of M or N must be multiple. Then one exhibits in $B(D)$ two cycles not separated by a loop. So $B(D)$ is of type A or B, and we are done.

In case (b), we have $x = y = 2$ by inadequacy. We have drawn above again into the diagram the loops and traces in $B(D)$ by thicker, and gray, lines. The property $x = y = 2$ (in $A(D)$) explains the small loops drawn near them in $B(D)$. Let us call these small loops V and W. The B-state loops X and Y close as indicated, because they touch a segment of the A-state loops neighbored to L on the outer cycle. With that it is easy to see that X is the only separating loop in $B(D)$. Now X has at most 3 multiple connections inside. (When augmenting multiplicity of some of the edges in $A(D)$, further small loops enter into $B(D)$, and multiple connections and their intertwining may only be spoiled, but never newly created.) Two of these connections, those from V and W, are intertwined with at most one connection outside X, the one to Y. Then we see that $IG(B)$ cannot have any cycle.

With this lemma 10.19 is proved. □

The following lemma makes a series of technical assumptions. However, these are often satisfied, and the argument will be needed repeatedly, so it is better to single it out.

Lemma 10.20 Consider a diagram D in case B, B' or C with the following properties:

- Assume $B(D)$ has a single separating loop X, and ≤ 3 multiple connections inside (resp. outside), each intertwined with at most 2 multiple connections outside (resp. inside) X.
- Let U,V be the regions of $A(D)$, which contain traces of all but at most one (in case B, B') connection of the outer cycle in their boundary. (For case C it is as in (51).) Assume further no attachment legs (of both (81) and (28)) touch upon the boundary of U or V.
- $B(D')$ has at least 7 loops, where D' is as in definition 9.2. (See also remark 10.3 after the lemma.)
- $IG(A(D))$ has at most 4 cycles, and at most one non-empty outer cycle loop.

Then such a diagram D cannot occur among our diagrams D_k.

Remark 10.3 Let us make the following remarks concerning the 7-loop condition on $B(D')$. (These points will be used implicitly when invoking the lemma several times below.) More precisely, as the proof will show, it is enough that $B(D)$ has at least 7 loops, when we assume the minimal admissible multiplicity for each edge of $A(D)$, which is not an edge between empty outer cycle loops.

Moreover, to find such 7 loops, it is enough to have one loop in $B(D)$ connected by a trace to X, whose connection to X is not intertwined with any connection on the opposite side of X. This is because we already have X, and at least 5 other loops are needed whose connections to create 2 cycles in $IG(B)$.

Proof. Now, by the second assumption, the boundaries of U,V correspond to loops in $B(D)$, which we denote again by U,V. The connection (U,V) between loops U and V in $B(D)$ is not intertwined with any other connection. Since, by the first condition, we can create at most two cycles in $IG(B)$, this means that (U,V) must be simple (else it would yield an isolated vertex in $IG(B)$).

Thus there exists at most one simple edge between empty outer cycle loops in $A(D)$. On the other hand, $IG(A(D))$ has at most 4 cycles, so there are at most two multiple edges between empty outer cycle loops in $A(D)$. Since the outer cycle has length at least 5, we see that it must have length exactly 5, and exactly two of the three edges between empty outer cycle loops in $A(D)$ must be multiple.

With an outer cycle in $A(D)$ of length 5, we have in $A(D)$ exactly 9 loops. Thus $B(D)$ has also 9 loops, and $c(D) = 20$ by (74). We are thus already down to a finite number of cases. Even those are easily dealt with by hand, assuming $B(D)$ has at least 7 loops, when all outer cycle edges between empty loops in $A(D)$ are simple.

Now we are supposed to have two multiple outer cycle edges between empty loops in $A(D)$. By connectivity reasons their multiplicities cannot be $(2,2)$. So making outer cycle edges multiple, we must add at least 3 traces in $A(D)$, and so at least 3 loops to $B(D)$. Then, however, $B(D)$ has at least 10 loops, a contradiction.

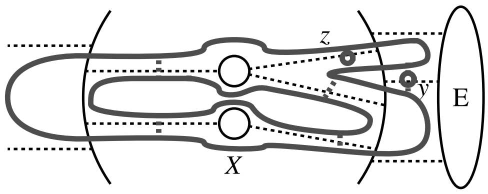
The completes the proof of lemma 10.20. □

Lemma 10.21 In cases B or C, there is no loop E of valence 3 attached (using (28)) outside to an outer cycle loop L.

Proof. Assume there is such E. We will rule gradually out all possibilities.

First observe that there must be three regions outside L touched by legs of loops attached inside L. Two of these regions are enclosed by L and E. (The third one is needed to avoid D becoming composite.)

Case B is easy. We already know that there is only one loop on the outer cycle or triangle with non-empty interior. The only option we have, avoiding inadequate loops, is



. (91)

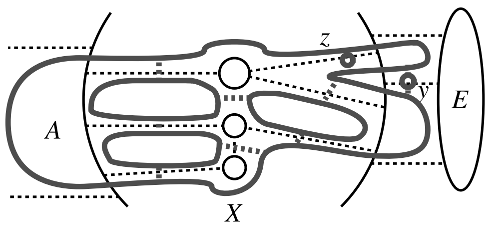
Here all (thin, black) edges in $A(D)$ are assumed to be of non-zero multiplicity. Again y and z are inadequate, so $y = z = 2$. We look at $B(D)$. Let Z be the loop between the two traces of z. We see that in $B(D)$ there is only one separating loop, call it X. It has at most two multiple connections inside, and one of them, to Z, is intertwined with at most two multiple connections outside X. Then $IG(B)$ has at most one cycle.

Consider case C. If we have a second outer cycle loop with non-empty interior, then in this interior must be the third attachment, using (28). However, it is easy to see that if this attachment has valence 2, then it is inadequate, while if it has valence 3, then $\gamma > 6$. So we can assume there is at most one outer cycle loop with non-empty interior. Let below L be this non-empty outer cycle loop.

Case 1. Inside L is the second attachment M (28) and the triangle (81).

Case 1.1. The valence of M is 3. If M is attached within some of the triangle loops, then using flatness and (63) we see that the other triangle loop must not intertwine anything outside L (i.e. the diagram on the right of (63) must become composite). Then, however, $IG(A)$ has no cycle. So M is of depth 1 within L, and both triangle loops are empty.

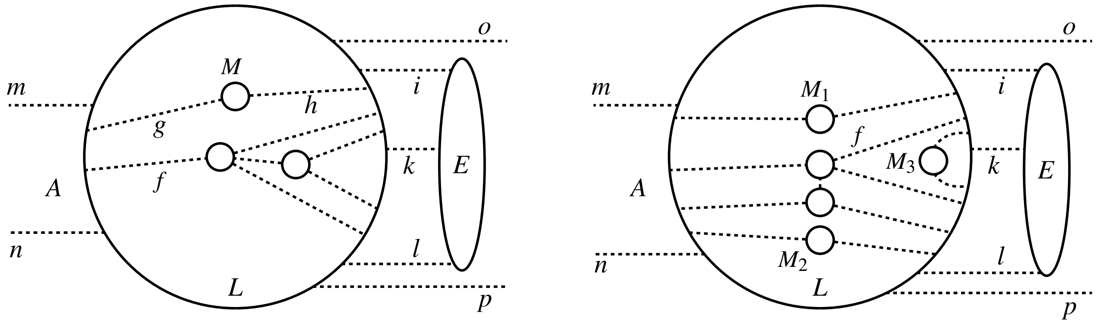
Then the legs of the triangle identify only two regions A, B outside L, and by lemma 10.3 only one of the 2 triangle loops is intertwined with other connections. This implies that region A must be between two edges of the same outside connection of L. So we have, up to mutation,



.

Then we apply the same argument as for (91).

Case 1.2. The valence of M is 2. We have up to mutations and symmetries two options:



(92)

$(X1)$ $(X2)$

In the second case we indicated the possible positions for M as M_i. Only one of these positions is used. An attachment M of depth 2 inside some of the triangle loops is prohibited in (X1) because $IG(A)$ will have no two cycles, and in (X2) because γ becomes > 4 (and otherwise by corollary 10.3).

The position M_3 in $(X1)$ is excluded, because $IG(A)$ has no cycle. For the same reason, adopting the nomenclature of (82) for the triangle edges, the edge f in (X1) has non-zero multiplicity. The same is true in (X2), because otherwise M_2 would give $\gamma = 2$, while $M_{1,3}$ would be inadequate. (We indicated only the edge f of (82) in the above diagrams, to call to mind that the orientation of the pattern (82) differs slightly.) Moreover, $M_{1,2}$ are symmetric in (X1).

Here is the first situation where it appeared more useful to apply a computer to check the cases, rather than discussing them manually one by one. However, as a preparation, we need still to argue out first some of them by hand.

Remember that to apply our program, and work with the simplest diagram according to lemma 10.15, we need to show that there is at most one multiple edge between empty outer cycle loops. (We already assured that there is at most one non-empty outer cycle loop.)

So we need to show that $IG(A)$ has at most 3 cycles, i.e. rule out the cases that $IG(A)$ has ≥ 4 cycles. (One could likely deal with the other cases using similar arguments, rather than using a computer, but this discussion would become too long and tedious.)

Sublemma 10.1 $IG(A)$ has at most 3 cycles.

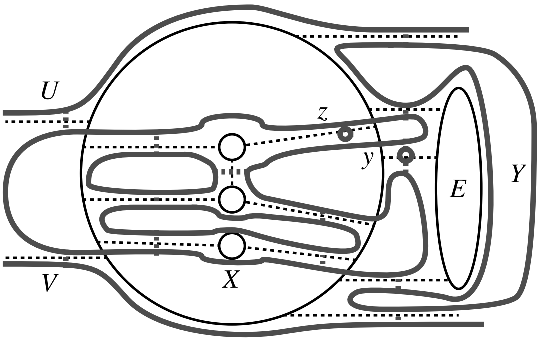
Proof. Now assume $IG(A)$ has ≥ 4 cycles. This means that there are 3 loops inside L, two from the triangle and one from (28), and the connections from L to them are intertwined with the one to E and the two outer cycle loops neighbored to L. Then $IG(A) = K_{3,3}$, the complete bipartite graph on $3+3$ vertices.

In both cases the assumed 4 cycles force A to be between edges of the same outside connection of L. Similarly the outside attached loop E is in a region B between edges of the same connection. Thus the multiplicities of all edges in (X1), (X2) are non-zero, *except* for the choice between M_i for M in (X2), and for some of the edges of the triangle.

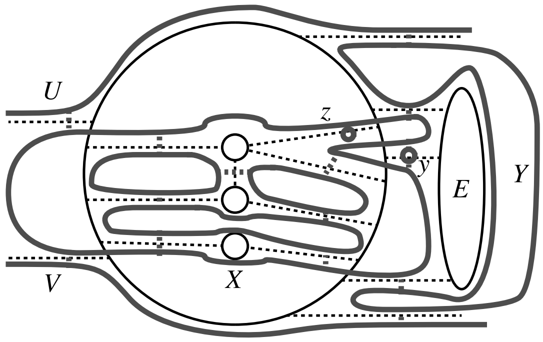
There are, up to mutations, 3 cases, see figure 6. (Here we assume all edge multiplicities to be non-zero.) The B-state loops X and Y are drawn closed since they are bounded by the outer cycle (A-state) loops neighboring to L. Observe again that all edges y and z in $(*)$ and $(**)$, are inadequate, so of multiplicity two.

These cases are ruled out by the following argument. Again there is a unique separating loop X in $B(D)$. It has at most 3 multiple connections inside, each intertwined with at most two connections outside. Also, $IG(A(D))$ has at most 4 cycles. Taking into account the (B-state loops coming from the multiplicity 2) inadequate edges y and z, we see that in all cases $B(D)$ has at least 7 loops, even when all outer cycle edges between empty loops in $A(D)$ are simple. Thus we can apply lemma 10.20. This completes the proof of sublemma 10.1. $\square$

With these cases ruled out, we justified the use of our computer algorithm on the patterns of (92).

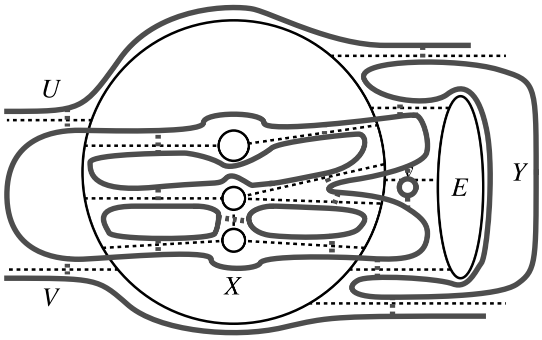


$(*)$



$(**)$

(93)



$(***)$

Figure 6

The edges are named as in (92), with the triangle edges following (82). To fix the remaining names for the edges, let also in (X2) g,h be the edges of M, and call the 3 edges between other empty outer cycle loops x,y,z (we use, as explained in §10.4.3, an outer cycle of length 5).

Let us specify what conditions we have on the multiplicities thus named. In case X1:

- $f>0$ (as argued above), $k=2$ (inadequate)
- $e=1$, otherwise it gives an isolated vertex in IG, and $\chi(IG(A))>-1$.
- $m,n,o,p>0$ (otherwise $IG(A)$ has at most 1 cycle) and not all are even (by connectivity)
- $x=y=z=1$ (because of $\chi(IG(A))=-1$, as for $e=1$).
- $i,l>0$ and not both even (by connectivity)
- not all of a,b,c,d,f even (by connectivity)
- the multiplicities for the valence-2 loop legs follow lemma 10.16

In case X2:

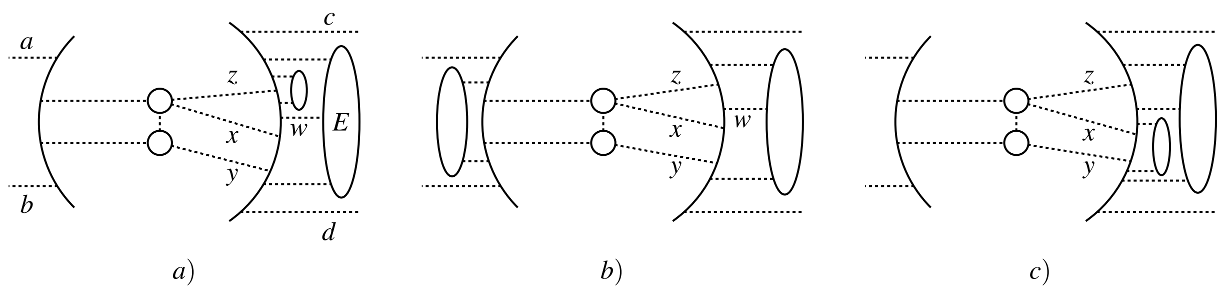
- $f>0$ (as argued above), $k=2$ (inadequate)
- $o,p>0$ and one of $m,n>0$ (w.l.o.g. up to mutation); m,n,o,p are not all even (by connectivity)
- $y=z=1$; x,e are not both >1, and if one is >1, then both $m,n>0$ (because of $\chi(IG(A))=-1$).

- $i, l > 0$ and not both even (by connectivity)
- not all of a, b, c, d, f even (by connectivity)
- the multiplicities for the valence-2 loop legs follow lemma 10.16

Calculation 10.2 Without strong effort at speed-up, the computer calculation took about 27 minutes, and ruled out all cases.

Case 2. Inside L is only the triangle, and outside L are the two attachements (28).

First note that then always $IG(A)$ has ≤ 3 cycles. To have at least 2 cycles, it is necessary that at least one outer cycle connection of L is intertwined by the triangle. There are 3 cases:



All edges are assumed to be of non-zero multiplicity, except possibly x. and/or at most one of a, b, c, d. (We labeled latter for simplicity only in case a).) If x is not there, y is inadequate, so $y = 2$.

Up to mutations, we may assume the intertwined outer cycle connection of L to be on either side. In other words, a region bounded by two edges of this connection is either where E is attached (so $c, d \neq 0$), or where the legs of the triangle loops inside L going to the left in the diagrams are touching L (here $a, b \neq 0$).

In case a), let us assume that the intertwined outer cycle connection is on the left of L in the diagram, i.e. $a \neq 0 \neq b$. Now z is inadequate, so $z = 2$. In $B(D)$ again the only separating loop X has at most 3 multiple connections inside. The one, coming from $z = 2$ in $A(D)$, is intertwined with at most one outside connection. Another connection comes from the trace in $B(D)$ dual to y, if $x = 1$ or $(x, y) = (0, 2)$. This connection is intertwined with at most two connections outside X. This already suffices to conclude that $\chi(IG(B)) > -1$.

In case b), assume w.l.o.g. $c, d \neq 0$. Now z, w are inadequate, and with the B-state loops coming from $z = w = 2$ we count already 7 loops. Again the loops U, V have a connection not intertwined with anything. Thus we can use the same argument of lemma 10.20 with which we ruled out (93).

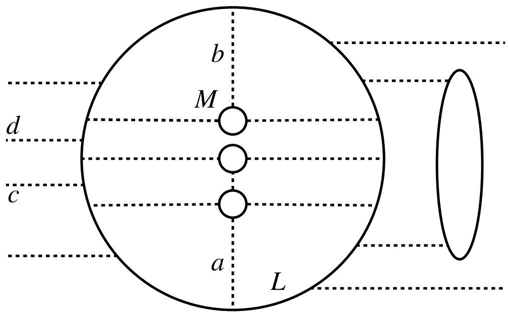
Finally, case c) is handled similarly to case a). We assume again $a \neq 0 \neq b$. The only difference to the argument for case a) is that the connection inside X coming from y (again if $x = 1$ or $(x, y) = (0, 2)$) is intertwined with at most one connection outside X, while the one coming from the inadequate edge $z = 2$ is intertwined with at most two connections outside X.

With this lemma 10.21 is proved. $\square$

Now we can go back to sublemma 10.1, which operated under the setting of lemma 10.21, and extend it fully using lemma 10.21.

Lemma 10.22 In case C, there are at most three cycles in $IG(A(D))$.

Proof. Assume $IG(A(D))$ has ≥ 4 cycles. As in the proof of lemma 10.21, we have then that $IG(A) = K_{3,3}$, with the triangle (81) and one loop (28) inside L, and the other loop (28) outside L. Now we know, by lemma 10.21, that the attachment outside L has valence 2. The only option we have then is



The edges a,b,c,d are optional, but using $\gamma = 4$ we see that exactly one of these four must be there (i.e. $\neq 0$). Again we assume all other edges have non-zero multiplicity, and the two edges that leave from L to the left resp. right connect it to the same outer cycle loop.

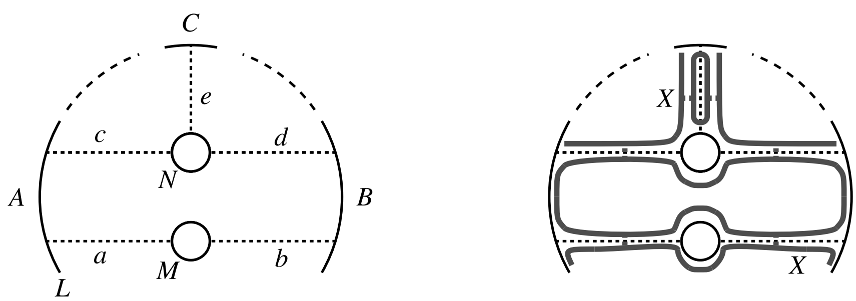
In fact, a or b may connect to a region between two outer cycle traces from L to the same neighbored outer cycle loop. Then, however, we can apply lemma 10.20. (Note that, because by inadequacy a or b is 2, and $c = d = 0$, by connectivity at least one of the other 6 legs of the 3 loops inside L is multiple, so we find 7 loops in $B(D)$ by remark 10.3.)

Now if $c = 0$, then we can use lemma 10.3 (where f is our a). So it remains to consider $c \neq 0$. This option is dealt with by the argument that was applied to the diagrams in figure 6. (Again there are two distinct B-state loops for the regions U and V in (51).) □

10.6. Case A

We argued that from the options (A1) and (A2) in (79), only (A2) is relevant. To have two cycles in $IG(A)$, both attachments we perform must be inside L, the common loop of the outer cycle and triangle. Both loops inside L must intertwine at least (the same) three connections outside L. In fact, if (A2) is drawn as in (79), we easily see that this is impossible without $\gamma \geq 6$. The minorly less trivial case results again from the situation in which we have the phenomenon (80) for the triangle, and/or the triangle lies between edges of the same outer cycle connection. Even with this complication, the case is relatively easy to settle, albeit with some computation.

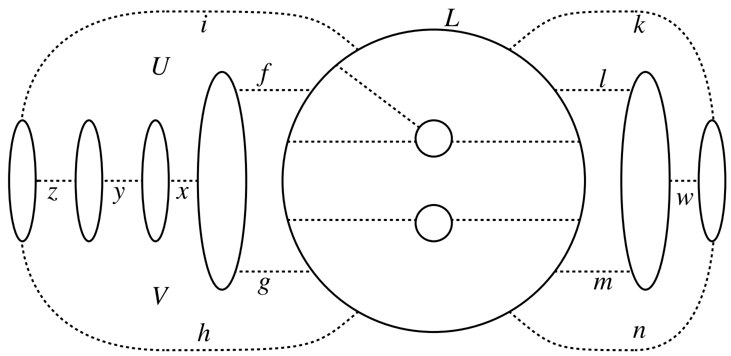
Using $\gamma = 4$ and the exclusion of inadequate loops, we see that from both loops inside L, one, N, must be of valence 3, and the other M of valence 2. The inadequate leg e of N is of valence 2. So we have (see (94)) in $B(D)$ a unique separating loop X, which has at most two connections inside intertwined with something outside, and both are intertwined with at most 3 connections outside X.



. (94)

Let a,b,c,d be the 4 other legs of M,N. Then since $e = 2$, one of a,b,c,d is > 1. So by lemma 10.20 (and remark 10.3), the region C outside L touched by e is among one of the two regions U,V in (51), and by symmetry we may assume $C = U$.

We have then the following picture. (The interior of L is as on the left of (94).)



(95)

Now we said that draw the outer cycle like (51). In the above diagram we made a unique exception to this rule. We choose, in deviation from elsewhere, the position of the outer cycle so that the curve giving the connected sum in (A2) goes through infinity. (This curve is supposed to intersect L in exactly two points, but should not give a connected sum decomposition of D, since it intersects traces of loops to be attached inside L.) This exception is made here merely for better display.

In (95), we have the following conditions on edge multiplicities:

- All edges are non-zero, except at most one of g,h,i,k,l,m,n.
- $f=2$ (inadequate); w.l.o.g. $y=z=1$, and x,w are not both >1 (because of $\chi(IG(A))=-1$)
- If one of g,h,i,k,l,m,n is zero, then $x=w=1$ (because of $\chi(IG(A))=-1$).
- $e=2$ (inadequate), c,d not both even (by connectivity)
- the multiplicities for the valence-2 loop legs a,b follow lemma 10.16

Calculation 10.3 The computer checked these cases within 40 seconds, and ruled all out. This finishes case A.

10.7. Cases B and B’

Now we are prepared enough to finish off case B completely. Later we move on to case C, which requires more work. However, let us first advance with the degenerate case B’ as far as we did with case B.

We use the diagram (80).

Lemma 10.23 There is only one non-empty outer cycle or triangle loop in case B’, which is w.l.o.g. L.

Proof. If both of L and M contain at most one attachment, then $IG(A)$ will have no cycles. So we can assume up to symmetry that w.l.o.g. L contains at least 2 of the 3 attachments, inside or outside.

Assume that L has 2 attachments, so the third attachment is inside some loop $N\neq L$. Note that then we have no outside attachments to L,N, so we have no different legs connecting to the same region. We distinguish the following 4 situations:

(i) The third attachment has valence 3, and the attachments inside L identify 2 of these 3 regions.

(ii) The attachments inside L identify 3 regions, and the one in N identifies 2 of the three regions.

(iii) The attachments both inside L and N identify the same 2 regions each, or

(iv) The attachments inside L,N identify the same triple of regions.

In case iii) the two pairs of identified regions must coincide, for otherwise the attachment within N would be inadequate. Cases i), ii), and iii) can then be reduced to one non-empty loop by mutations.

In case iv), if $N = M$, then one has a modification of (64). (Now X in (64) is L here, Y is M, and A,B may not be those of (80).) Now one sees that the attachments inside L intertwine at most two connections (to M and N), and the attachments inside L,M intertwine at most one common connection (the one between L and M). Thus $IG(A)$ cannot have 2 cycles.

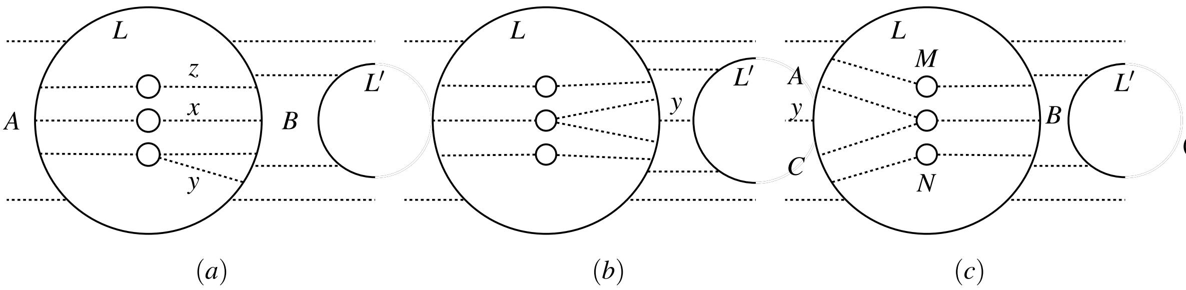
If $N \neq M$, then by the argument after (80), we may assume that A,B of (80) are among the A,B,C of (64). However, A,B are not simultaneously neighbored to any other loop than L, a contradiction.

Thus we have only one non-empty loop on the outer cycle/triangle. If this loop is different from L, we can reduce case B' to case B by mutations. So we can assume that it is L, and the lemma is proved. □

Since now M is empty in (80), we can assume that attachments inside L touch, or attachments outside L are made within, both regions A and B in (80).

Lemma 10.24 In case B', $IG(A)$ has at most 3 cycles.

Proof. Assume we have ≥ 4 cycles. Then all 3 attachments are inside L, and $IG(A) = K_{3,3}$, with exactly 4 cycles. We have the following three options, under exclusion of inadequate loops. (We draw for space reasons the refinement of (80) only until the left half of L'.) In case (a), we allow the edge y to connect to any region outside L different from A,B.



(96)

Let us first deal with (a). Since y is inadequate, $y = 2$, and then w.l.o.g. we have $x,z \geq 2$ by connectivity. Then, however, it is easy to see that $B(D)$ has two cycles, which share a single loop, but are not on opposite sides of it. Thus $B(D)$ is of type A, and we are done.

Case (c) is similarly simple. Again $y = 2$ by inadequacy. Then we observe that $B(D)$ has a single separating loop X. It has again at most two multiple connections inside, each intertwined with at most two multiple connections outside X. This implies that $IG(B)$ has at most one cycle. There is also a modification of this case, in which, say, M connects B,C rather than A,B, and is parallel to N. This is handled by the argument for case (a).

Case (b) is handled as case (c). □

Corollary 10.4 In case B', there exists at most one multiple edge between empty outer cycle loops.

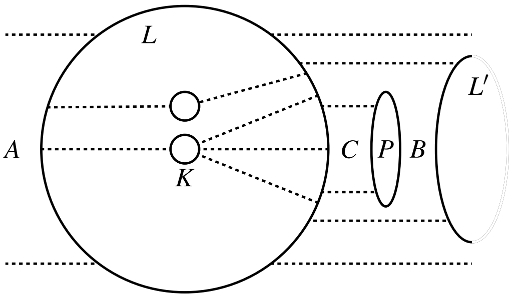
Proof. As before, using lemma 10.13, lemma 10.24, and $\chi(IG(A)) = -1$. □

Lemma 10.25 In case B', there is no pair of distinct attachment legs touching the same region.

Proof. Let P be an attachment to L, on either side of which there are legs touching from the opposite side of L.

First, again P cannot be inside L. If so, and L had only one attachment outside, P would intertwine at most one connection, and $IG(A)$ has no cycle. If L had both other attachments outside, D will be composite.

So P is outside L. Again because $\gamma = 4$ and D is prime, P has valence 2. Recall that legs inside L should touch A and B in (80). Since they can touch at most 3 regions, this means that P must be attached within A or B, and legs inside L touch only A, B and the region C between L and P. So by a mutation we can assume w.l.o.g. that P is attached within B.



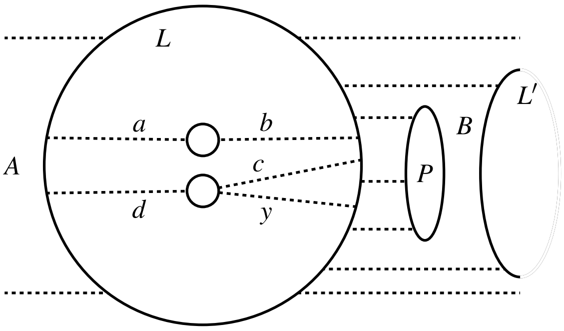
(a) (b) (97)

Clearly, given P is outside, it is the only outside attached loop. Now P is inadequate, unless there is an attachment (of depth 2) inside K in case (a) of (97), or both $K_{1,2}$ in case (b). But then after 3 attachments $IG(A)$ will have no cycle. □

In particular, we assured now that in case B' too, all attachments have valence 2 or 3.

Lemma 10.26 In case B', there is no attachment outside L of valence 3.

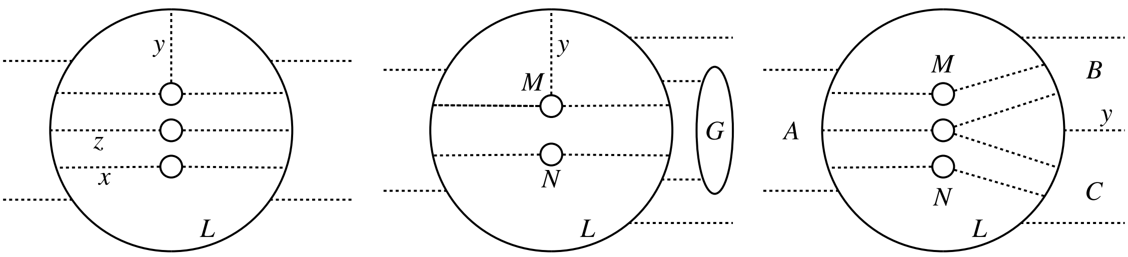
Proof. Assume there were such an attachment P. The three regions identified by legs inside L must be then, w.l.o.g. up to mutation, A of (80), and the two regions between P and L, whereby P is attached inside B of (80). Up to symmetries, the only option we have, avoiding inadequate loops, is



Again y is inadequate, so $y = 2$. Then $B(D)$ has a separating loop X, with at most two multiple connections inside. One of those, the one dual to y, is intertwined with at most two connections outside X. So $IG(B)$ has no two cycles. □

With this all work until the end of §10.5 is extended to case B'. We continue treating B and B' simultaneously.

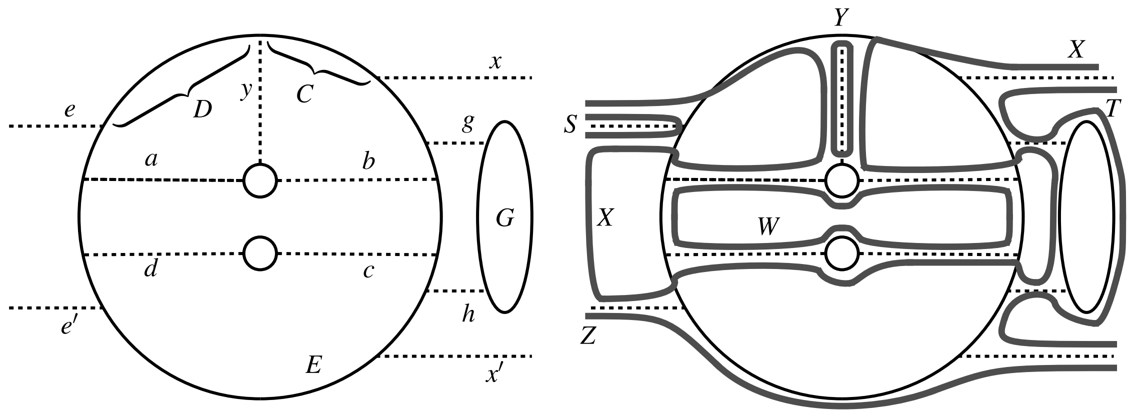
Using lemma 10.6, 10.10 and 10.21, we are left with the following choices to perform 3 attachments (28) in case B, so as to create 2 cycles in $IG(A)$, have $\gamma = 4$, and avoid inadequate loops. The region outside L touched by y in (a) and (b) of (98) may *a priori* be arbitrary.



(a) (b) (c) (98)

Now we can treat case B' uniformly, since we know that we must connect the regions A,B in (80). The modification to (98) is the that traces going out from L on the right are supposed to connect to both L' and M in (80), with the innermost two (excluding y in (98) (c)) connecting to L'. (So regions B,C in (98) (c) lie between L and L'. Only in case (b) we could place the right legs of M,N so that G is intertwined with only one of them, but then G would be inadequate.)

Cases (a) and (c) are ruled out as in the proof of lemma 10.24. Thus, for the rest of this subsection, we consider case (b) in (98). We draw in (99) the A- and B-state (we waived, though, on drawing the traces in $B(D)$, which are obvious). For the A-state, our understanding is that $e.e'$, as well as x,x', connect L to the same outer cycle or triangle loop. All drawn edges have non-zero multiplicity. To keep the case general, we allow then that loops end from outside on the segments C,D of L, as well as on the lower one E between the endpoints of x' and e'.



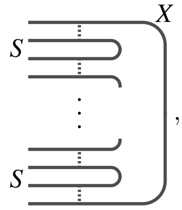
(99)

Note that we drew the B-state assuming there are no additional edges on C,D,E in the A-state. For E adding edges will not alter the following argument. For C,D we will just see that in fact there must not be any further edges.

Namely, we have again only one separating loop X in $B(D)$. Again, X closes on the left as shown because e,e' connect to the same loop in $A(D)$. X contains in its interior only two multiple connections, to W and Y, the loop obtained from the inadequate edge $y=2$ in $A(D)$.

Moreover, X has at most 3 multiple connections in its exterior, which can be intertwined with Y or W, the connections to Z, T and S. In particular, y does not end on a segment of L between edges that connect L from outside to the same loop; otherwise Y intertwines at most two connections outside X, and $IG(B)$ has no two cycles. Moreover, in order T to intertwine with both Y,W, we see that C must be empty, $x=1$, and at least one of g,h, say g, is 1. We call this condition *y has distance 2 from G*. Also, in order S to intertwine, we must have either that D is empty and $e=2$, or $e=1$, and D contains exactly one endpoint of a trace that connects L to a loop different from the one e does.

Note that we could also have something like

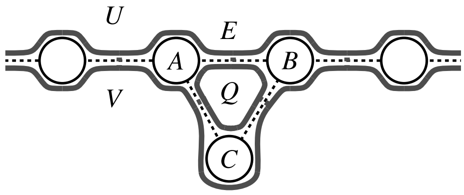


,

but this is excluded because D is prime, and X is the only separating loop in $B(D)$, so S is empty on the side opposite to X.

Note that the loop T closes on the right, as X closes on the left. We have then a cycle in $B(D)$ that contains X,T,Z.

Consider now the B state of the diagram $\tilde{D}$ of (B) in (79) before the attachments (28) are made to L. We draw the connections of x,x' and e,e' as simple edges, i.e. the diagram $(\tilde{D})'$ (see definition 9.2).



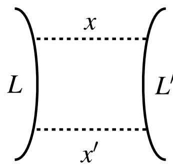
(100)

This diagram $\tilde{D}'$ already has a cycle in the B-state, call it $R=(Q,V,U)$. Now the effect of the attachments (28) on $B(D)$ is that T and one of X or S is added, and X obtains 2 new loops in its interior, Y,W. Again by connectivity, since $y=2$, we have some of $a,b,c,d>1$, so that $B(D)$ contains a cycle in the interior of X.

There is also the option of case B' that A,B are connected by traces e'' below C in (100). We will soon rule out this option, but for the moment just observe that the arguments below apply to case B' in the same way as for case B.

If now y connects to a region outside L different from U,V, then we can use lemmas 10.23 and 10.24 and apply lemma 10.20: we have at least 7 loops in $B(D)$ (namely, X, and at least 3 loops on its either side, taking into account $\max(a,b,c,d)>1$), and the loops U,V of (100) remain non-separating, with a multiple edge. Thus the attachments inside L are done so as y to touch U or V.

Next note that under the attachments inside L, the cycle R in (100) also persists in D, unless legs inside L touch between x and x' in



, (101)

and x, x' lie in the common connection E of the outer cycle and triangle in $A(D)$ in (100). Otherwise the cycle X,T,Z in (99) does not contain edges dual to those in E, and so it is not part of R. Then, counting the cycle inside X, the addition of T in $B(D)$ under the attachments in $A(D)$ to L creates a third cycle in $B(D)$.

Thus we can assume that x in (101) belongs to the connection E, that is, (w.l.o.g. up to symmetry) $L=A$ and $L'=B$ for A,B in (100). With the previously observed properties on the edges e, x and y, we are left with 5 options for the attachments (28), see figure 7.

The patterns of figure 7 we have now are very explicit. Additionally, we have strong restrictions on edge multiplicities:

- $m=n=o=p=1$ (where $m=1$ applies only if m is drawn), and all not drawn outer cycle connections are simple (because $\chi(IG(A))=-1$),
- w.l.o.g. $g=1$ (by the arguments on C,D in (99)); for (B1), $e=2$, $x=1$, for (B2) $x=1$, for (B3) $e=1$, $x=2$, for (B4) $e=x'=1$, for (B5) $x=1$ (for the same reasons),
- $y=2$ (by inadequacy),
- one of a,b,c,d is even, but not both (a,b) or both (c,d) are even (by connectivity).

Calculation 10.4 The computer checked these cases (though still to many to write one by one here) within a few seconds. With the explanation of §10.4.3, the sequences our computation finds are those of figure 10 for $i=1,2,8,9$. This concludes case B.

In case B', i.e. if the above named potential traces e'' exist, we must have a trace ending from inside L between n and n' in (B2) or (B5), following the remark after (80). If we attach G somewhere else than between n and n', the three regions identified by traces inside L are determined, and it is impossible to install y so that it connects U or V in (100). Thus we should try to attach G between n and n'. However, with e'' there, it is then impossible to install y to have distance 2 from G and to connect U or V. With this case B' is finished, too.

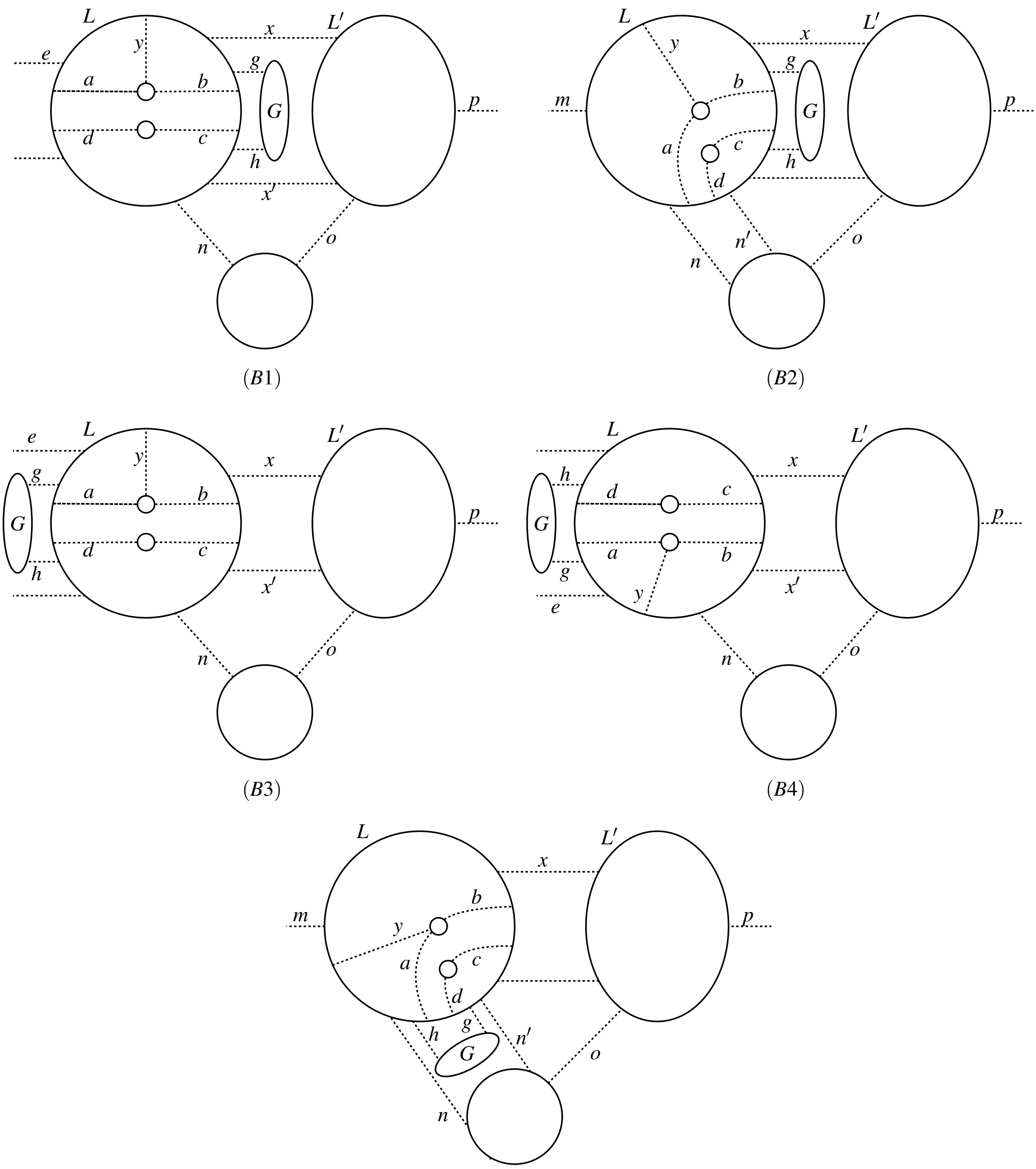


Figure 7: Patterns in cases B1 to B5.

10.8. Case C

First we try again to clear the way for applying computations, by establishing the premise of lemma 10.15. We need the below lemma 10.27.

Lemma 10.27 There is at most one multiple edge between empty outer cycle loops.

Proof. This lemma is immediately clear from lemmas 10.22 and 10.5. □

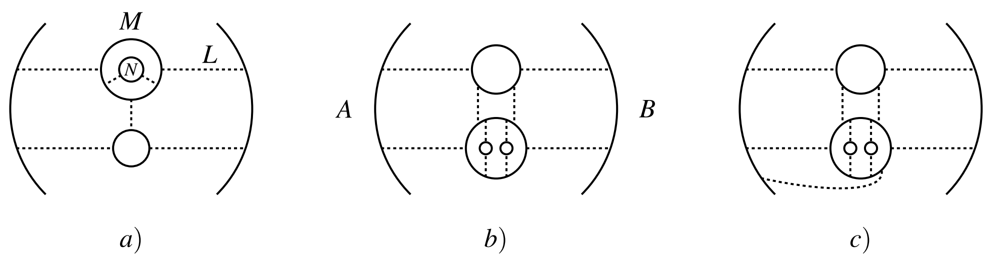
Next we eliminate loops of depth ≥ 2 completely.

Lemma 10.28 There are no depth-2 loops.

Proof. With corollary 10.3, we are left only with depth-2 loops inside triangle loops.

If there is a depth-2 loop of valence 3, then we can argue using (63) and $\gamma = 4$.

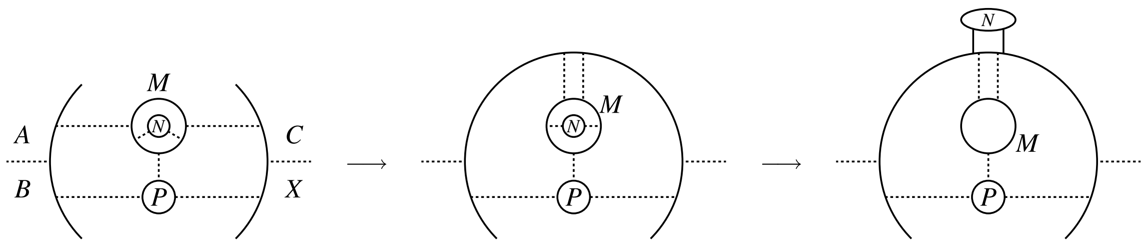
Next assume one of the triangle loops M has exactly one valence-2 loop N inside. Then we have by flatness the case (a) of (102).



(102)

Note that the other triangle loop must be empty, otherwise we cannot create 2 cycles in $IG(A)$.

Now if N is not supposed to be inadequate, we need traces from outside L like on the left below

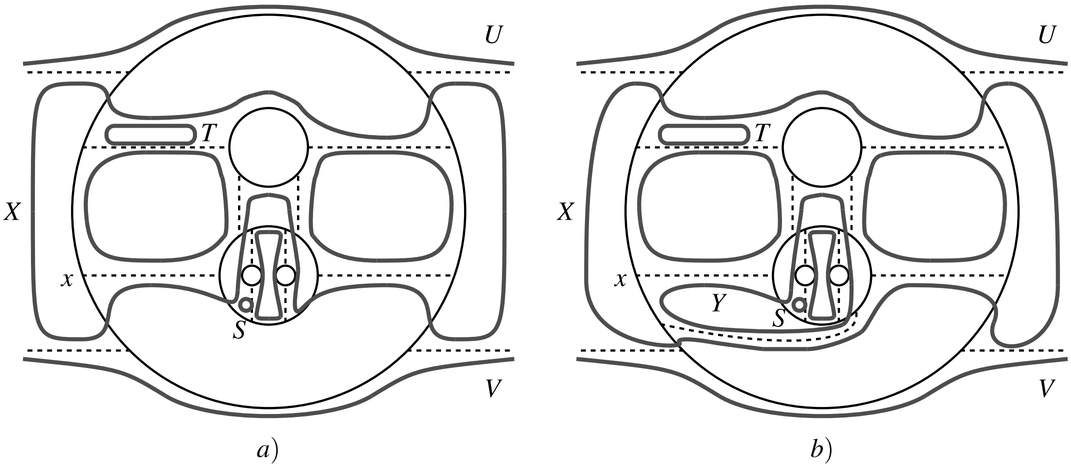


(103)

Now if there are other traces from outside L both between A and C, and between B and X, then $\gamma \geq 6$. Thus w.l.o.g. (up to flyping N into P), $A = C$. Then the legs of M become parallel, and we can mutate N, as shown above, into a loop of depth 0, in contradiction to flatness.

The last option is to have two valence-2 loops attached inside a triangle loop M. Diagrams (b) and (c) and (102) display the only options we have, if one is to create two cycles in $IG(A)$ and avoid inadequate loops. Because of $\gamma = 4$, the regions A, B outside L cannot be separated by traces as on the left of (103). On the other hand, the regions A and B must be between edges that connect L to the same neighboring loop on the outer cycle, in order to have 2 cycles in $IG(A)$.

The B-state for diagram (b) of (102) looks then as diagram (a) in (104).



(104)

Here again the extra loops S,T result from edges in $A(D)$ which must be multiple for connectivity reasons. (T may be, instead of its displayed position, also connected by the dual 2 edges to $x=2$.) Then this is ruled out by lemma 10.20.

The case c) of (102) leads to the diagram b) of (104) (with the same remarks on S,T as (104) a)). For this diagram, $B(D)$ has two separating loops, X and Y, but each one has at most one multiple connection on one side. Thus $IG(B)$ has no cycle. □

Lemma 10.29 Assume M is a loop of an attachment, and M has two edges. Then, with the exception of the 2 sequences of figure 10 for $i=4,10$, neither of them has multiplicity 4 or 5. In particular, the list of options of lemma 10.16 reduces to $(1,1)$ and $(1,2)$.

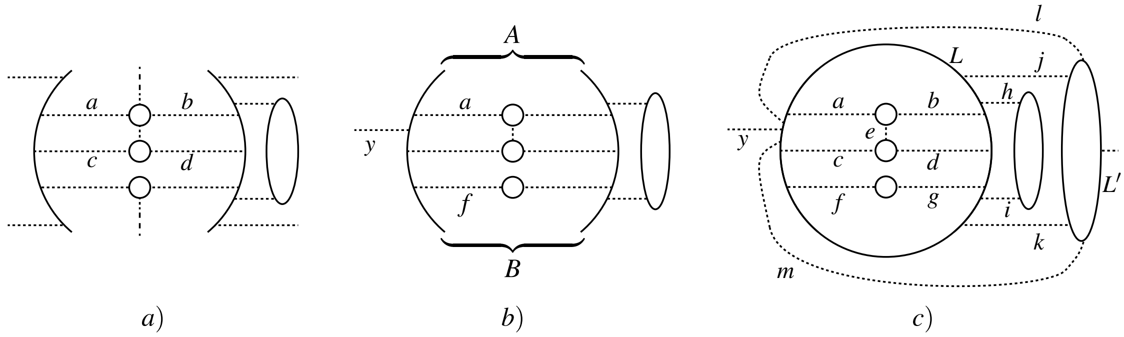
Proof. If there were such an edge, then we would have in $B(D)$ an outer cycle with at most one multiple connection. Thus the claim then follows from the next lemma. □

Lemma 10.30 The outer cycle has at least 2 multiple connections, except the sequences of figure 10 for $i=4,10$.

Proof. Assume it has at most one multiple connection. Using lemma 10.21, avoiding inadequate loops, we are left with only few possibilities to create two cycles in $IG(A)$. Clearly the outer cycle loop L with non-empty interior must be among those of the multiple connection. Moreover, we need at least one attachment outside L.

Case 1. Assume first there is one attachment (28) outside L, and one inside.

Then we have two options, up to mutation, which are shown below as a) and b).



(105)

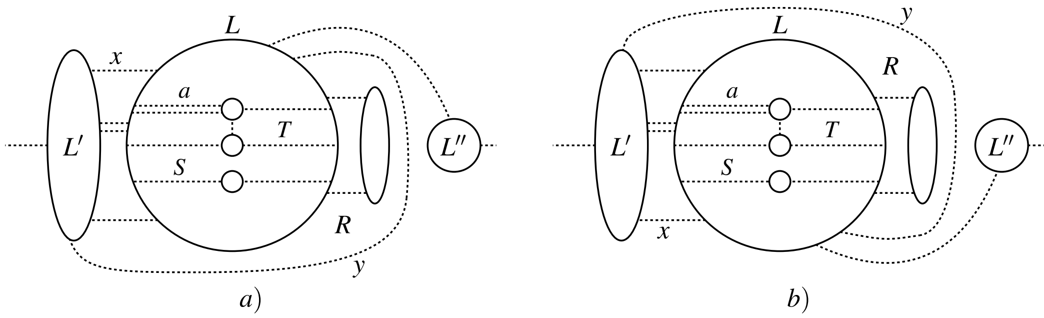
All legs are assumed non-empty, except a few in a). There, among the two vertical legs, exactly one of the two is supposed to be there (because of $\gamma=4$). Also, some outer cycle edges may be zero. (It is not implied here that edges going out to the left/right from L connect to the outer cycle loop on the same side of L. Also, it must be allowed that an outgoing trace stands in fact for several edges, whose traces have neighbored basepoints on L.)

Case 1.1. In case (105) a), we must have all of a,b,c,d non-zero to have 2 cycles in $IG(A)$. But then we can apply lemma 10.3.

Case 1.2. Now consider case b). Let L' resp. L'' be the neighbors of L in the outer cycle. We assume that the connection (L,L') is the multiple one, and (L,L'') is the simple one. In order to have two cycles in $IG(A)$, on both segments A and B of (105) b) there must be a trace connecting L to L'.

Case 1.2.1. If y consists of only one edge, one has first $y=2$ by inadequacy, and also $a=2$ the same way. Now it is easy to see that $B(D)$ has a single separating loop X, with at most 3 multiple connections inside.

Case 1.2.1.1. Assume first the leftmost (i.e. neighbored to a resp. f in (105) b)) traces outside L on the segments A and B are both part of the multiple outer cycle connection (L,L'). Then we have diagram a) or b) of (106), with the convention that all edges drawn are non-zero, except possibly y. All 3 multiple connections inside X in $B(D)$ are intertwined each with at most 2 connections outside X, and we can apply lemma 10.20.



(106)

Case 1.2.1.2. So one of the two leftmost traces outside L on segments A or B of (105) b) is the simple outer cycle connection (L,L''), as in diagram a) or b) of (106) with $x=0$ (and all other edges non-zero). In either case again a look at the B state shows that the unique separating loop X has at most 3 multiple connections inside, coming from regions S,T of $A(D)$, and one, W, dual to $a=2$. Each of S,T,W is intertwined with at most 2 connections outside X. However, the pair of intertwined outside connections is not the same for all of S,T,W: the connection of X to the loop (in $B(D)$) of the region R (in $A(D)$) intertwines one of S and W, but not the other. So again $IG(B)$ has no two cycles.

Case 1.2.2. However, if y consists of several edges, the situation is less easy. Then we have c) of (105). By assumption, a single trace from y connects L to the outer cycle loop L'' (different from L' on the right).

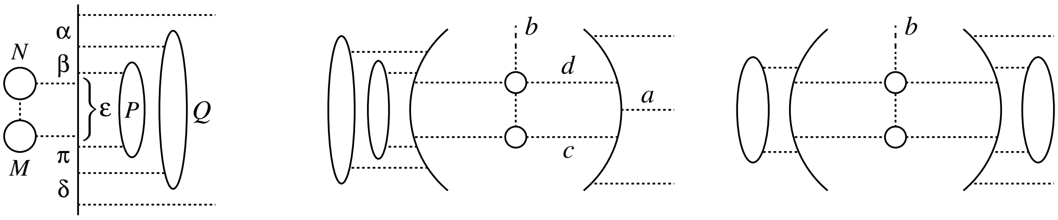
All edges drawn in (105) c) are assumed non-zero, except possibly l,m. It is easy to see that one of l,m must be 0 by lemma 10.20, but of course (by assumption) not both $l=m=0$. Also, (h,i) follow lemma 10.16, and $a=2$ by inadequacy (independently on whether $l=0$ or $l\neq 0$). By looking at $B(D)$, we can assume w.l.o.g. that $b=c=f=g=1$, otherwise $B(D)$ is of type B. Next $e=1$, because otherwise we would need 3 cycles in $IG(A(D))$, which we cannot create. Furthermore, all outer cycle edges, except possibly j,k (and l,m), have multiplicity one.

Calculation 10.5 This input was again processed by the computer. The result, after a few seconds, are the sequences of figure 10 for $i=4,10$.

Case 2. Assume next we have two attachments (28) outside L (and none inside). By lemma 10.19, we have two options.

Case 2.1. The two outside loops are parallel.

Case 2.1.1. The two outside loops are attached in a region between edges of the multiple outer cycle connection of L. See diagram (a) in (107).



$a)$ $b)$ $c)$ (107)

Clearly for two cycles in $IG(A)$, both M,N must be intertwined with P,Q, so they have legs on the segment ε of L. If none of M,N had a leg on one of α,β,π,δ, then using that some of the legs of P,Q must be multiple by connectivity reasons, we see that $B(D)$ is of type A. Let thus $\nu \in \{\alpha,\beta,\pi,\delta\}$ be this segment touched. (The following argument immediately clarifies that ν must be unique.) Now, since M,N must be intertwined also with the (only multiple) outer cycle connection of L, there must be a unique (because of $\gamma = 4$) third region $\mu \notin \{\alpha,\beta,\pi,\delta,\varepsilon\}$ outside L touched by a leg of M,N, and because of $\chi(IG) = -1$, it is touched by legs of both M,N. Then lemma 10.3 applies with $f = 2$.

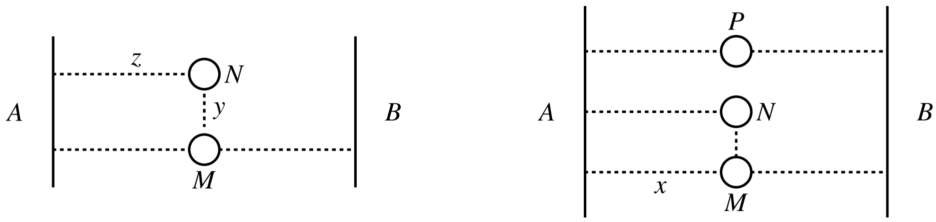
Case 2.1.2. The two outside loops are attached in a region between an edge of the multiple outer cycle connection of L and the edge of the simple connection. See diagram (b) in (107). Because of $\chi(IG) = -1$, all edges of M,N are there, except possibly b. By lemma 10.3, there must be an edge a at the indicated position. Then by $\gamma = 4$, we see $b = 0$. Now, by inadequacy we have $a = c = d = 2$. Then we can apply lemma 10.20 to exclude this case.

Case 2.2. The two outside loops are in different regions outside L. By $\gamma = 4$ and $\chi(IG(A)) = -1$ we see that we have the diagram c) in (107), and all 6 edges of the triangle attachment must be there. Then we are done by lemma 10.3. □

10.8.1. Case C1

We distinguish two subcases of case C, depending on whether the triangle loops have legs that (jointly) touch 2 or 3 regions out side L. We consider first the case C1 of 2 regions A,B.

We have then a triangle attachment of the type (82) with $f = 0$. By lemma 10.3, one of a,b,c,d in (82) must be 0. The triangle attachment looks then like diagram (a) in (108). This implies in particular that for a cycle in $IG(A)$ we need at least one more attachment inside L, so at most one attachment is outside L.



$a)$ $b)$ (108)

It will be useful to restrict first the valence-2 attachments (28), touching upon A,B.

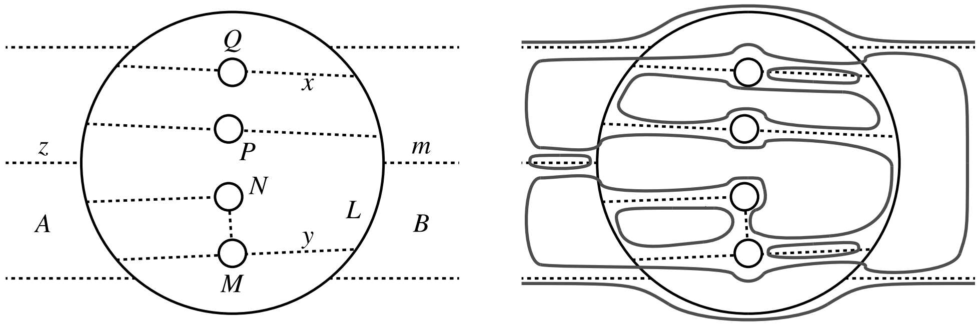
Lemma 10.31 In diagram (b) of (108), there is no parallel loop to P, and all legs of M,N,P are simple, except possibly x.

Proof. Otherwise $B(D)$ is of type A or B. □

Next we can eliminate parallel loops completely.

Lemma 10.32 There is no pair of parallel (valence-2) loops.

Proof. Assume there were two such loops P,Q. By the previous lemma, at least one of the two regions outside L they touch is different from A,B. We have, up to mutation, diagram a) in (109), with the option that $m=0$ if P,Q touch one of A,B, and $m\neq 0$ otherwise.



a) b) (109)

Note that, by assumption, and in order to have 2 cycles in $IG(A)$, all 5 other edges drawn outside L, different from m, must be there. Thus again the regions U,V of (51) remain untouched. Now by inadequacy, $y=2$, and by connectivity, w.l.o.g. up to mutation, $x\geq 2$.

If P,Q touch one of A,B, i.e. $m=0$, we see that again we can apply lemma 10.20. The diagram (b) in (109) shows the B-state loops, taking account of $x\geq 2$ and $z=y=2$, since z is also inadequate if $m=0$.

In case P,Q identify a pair of regions disjoint from $\{A,B\}$, we have $m\neq 0$. Then we see that $B(D)$ has a pair of separating loops, each with at most one multiple connection on one side. Thus $IG(B)$ cannot contain any cycle. □

Lemma 10.33 In diagram (a) in (108), we can assume that $y=z=1$.

Proof. With lemma 10.29, it remains to rule out the option $(y,z)=(1,2)$. (The case $y=z=2$ is out by connectivity.) For this refine the complexity of diagrams in case C, by saying that among type C states $A(D)$ and $B(D)$, one is simpler, if it has a triangle loop with only two edges connected to it, but the triangle in the other state has no such loop. With this set up, for $(y,z)=(1,2)$ in $A(D)$, go over to $B(D)$. □

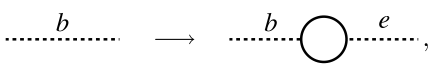
With all these preparations, we have now again, as in figure 4, a pattern that unifies all remaining possibilities for attachments, see figure 8.

The left part shows L and its attachments. There are now 3 triangles, so obviously our understanding is that exactly one of them is there. Also, we argued in the beginning of §10.8.1 that among the outside attachments D,E,F at most one is there. If one is, we have to choose one, otherwise two, among the 3 valence-2 attachments A,B,C and the valence-3 attachment H inside L.

The legs in a,b,c go to the outer cycle loops neighbored to L. This is shown on the right diagram. Again we can assume that traces going from L on the left resp. right connect L to L' resp. L''. Thus the traces that appear as b on the left may belong to two edges $b_{1,2}$, connecting L to L' and L''. The same remark applies to c and $c_{1,2}$. The option that the regions outside L into which D,E,F are attached are between edges from L to L' (say) only, is again recurred to the others by a flype.

We can assume again an outer cycle of length 5, with at most one multiple edge between empty loops (lemma 10.27).

The multiplicities of edges conform to lemmas 10.31, 10.29, and 10.33. Those of the outer cycle edges are as on the left of part (a) of figure 4 (page 51), except that we change, in the notation of that figure,



and set $b=e=1$, w.l.o.g. up to mutation. (Otherwise still $a,c,d\geq 1$ and $x,y,z\geq 0$, with the same further restrictions as in §9.7.)

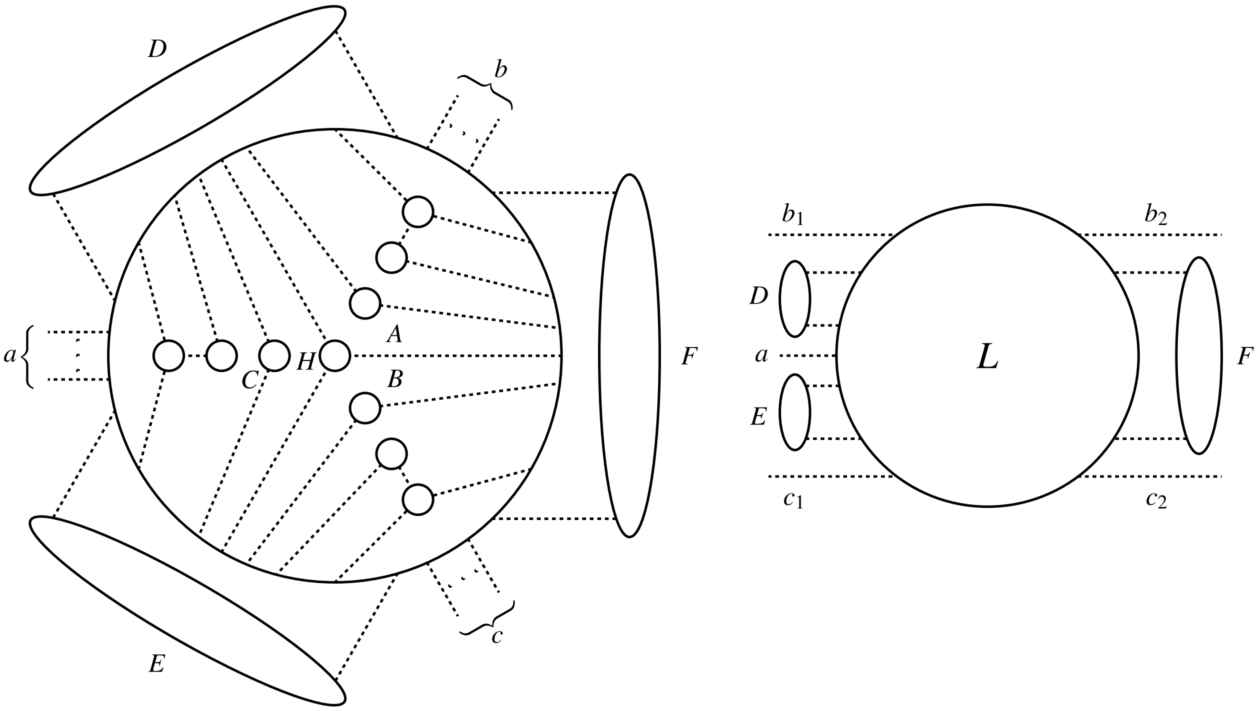


Figure 8: The pattern in case C1.

Calculation 10.6 Now we felt the diagrams are explicit enough to be checked by computer. We also implemented a direct calculation of $\chi(IG(A))$ during the successive attachments (rather than calculating it from the diagram), in order to save the verification of unnecessary cases. With this we could reduce the calculation to 3 minutes, and it ruled out all possibilities.

10.8.2. Case C2

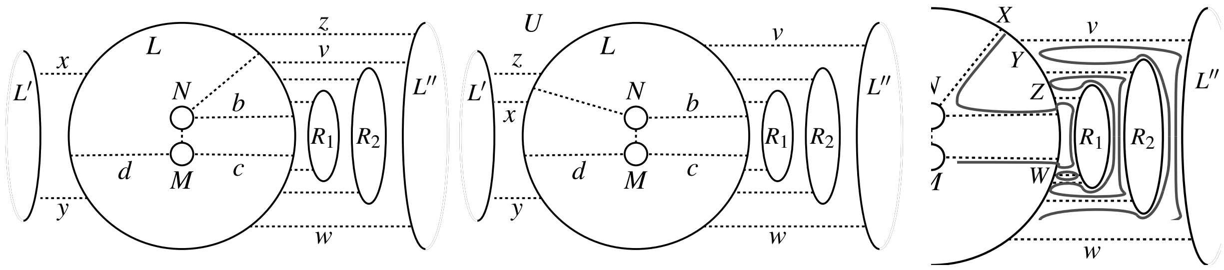
The final case, C2, is when the triangle inside L has legs touching on 3 regions outside L. The triangle is then of the type (82), with $f \neq 0$.

This case must again be dealt with by computer, but first we simplify it.

Lemma 10.34 There is at most one outside attachment to L.

Proof. Assume we had two outside attached loops $R_{1,2}$.

Case 1. The two outside attachments are parallel. The two triangle loops M,N are the only loops inside L, and $f=2$ by inadequacy. Then by lemma 10.3 with $f=2$, one of a,b,c,d must be 0. Still we need that both M,N are intertwined with at least 3 connections outside L. Up to symmetry assume $R_{1,2}$ are attached in region B of (82). This easily rules out the case that b, c or d is 0, so assume w.l.o.g. $a = 0$. We have then two situations, shown by a) and b) of (110):



a) b) c)

(110)

It is *a priori* to allow some of v,w,x,y,z to be 0.

Consider first case a) in (110). We must have $x+z > 0$ (otherwise $\gamma = 2$) and $y+w > 0$ (otherwise M intertwines only two connections) and $v > 0$ (otherwise N intertwines only two connections).

The multiplicities of the legs of $R_{1,2}$ are three times 1 and once 2. For this use lemma 10.29; four simple legs are excluded by connectivity, two times (2,1) because $B(D)$ becomes of type B. So assume w.l.o.g. up to mutation that the lower leg of R_1 has multiplicity two.

Now consider a fragment of the B-state, as shown in part (c) of (110). (The lower part of the loop Y is not drawn to remind that w may be not there.) It still shows that there exist two cycles in $B(D)$, given by XZY and XZW. Here we use that by $x+z > 0$ and $y+w > 0$, we have $X \neq Y$. Thus $B(D)$ is of type B.

Consider then case b) in (110). Then again $y+w > 0$ (else M intertwines only with 2) and now $z+v > 0$ (else N intertwines only with 2); also $x > 0$ (because else $\gamma = 2$), and so $x = 2$ by inadequacy. Then the same argument as for case a) applies. (The cycle in $B(D)$ formed by X,Z,Y may contain an extra loop from the region U in $A(D)$, if both z and v are non-zero.)

Case 2. The two outside attachments $R_{1,2}$ are not parallel. Then, by lemma 10.19, they do not mutually enclose.

We use diagram (82). Since the triangle is the only attachment inside L, by lemma 10.3 for $f = 2$, we must have some of a,b,c,d vanishing.

If now c or d is 0, or both of a,b are 0, then one of the triangle loops M,N in intertwined with at most one connection outside L. Then we cannot have 2 cycles in $IG(A)$. So w.l.o.g., $a = 0$, and then b,c,d are non-zero. Then c is inadequate, so $c = 2$ (and also $f = 2$).

Now the triangle loops M,N have each two legs, but must intertwine with at least 3 connections outside L. This easily implies that none of the regions A,B,C in (60) touched by legs of M,N can be any of the regions U,V in (51). For the same reason, at least one of $R_{1,2}$, say R_1, must be attached outside L in a region different from U,V, i.e. between edges x,x' that connect L to the same loop L' on the outer cycle.

Then we have at least 7 loops in $B(D)$: a separating loop X, 3 loops inside X (two of which from $c = f = 2$), and 3 outside X (those along U,V, and one formed by R_1, its legs, x, x' and arcs of L,L'). So with lemma 10.20 we are done. □

Lemma 10.35 There are no parallel loops.

Proof. Parallel loops outside L were ruled out in lemma 10.34. For parallel loops inside L, note that one could obtain (60) by deleting edges and valence-2 loops like ···○··· ⟶ ······. Then the argument in the proof of lemma 9.12 modifies to show that $B(D)$ is of type A or B. □

With this we can already specify a common superconfiguration of $A(D)$. Before we do so, though, let us make one final simplification.

Lemma 10.36 In the attachment (82), we have $e = 1$.

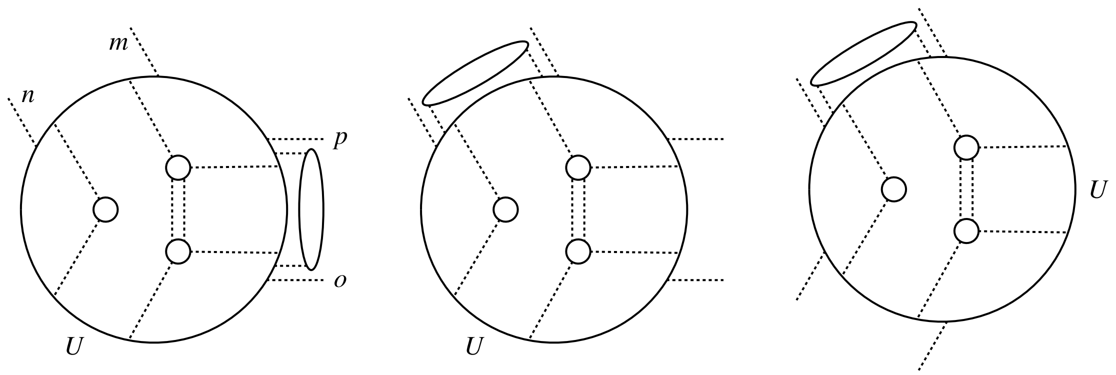
Proof. In (82), assume $e \geq 2$. Since this then gives an isolated vertex in $IG(A)$, we need at least 3 cycles therein. Moreover, by lemma 10.34, we know that we must do an attachment (28) inside L, call it P, which must be then of valence 2. It can be specified by choosing two of the three regions A,B,C touched by legs of the triangle loops M,N.

If all of a,b,c,d are non-zero in (82), we use lemma 10.3.

If now c or d is 0, or both of a,b are 0, then one of the triangle loops M,N in intertwined with at most one connection outside L. Then we cannot have 3 cycles in $IG(A)$. So w.l.o.g., $a=0$, and then b,c,d are non-zero.

When $a=0$, among the three choices for P, only the one identifying A and C avoids that $B(D)$ becomes of type A or B. This, together with lemma 10.35, shows in particular that P must be the only attachment (28) inside L, so that the other one is outside L.

Next, by lemma 10.20 we can assume that one of A,B,C coincides with one of the regions U,V in (51), say w.l.o.g U. (Again $B(D)$ has only one separating loop, X, we need at least 5 others to have 2 cycles in $IG(B)$, and there is an extra loop inside X, between the two edges of e, which does not intertwine with anything. Thus again we have at least 7 loops.) Then we have the following 3 choices:



(Here we can assume that o,p connect L to L'', and m,n connect L to L'.) In all cases, though, there is a loop inside L intertwined with at most one connection outside L, and so we cannot have 3 cycles in $IG(A(D))$. □

The promised common superconfiguration of all options we have for $A(D)$ is obtained now by replacing in figure 8 the left diagram by the one on the left of figure 9.

The reading of figure 9 is as follows. Let e,e',e'' resp. be the edges between H and A,B,C resp. Then we demand that exactly one of e,e',e'' is there, and then it is 1. It determines the orientation of the attachment in (82). For example, if $e=1$, then B,C and the traces touching upon them are not there.

With the choice between e,e',e'', which gives the edge e of (60), also the edges a,c,b,d,f are determined, and the following conditions apply:

- $f \neq 0$ (else case C1), but one of a,b,c,d is 0 (by lemma 10.3),
- $b+d>0$, $a+c>0$ (else case C1), and $c+d>0$ (else e is isolated and D composite),
- if $a=b=0$, then $f=1$; if one of c or d is zero, the other is 1 (otherwise the connection between L and one of M,N is an isolated vertex in $IG(A)$, and one cannot create 3 cycles therein with the other attachments), and
- not all of a,b,c,d,f are even (by connectivity).

Moreover, at most one of D,E,F is attached (lemma 10.34). If an outside attachment is there, we have 1, otherwise 2, of the three valence-2 loops inside L.

The same remark as in case C1 on the outer cycle edges and multiplicities apply.

Calculation 10.7 At this stage we applied a computer to check the pattern. For speed-up, again we kept track of $\chi(IG(A))$ during the successive attachments, rather than calculating it from the (completely) constructed diagram. With all theoretical input and optimization, the calculation time could be reduced to 18 minutes.

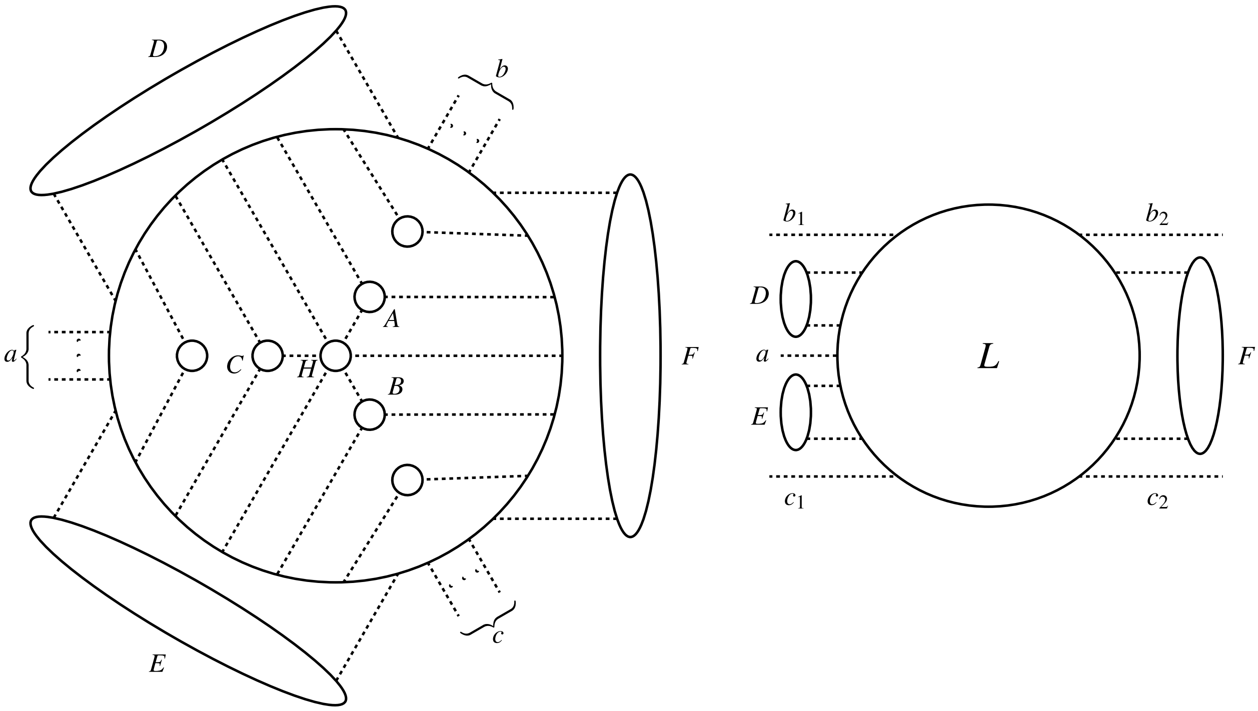


Figure 9: The pattern in case C2.

After adjusting the multiplicity of the twist edge to 3, so that (74) holds, we obtain a list of 56 diagrams $\hat{D}_0$, from which we construct 56 sequences of diagrams $\hat{D}_k$. However, it turns out that only 7 sequences are distinct up to flypes. (Again diagrams appear repeatedly, since it is too hard to remove all symmetries in advance.) These indeed then represent distinct sequences of knots, of which 3 consist of amphicheiral ones. The knots for $k=1$ are shown in figure 10 as series $\hat{D}^{(1)}$ to $\hat{D}^{(7)}$.

10.9. Concluding part: writhe and Vassiliev invariant test

To sum up the proof so far, after check of semiadequacy invariants, we are left with only 10 sequences $\{\hat{D}_k^{(i)}\}_{k\in\mathbb{N}}$ for $i=1,\dots,10$ of diagrams that could match a hypothetical $16+4k$-crossing diagram D_k of our amphicheiral knots K_k.

Figure 10 shows the diagrams $\hat{D}$ 1 of these sequences for $k=1$. (We omit the serial number superscript i when it is fixed.) The diagrams of higher k are obtained by adding a full twist (2 crossings) at each twist class of 3 or 4 crossings.

In order to distinguish D_k from $\hat{D}_k^{(i)}$, we use first the writhe. The diagrams in the series for $i=1,2,8,9$ all have non-zero writhe, and are excluded.

For the other 6 series, we use again Vassiliev invariants to distinguish $\hat{D}_k$ from our hypothetic diagrams D_k. Considering v_2 and v_3 of (8), we have 5 distinct sequences of value pairs, up to negating v_3 (which is the result of taking mirror

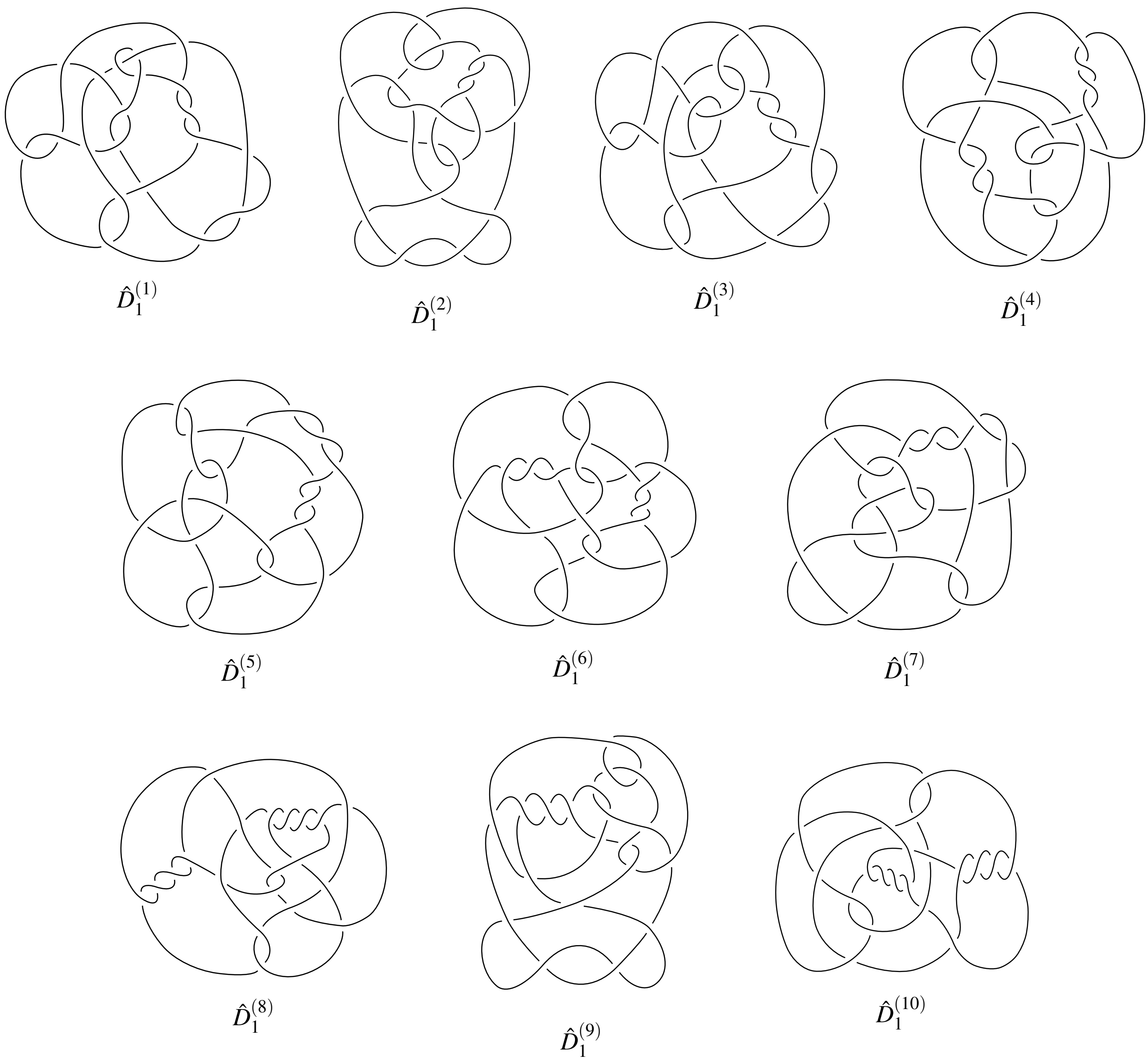


Figure 10: The diagrams $\hat{D}_k^{(i)}$ for $k = 1$. Here the parenthesized superscript refers to the number i of the series. The subscript stands for the number of the diagram in the series, so that $\hat{D}_k$ has $16 + 4k$ crossings.

images). The values for $k = 1, \dots, 4$ are given in the following table, in comparison to those for D_k in the last row:

k	1	2	3	4	1	2	3	4	
v_2	-4	-7	-12	-19	0	0	0	0	v_3
	-5	-7	-9	-11	0	0	0	0	
	-5	-9	-13	-17	1	2	3	4	
	-6	-13	-22	-33	0	0	0	0	
	-8	-15	-24	-35	0	0	0	0	
D_k	-7	-11	-15	-19	0	0	0	0	

(111)

By determining the braiding polynomial of v_3, one easily finds that for the sequence with $v_3(\hat{D}_k) \neq 0$ when $k \leq 4$, the same holds also if $k \geq 5$.

A similar check of the braiding polynomial of v_2 of the remaining 4 sequences $\hat{D}_k$ shows that the only coincidence with $v_2(D_k)$ occurs for $k = 4$ in the first sequence in (111), where $v_2 = -19$. In this case, one can use v_4 of [St9]. It can be again evaluated on D_4 and $\hat{D}_4$ either directly, or (since these diagrams have 32 crossings) from its braiding polynomial obtained by examining simpler diagrams for smaller k. We find that whenever $v_2(\hat{D}_4) = -19$, we have $v_4(\hat{D}_4) = 432$. On the other hand, $v_4(D_4) = 346$.

This completes the proof of theorem 10.1, and then also of theorem 1.1. $\square$

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