Hoste's conjecture and roots of the Alexander polynomial

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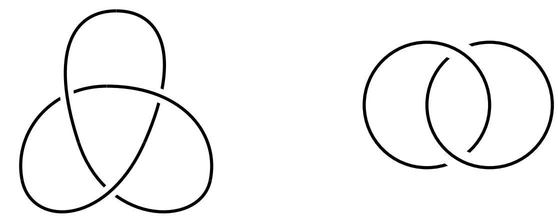
Contents

Links and Alexander polynomial

knot K $\qquad S^1 \hookrightarrow S^3$

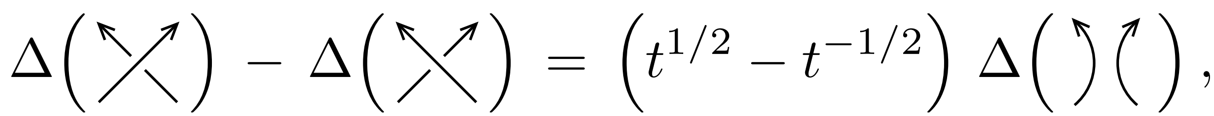
link L $\qquad \underbrace{S^1 \cup \ldots \cup S^1}_{n \text{ components}} \hookrightarrow S^3$



$K \qquad\qquad L$

Alexander polynomial $\Delta : \{ \text{ knots and links } \} \to \mathbb{Z}[t^{\pm 1/2}]$,

determined by (and studied below using) the *skein relation*



(1)

and (here) with the (common) normalization

$$\Delta\left(\bigcirc\right) = 1\,.$$

(Alternative approach using Seifert matrices, Fox calculus, etc.)

Remark 1. We have $\Delta(L) \in \mathbb{Z}[t^{\pm 1}]$ for links of odd (number of) components (in particular, for *knots*), and $\Delta(L) \in t^{1/2}\mathbb{Z}[t^{\pm 1}]$ for even components.

Remark 2. Alexander polynomial *a priori* an *oriented* link invariant. Invariant when orientation of *all components* reversed $\Rightarrow$ for *knots* orientation does not matter, but is *does* a lot for *links*.

The Alexander polynomial is of profound importance. *Roots* are studied among others for

- monodromy and dynamics of surface homeomorphisms (*cf.* Rolfsen "Knots and links"; Silver-Williams),
- divisibility of knot groups (Murasugi),
- orderability of knot groups (Perron-Rolfsen),

- statistical mechanical models of the Alexander polynomial (Lin-Wang),
- Mahler measure and Lehmer's question (Ghate-Hironaka, Silver-Williams).

Conway version of Δ, *Conway polynomial* $\nabla(z) \in \mathbb{Z}[z]$

$$\nabla(L)(t^{1/2} - t^{-1/2}) = \Delta(L)(t)$$

For an n-component link,

$$\nabla(L) \in z^{n-1}\mathbb{Z}[z^2]\,. \qquad (2)$$

It is known what are Alexander polynomials of knots.

Theorem 3. *(Levine, Kondo, ...; late 60's)*
$\Delta \in \mathbb{Z}[t^{\pm 1}]$ is Alexander polynomial of a knot iff Δ satisfies

(1) $\Delta(t) = \Delta(1/t)$ *(*reciprocity*)*

(2) $\Delta(1) = 1$ *(*unimodularity*)*

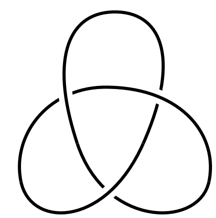
There is also a corresponding theorem for n-component links (e.g., easy consequence of Kondo's proof for knots). The conditions are *superreciprocity*

$$\Delta(t) = (-1)^{n-1}\Delta(1/t)\,,$$

and a divisibility property, following from (2).

How about alternating knots and links?

A knot (or link) is *alternating* if it has a diagram where (along each component) one passes strands under-over.



Alexander polynomial of alternating links

Problem 4. *Characterize the Alexander polynomials of* alternating *knots (or links).*

Even though in general alternating knots are much better understood, this problem seems very difficult. A complete solution is likely *impossible*!

What is known

Below some classes of knots in relation to alternating knots (similarly links).

rational (2-bridge) knots	$\subset$	Montesinos knots	$\subset$	algebraic (arborescent) knots
$\cap$				
alternating knots	$\supset$	special alternating knots		

Let $[\Delta]_k$ for $k \in \mathbb{Z} \cdot \dfrac{1}{2}$ be the coefficient of t^k in Δ.

$$\textit{maximal degree} \qquad \max\deg\Delta \;=\; \max\left\{\, k \in \mathbb{Z}\cdot\tfrac{1}{2} \;:\; [\Delta]_k \neq 0 \,\right\}$$

$$\textit{minimal degree} \qquad \min\deg\Delta \;=\; \min\left\{\, k \in \mathbb{Z}\cdot\tfrac{1}{2} \;:\; [\Delta]_k \neq 0 \,\right\}$$

Remark 5. Degrees make sense if $\Delta \neq 0$. Unimodularity $\Rightarrow \Delta \neq 0$ for all knots ($\Delta(-1) \neq 0$, odd). But $\exists$ links with $\Delta = 0$! However, for an *alternating* link L,

$$\Delta \neq 0 \quad \Longleftrightarrow \quad L \textit{ non-split(table)}.$$

(There is no hyperplane in $\mathbb{R}^3$ which can separate L non-trivially.) We thus assume alternating *links are non-split*.

(super)reciprocity $\Rightarrow \min\deg \Delta = -\max\deg \Delta$.

Definition 6.

- We call a coefficient $[\Delta]_k$ *admissible* if $\min\deg \Delta \leq k \leq \max\deg \Delta$ and $k - \min\deg \Delta$ (or $\max\deg \Delta - k$) is an integer.

- We call Δ *positive/negative* if all its admissible coefficients are positive/negative (and in particular non-zero).

- We call $\Delta(t)$ *alternating* if $\Delta(-t)$ is positive or negative.

Remark 1 $\Rightarrow [\Delta]_k \neq 0$ only if $[\Delta]_k$ is admissible.

Theorem 7 (Crowell-Murasugi '59-'61). *L alternating knot or (non-split) alternating link $\Rightarrow$ $\Delta_L(t)$ is alternating.*

Crowell-Murasugi: if K knot, then $\max\deg\Delta = g(K)$, *genus* of K. (For L link, $\dfrac{1-\chi(L)}{2}$.)

Fox conjectured more:

Conjecture 8 (Fox's Trapezoidal conjecture). *K alternating knot $\Rightarrow \exists$ a number $0 \le n \le g(K)$ such that for $\Delta_{[k]} := \big|[\Delta_K]_k\big|$ we have*

$$\begin{array}{ll} \Delta_{[k]} = \Delta_{[k-1]} & \text{for } 0 < |k| \le n, \\ \Delta_{[k]} < \Delta_{[k-1]} & \text{for } n < |k| \le g(K)\,. \end{array} \tag{3}$$

(n is half-length of 'upper base of trapezoid')

Trapezoidal conjecture was verified for

- rational (2-bridge) knots (Hartley '79)

- some more algebraic knots (Murasugi '85)

The *signature* of a knot $\sigma(K)$ is even and satisfies

$$|\sigma(K)| \leq 2g(K)\,. \tag{4}$$

Extension (first) of Trapezoidal conjecture (S. '05): for n in (3),

$$n \leq |\sigma(K)|/2\,,$$

where σ is the signature. (Extended Trapezoidal conjecture)

(In particular $\sigma(K) = 0 \Rightarrow n = 0$, i.e., Δ is a 'triangle'; Murasugi conjectured independently this case.)

Recent partial results toward the (Extended) Trapezoidal conjecture:

- S.: knots of genus $g(K) \leq 4$, using a combinatorial method developed from S.-Vdovina (I.D. Jong '08 for genus $g \leq 2$ using same method);

- Ozsváth-Szabó: $g(K) \leq 2$, and $|k| = g(K)$ in (3) for general case (knot Floer homology)

(Ozsváth-Szabó obtain more generally certain inequalities on the coefficients of Δ for an alternating knot.)

Second extension of Trapezoidal conjecture (S. '05)

Polynomial X *log-concave*, if $[X]_k$ are log-concave, i.e.

$$[X]_k^2 \geq [X]_{k+1}[X]_{k-1} \geq 0 \tag{5}$$

for all $k \in \mathbb{Z}$. ('≥ 0', because want to regard only positive and alternating polynomials as log-concave.)

Conjecture 9 (log-concavity conjecture, S. '05). *If K is an alternating knot, then $\Delta_K(t)$ is log-concave.*

log-concavity conjecture $\Rightarrow$ Trapezoidal conjecture

Refined log-concavity conjecture: equality in (5) for admissible $[\Delta]_k$ only if

$$[\Delta]_k = [\Delta]_{k-1} = [\Delta]_{k+1} \,.$$

Using method related to S.-Vdovina, I verified the (refined) log-concavity conjecture for genus $g(K) \leq 4$ (and $\chi(L) \geq -7$ for links L).

Hoste's conjecture

Hoste, based on computer verification, made the following conjecture about 10 years ago.

Conjecture 10 (Hoste's conjecture). *If $t \in \mathbb{C}$ is a root of the Alexander polynomial Δ of an alternating knot, then $\Re e\, t > -1$.*

Not much is known.

1. Crowell-Murasugi: Since Δ is alternating, *real* $t < 0$ is never a root. Thus Hoste's conjecture is true if all roots of Δ are real.

2. Let
$$S^1 := \{\, t \in \mathbb{C} \,:\, |t| = 1 \,\}\,.$$

There's the following 'folklore' inequality (Riley)

$$\#\{\text{ zeros } t \text{ of } \Delta \text{ on } S^1 \text{ with } \Im m\ t > 0\ \} \geq \frac{|\sigma(K)|}{2}\,. \tag{6}$$

$$\begin{aligned} K \text{ special alternating} \quad &\Longleftrightarrow \quad \text{(4) is an equality (Murasugi)} \\ &\Longrightarrow \quad \text{all roots of } \Delta \text{ are on } S^1 \\ &\Longrightarrow \quad \text{Hoste's conjecture for sp. al. knots.} \end{aligned}$$

3. Using (6), S-V, and a test based on Rouché's theorem, I verified the Hoste conjecture for $g(K) \leq 4$.

Example 11. (Mizuma; as quoted by Murasugi) The (unimodular symmetric) polynomial

$$t^{-6} - 2t^{-5} + 4t^{-4} - 8t^{-3} + 16t^{-2} - 32t^{-1} + 43 - 32t + 16t^2 - 8t^3 + 4t^4 - 2t^5 + t^6$$

is trapezoidal (and log-concave), but has zero t with $\Re e\ t < -1$.

Thus trapezoidality (or log-concavity) of Δ does not imply Hoste's conjecture. In fact, they are *almost unrelated*:

Theorem 12 (S. '11). *Zeros of log-concave (even monic) alternating Alexander knot polynomials are dense in* $\mathbb{C}$.

Monic: leading coefficient is ± 1; can be realized by a *fibered* (hyperbolic) knot.

Remark 13. Minor relations, e.g.,

- an alternating polynomial cannot have a real negative zero.
- conditions when restricting the degree. E.g., when $\max\deg\Delta = 2$, then Δ alternating $\Rightarrow$ Hoste's conjecture (Murasugi).

Results for 2-bridge links

Rational (2-bridge) links are one important class of alternating links.

Schubert's ('56) form $L = S(q,p)$, for p and q coprime integers with $0 < p < q$; determines L up to mirror image up to ambiguity

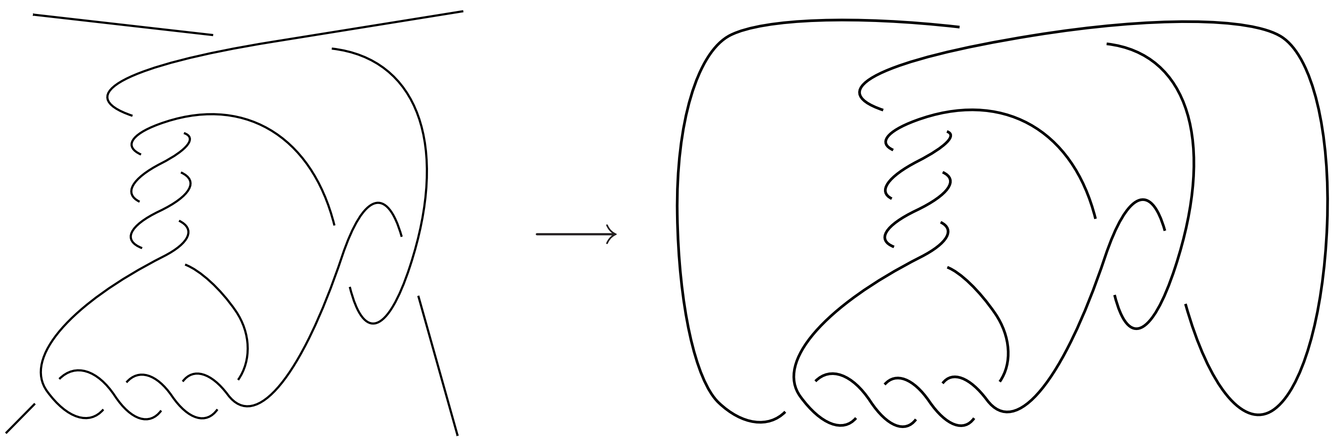
$$\pm p^{\pm 1} \in \mathbb{Z}_q^* \,. \tag{7}$$

Continued fraction expansion of $p/q \in \mathbb{Q}$:

$$\frac{p}{q} = (b_1, \ldots, b_n) \;=\; \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ldots \cfrac{1}{b_n}}} \tag{8}$$

Ambiguity (7) allows for special types: positive fraction expansion, <u>even fraction expansion</u> (below).

How to join twists into a rational tangle and close up. (Twists composed in a non-alternating way when the sign of b_i changes.)



$$(1,2,4,-4) = \frac{34}{49} \longrightarrow S(49,34)$$

rational tangle $\longrightarrow$ rational link

Lyubich-Murasugi (arXiv '11) examine roots of Δ of a 2-bridge (rational) knot or link, by studying stability of Seifert matrix. One of their results:

Theorem 14 (Lyubich-Murasugi). *L 2-bridge knot or link, t root of $\Delta(L)$ $\Rightarrow -3 < \Re e\ t < 6$.*

Theorem 15 (S.). *If L 2-bridge, $\Delta(L)(t) = 0$, then*

$$\left|t^{1/2} - t^{-1/2}\right| < 2. \tag{9}$$

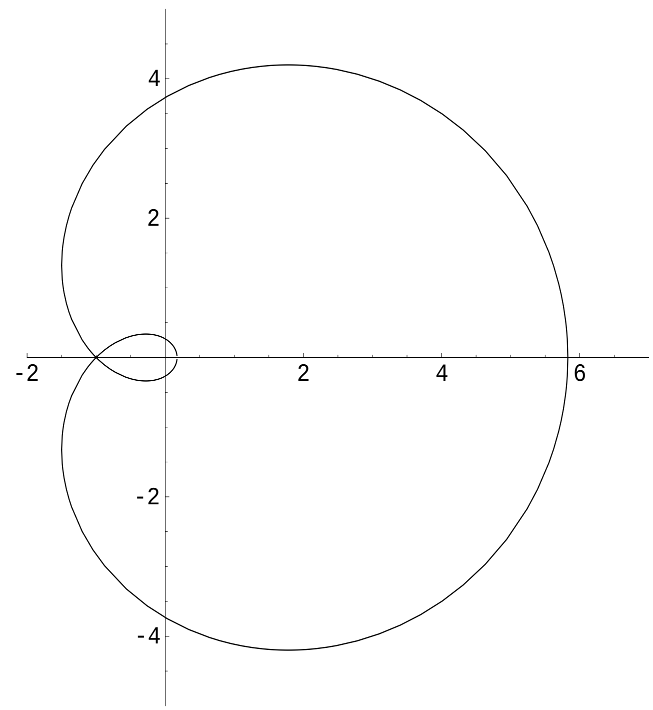
Or:

$$\nabla(L)(z) = 0 \Rightarrow |z| < 2\,. \tag{10}$$

Let

$$\begin{array}{rcl} t \in \mathbb{C} \setminus \{0\}\ \textit{internal} & :\iff & t \text{ satisfies (9)}, \\ \textit{external} & & \text{otherwise.} \\ \mathcal{D} & := & \{\, t \in \mathbb{C} \setminus \{0\} \,:\, t \text{ internal } \}. \end{array}$$

The domain $\mathcal{D}$ is bounded by the graphs of the four functions



$$\pm f_{\pm}(x) \;=\; \pm\sqrt{-x^2 + 2x + 7 \pm 4\sqrt{2x+3}}\,.$$

$f_{\pm}$ defined on $\left[-\frac{3}{2}, 3 \pm 2\sqrt{2}\right]$.

A few special values are

$$f_\pm\left(-\frac{3}{2}\right) = \frac{\sqrt{7}}{2}\,, \qquad \begin{aligned} f_-(-1) &= 0\,, \\ f_+(-1) &= \sqrt{8}\,, \end{aligned} \qquad f_\pm(3 \pm 2\sqrt{2}) = 0\,. \tag{11}$$

Thus (9) $\Longrightarrow$

- $$-\frac{3}{2} < \Re e\ t\,. \tag{12}$$

  (improves lower bound in L-M)

- $$(\Re e\ t \leq) \quad |t| < 3 + 2\sqrt{2} \approx 5.8284 \tag{13}$$

  (improves upper bound in L-M)

L-M prove the conjecture for certain 2-bridge links:

Theorem 16 (Lyubich-Murasugi). *Consider the* even *expansion* (8), *with*

$$b_i = 2a_i \quad (a_i \in \mathbb{Z} \setminus \{0\})\,. \tag{14}$$

- If no <u>two</u> consecutive $a_i = \pm 1$, then H.'s conjecture holds.

- If no $a_i = \pm 1$, then $-1 < \Re e\ t < 3$.

I improved this:

Proposition 17 (S.). *Under the previous assumption,*

- if no <u>three</u> consecutive $a_i = \pm 1$, then H.'s conjecture holds;
- if no $a_i = \pm 1$, then $|z| < 1$ in (10). In particular,

$$\frac{3}{8} < \Re e\ t \quad \text{and} \quad |t| < \frac{3+\sqrt{5}}{2}\,.$$

Interestingly, the two approaches – Seifert matrix (L-M) and skein relation (S.) – meet similar difficulties.

Here skein relation does better, but L-M have further results, not skein theoretically recovered. E.g.:

Proposition 18 (L-M). *If all* $a_i > 0$ *in* (14), *then all zeros of* Δ *are real* *(⇒ H.'s conjecture).*

On the other hand, the skein approach does more:

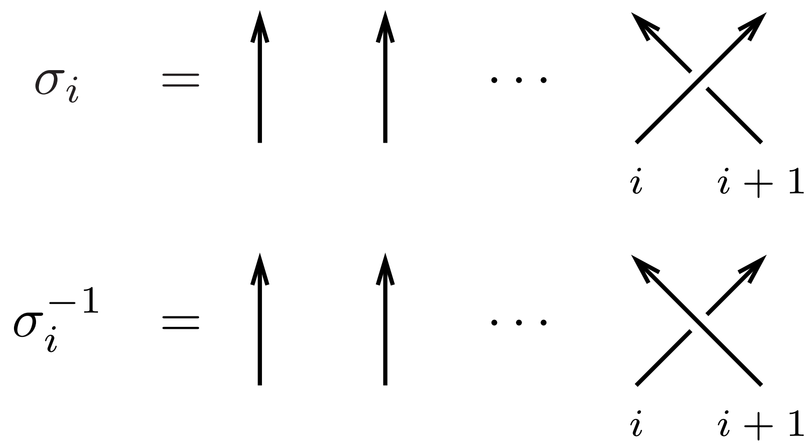
- condition on the zeros of the *skein (HOMFLY-PT)* polynomial of a 2-bridge link (skipped), and
- for Δ of more general links (below).

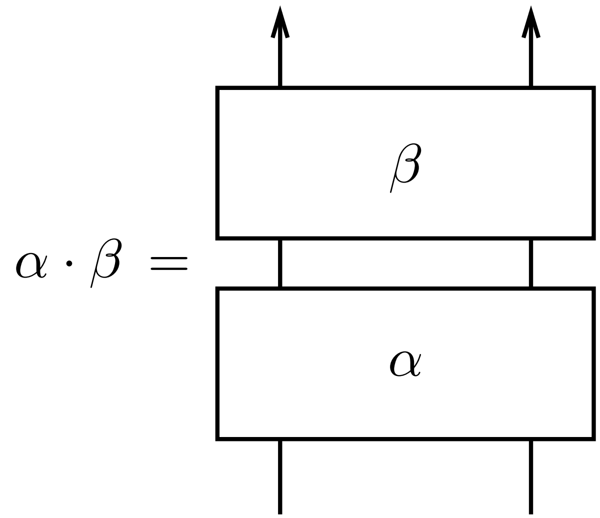
3-braid alternating links

Definition 19. *The* braid group B_n *on* n *strands:*

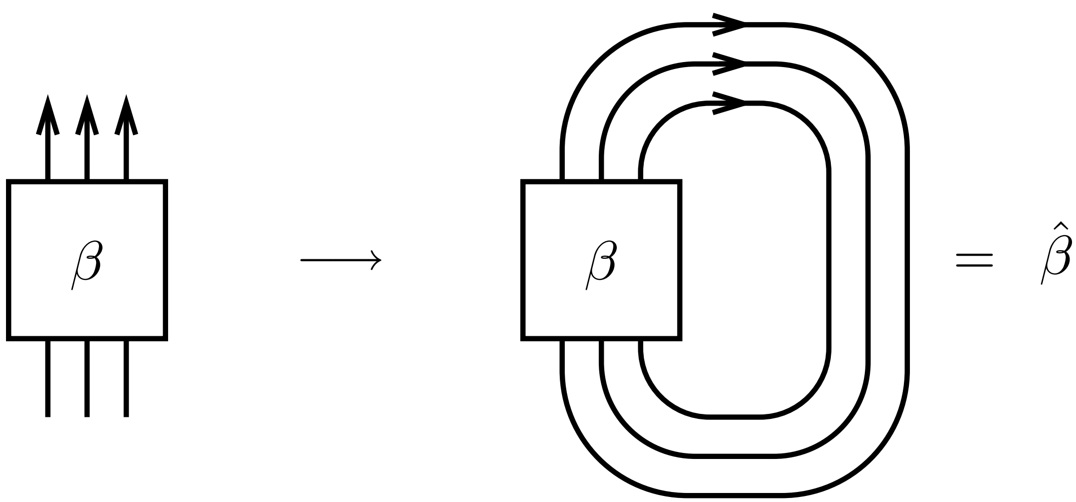
$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i & |i-j| = 1 \end{array} \right\rangle$$

σ_i – Artin standard generators. *An element* $\beta \in B_n$ *is an* n-braid.





Braid closure $\hat{\beta}$:



is a knot $S^1 \hookrightarrow S^3$ *or (more generally)* link $S^1 \cup \cdots \cup S^1 \hookrightarrow S^3$.

Alexander's theorem: all links arise this way.

We say that a braid (word)

$$\beta = \prod_{i=1}^{n} \sigma_{p_i}^{q_i} \tag{15}$$

(with $q_i \neq 0$) is *alternating* if $q_i q_j \cdot (-1)^{p_i - p_j} > 0$ whenever $i \neq j$.

We consider here $\beta \in B_3$.

Theorem 20 (S. '03). *If L alternating link is closed 3-braid, then L is either*
(a) closed alternating 3-braid, or
(b) a (special alternating) pretzel link $P(1, p, q, r)$ $(p, q, r > 0$; see below).

For special alternating links, H.'s conjecture is true, so look at alternating 3-braids.

Theorem 21. *If L closed alternating 3-braid, $\Delta(L)(t) = 0$, then*

$$\left|t^{1/2} - t^{-1/2}\right| < 2.45317\,,$$

i.e., $|z| < 2.45317$ *in* (10). *(All decimal constants rounded.)*

The proof (and the constant) is more technical.

A discourse on *non-alternating* braids (and links):

Proposition 22. *If L closed positive 3-braid (in* (15) *all $q_i > 0$),*

$$\left|t^{1/2} - t^{-1/2}\right| < 3.274601\,.$$

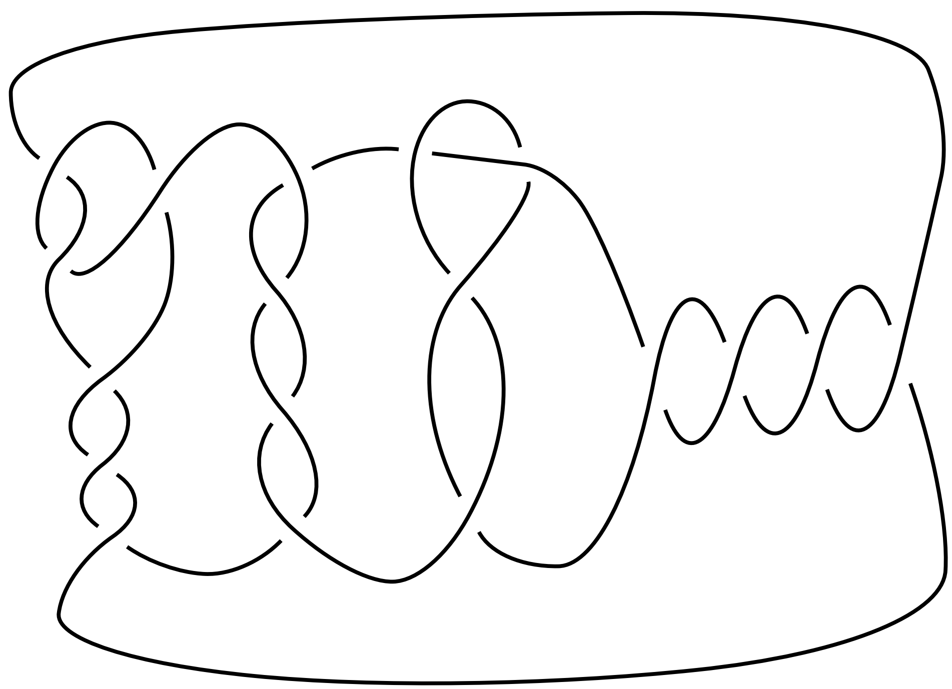
Remark 23. S. '05: if L closed positive braid and closed 3-braid $\Rightarrow$ L is closed positive 3-braid. Not true for 4-braids ($\exists$ counterexamples).

Example 24. (Hirasawa) 10_{152} is closed positive 3-braid, but Δ has (real) root $t \approx -1.85 \Rightarrow$ Hoste's conjecture (and (9)) fails for positive 3-braid links.

Montesinos links

A Montesinos link has the presentation

$$L = M(e, p_1/q_1, \ldots, p_n/q_n)\,. \tag{16}$$



$M(4, 3/11, -1/4, 2/5)$

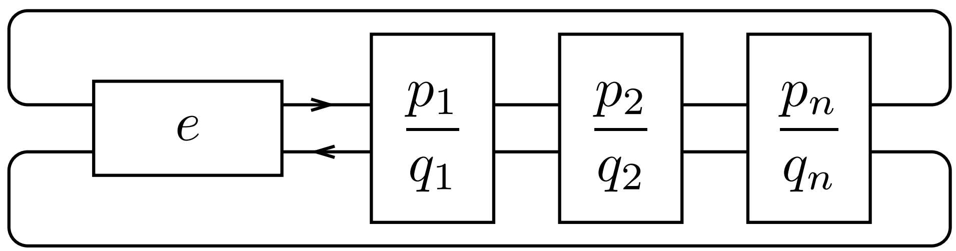
Terminology: e *integer part*, and p_i/q_i *fractional parts*. Their number n *length*.

Special cases:

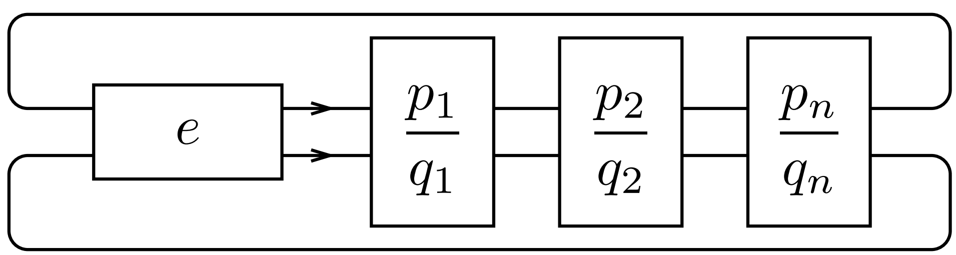
- If $n \leq 2$, Montesinos link is rational.
- When all $p_i = \pm 1$, we stipulate that we sign q_i so that $p_i = 1$, and have *pretzel link*

$$L = M(1/q_1, \ldots, 1/q_n) \; = \; P(q_1, \ldots, q_n)\,. \tag{17}$$

Here orientation issues become essential, and we distinguish:



reverse M link



parallel M link

Our result regarding M links is:

Theorem 25. *L alternating Montesinos link, $\Delta(L)(t) = 0$ and $\underline{t \notin \mathcal{D}}$ (i.e., t external).*

- If L is reverse, then $\Re e\ t > 0$.
- If L is parallel, then $\Re e\ t > -1$ and t satisfies (13).

Corollary 26. *If L alternating Montesinos link, Hoste's conjecture holds* for external zeros*; in particular $\Delta(L)(t) = 0 \Rightarrow$ (12).*

One more specific statement possible for reverse links (works also for many *non-alternating* ones):

$$\text{if } t \notin \mathcal{D}, \quad \text{then} \quad \left| \Re e \left(z^2 \right) \right| < |z| \qquad (\text{with } z = t^{1/2} - t^{-1/2}),$$

i.e., (roughly) when $|t|$ large, $|\Im m\ t|$ or $|\Re e\ t|$ small.

But no bound on $|t|$ (Murasugi has examples, where $t \in \mathbb{R}$, $t \to \infty$).

Proofs: rational links 1 page (easy), 3-braid links 5 pages (tricky but manageable), Montesinos links 20+ pages (still not deep, but very painful).

Open questions

- H.'s conjecture remains open for everything (in general): rational knots, 3-braid knots, Montesinos knots (and links)!
- Statements about non-alternating links. $\exists$ non-split (non-alternating) 3-braid or Montesinos links with $\Delta = 0$. (For 3-braid at least there's a full description; S. '06.) How bypass them?
- (S. '07) Is there a condition (at all, except general ones) on the Alexander polynomial of an arbitrary Montesinos link? (No for arborescent links, yes – obviously – for rational links.)

Thank you!

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