Exchange moves and non-conjugate braid representatives of links

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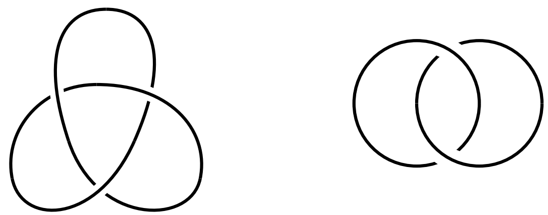
Contents

Braid closure and braid index

knot K $\quad S^1 \hookrightarrow S^3$

link L $\quad \underbrace{S^1 \cup \ldots \cup S^1}_{n \text{ components}} \hookrightarrow S^3$

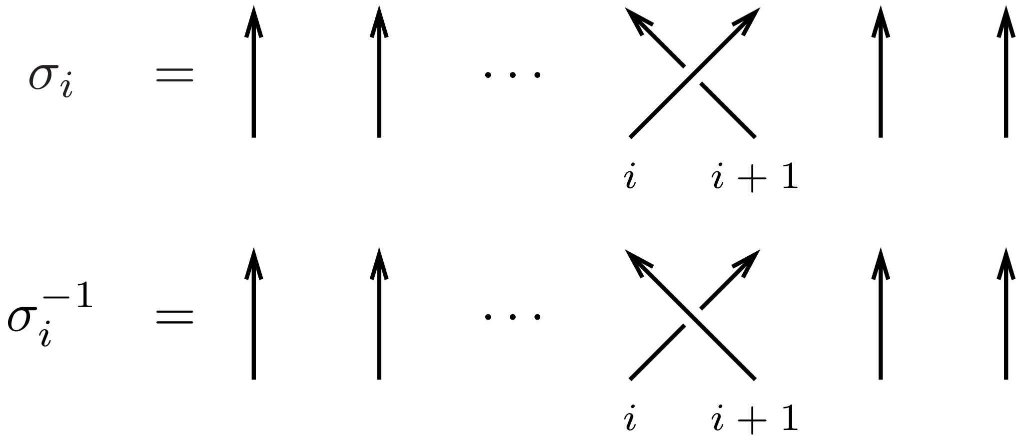


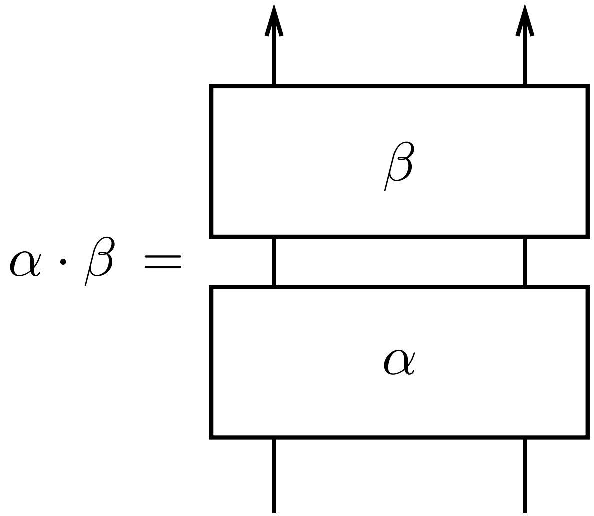
$K \qquad L$

Definition 1 *The* braid group B_n *on* n *strands:*

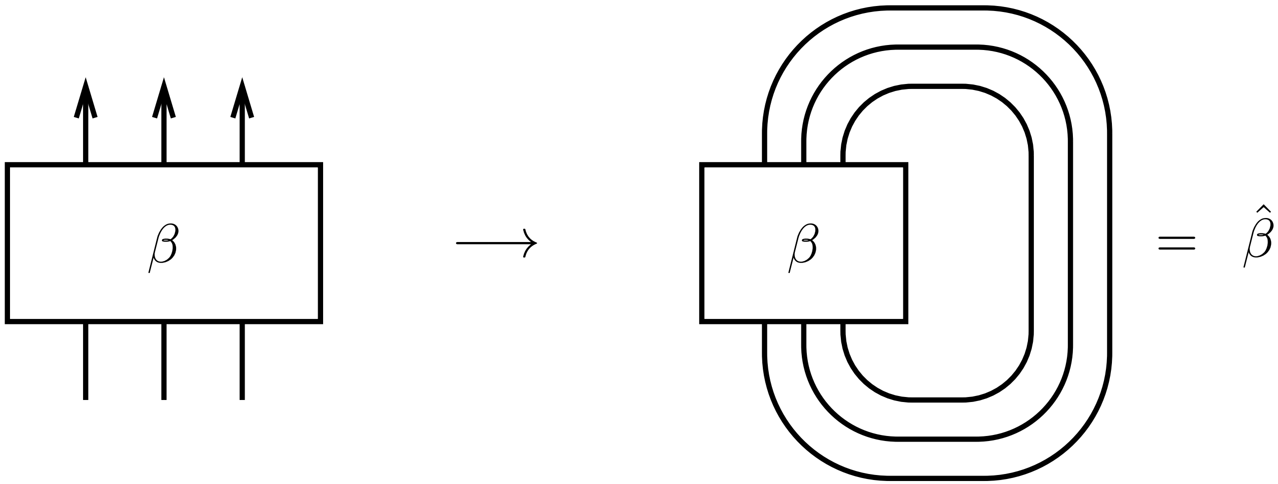
$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i & |i-j| = 1 \end{array} \right\rangle$$

σ_i – Artin standard generators. *An element* $\beta \in B_n$ *is an* n-braid.





Braid closure $\hat{\beta}$:



is a knot $S^1 \hookrightarrow S^3$ *or (more generally)* link $S^1 \cup \cdots \cup S^1 \hookrightarrow S^3$.

Note: # of components of $\hat{\beta}$ = # of cycles of *associated permutation* $\pi(\beta)$,

$$\pi \;:\; B_n \to S_n\,, \qquad \pi(\sigma_i) = \tau_i \;=\; (i\;i+1)\,.$$

We call β a *pure braid* if $\pi(\beta) = Id$.

Fact: The *center* of B_n (elements that commute with all B_n) is infinite cyclic and generated by

$$\Delta^2 = (\sigma_1 \ldots \sigma_{n-1})^n\,.$$

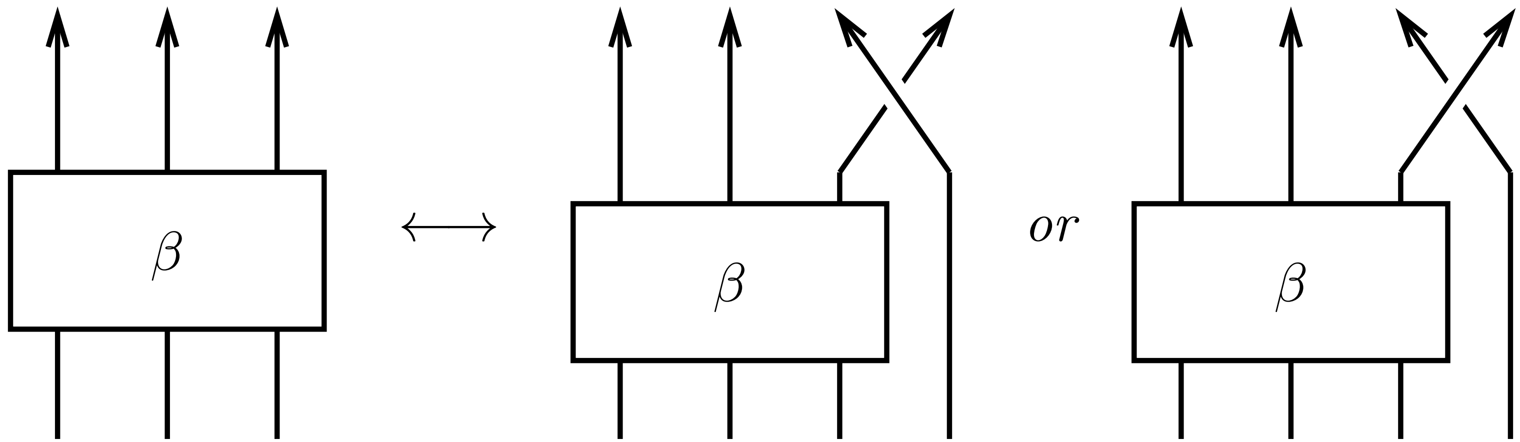
Theorem 2 *(Alexander '23) Any link L is the closure of a braid β.*

Definition 3 $b(L) \;:=\; \min\left\{ n \in \mathbb{N} \;:\; \exists\beta \in B_n,\, \hat{\beta} = L \right\}$ braid index *of* L.

We call β a *braid representative* of L. If $n = b(L)$ then β is called *minimal*.

Theorem 4 *(Markov '35, Birman '76) If $\hat{\beta}_1 = \hat{\beta}_2$, then $\beta_{1,2}$ are related by a sequence of*

1. *conjugacies in the braid group* $\beta \longmapsto \alpha\beta\alpha^{-1}$
2. *(de)stabilizations* $B_n \ni \beta \longleftrightarrow \beta\sigma_n^{\pm 1} \in B_{n+1}$



First move $\Rightarrow \{\, \beta \in B_n \;:\; \hat{\beta} = L \,\}$ is a union of conjugacy classes. Let

$$c(n, L) \;:=\; \# \text{ of conjugacy classes}\,.$$

Conjugacy is B_n is well-understood (Garside, ...) $\Rightarrow$ 'easy' part.
Second move (difficult part): how does it relate different conjugacy classes?

In this talk we will be mainly concerned with:

Question 5 *Fix a link L and $n \geq b(L)$. What is $c(n, L)$? In particular, when is $c(n, L) = \infty$?*

Exponent sum and Jones conjecture

How to distinguish conjugacy classes? Simplest invariant: exponent sum.

Definition 6 *$H_1(B_n) = B_n/[B_n, B_n] \simeq \mathbb{Z}$; homomorphism $[\,.\,]$ given by $\sigma_i \mapsto [\sigma_i] := 1$. $[\beta]$ is* writhe *or* exponent sum *of β.*

For a link L consider

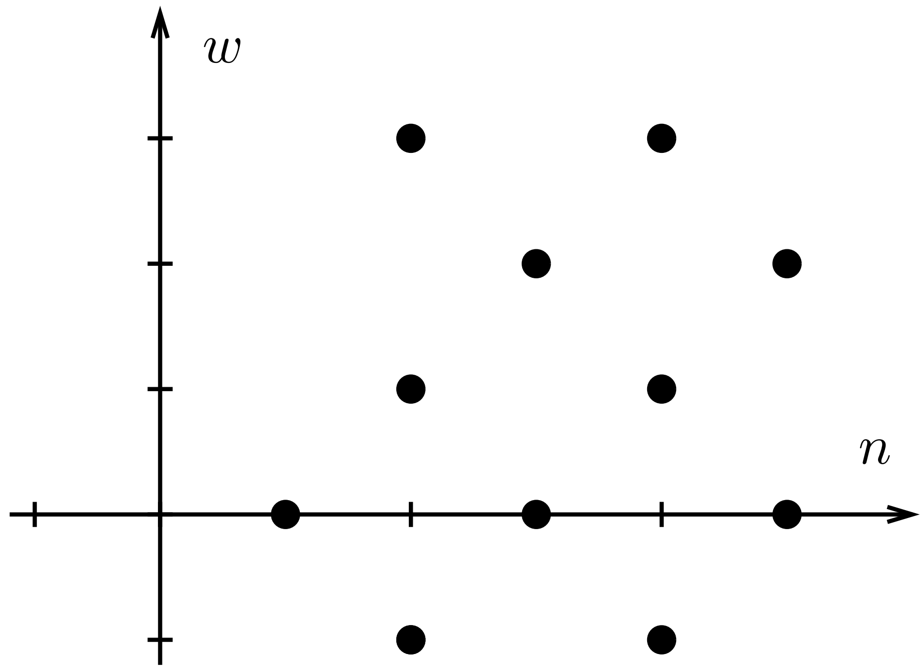
$$W_L := \{ (n, w) \in \mathbb{N} \times \mathbb{Z} \,:\, \exists\, \beta \in B_n \,:\, [\beta] = w\,,\ \hat{\beta} = L \}$$

and

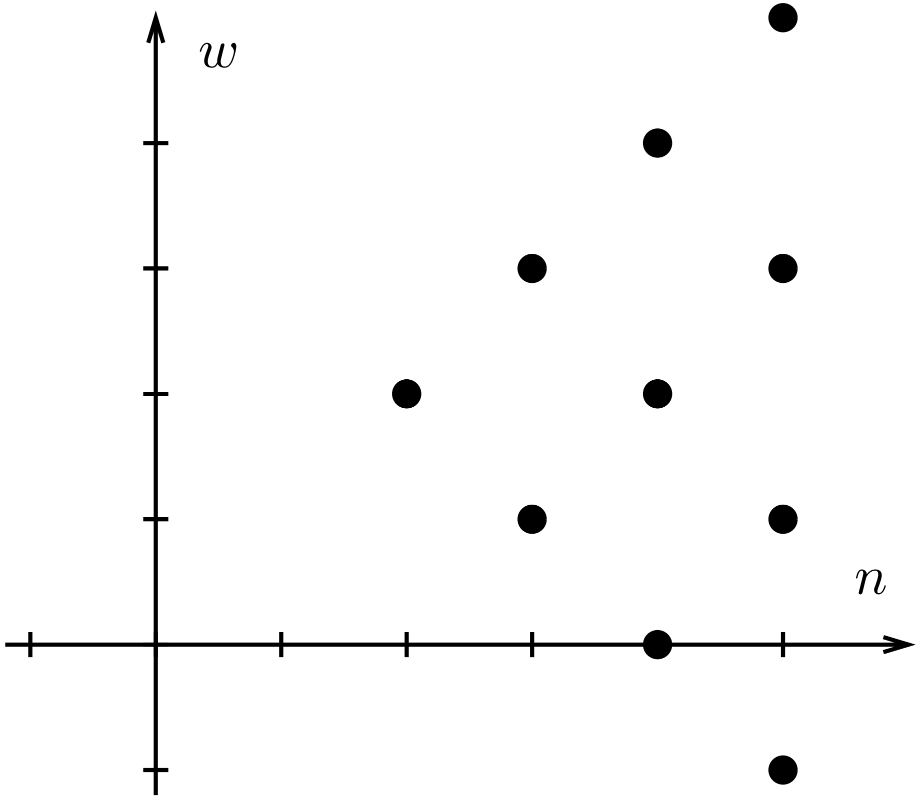
$$w(n, L) := \#\{ w : (n, w) \in W_L \}$$

(the number of exponent sums of n-braid representatives of L).

For example, if L knot $\Longrightarrow$ $n + w$ odd, so $W_L \subset$ right diagram.



Stabilization shows that when $(n, w) \in W_L$, then W_L contains the "cone" $C(n, w)$ of (n, w), made up of all $(n+k, w+m)$, for $k > 0$, $|m| \leq k$ with $m+k$ even:



Corollary 7

$$w(n, L) \geq n - b(L) + 1\,. \qquad (1)$$

Corollary 8 $c(n, L) \geq n - b(L) + 1\,.$

Exponent sum distinguishes *some* conjugacy classes when $n > b(L)$.

Conjecture 9 *(Jones '87)*
1. (weaker version) (1) *is exact for* $n = b(L)$*, i.e.* $W_L \cap (\{b(L)\} \times \mathbb{Z})$ *is one point* $=: (b(L), w(L))$ *('Exponent sum of minimal representations is unique')*
2. (stronger version) (1) *is exact for all* $n \geq b(L)$*, i.e.* $W_L = C(b(L), w(L))$*.*

It is now proved (Lafountain and Menasco 2014, Dynnikov and Prasolov 2013) It was previously known for certain classes of links. For example, Birman-Menasco proved:

Theorem 10 *(Birman-Menasco '93) Up to 3 strands almost always only one minimal conjugacy class, and the conjugacy classes of its stabilizations, have same closure link.*

$\Rightarrow$ at most 3 classes have same closure link.
$\Rightarrow$ Jones conjecture is true when $3 \geq n \geq b(L)$. (I later showed $(n >)3 \geq b(L)$.)

One main tool related to the Jones conjecture: *Morton-Franks-Williams inequality*. (We will not discuss it further in detail here.) It shows:

Corollary 11 *For every link L and any $n \geq b(L)$, we have $w(n, L) < \infty$.*

So the exponent sum is (not surprisingly) too weak to distinguish infinitely many conjugacy classes of n-braid representatives.

Constructing infinitely many non-conjugate braids

For $n > b(L)$ (non-minimal braids), the first result is due to Morton:

Theorem 12 *(Morton '78) There is an infinite sequence of conjugacy classes of 4-braids β with closure $\hat{\beta} = \bigcirc$ (unknot).*

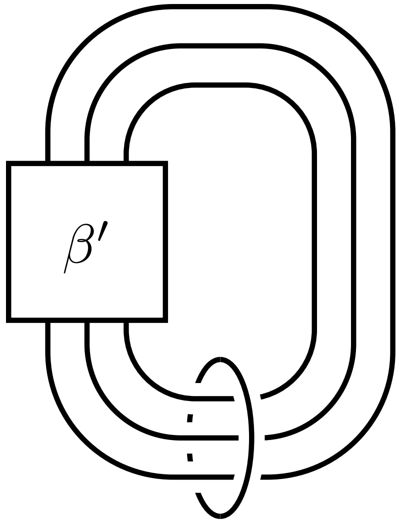
Then,. . . . . . ,knots (Shinjo):

Theorem 13 *(Shinjo '05)* K knot, $n \geq 4$, $\exists$ $n-1$*-strand braid* β *with closure* $\hat{\beta} = K \Rightarrow \exists$ *infinitely many non-conjugate* β' *of* n *strands with* $\hat{\beta}' = K$.

Proof. Conjugate β and stabilize

$$\beta' = \alpha\beta\alpha^{-1}\sigma_n \,. \tag{2}$$

Consider *axis addition link* of β':



Conjugate braids have the same axis addition link. If one can distinguish axis addition links by some link invariant, then braids are not conjugate. Use (the second lowest term of) the *Conway polynomial* (and choose α well). □

We have now, after several more steps,…

Theorem 14 *(S. '10) If* $\beta \in B_{n-1}$ *with* $L = \hat{\beta}$ *and* $n \geq 4$ *is not central, then* $c(n, L) = \infty$*:*

$$\{ \gamma \beta \gamma^{-1} \sigma_{n-1} \; : \gamma \in B_{1,n-1} \} \tag{3}$$

contains infinitely many non-conjugate braids.

Proof. Application of some Lie group approach showing the density of the image of braid representatives in unitary groups under the Burau and Lawrence-Krammer representation. □

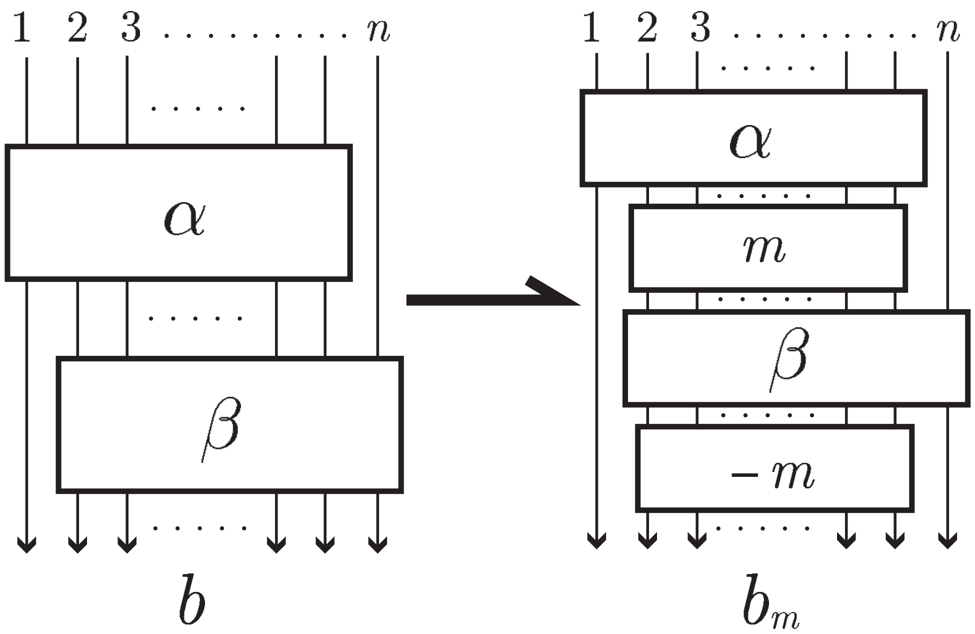
Exchange move

The case $n = b(L)$ of minimal conjugacy classes is *far* more difficult.

- Birman '69 conjectured $c(b(L), L) = 1$ for all L. Murasugi-Thomas '72 disproved this.
- But $c(b(L), L) = 1$ for $L =$ unlink (remark in problem 18) and ∃ further example (Ko-Lee '98).

- $c(b(L), L) \leq 2$ if $b(L) \leq 3$ follows from Theorem 10.
- So *some* links have $c(b(L), L) < \infty$. But many links have $c(b(L), L) = \infty$.

Birman-Menasco '92: *exchange move*,



(4)

Here m is some non-zero integer, and the boxes labeled $\pm m$ represent the full twists $\Delta_{[2,n-1]}^{\pm 2m}$ respectively, acting on the middle $n-2$ strands. (Thus a positive number of full twists are understood to be right full twists, and $-m$ full twists mean m full left-handed twists.)

Birman and Menasco proved that an exchange move necessarily underlies the switch between many conjugacy classes of braid representatives of L.

Theorem 15 (Birman-Menasco '92) *The n-braid representatives of a given link decompose into a finite number of classes under the combination of exchange moves and conjugacy.*

Theorem 16 (Shinjo-S. '18) *Let a link L have an n-braid representative β admitting an exchange move, such that the permutation $\pi(\beta)$ satisfies*

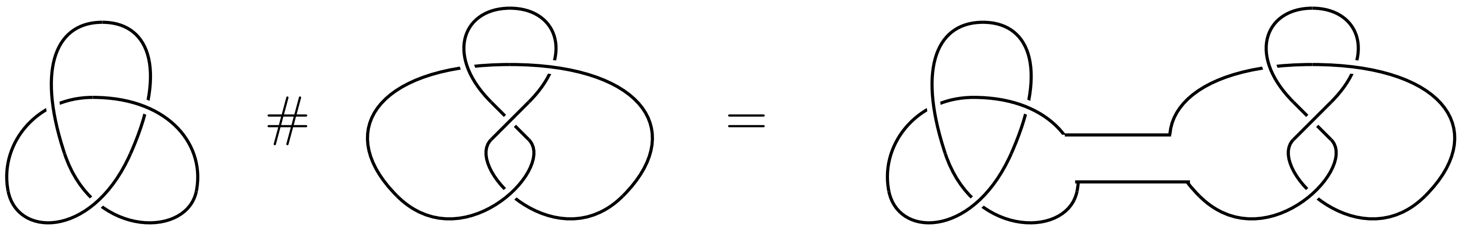
$$\pi(\beta)(1) \neq 1 \quad \textit{and} \quad \pi(\beta)(n) \neq n\,. \tag{5}$$

Then iterated exchange moves on β generate infinitely many non-conjugate braid representatives of L.

This follows Shinjo's method for Theorem 13.

Corollary 17 *Let L be a knot or a link without trivial components, and $n \geq 4$. Then L has infinitely many conjugacy classes of n-braid representatives if and only if it has one admitting an exchange move.*

E.g. L *composite* knot with $b(L) \geq 4$ has $c(b(L), L) = \infty$.



Open questions

Theorem 14 is the maximum one can get out of (2). It settles question 5 almost fully for $n > b(L)$. The most interesting missing case is:

Problem 18 *Let L be the trivial $n-1$-component link* $\underbrace{\bigcirc\bigcirc\cdots\bigcirc}_{n-1}$ *(*unlink*),* $n \geq 4$*. Is a braid representative $\beta' \in B_n$ of L conjugate to $\sigma_1^{\pm 1}$? (Remark: Birman-Menasco $\Rightarrow$ $\beta \in B_{n-1}$ is trivial.)*

Problem 19 *We don't know about* β *with* $n > b(L)$ *which are not conjugate to stabilizations (*Markov irreducible*): known only for* $n = 4$ *and* $L =$ *unknot (Morton, Fiedler).*

Thank you!

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