Everywhere equivalent and everywhere different knot diagrams

Alexander Stoimenow

Department of Mathematics, Keimyung University, Daegu, Korea

계명대학교 자연과학대학 수학과

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Workshop on Knots and Spatial graphs

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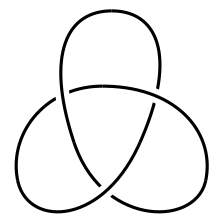
Contents

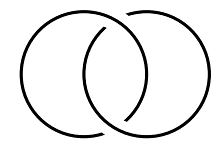
Everywhere different knot diagrams

knot K $\qquad S^1 \hookrightarrow S^3$

link L $\qquad \underbrace{S^1 \cup \ldots \cup S^1}_{n \text{ components}} \hookrightarrow S^3$



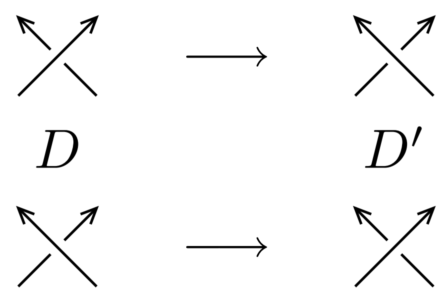
K



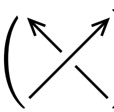
L

(knots/links and their diagrams usually oriented)

crossing switch



(1)

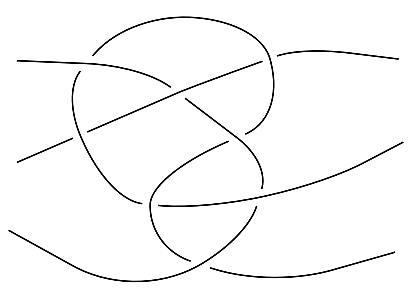
A diagram is *positive* if all crossings are positive ().

Definition 1. *(Askitas-S., Taniyama)*

D everywhere (1-)trivial	$:\Longleftrightarrow$	*all D' represent the unknot*
D everywhere equivalent (EE)	$:\Longleftrightarrow$	*all D' represent the same knot (or link)*
D everywhere different	$:\Longleftrightarrow$	*all D' represent different knots (or links)*

For a given diagram D it is (generally) easy to check that (if) it is everywhere different.

Question 2. (Taniyama; independently Ishii for alternating diagrams) Do infinitely many everywhere different diagrams exist?



T

alternating diagram D_n of $8+2n$ crossings: tangle T on left + braid tangle $(\sigma_1\sigma_2^{-1})^n$ + close up.

Theorem 3. *For almost all $n = 3k+1$, the diagram D_n is everywhere different.*

This example was chosen for a short proof: semiadequacy formulas for Jones polynomial + Menasco-Thistlethwaite

We consider an example studied by Shinjo and Taniyama.

D_n = compose T with n copies of T' and close up. Shinjo and Taniyama had verified that D_1 is everywhere different.

Theorem 4. *For almost all n, the diagram D_n is everywhere different.*

Proof based on the Temperley-Lieb category. Choose a value of the Kauffman bracket + diagonalization and eigenvalue estimates.

Works also for a *non-alternating* version of D_n.

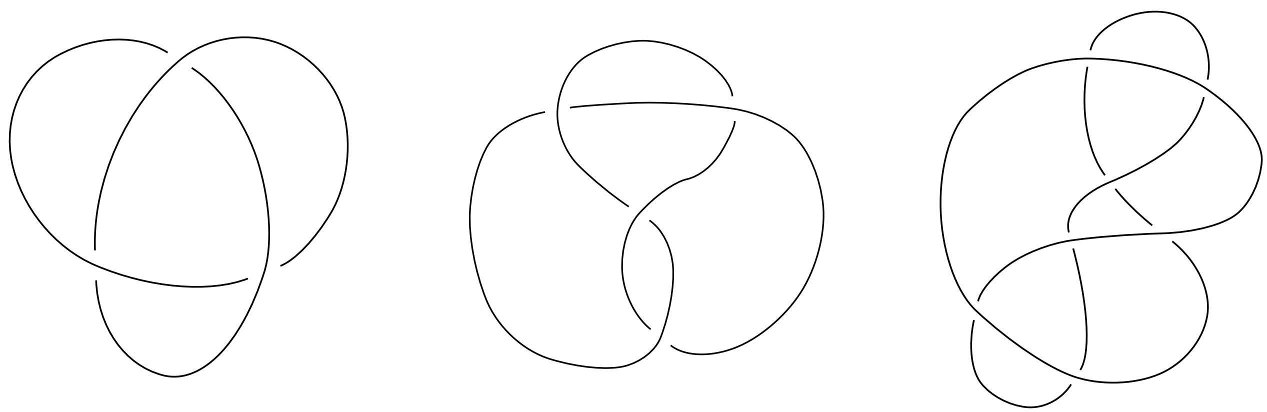
Everywhere trivial knot diagrams

Important special case of everywhere equivalent (EE) knot diagrams.

D *everywhere trivial* $:\Longleftrightarrow$ all D' represent the unknot

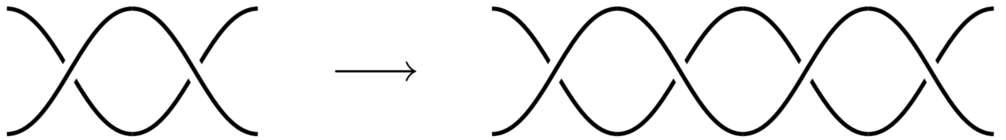
studied by Askitas-S. '03 (called "everywhere 1-trivial")

Example 5. Some simple everywhere trivial diagrams.



Question 6. (A-S) Can one describe everywhere trivial diagrams?

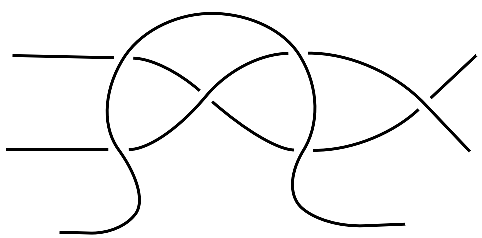
There are *many* everywhere trivial unknot diagrams! One can produce more by adding trivial clasps beside a given one:



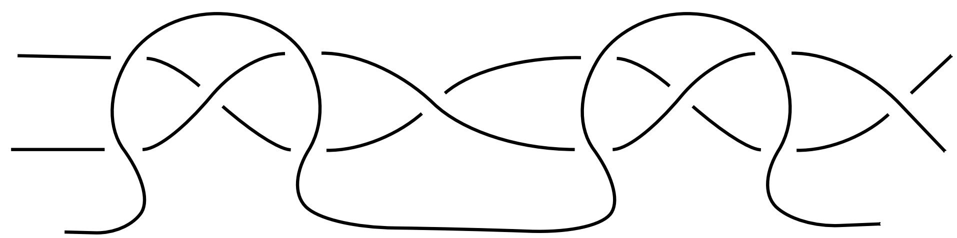
But it goes without trivial clasps:

Proposition 7. *For every crossing number* ≥ 11 *there are prime everywhere trivial unknot diagrams without a trivial clasp.*

Proof. (Uses an idea of Shinjo and Taniyama) Apply $T \to T^n$ (and modifications)

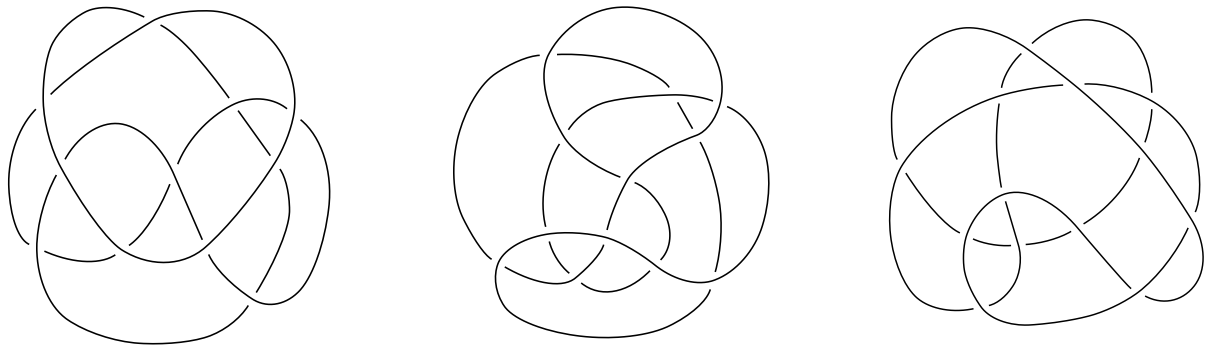


T



T^2

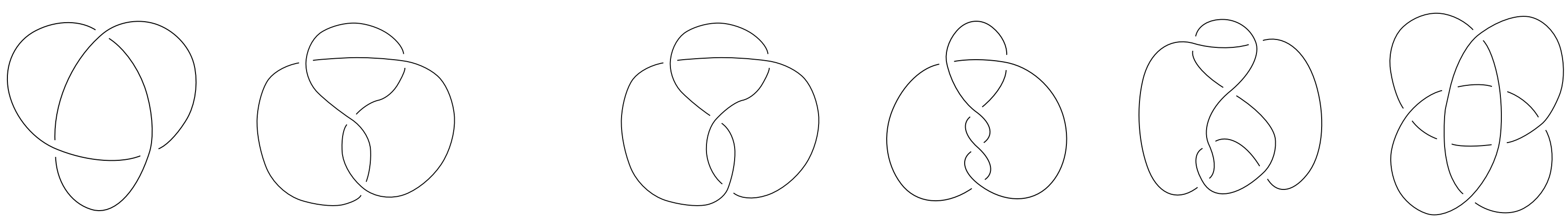
on suitably chosen (and computationally found) 11 to 16 crossing diagrams, e.g.,

 . ☐

Thus the part of question 6 for unknotted D is likely too complicated.

<u>How about D knotted?</u>

A.-S. found six (two trefoil and four figure-8-knot) diagrams:

 . (2)

trefoil figure-8-knot

Question 8. (A.-S.) Are these all?

Verification (part of more general results discussed later):

- up to 14 crossings (A.-S. '03), later 18 crossings (S. '11)
- for rational and Montesinos diagrams follows from the classification of rational and Montesinos knots (not done in every detail, but not too interesting)
- diagrams of genus ≤ 3 (using generator approach)
- 3-braid diagrams

Everywhere equivalent knot diagrams

D *everywhere equivalent (EE)* $:\iff$ all D' represent the same knot

Question 9. (Taniyama) How do EE diagrams look like?

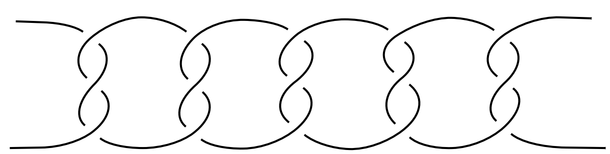
It is helpful to distinguish:

$$D \text{ strongly everywhere equivalent (SEE)} \quad :\Longleftrightarrow$$
$$D \text{ is EE and } D' \text{ represents the same knot as } D$$
$$D \text{ weakly everywhere equivalent (WEE)} \quad :\Longleftrightarrow$$
$$D \text{ is EE and } D' \text{ represents a different knot from } D$$

We (suggestively) focus here on the case that D' is knotted. Let us also assume D is *prime*.

Some general constructions:

pretzel tangle diagram $P(p,q) = (\underbrace{p,p,\dots,p}_{q \text{ times}})$.

$P(3,5) \; =$  (3)

Proposition 10. *EE* knot *diagrams:*

1. *The pretzel knot diagram $\hat{P}(p,q)$ with $p \geq 1, q \geq 3$ both odd (obtained from $P(p,q)$ as in* (3) *by closing the two top and two bottom ends).*

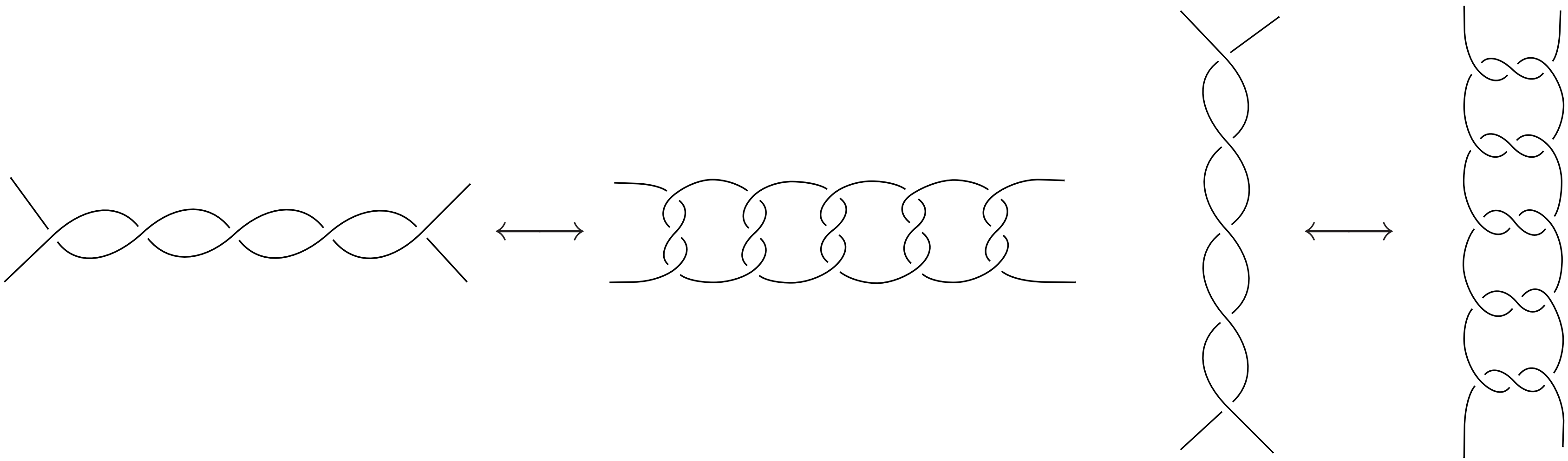
2. *In the following $k \geq 2$.*

   2.a. *The diagram of the closed 3-braid $(\sigma_1^l \sigma_2^l)^k$ (l odd, $3 \nmid k$), and*

   2.b. *diagram of closed braid $(\sigma_1 \sigma_2)^k$, in which each crossing replaced (disregarding braid orientation) by l positive half-twists in direction* not *coinciding with the one of the braid ($l \geq 1$, and $3 \nmid k$ for l odd).*

3. *The arborescent diagram $(\underbrace{P(3,p),\dots,P(3,p)}_{q\ times})$ for $p,q \geq 3$ odd.*

4. *A diagram obtained from those in type 2 by replacing (respecting direction of twists; see* (4) *below) each twist of l crossings by $P(3,l)$ for $l \geq 2$.*



$$(4)$$

Remark 11. All these diagrams are positive ($\Longrightarrow$ only WEE).

Question 12.

- Is the construction (for D' knotted + (2) for D' unknotted) exhaustive for prime WEE diagrams?
- D is SEE $\Longrightarrow D$ (and D') unknotted?
- (consequence of previous two + Remark 11) D' knotted $\Longrightarrow D$ positive?

Theorem 13. *All is true for*

- *diagrams up to 18 crossings,*
- *diagrams up to genus 3,*
- *genus 4 diagrams which are (at least) one of ≤ 25 crossings, positive, SEE, or alternating.*

Remark 14. Also true for

- rational and Montesinos diagrams (with minor '?'; as explained)
- 3-braid diagrams (later)

Proof. Use generator description. Parametrize a diagram in the series of $\hat{D}$ with n $\sim$-equivalence classes by a twist vector $\mathbf{v} \in \mathbb{Z}^n$.

Test Vassiliev invariants v_i on $\mathbf{v}$. The degree-2 invariant gives an affine lattice in $\mathbb{Z}^n$ (which is empty for many generators). Then test higher degree invariants until you are left with what you need. □

Observation 15. Proposition 10 yields diagrams of crossing numbers $\neq 2 \cdot 3^l$.

Question 16. Are there any prime EE *knotted* diagrams of $2 \cdot 3^l$ crossings?

One of 6 crossings is in (2), but indeed there is none for 18 (not at all obvious!). How about 54?

Constructions of everywhere equivalent link diagrams

Here component orientation is important, thus:

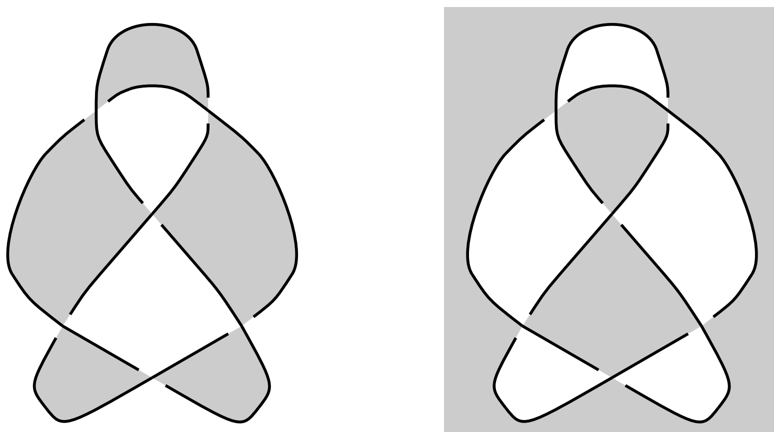
Definition 17. *D link diagram*

D unorientedly everywhere equivalent $:\Longleftrightarrow$
all D' represent the same unoriented link

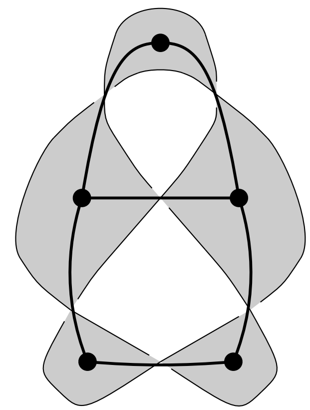
D orientedly everywhere equivalent $:\Longleftrightarrow$
all D' represent the same oriented link
(may allow reversing simultaneously orientation of all components)

First consider *unoriented* EE: an idea how to create such diagrams comes via the checkerboard graph.

unoriented link diagram $D \longrightarrow$ *checkerboard graph* $G = G(D)$ (up to duality)



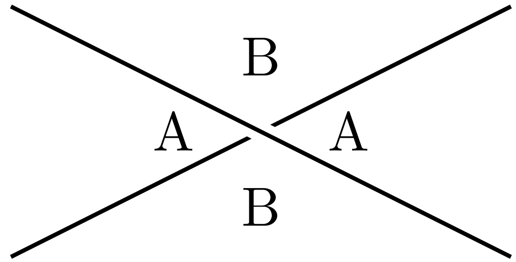
two checkerboard colorings



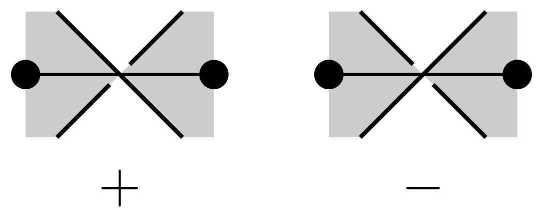
the checkerboard graph
of the first coloring

Graph is *signed* (for non-alternating diagrams).

Kauffman sign: crossing c of D is *Kauffman positive* (resp. *Kauffman negative*) if the A-corners (resp. B-corners) of c



lie (say; it's convention) in black region of checkerboard coloring.



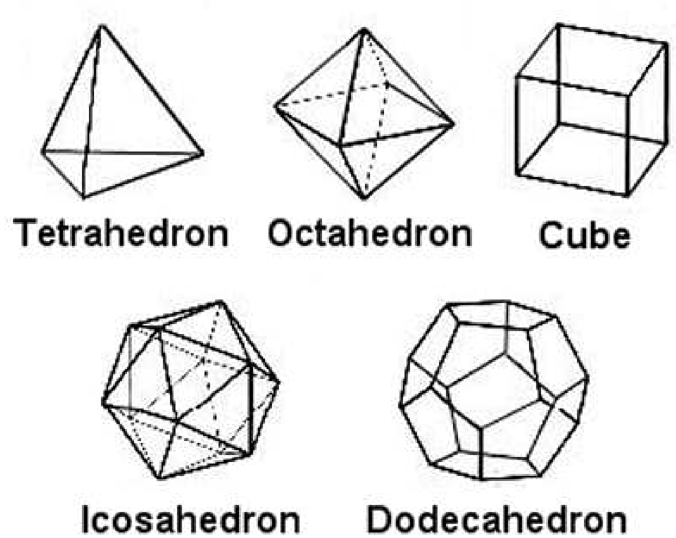
Kauffman signs are unoriented and *different* from skein signs in (1).

Definition 18. *A graph is* edge transitive *if for every two edges* e, e' *there is a symmetry mapping* e *to* e'*.*

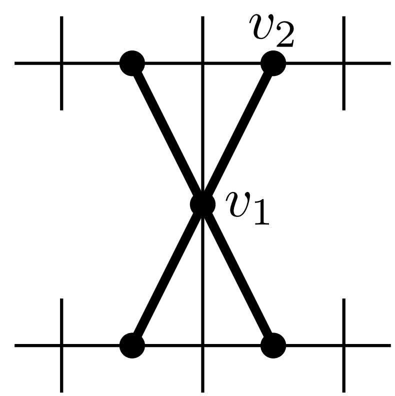
Studied in combinatorics for some time.

For example, it is well-known that there are only nine finite edge transitive *tesselations* (3-connected and dually 3-connected):

- nets (1-skeletons) of the 5 Platonic solids



- cuboctahedron, *median graph* of the cube net,



.

and icosidodecahedron (of the dodecahedron net)

- the planar duals of the latter two.

The other (non-tesselation) cases are also known (Fleischner-Imrich '79).

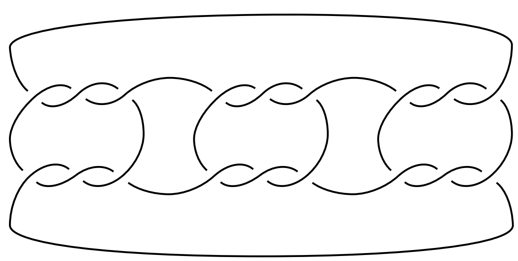
edge transitive checkerboard graph $\longrightarrow$ EE diagram

Construction 19. *G cut-free edge transitive graph, $p = 1, 3$, $q \geq 1$. Build alternating diagram $D_i(G; p, q)$ by replacing each edge e of G by $P(p, q)$ either along ($i = 1$), or opposite to ($i = 2$), the direction of e.*

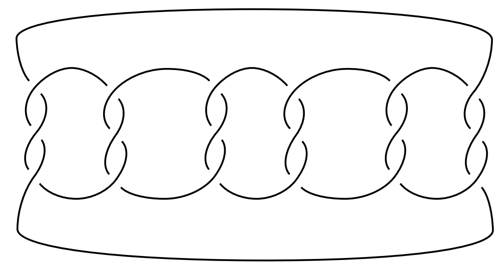
*When G has a reflection symmetry that reflects an edge (exchanges its endpoints) consider also $D_1(G; p, 2)$ for $p \geq 1$ (*reflective case*).*

Remark 20. G has an edge-*reflecting* symmetry $\iff$ G^* has an edge-*fixing* one. Keep both types apart!

Example 21. $G = \theta$ theta-curve, $p = 3$ and $q = 2$.



$D_1(M_3; 3, 2)$
(3 components)



$D_2(M_3; 3, 2)$
(2 components)

Now recall that checkerboard graph has duality ambiguity.

Definition 22. *G has* dual *G^*. Each set $E \subset E(G)$ of edges of G has dual set $E^* \subset E(G^*)$.*

Thus one can produce more EE diagrams.

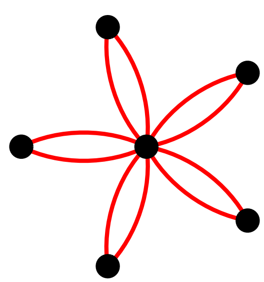
Definition 23. G dually edge transitive *if*

- G *is* self-dual, $G = G^*$

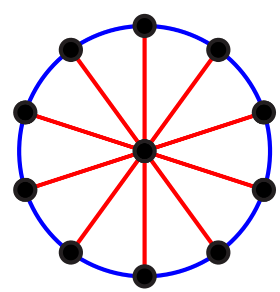
- $\exists$ *edge partition* $E(G) = E_1 \Cup E_2$*:*
  - *if* $e, e' \in E_i$, $\exists$ *symmetry* s *of* G *with* $s(E_i) = E_i$ *and* $s(e) = e'$,
  - *if* $e \in E_i$, $e' \in E_j$, $\exists$ *symmetry* s *of* G *with* $s(E_i^*) = E_j$ *and* $s(e^*) = e'$

Example 24. This is a bit technical, so a few examples.

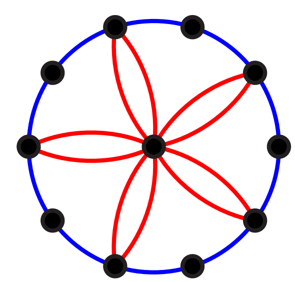
- *wheel (graph)* W_n: connect all vertices of an n-cycle C to an extra *central* vertex v ($E_1 = \star v$, $E_2 = C$).
- *twofold wheel* (similar)
- double star ($E_1 = E(G)$, $E_2 = \varnothing$; not cut-free)



a double star



wheel W_{10}



twofold wheel

Remark 25. One can exchange $E_1 \leftrightarrow E_2 = E(G) \setminus E_1 =: \overline{E_1}$. For $G = \tau$ tetrahedral graph $\exists$ further ambiguity, so better write (G, E_1).

Construction 26. *(G, E_1) cut-free dually edge transitive, $p = 1, 3$ and $q \geq 1$. Build $D(G, E_1; p, q)$ by replacing edge $e \in E_i$ by $P(p, q)$ in ($i = 1$; Kauffman positive crossings) or opposite ($i = 2$; Kauffman negative crossings) to the direction of e.*

Remark 27. The case (like $G =$ double star) of some $E_i = \varnothing$ is of (self-dual) edge transitive G, which is nothing new: $D(G, E(G); p, q) = D_1(G; p, q)$ and $D(G, \varnothing; p, q) = D_2(G; p, q)$ of construction 19. Thus let $E_i \neq \varnothing \implies |E_1| = |E_2|$.

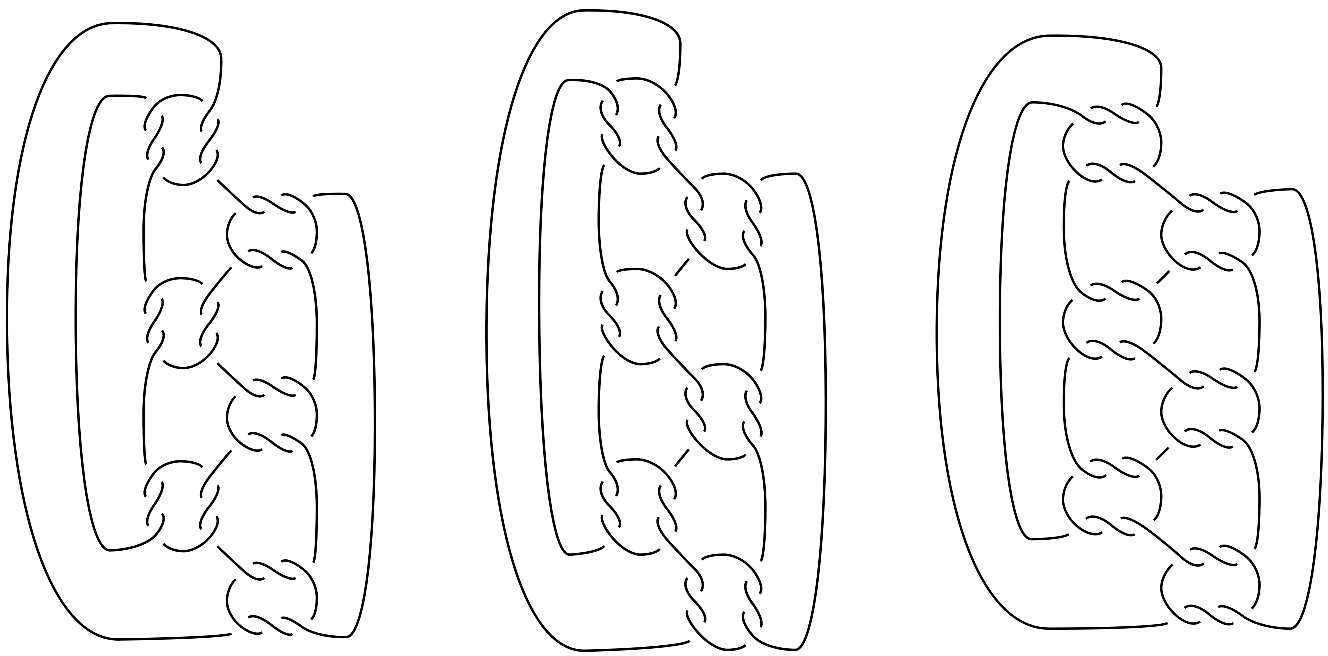
If G has an edge-reflecting symmetry along an edge $e \in E_1$, consider additionally $D(G, E_1; p, 2)$ for $p \geq 1$ (and again call it the reflective case).

Example 28. tetrahedral graph $G = \tau$ has extra peculiarity:

- $(G, E(G))$ is dually edge-transitive (because G is edge-transitive and self-dual), and

- $(G, \star v)$ is so (for any vertex v, because $\tau = W_3$)

This yields three different types of diagrams: $D_1(\tau; p, q) = D_2(\tau; p, q)$ because of self-duality, but $D(\tau, \star v; p, q) \neq D(\tau, \overline{\star v}; p, q)$



$D_{1,2}(\tau; 3, 2)$ $\quad$ $D(\tau, \star v; 3, 2)$ $\quad$ $D(\tau, \overline{\star v}; 3, 2)$

Remark 29. Again, as in remark 20, the reflective case is not duality invariant.

Proposition 30. *(a bit disappointing) These constructions yield no new* knot *diagrams!*

Question 31. (speculative) Are constructions exhaustive (say, for links)?

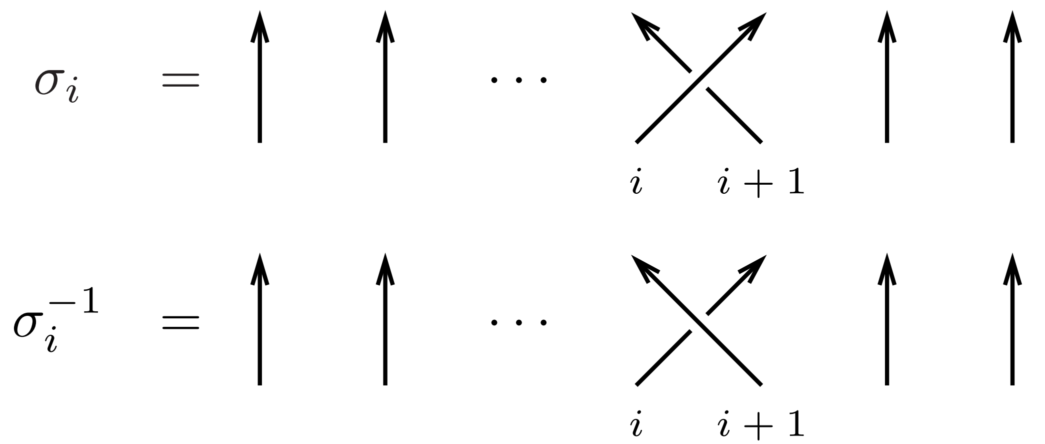
Answer: NO! There are totally asymmetric (and thus totally different) examples. (But 'YES' in another case. . .)

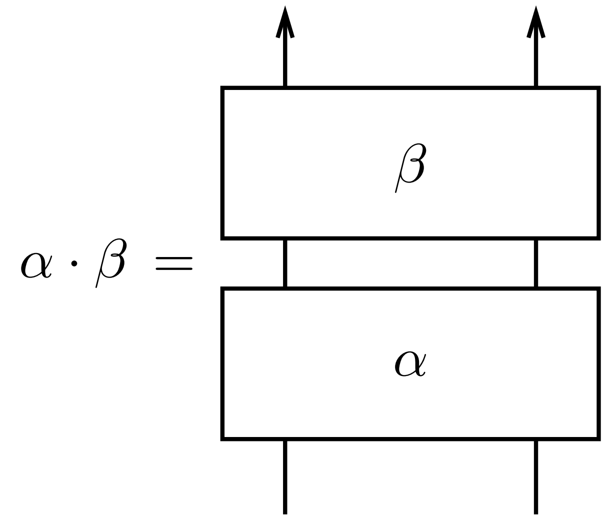
3-braids

Definition 32. *The* braid group B_n *on* n *strands:*

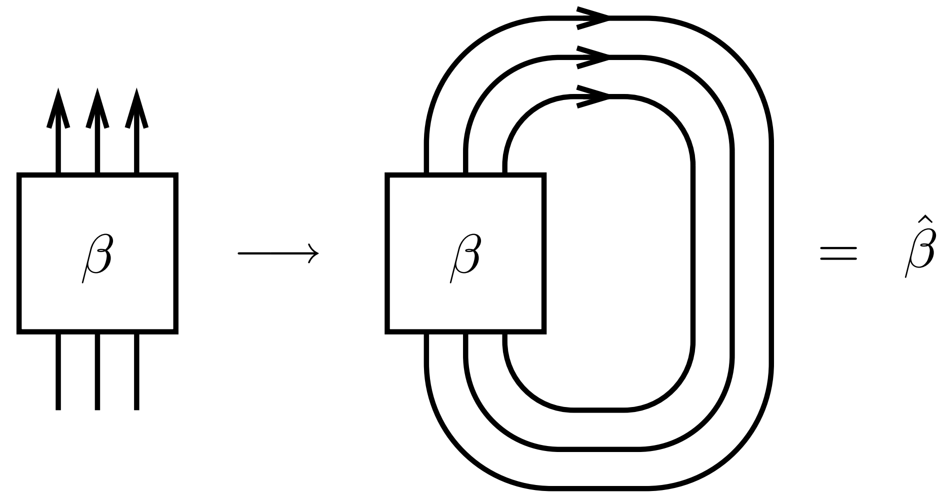
$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i & |i-j| = 1 \end{array} \right\rangle$$

σ_i – Artin standard generators. *An element* $\beta \in B_n$ *is an* n-braid.





Braid closure $\hat{\beta}$:



β	$\longmapsto$	$\hat{\beta}$
a braid	$\longmapsto$	*a* knot $S^1 \hookrightarrow S^3$ *or (more generally)* link $S^1 \cup \cdots \cup S^1 \hookrightarrow S^3$
a braid word *(possibly up to cyclic permutations + symmetries)*	$\longmapsto$	*a link* diagram

[Alexander's theorem: all links arise this way.]

Square of half-twist element Δ, the *full twist*

$$\Delta^2 = (\sigma_1 \sigma_2 \dots \sigma_{n-1})^n,$$

is the generator of the *center* of B_n.

We consider here $\beta \in B_3$.

Theorem 33. *3-braid word β gives an EE diagram $\iff$ (up to equivalence) in following four families:*

1) *the words* $(\sigma_1\sigma_2^{-1})^k$ *or* $(\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1})^k$ *for* $k = 1, 2$, *and* $\sigma_1\sigma_2^{-1}\sigma_1^{-2}\sigma_2^2$ *(*non-positive *case),*

2) any *positive (or negative, or the trivial) word representing a central element* Δ^{2k}, $k \in \mathbb{Z}$ *(*central *case),*

3) *the words* $(\sigma_1^l\sigma_2^l)^k$ *for* $k, l \geq 1$ *(*symmetric *case), and*

4) *the words* σ_1^k *for* $k > 0$ *(*split *case).*

Proof uses the relations

$$\text{Burau representation} \longleftrightarrow \text{Jones polynomial} \longleftarrow \text{adequacy}$$

Remark 34.

- Type 4 is uninteresting,
- type 1 (essentially (2)) and type 3 (proposition 10) were (largely) expected,

- but type 2 was (except the trivial word) totally surprising, and suggests the following more general construction.

Lemma 35. *When $\beta \in B_n$ is central, then every positive word of β is everywhere equivalent.*

Proof. All β' represent $\sigma_i^{-2}\beta$, and all are conjugate. □

Thus ∃ examples of words (and diagrams) lacking any symmetry: *Every positive word* in B_n is subword of a positive central word.

Remark 36. Braids come with orientation, but one can argue that theorem 33 holds for unoriented EE.

Thus the situation is already complicated even in special cases.

2-component links

Finally, there is one large case, which can be completely resolved, and the answer (as well as its argument) is rather simple.

To leave the subtleties for knots to their own merit, *assume all link diagrams are non-split*.

Theorem 37. *D is orientedly everywhere equivalent (non-split) 2-component link diagram $\Longleftrightarrow$ among the following families:*

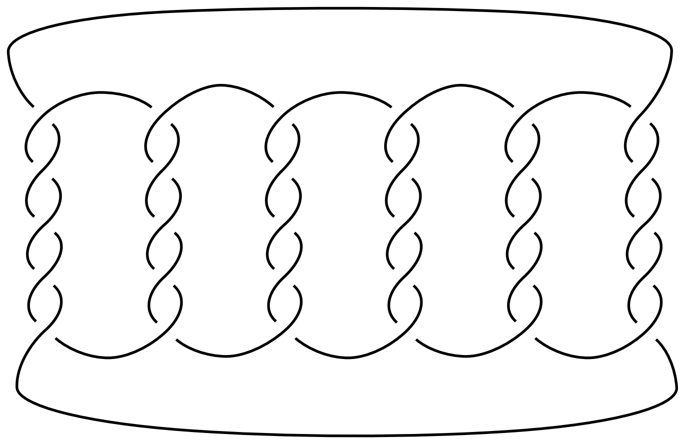
1. *the pretzel link diagrams $\hat{P}(p,q) = (\underbrace{p,p,\dots,p}_{q \text{ times}})$ for $p,q > 0$, p odd and q even, or*

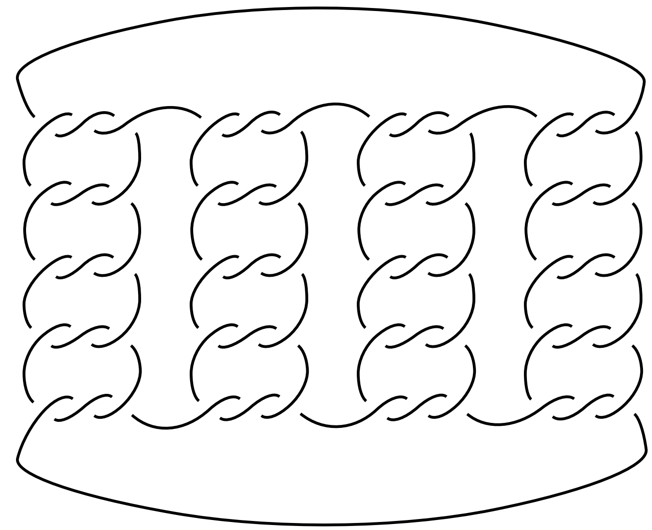
2. *the arborescent link diagrams $\hat{P}(q,3,p) = (\underbrace{P(3,p),\dots,P(3,p)}_{q \text{ times}})$, for $p > 1$ odd and $q > 2$ even.*

This says that constructions 19 and 26 are exhaustive (at least) here.

First family includes, for $p = 1$, the $(2,q)$-torus links. For $p = 1$, second family reduces to the $(2,3q)$-torus links, and for $q = 2$ to $P(3,2p)$, which is why we excluded these values.

Typical examples, for $q = 4, 6$ and $p = 5$:





$\hat{P}(5, 6)$ $\qquad$ $\hat{P}(4, 3, 5)$

Proof. Uses an observation that diagrams are special + positive $\Rightarrow$ alternating, and then shoots with Menasco-Thistlethwaite (Flyping theorem) and Kauffman-Thistlethwaite (Jones polynomial of alternating links). ☐

Remark 38. Oriented EE is essential (in proof). For unoriented EE, diagrams are special, but no longer positive (and alternating).

Thank you!

Alexander Stoimenow

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August 14, 2012

KAIST, Daejeon Korea