Concordance of positive knots

Alexander 스토이메노프

School of General Studies, GIST College,

Gwangju Institute of Science and Technology, Korea

Friday, August 23, 2014

Knots and Low Dimensional Manifolds
Satellite Conference of Seoul ICM 2014,

BEXCO Convention & Exhibition Center, Busan, Korea

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1. Intro to Korean

Notation: Koreans replace Chinese characters by their own letters.

本 본	松 송	川 천	藤 등	秋 추	志 지	小 소
村 촌	男 남	佐 좌	山 산	中 중	啓 계	尚 상
加 가	廣 광	生 생	杉 삼	葉 엽	堯 요	吉 길
谷 곡	夫 부	宏 굉	澤 택	明 명	平 평	春 춘
俊 준	邦 방	河 하	内 내	史 사	子 자	東 동
司 사	拓 척	丈 장	野 야	田 전	横 횡	

Exercise 1. What is 동횡 hotel (where I stay)?

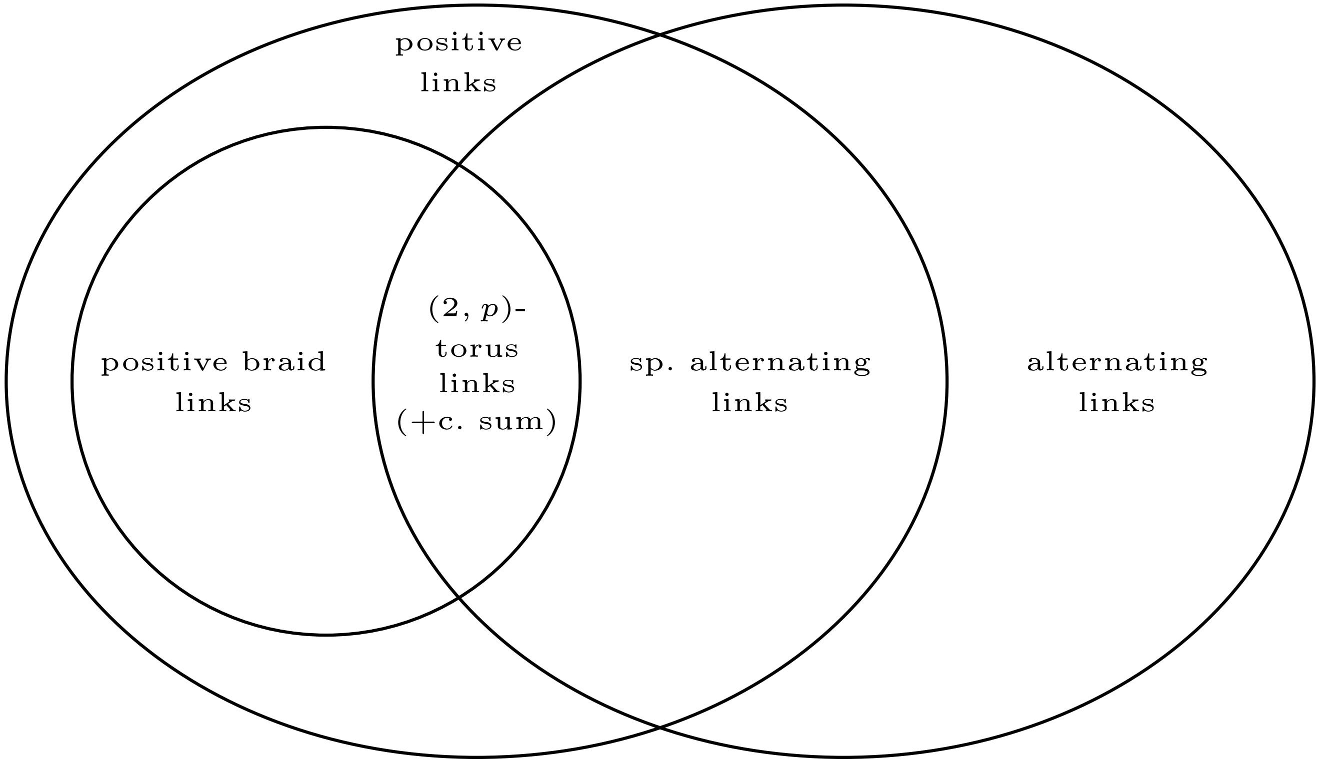
2. positive knots and links

oriented diagrams

positive negative

Definition 2. A diagram D is positive if all crossings are positive, a link L is positive if it has a positive diagram.

Occur in dynamical systems (Bir.-Williams), algebraic curves (Rudolph), and singularity theory (A'Campo, Boi.-Weber)



Remark 3. (S.,중촌)

$$\{sp.alternating\} = \{positive\} \cap \{alternating\}$$

(studied by 촌삼)

almost positive: not positive but 1 negative crossing.

The talk will center around the following conjecture.

Conjecture 4. (Positive concordance conjecture, PCC) Any concordance class of knots contains only finitely many (almost) positive ones.

(We usually talk about positive case.)

3. Types of concordance

What type of concordance?

$$algebraic \Longleftarrow topological \Longleftarrow smooth$$

- algebraic (Levine); invariants of the Seifert form

  Let M be a Seifert matrix, $\xi \in S^1$ (i.e., unit norm comlex number).

$$M_\xi(L) \;:=\; (1-\xi)M + (1-\bar{\xi})M^T\,.$$

  Hermitian, so diag'ble, and all eigenvalues real; let

$$\sigma_\xi(L) := \sigma(M_\xi) \qquad \nu_\xi(L) := \mathrm{null}(M_\xi)\,,$$

  T-L *signature* (sum of signs of eigenvalues) and *nullity* (dim ker).

  Let $g(K)$ be the *genus* of K, given by

$$g(K) \;=\; \min\{\, g(S) \;:\; S \text{ is a Seifert surface of } K \,\}\,,$$

  g_c for canonical (minimum of $g(D)$ genus of canonical sufrace of D), g_s for smooth $\subset B^4$, g_t for top. (locally flat) $\subset B^4$. Then

$$g_t \le g_s \le g \le g_c\,.$$

  Tri.-촌삼 inequality: ξ is a prime power root of unity, $n(L)$ number of components,

$$\big|\,\sigma_\xi(L)\,\big| + \nu_\xi(L) \;\le\; 2g_t(L) + n(L) - 1\,. \tag{1}$$

Consequence: for K *knot*, $\sigma_\bullet(K)$ is a concordance invariant *outside* the zeros of the Alexander polynomial Δ. In particular, true (as $\Delta_K(-1) \neq 0$) for *classical (촌삼) signature*:

$$\sigma(K) \,=\, \sigma_{-1}(K)\,.$$

- top. concordance: much (above alg. conc. invariants) turns around Freedman's result $\Delta_K = 1 \Rightarrow K$ is top. slice
- smooth: Bennequin-Rudolph machinery (+ Ozsvath-Szabo, Rasmussen &…) if D is a *knot* diagram (for simplicity) with l negative crossings,

$$g(K) \geq g(D) - l \quad \text{(Benn.)} \qquad \Longleftarrow \qquad g_s(K) \geq g(D) - l \quad \text{(later R.)}$$

consequence: $(l = 0)$ K positive $\Rightarrow$ $g_s(K) = g(K) = g_c(K)$
But *not* g_t!

algebraic	$\neq$	top.	$\neq$	smooth
	Cas.-Go.		Casson (using F.+Donaldson), later Rudolph	

Remark 5. (Cromwell) For a positive braid knot K, $g_s(K) = g(K) \geq c(K)/4 \Rightarrow$ PCC true smoothly for positive braid knots (similarly links)

top.? Let's look the simplest invariant!

4. Signature of positive knots and links

Consider $\sigma(K) = \sigma_{-1}(K)$. It satisfies for knots

$$\sigma(K) \,\leq\, 2g_t(K) \,\leq\, 2g(K)\,.$$

Theorem 6. *(Co-Gompf) K positive* $\Rightarrow \sigma(K) > 0$

In particular, K is not slice: this is necessary for PCC.

For sp. alternating: follows from M.'s result:

$$\sigma(K) = 2g(K)\,. \tag{2}$$

For positive braid: also proved by Rudolph (previously) and Traczyk (independently; removing error in his general case proof).

Theorem 7. *(Prz.-곡산+α)* $\sigma = 2 \iff g = 1$. *(i.e., $g \geq 2 \Rightarrow \sigma \geq 4$)*

Next case: $\sigma = 4$?

Let first

$$\mathcal{P}_{g,n} := \{\, K \,:\, K \text{ positive}, \, g(K) = g, \, c(K) = n \,\}\,.$$

It is known that

$$\#\mathcal{P}_{g,n} \sim_n n^{6g-4}\,. \tag{3}$$

Remark 8. Here really crossing number $c(K)$ of the knot is meant (and not crossing number $c(D)$ of a positive diagram D). For $g \geq 3$ $\exists$ positive knots with *no* positive minimal (crossing) diagram.
Contrast K.-M.-T.: *all* (reduced) *alternating* diagrams have minimal crossing number!
But (S., using Th.+횡 전): for D positive diagram of L

$$c(L) \geq c(D) + \chi(D)$$

(χ Euler char., $= 1 - 2g$ for knots)

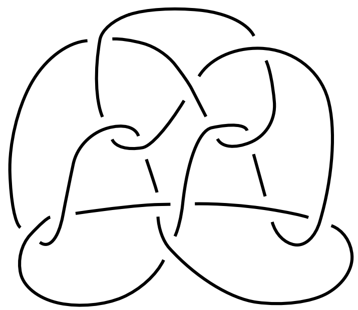
Theorem 9. *(S.) The positive knots of $\sigma = 4$ are:*

1) *all genus 2 knots,*

2) *an infinite family of genus 3 knots, which is scarce, in the sense for $n \to \infty$*

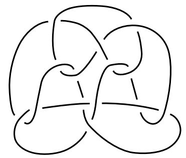
$$\frac{\#\{\, K \,:\, K \text{ positive},\ g(K) = 3,\ \sigma(K) = 4,\ c(K) = n \,\}}{\#\,\mathcal{P}_{3,n}} = O\left(\frac{1}{n^{10}}\right), \qquad (4)$$

(with $\#\,\mathcal{P}_{3,n} \sim n^{14}$) and

3) *the knot 14_{45657} of genus 4:* 

This suggests in general: for given σ, finitely many g. Or:

Conjecture 10. (Positive signature conjecture, PSC) K positive $\Rightarrow \sigma(K) \geq f(g(K))$, f increasing

σ \ g	0	1	2	3	4	5	6	7
0	$\bigcirc$	$\varnothing$(CG& Co)						
2		all $g = 1$	$\varnothing$(P-T+α)					
4			all $g = 2$	$O(n^{-10})$ of $\{g = 3\}$		$\varnothing$(S)		
6				‘most’ $g = 3$ $\supset$ sp.al.	‘few’ $g = 4$	???		
8					$\supset$ sp.al. $g = 4$	???		
10						$\supset$ sp.al. $g = 5$	???	

Further evidence:

- Clearly true for sp. alternating by (2)
- Proved for pos. braid (S.) $\Rightarrow$ PCC in alg./top. category (also links)
- (S.+S.-Vdovina) For fixed genus: $\{sp.alternating\} \subset \{positive\}$ are asymptotically dense:

$$\lim_{n\to\infty} \frac{\#\{\,K\;:\;K \text{ sp. alternating, } g(K)=g,\, c(K)=n\;\}}{\#\,\mathcal{P}_{g,n}} = 1\,,$$

consequence: *average* σ for fixed genus is asymptotically maximal:

$$\lim_{n\to\infty} \frac{1}{\#\mathcal{P}_{g,n}} \sum_{K\in\mathcal{P}_{g,n}} \sigma(K) \;=\; 2g\,.$$

5. Main results

Return to PCC. We saw it's true (in all cat.) for positive braid knots.

Theorem 11. *PCC is true (in all cat.) for sp. alternating knots. I.e., only finitely many sp. alternating knots are concordant.*

More precisely: ... have the same T-L signature *jump* function:

$$j_\xi(L) := \lim_{\epsilon \searrow 0} \sigma_{\xi e^{i\epsilon}}(L) - \lim_{\epsilon \nearrow 0} \sigma_{\xi e^{i\epsilon}}(L)\,. \tag{5}$$

In smooth cat. more:

Theorem 12. *Any sp. alternating knot is smoothly concordant to only finitely many* positive *knots.*

I.e., $\{K_i\}$ infinite sequence of smoothly concordant positive knots $\Rightarrow$ no K_i is (special) alternating.

(Both results hold for links also, and the first has extensions to alm. positive.)

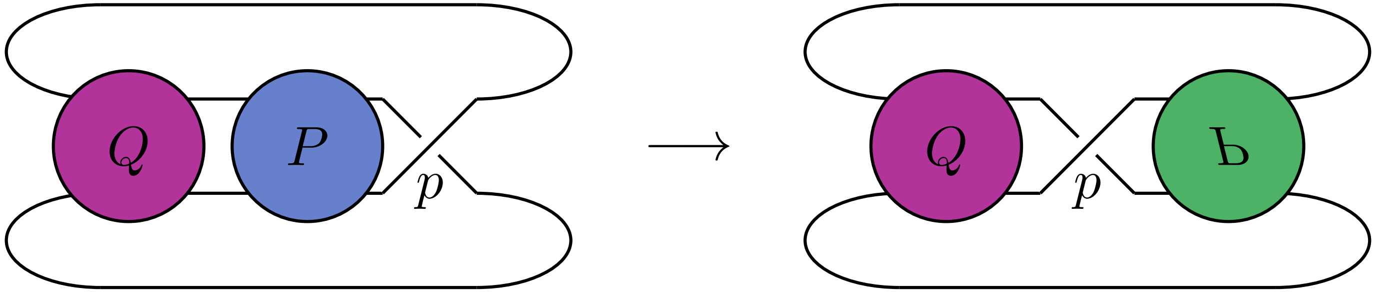
Ingredients of proof:

- T-L signature jump

- generator theory for given (canonical) genus (details below)
- signature and zeros of the Alexander polynomial on S^1 (details below)

6. Crossing equivalence, generators

A *flype* is the move



Definition 13. A $\bar{t}'_2$ *move* is a move creating a pair of crossings reverse twist ($\sim$-)equivalent to a given one:

Alternating diagram *generating*: $\iff$ irreducible under flypes and reverse of $\bar{t}_2'$ moves. For such diagram D,

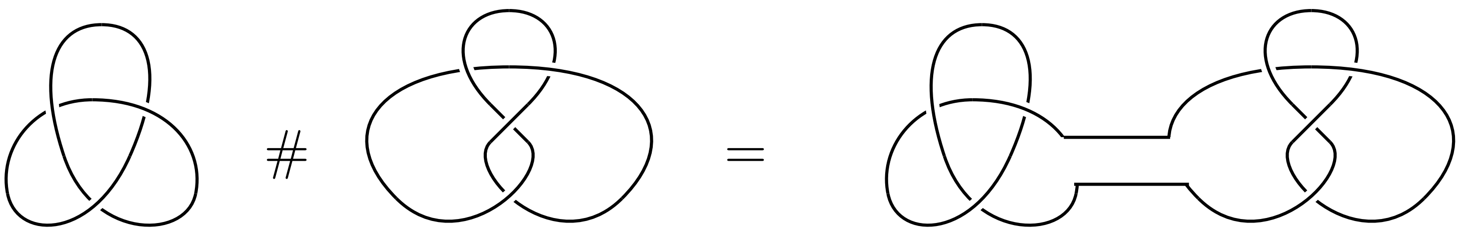
$$\textit{(generating) series } \text{of } D := \left\{ \begin{array}{c} \text{diagrams obtained by flypes} \\ \text{and } \bar{t}_2' \text{ moves on } D \end{array} \right\}.$$

generator:= alternating knot whose alternating diagrams are generating.

Theorem 14. *(S; Brittenham) The number of generators of given genus is finite.*

More precisely (S.): they have $\le 6g-3$ $\sim$-equivalence classes. (For links -3χ, except $\chi = 0$, where the Hopf link is the only generator.) This has very much to do with the exponent in (3)!

Let us discard generators which are *composite knots*:



$g = 1$ gen. $\qquad$ $g = 1$ gen. $\qquad$ $g = 2$ gen.

(Similarly, discard composite and split links.)

Thus we consider only prime generators. We calculated (for *knots*):

genus	1	2	3	4	5
# prime generators	2	24	4,017	3,414,819	???

S.-Vdovina: (Exponential) growth rate is ≥ 400!

7. Signature and zeros of the Alexander polynomial

For a link L, let Δ_L be the 1-variable Alexander polynomial

$$\Delta_L \in t^{(n(L)-1)/2}\mathbb{Z}[t^{\pm 1}]$$

(balance degrees: $\min\deg\Delta = -\max\deg\Delta$).

Definition 15. Count zeros of a Laurent polynomial X over some complex domain S with multiplicity:

$$\zeta(X, S) := \sum_{\substack{\xi \in S \setminus \{0\} \\ X(\xi) = 0}} \operatorname{mult}_\xi(X)\,.$$

Observe that

$$\zeta(X, \mathbb{C}) = \operatorname{span} X \;(:= \max\deg X - \min\deg X\,)\,. \tag{6}$$

Moreover, there is the complex-analytic integral formula

$$2\pi i \cdot \zeta(X, S) = \oint_{\partial S} \frac{X'(z)}{X(z)}\, dz\,, \tag{7}$$

valid when $S \not\ni 0$, at least piecewise smooth boundary ∂S (oriented counterclockwise) $\not\ni$ roots of X.

Theorem 16. *For a link L with $\Delta_L \neq 0$,*

$$\big|\,\sigma(L)\,\big| \leq \zeta(\Delta_L, S^1)\,. \tag{8}$$

Remark 17. Clearly with any zero $z \in S^1$ of Δ, the conjugate $\bar{z}$ is also one. Moreover, for a *knot* K, there is no overlap because of

$$\Delta(\pm 1) \neq 0\,. \tag{9}$$

Thus

$$\zeta(\Delta_K, S^1_+) \;=\; \frac{1}{2}\,\zeta(\Delta_K, S^1)\,, \tag{10}$$

for $S^1_+ \;:=\; S^1 \cap \{\,\Im m > 0\,\}$.

Many accounts on this theorem have been *a mess*.
Several special cases (i.p., knots) follow from old results (e.g., 송본's identification of Milnor's signature μ_θ).
But for links the failure of (9), among others, made things tricky, and the full story was completed only by an arg. given very recently Gilmer-Livingston (to appear).

However, it is 'easy' when L is regular (and this is enough here).

Definition 18. Define a knot or link L to be *regular* if

$$1 - \chi(L) \;=\; 2\max\deg\Delta(L)\,(\,=\;\operatorname{span}\Delta_L\,)\,.$$

Remark 19. L is regular $\iff$ L has a regular ($\det \neq 0$) Seifert matrix.

Theorem 20. *(Cromwell, easy using 촌삼+M.-Prz.) K positive, (S., more work) K alm. positive $\Rightarrow$ K regular*

In the following graphic we use one-sided signature jumps as follows:

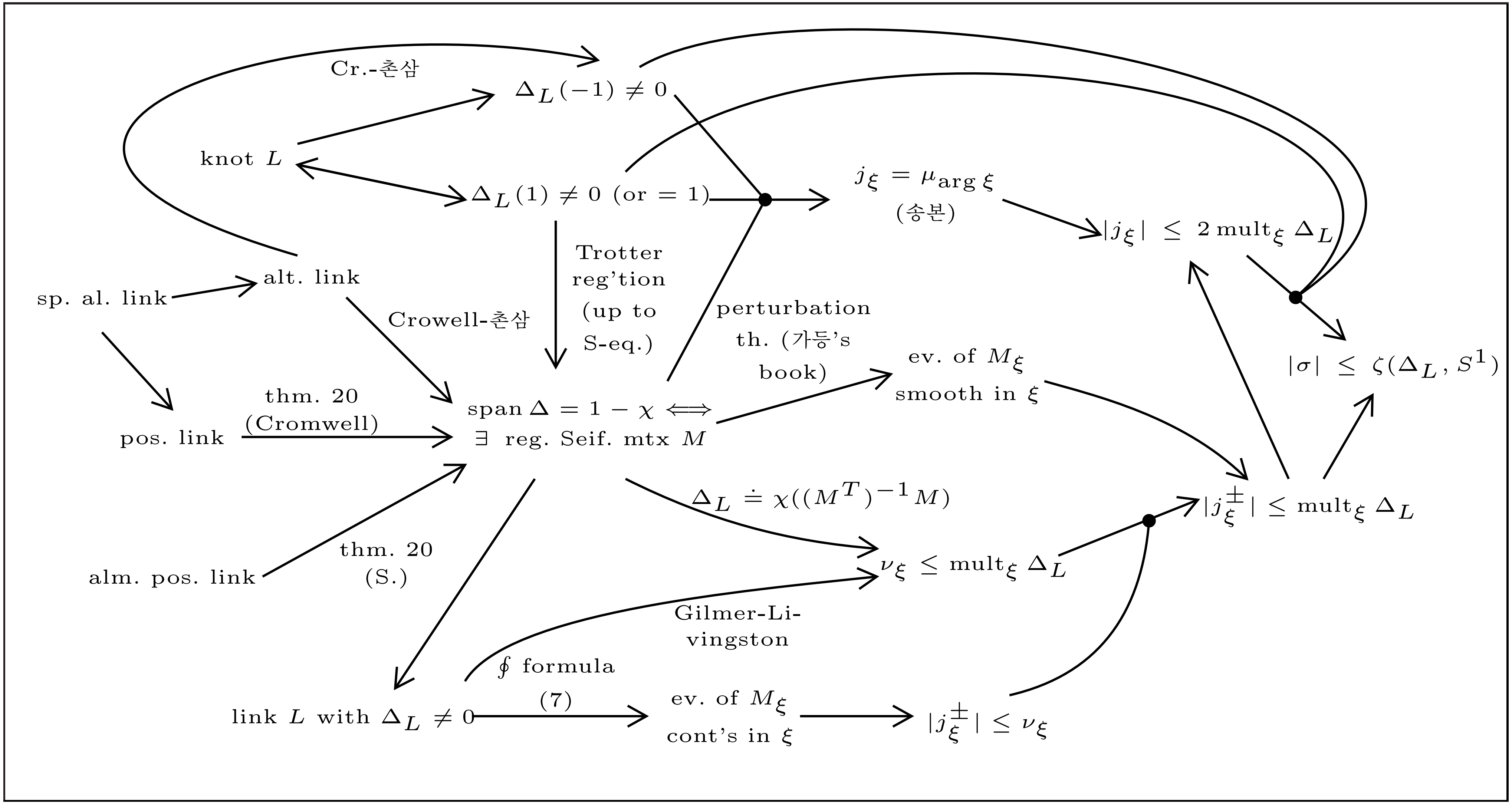
$$j_\xi^+(L) := \lim_{\epsilon \searrow 0} \sigma_{\xi e^{i\epsilon}}(L) - \sigma_\xi(L)\,, \quad \text{and} \quad j_\xi^-(L) := \lim_{\epsilon \nearrow 0} \sigma_{\xi e^{i\epsilon}}(L) - \sigma_\xi(L)\,.$$

Then $j_\xi(L) = j_\xi^+(L) - j_\xi^-(L)$.

We emphasize one special case of $\left| j_\xi \right| \leq 2 \operatorname{mult}_\xi \Delta$:

$$\left| j_{\xi_0}(K) \right| = 2 \quad \text{when} \quad \operatorname{mult}_{\xi_0} \Delta_K = 1\,. \tag{11}$$

This is essentially a consequence of the Implicit Function Theorem applied on $f(\xi, \alpha) = \det(M_\xi - \alpha \cdot Id)$.



Cr.-촌삼
$\Delta_L(-1) \neq 0$
knot L
$\Delta_L(1) \neq 0$ (or $= 1$)
$j_\xi = \mu_{\arg \xi}$
(송본)
$|j_\xi| \leq 2\,\mathrm{mult}_\xi\,\Delta_L$
Trotter
reg'tion
(up to
S-eq.)
alt. link
sp. al. link
Crowell-촌삼
perturbation
th. (가등's
book)
$|\sigma| \leq \zeta(\Delta_L, S^1)$
ev. of M_ξ
smooth in ξ
thm. 20
(Cromwell)
pos. link
span $\Delta = 1 - \chi \iff$
$\exists$ reg. Seif. mtx M
$|j^\pm_\xi| \leq \mathrm{mult}_\xi\,\Delta_L$
$\Delta_L \doteq \chi((M^T)^{-1}M)$
thm. 20
(S.)
alm. pos. link
$\nu_\xi \leq \mathrm{mult}_\xi\,\Delta_L$
Gilmer-Li-
vingston
$\oint$ formula
(7)
link L with $\Delta_L \neq 0$
ev. of M_ξ
cont's in ξ
$|j^\pm_\xi| \leq \nu_\xi$

8. Outline of proof

For theorem 11:

$$
\begin{array}{llll}
\{K_i\}_{i=1}^{\infty} \text{ conc.} & \Rightarrow & \sigma(K_i) \text{ same} & \\
& \Rightarrow & g(K_i) \text{ same} & (\sigma = 2g) \\
& \Rightarrow & \text{w.l.o.g. } D_i = D(p_{1,i}, \dots, p_{n,i}) & (\text{theorem } 14) \\
& \Rightarrow & \sigma_\xi(D_{p_1}) \text{ constant} & (\text{by skein monotonicty}) \\
& & \text{Now all zeros of } \Delta_{p_1} \text{ are on } S^1 & (8) \\
& & \textit{simple} \text{ zeros are detected by } j_\bullet & (11) \\
& \Rightarrow & \exists \text{ linear pregression of Alexander} & \\
& & \text{polynomials with fixed simple factors} & \\
& \Rightarrow & \Delta(k),\ k \in \mathbb{N}_+, \text{ linear progression} & \\
& & \text{with finitely many simple primes} & \\
& \Rightarrow & \text{contradiction} & (\text{Dirichlet}) \qquad \square
\end{array}
$$

For theorem 12:
If $\{K_i\}$ smoothly concordant $\Rightarrow g_s(K_i) = g(K_i)$ same and $\sigma(K_i)$ same. If some K_i is sp. alternating, $\sigma(K_i) = 2g(K_i)$, etc.

9. Computations and problems

Using the outlined strategy one can extend theorems 11 and 12:

Proposition 21. *If $\{K_i\}$ infinite $*$ conc. class of positive knots, then*

- *Not all K_i have $\sigma \geq 2g - 2$ (if $* =$ alg./top.).*
- *No K_i has $\sigma \geq 2g - 2$ (if $* =$ smooth).*

Applying theorem 9 (& etc.)

Corollary 22. *Both claims are true for $g \leq 4$.*

Making computer checks in Hoste-Th.'s table, can settle smooth case there (analogue of statement for alg./top. trivial on a finite set):

Corollary 23. *If $\{K_i\}$ infinite smooth conc. class of positive knots, all (incl. composite!) K_i have (genus at least 5 and) at least 17 crossings.*

Example 24. (??) *Possible* instance of failure (i.e., *may* $\in$ infinite positive *smooth* conc. class): $3_1\#K$ for a positive knot K with $\mathrm{mult}_{\Delta(3_1)}\Delta(K)=1$ and $j_{e^{\pi i/3}}(K)=-2$. But so far I cannot find this K (or any other example).

Using 송본 (1977), Kearton (1979): although for torus knots often $\sigma \ll 2g$,

Corollary 25. *No connected sum of (positive) torus knots smoothly concordant to ∞ly many positive knots.*

Problems (how to extend proofs?)

- If PSC is true, can use series and $\bar{t}_2'$ twisting also for top.
- Other upper bound on $g(K)$ of a positive knot in a topological concordance class?
- How to manage zeros of Δ off S^1 in terms of concordance invariants?
- COMPUTABLE concordance invariants on series?

Thank you!

Alexander 스토이메노프

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